mathematics

MDPI

Article

On a Harry–Dym-Type Hierarchy: Trigonal Curve and Quasi-Periodic Solutions

Qi Feng [1,*] and Lihua Wu [2]

[1] College of Science, Zhongyuan University of Technology, Zhengzhou 450007, China
[2] School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China; wulihua@hqu.edu.cn
* Correspondence: 5777@zut.edu.cn

Abstract: Resorting to the characteristic polynomial of Lax matrix for a Harry–Dym-type hierarchy, we define a trigonal curve, on which appropriate vector-valued Baker–Akhiezer function and meromorphic function are introduced. With the help of the theory of trigonal curve and three kinds of Abelian differentials, we obtain the explicit Riemann theta function representations of the meromorphic function, from which we obtain the quasi-periodic solutions for the entire Harry–Dym-type hierarchy.

Keywords: Harry–Dym-type hierarchy; trigonal curve; quasi-periodic solutions

Citation: Feng, Q.; Wu, L. On a Harry–Dym-Type Hierarchy: Trigonal Curve and Quasi-Periodic Solutions. *Mathematics* **2022**, *10*, 314. https://doi.org/10.3390/math10030314

Academic Editor: Jan L. Cieśliński

Received: 16 December 2021
Accepted: 17 January 2022
Published: 20 January 2022

1. Introduction

The Harry–Dym equation

$$u_t = (u^{-\frac{1}{2}})_{xxx}, \tag{1}$$

was first discovered in an unpublished work by Harry–Dym [1] and rediscovered in a more general form within the classical string problem by Sabatier [2] and Li [3]. It was shown that the Harry–Dym equation admits many properties typical for soliton equations, such as inverse scattering transform, bi-Hamiltonian structure, and an infinite number of conservation laws and symmetries (see [4–7] and references therein). The quasi-periodic and involutive solutions of Harry–Dym equation were also discussed in [8–11]. Moreover, the integrable extensions of the Harry–Dym equation have attracted the attention of many researchers [12–21].

Konopelchenko and Dubrovsky [12,14] found the following Harry–Dym-type equation

$$v_t = (v^{-\frac{2}{3}})_{xxxxx}, \tag{2}$$

from the reduction of a 2 + 1 dimensional system. The authors in Refs. [22,23] derived a hierarchy of Harry–Dym-type equations and discussed their parametric solutions through the method of nonlinearization. Furthermore, the Harry–Dym-type equation can be linked with the Kaup–Kupershmidt or Sawada–Kotera equation by hodograph and Miura transformations [24]. The principal aim of the present paper is to study the algebro-geometric constructions and quasi-periodic solutions [25–29] of the Harry–Dym-type hierarchy, with the aid of the theory of trigonal curve [30,31].

The outline of the present paper is as follows. In Section 2, in view of the Lenard recursion equations and the zero-curvature equation, we derive the Harry–Dym-type hierarchy. In Section 3, we introduce the vector-valued Baker–Akhiezer function and the associated meromorphic function, from which a trigonal curve $\mathcal{K}_{m-1}$ of arithmetic genus $m-1$ is defined with the help of the characteristic polynomial of Lax matrix for the Harry–Dym-type hierarchy. After this, the Harry–Dym-type hierarchy is decomposed into a system of Dubrovin-type equations. In Section 4, by introducing three kinds of Abelian differentials, we present the Riemann theta function representations of the meromorphic function, and in particular, that of the potential for the entire Harry–Dym-type hierarchy.

2. Harry–Dym-Type Hierarchy

In this section, we shall recall the derivation of the Harry–Dym-type hierarchy, in which the first nontrivial member includes Equation (2). To this end, we introduce two sets of Lenard recursion equations

$$Kg_{j-1} = Jg_j, \quad j \geq 0, \tag{3}$$

$$K\hat{g}_{j-1} = J\hat{g}_j, \quad j \geq 0, \tag{4}$$

with two starting points

$$g_{-1} = \begin{pmatrix} v^{-\frac{2}{3}} \\ 0 \end{pmatrix}, \quad \hat{g}_{-1} = \begin{pmatrix} -\frac{1}{3}v^{-1}(v^{-\frac{1}{3}})_{xx} + \frac{1}{6}v^{-\frac{2}{3}}[(v^{-\frac{1}{3}})_x]^2 \\ v^{-\frac{1}{3}} \end{pmatrix},$$

and two operators are defined as

$$K = \begin{pmatrix} -\frac{1}{3}\partial^5 & 0 \\ 2\partial v + v\partial & \partial^3 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -(\partial v + 2v\partial) \\ 2\partial v + v\partial & \partial^3 \end{pmatrix}.$$

It is easy to see that

$$\ker J = \{\alpha_0 g_{-1} + \beta_0 \hat{g}_{-1} \mid \forall \alpha_0, \beta_0 \in \mathbb{R}\}.$$

In order to generate a hierarchy of Harry–Dym-type equations associated with the 3×3 matrix spectral problem [22,23]

$$\psi_x = U\psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda v & 0 & 0 \end{pmatrix}, \tag{5}$$

where $v(\neq 0)$ is a potential and λ is a constant spectral parameter, we solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \quad V = \lambda \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \tag{6}$$

which is equivalent to

$$\begin{aligned}
&V_{11,x} + \lambda v V_{13} - V_{21} = 0, \\
&V_{12,x} + V_{11} - V_{22} = 0, \\
&V_{13,x} + V_{12} - V_{23} = 0, \\
&V_{21,x} + \lambda v V_{23} - V_{31} = 0, \\
&V_{22,x} + V_{21} - V_{32} = 0, \\
&V_{23,x} + V_{22} - V_{33} = 0, \\
&V_{31,x} - \lambda v(V_{11} - V_{33}) = 0, \\
&V_{32,x} - \lambda v V_{12} + V_{31} = 0, \\
&V_{33,x} - \lambda v V_{13} + V_{32} = 0,
\end{aligned} \tag{7}$$

where each entry $V_{ij} = V_{ij}(a, b)$ is a Laurent expansion in λ:

$$\begin{aligned}
&V_{11} = -\tfrac{1}{3}\partial^2 a - \lambda\partial b, \;\; V_{12} = \partial a + \lambda b, \quad V_{13} = -2a, \\
&V_{21} = -\tfrac{1}{3}\partial^3 a - \lambda\partial^2 b - 2\lambda v a, \;\; V_{22} = \tfrac{2}{3}\partial^2 a, \;\; V_{23} = -\partial a + \lambda b, \\
&V_{31} = -\tfrac{1}{3}\partial^4 a + \lambda^2 v b, \;\; V_{32} = \tfrac{1}{3}\partial^3 a - \lambda\partial^2 b - 2\lambda v a, \;\; V_{33} = -\tfrac{1}{3}\partial^2 a + \lambda\partial b.
\end{aligned} \tag{8}$$

Substituting (8) into (7) and expanding the functions a and b into the Laurent series in λ

$$a = \sum_{j\geq 0} a_{j-1}\lambda^{-2j}, \quad b = \sum_{j\geq 0} b_{j-1}\lambda^{-2j}, \tag{9}$$

we obtain the recursion equations

$$KG_{j-1} = JG_j, \quad JG_{-1} = 0, \ \ j \geq 0, \tag{10}$$

with $G_j = (a_j, b_j)^T$. Since equation $JG_{-1} = 0$ has the general solution

$$G_{-1} = \alpha_0 g_{-1} + \beta_0 \hat{g}_{-1}, \tag{11}$$

G_j can be expressed as

$$G_j = \alpha_0 g_j + \beta_0 \hat{g}_j + \cdots + \alpha_j g_0 + \beta_j \hat{g}_0 + \alpha_{j+1} g_{-1} + \beta_{j+1} \hat{g}_{-1}, \quad j \geq 0, \tag{12}$$

where α_j and β_j are arbitrary constants. Let ψ satisfy the spectral problem (5) and an auxiliary problem

$$\psi_{t_r} = \widetilde{V}^{(r)} \psi, \quad \widetilde{V}^{(r)} = \lambda \begin{pmatrix} \widetilde{V}_{11}^{(r)} & \widetilde{V}_{12}^{(r)} & \widetilde{V}_{13}^{(r)} \\ \widetilde{V}_{21}^{(r)} & \widetilde{V}_{22}^{(r)} & \widetilde{V}_{23}^{(r)} \\ \widetilde{V}_{31}^{(r)} & \widetilde{V}_{32}^{(r)} & \widetilde{V}_{33}^{(r)} \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} &\widetilde{V}_{ij}^{(r)} = V_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)}), \quad \tilde{a}^{(r)} = \sum_{j=0}^{r} \tilde{a}_{j-1} \lambda^{2(r-j)}, \quad \tilde{b}^{(r)} = \sum_{j=0}^{r} \tilde{b}_{j-1} \lambda^{2(r-j)}, \\ &(\tilde{a}_j, \tilde{b}_j)^T = \tilde{\alpha}_0 g_j + \tilde{\beta}_0 \hat{g}_j + \cdots + \tilde{\alpha}_j g_0 + \tilde{\beta}_j \hat{g}_0 + \tilde{\alpha}_{j+1} g_{-1} + \tilde{\beta}_{j+1} \hat{g}_{-1}, \quad j \geq -1, \end{aligned} \tag{14}$$

and the constants $\tilde{\alpha}_j, \tilde{\beta}_j$ are independent of α_j, β_j. Then the compatibility condition of (5) and (13) yields the zero-curvature equation, $U_{t_r} - \widetilde{V}_x^{(r)} + [U, \widetilde{V}^{(r)}] = 0$, which is equivalent to

$$\begin{pmatrix} v_{t_r} \\ 0 \end{pmatrix} = K \begin{pmatrix} \tilde{a}_{r-1} \\ \tilde{b}_{r-1} \end{pmatrix} = J \begin{pmatrix} \tilde{a}_r \\ \tilde{b}_r \end{pmatrix}. \tag{15}$$

The first component in (15) gives rise to a hierarchy of nonlinear evolution equations

$$v_{t_r} = -\frac{1}{3} \partial^5 \tilde{a}_{r-1} = -(\partial v + 2v\partial) \tilde{b}_r, \tag{16}$$

in which the first nontrivial member is

$$v_{t_0} = -\frac{1}{3} \partial^5 \{ \tilde{\alpha}_0 v^{-\frac{2}{3}} + \tilde{\beta}_0 [-\frac{1}{3} v^{-1} (v^{-\frac{1}{3}})_{xx} + \frac{1}{6} v^{-\frac{2}{3}} (v^{-\frac{1}{3}})_x^2] \}. \tag{17}$$

If choosing $\tilde{\alpha}_0 = -3, \tilde{\beta}_0 = 0, t_0 = t$ in the Equation (17), it reduces to the Harry–Dym-type Equation (2).

Remark 1.

1. *The second component of* (15) *infers that* $\tilde{b}_r = (-2\partial^{-2} v - \partial^{-3} v \partial) \tilde{a}_r$. *Substituting it into* (16), *two Hamiltonian operators for the Harry–Dym-type hierarchy read as* $H_1 = \partial^5$ *and* $H_2 = \partial v \partial^{-3} v \partial + 4v\partial^{-1} v + 2v\partial^{-2} v \partial + 2\partial v \partial^{-2} v$, *which immediately gives recursion operator* $R = H_1 H_2^{-1}$.
2. *Here, we only show the first nontrivial member* (17) *of the Harry–Dym-type hierarchy* (16) *for* $r = 0$, *since the higher flows for* $r \geq 1$ *are very complicated. If you are interested, you can use some mathematical softwares to compute.*

3. The Trigonal Curve and Dubrovin-Type Equations

In this section, we shall introduce the vector-valued Baker–Akhiezer function, meromorphic function and trigonal curve for the Harry–Dym-type hierarchy. Then, we derive a system of Dubrovin-type differential equations.

We introduce the vector-valued Baker–Akhiezer function

$$\begin{aligned}
&\psi_x(P,x,x_0,t_r,t_{0,r}) = U(v,\lambda)\psi(P,x,x_0,t_r,t_{0,r}),\\
&\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{V}^{(r)}(v,\lambda)\psi(P,x,x_0,t_r,t_{0,r}),\\
&\lambda^{-1}V^{(n)}(v,\lambda)\psi(P,x,x_0,t_r,t_{0,r}) = y(P)\psi(P,x,x_0,t_r,t_{0,r}),\\
&\psi_1(P,x_0,x_0,t_{0,r},t_{0,r}) = 1, \quad P=(\lambda,y),\ x,t_r\in\mathbb{C}.
\end{aligned} \tag{18}$$

Here $V^{(n)} = \lambda\left(V_{ij}^{(n)}\right)_{3\times 3}$, $V_{ij}^{(n)} = V_{ij}(a^{(n)},b^{(n)})$, and

$$a^{(n)} = \sum_{j=0}^{n} a_{j-1}\lambda^{2(n-j)}, \quad b^{(n)} = \sum_{j=0}^{n} b_{j-1}\lambda^{2(n-j)}.$$

The compatibility conditions of the first three equations in (18) yield that

$$U_{t_r} - \widetilde{V}_x^{(r)} + [U,\widetilde{V}^{(r)}] = 0, \tag{19}$$

$$-V_x^{(n)} + [U,V^{(n)}] = 0, \tag{20}$$

$$-V_{t_r}^{(n)} + [\widetilde{V}^{(r)},V^{(n)}] = 0. \tag{21}$$

A direct calculation shows that $yI-\lambda^{-1}V^{(n)}$ satisfies (20) and (21), which ensures that the characteristic polynomial of Lax matrix $\lambda^{-1}V^{(n)}$ for the Harry-Dym-type, i.e., $\mathcal{F}_m(\lambda,y) = \det(yI-\lambda^{-1}V^{(n)})$, is a constant independent of variables x and t_r. Hence, $\mathcal{F}_m(\lambda,y)=0$ naturally leads to a trigonal curve

$$\mathcal{K}_{m-1}: \ \mathcal{F}_m(\lambda,y) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0, \tag{22}$$

where

$$\begin{aligned}
S_m(\lambda) &= \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{11}^{(n)} & V_{13}^{(n)} \\ V_{31}^{(n)} & V_{33}^{(n)} \end{vmatrix} + \begin{vmatrix} V_{22}^{(n)} & V_{23}^{(n)} \\ V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix},\\
T_m(\lambda) &= \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} & V_{13}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} & V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix} = \begin{cases} \beta_0^3\lambda^{6n+4}+\cdots, & \beta_0\neq 0, \alpha_0\in\mathbb{R},\\ -8\alpha_0^3\lambda^{6n+2}+\cdots, & \beta_0=0,\alpha_0\neq 0. \end{cases}
\end{aligned} \tag{23}$$

It is evident that $T_m(\lambda)$ is a polynomial of degree $m=6n+4=3(2n+1)+1$ and $m=6n+2=3(2n)+2$ as $\beta_0\neq 0$, $\alpha_0\in\mathbb{R}$ and $\beta_0=0,\alpha_0\neq 0$, respectively. We compactify the trigonal curve $\mathcal{K}_{m-1}$ to be a three-sheeted Riemann surface of arithmetic genus $m-1$, and still denote it by the same symbol $\mathcal{K}_{m-1}$.

A meromorphic function $\phi(P,x,t_r)$ on $\mathcal{K}_{m-1}$ is defined as

$$\phi(P) = \phi(P,x,t_r) = v^{-\frac{1}{3}}\frac{\psi_{1,x}(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})} = v^{-\frac{1}{3}}\frac{\psi_2(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})}, \quad P\in\mathcal{K}_{m-1}, \tag{24}$$

which, together with (18), infers that

$$\phi(P) = v^{-\frac{1}{3}}\frac{yV_{23}^{(n)}+C_m}{yV_{13}^{(n)}+A_m} = \frac{v^{-\frac{1}{3}}F_m}{y^2V_{23}^{(n)}-yC_m+D_m} = \frac{y^2V_{13}^{(n)}-yA_m+B_m}{v^{\frac{1}{3}}E_{m-1}}, \tag{25}$$

where

$$\begin{aligned}
A_m &= V_{12}^{(n)} V_{23}^{(n)} - V_{13}^{(n)} V_{22}^{(n)}, \quad C_m = V_{13}^{(n)} V_{21}^{(n)} - V_{11}^{(n)} V_{23}^{(n)}, \\
B_m &= V_{13}^{(n)} (V_{11}^{(n)} V_{33}^{(n)} - V_{13}^{(n)} V_{31}^{(n)}) + V_{12}^{(n)} (V_{11}^{(n)} V_{23}^{(n)} - V_{13}^{(n)} V_{21}^{(n)}), \\
D_m &= V_{23}^{(n)} (V_{22}^{(n)} V_{33}^{(n)} - V_{23}^{(n)} V_{32}^{(n)}) + V_{21}^{(n)} (V_{13}^{(n)} V_{22}^{(n)} - V_{12}^{(n)} V_{23}^{(n)}), \\
E_{m-1} &= (V_{13}^{(n)})^2 V_{32}^{(n)} + V_{12}^{(n)} V_{13}^{(n)} (V_{22}^{(n)} - V_{33}^{(n)}) - (V_{12}^{(n)})^2 V_{23}^{(n)}, \\
F_m &= (V_{23}^{(n)})^2 V_{31}^{(n)} + V_{21}^{(n)} V_{23}^{(n)} (V_{11}^{(n)} - V_{33}^{(n)}) - V_{13}^{(n)} (V_{21}^{(n)})^2.
\end{aligned} \tag{26}$$

Taking (22) and (25) into account, we arrive at some important identities among polynomials $A_m, B_m, C_m, D_m, E_{m-1}, F_m, S_m, T_m$:

$$\begin{aligned}
V_{13}^{(n)} F_m &= V_{23}^{(n)} D_m - (V_{23}^{(n)})^2 S_m - C_m^2, \\
A_m F_m &= (V_{23}^{(n)})^2 T_m + C_m D_m,
\end{aligned} \tag{27}$$

$$\begin{aligned}
V_{23}^{(n)} E_{m-1} &= V_{13}^{(n)} B_m - (V_{13}^{(n)})^2 S_m - A_m^2, \\
C_m E_{m-1} &= (V_{13}^{(n)})^2 T_m + A_m B_m,
\end{aligned} \tag{28}$$

$$\begin{aligned}
& V_{23}^{(n)} B_m + V_{13}^{(n)} D_m - V_{13}^{(n)} V_{23}^{(n)} S_m + A_m C_m = 0, \\
& V_{13}^{(n)} V_{23}^{(n)} T_m + V_{23}^{(n)} A_m S_m + V_{13}^{(n)} C_m S_m - B_m C_m - A_m D_m = 0, \\
& V_{23}^{(n)} A_m T_m + V_{13}^{(n)} C_m T_m - B_m D_m + E_{m-1} F_m = 0,
\end{aligned} \tag{29}$$

$$\begin{aligned}
E_{m-1,x} &= -2 V_{13}^{(n)} S_m + 3 B_m, \\
V_{23}^{(n)} F_{m,x} &= -3 V_{22}^{(n)} F_m + V_{21}^{(n)} (2 V_{23}^{(n)} S_m - 3 D_m).
\end{aligned} \tag{30}$$

In what follows, we shall present some properties of the meromorphic function $\phi(P, x, t_r)$ without proofs.

$$[v^{\frac{1}{3}} \phi(P)]_{xx} + 3 v^{\frac{1}{3}} \phi(P) [v^{\frac{1}{3}} \phi(P)]_x + v \phi^3(P) = \lambda v, \tag{31}$$

$$[v^{\frac{1}{3}} \phi(P)]_{t_r} = \lambda \Big(\widetilde{V}_{11}^{(r)} + v^{\frac{1}{3}} \widetilde{V}_{12}^{(r)} \phi(P) + \widetilde{V}_{13}^{(r)} [(v^{\frac{1}{3}} \phi(P))_x + v^{\frac{2}{3}} \phi^2(P)] \Big)_x, \tag{32}$$

$$v \phi(P) \phi(P^*) \phi(P^{**}) = -\frac{F_m}{E_{m-1}}, \tag{33}$$

$$v^{\frac{1}{3}} [\phi(P) + \phi(P^*) + \phi(P^{**})] = \frac{E_{m-1,x}}{E_{m-1}}, \tag{34}$$

$$\tfrac{1}{\phi(P)} + \tfrac{1}{\phi(P^*)} + \tfrac{1}{\phi(P^{**})} = v^{\frac{1}{3}} \Big[-3 \tfrac{V_{22}^{(n)}}{V_{21}^{(n)}} - \tfrac{V_{23}^{(n)} F_{m,x}}{V_{21}^{(n)} F_m} \Big]. \tag{35}$$

Here $P = (\lambda, y)$, $P^* = (\lambda, y^*)$, $P^{**} = (\lambda, y^{**})$, and y, y^*, y^{**} denote three branches of y satisfying $\mathcal{F}_m(\lambda, y) = 0$.

Lemma 1. *Assume that* (18) *and* (19) *hold, and let* $(\lambda, x, t_r) \in \mathbb{C}^3$. *Then*

$$E_{m-1,t_r} = \lambda E_{m-1,x} [\widetilde{V}_{12}^{(r)} - \tfrac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{12}^{(n)}] + 3 \lambda E_{m-1} [\widetilde{V}_{11}^{(r)} - \tfrac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{11}^{(n)}], \tag{36}$$

$$F_{m,t_r} = \lambda F_{m,x} [\widetilde{V}_{23}^{(r)} - \tfrac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}} V_{23}^{(n)}] + 3 \lambda F_m [\widetilde{V}_{22}^{(r)} - \tfrac{\widetilde{V}_{21}^{(r)}}{V_{21}^{(n)}} V_{22}^{(n)}]. \tag{37}$$

Proof. Differentiating (34) with respect to t_r and using (32) and (34), we have

$$\partial_x \partial_{t_r} (\ln E_{m-1}) = \lambda \big[(\widetilde{V}_{12}^{(r)} - \tfrac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{12}^{(n)}) \tfrac{E_{m-1,x}}{E_{m-1}} + 3 (\widetilde{V}_{11}^{(r)} - \tfrac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}} V_{11}^{(n)}) \big]_x.$$

Integrating the above equation with respect to x, and choosing the integration constant as zero to imply (36). Differentiating (33) with respect to t_r, an analogous process shows (37). □

By observing (12) and (26), we find that E_{m-1} and F_m are polynomials with respect to λ of degree $m-1$ and m, respectively. Therefore,

$$E_{m-1}(\lambda, x, t_r) = -\epsilon(m)v^{-1}\prod_{j=1}^{m-1}(\lambda - \mu_j(x, t_r)), \tag{38}$$

$$F_m(\lambda, x, t_r) = \epsilon(m)\prod_{l=0}^{m-1}(\lambda - \nu_l(x, t_r)), \tag{39}$$

with

$$\epsilon(m) = \begin{cases} \beta_0^3, & m = 6n+4, \\ 8\alpha_0^3, & m = 6n+2. \end{cases}$$

Let us denote

$$\begin{aligned} \hat{\mu}_j(x, t_r) &= \Big(\mu_j(x, t_r), y(\hat{\mu}_j(x, t_r))\Big) \\ &= \Big(\mu_j(x, t_r), -\tfrac{A_m(\mu_j(x,t_r),x,t_r)}{V_{13}^{(n)}(\mu_j(x,t_r),x,t_r)}\Big) \in \mathcal{K}_{m-1}, \;\; 1 \le j \le m-1, \end{aligned} \tag{40}$$

$$\begin{aligned} \hat{\nu}_l(x, t_r) &= \Big(\nu_l(x, t_r), y(\hat{\nu}_l(x, t_r))\Big) \\ &= \Big(\nu_l(x, t_r), -\tfrac{C_m(\nu_l(x,t_r),x,t_r)}{V_{23}^{(n)}(\nu_l(x,t_r),x,t_r)}\Big) \in \mathcal{K}_{m-1}, \;\; 0 \le l \le m-1, \end{aligned} \tag{41}$$

then, it is easy to see that the following Lemma holds.

Lemma 2. *Suppose the zeros $\{\mu_j(x, t_r)\}_{j=1}^{m-1}$ and $\{\nu_l(x, t_r)\}_{l=0}^{m-1}$ of $E_{m-1}(\lambda, x, t_r)$ and $F_m(\lambda, x, t_r)$ remain distinct for $(x, t_r) \in \Omega_\mu$ and $(x, t_r) \in \Omega_\nu$, respectively, where $\Omega_\mu, \Omega_\nu \subseteq \mathbb{C}^2$ are open and connected. Then, $\{\mu_j(x, t_r)\}_{j=1}^{m-1}$ and $\{\nu_l(x, t_r)\}_{l=0}^{m-1}$ satisfy the Dubrovin-type equations*

$$\mu_{j,x} = \frac{vV_{13}^{(n)}(\mu_j)[3y^2(\hat{\mu}_j)+S_m(\mu_j)]}{\epsilon(m)\prod_{k=1, k\neq j}^{m-1}(\mu_j-\mu_k)}, \quad 1 \le j \le m-1, \tag{42}$$

$$\mu_{j,t_r} = \frac{v\mu_j[V_{13}^{(n)}(\mu_j)\widetilde{V}_{12}^{(r)}(\mu_j)-\widetilde{V}_{13}^{(r)}(\mu_j)V_{12}^{(n)}(\mu_j)][3y^2(\hat{\mu}_j)+S_m(\mu_j)]}{\epsilon(m)\prod_{k=1, k\neq j}^{m-1}(\mu_j-\mu_k)}, \quad 1 \le j \le m-1, \tag{43}$$

$$\nu_{l,x} = \frac{V_{21}^{(n)}(\nu_l)[3y^2(\hat{\nu}_l)+S_m(\nu_l)]}{\epsilon(m)\prod_{k=0, k\neq l}^{m-1}(\nu_l-\nu_k)}, \quad 0 \le l \le m-1, \tag{44}$$

$$\nu_{l,t_r} = \frac{\nu_l[V_{21}^{(n)}(\nu_l)\widetilde{V}_{23}^{(r)}(\nu_l)-\widetilde{V}_{21}^{(r)}(\nu_l)V_{23}^{(n)}(\nu_l)][3y^2(\hat{\nu}_l)+S_m(\nu_l)]}{\epsilon(m)\prod_{k=0, k\neq l}^{m-1}(\nu_l-\nu_k)}, \quad 0 \le l \le m-1. \tag{45}$$

Proof. We just need to prove that (42) for the proofs of (43)–(45) are similar to (42). Substituting $\lambda = \mu_j$ into the first expression in (30), and using (28) and (41), we obtain

$$E_{m-1,x}(\mu_j, x, t_r) = V_{13}^{(n)}(\mu_j, x, t_r)[3y^2(\hat{\mu}_j) + S_m(\mu_j)]. \tag{46}$$

On the other hand, differentiating (38) with respect to x and inserting $\lambda = \mu_j$ into it give rise to

$$E_{m-1,x}(\mu_j, x, t_r) = \epsilon(m)v^{-1}\mu_{j,x}\prod_{k=1, k\neq j}^{m-1}(\mu_j - \mu_k). \tag{47}$$

A comparison of (46) and (47) yields (42). □

4. Quasi-Periodic Solutions

In this section, we shall derive Riemann theta function representations of the meromorphic function and potentials for the entire Harry–Dym-type hierarchy.

Equip the Riemann surface $\mathcal{K}_{m-1}$ with the canonical basis of cycles $\{\mathfrak{a}_j, \mathfrak{b}_j\}_{j=1}^{m-1}$, which admits intersection numbers

$$\mathfrak{a}_j \circ \mathfrak{b}_k = \delta_{j,k}, \quad \mathfrak{a}_j \circ \mathfrak{a}_k = 0, \quad \mathfrak{b}_j \circ \mathfrak{b}_k = 0, \quad j,k = 1, \cdots, m-1,$$

and the basis of holomorphic differentials

$$\begin{aligned} \widetilde{\omega}_l(P) &= \tfrac{1}{3y^2(P)+S_m(\lambda)} \begin{cases} \lambda^{l-1} d\lambda, & 1 \leq l \leq m-2n-2, \\ y(P)\lambda^{l+2n-m+1} d\lambda, & m-2n-1 \leq l \leq m-1, \end{cases} \quad m = 6n+4, \\ \widetilde{\omega}_l(P) &= \tfrac{1}{3y^2(P)+S_m(\lambda)} \begin{cases} \lambda^{l-1} d\lambda, & 1 \leq l \leq m-2n-1, \\ y(P)\lambda^{l+2n-m} d\lambda, & m-2n \leq l \leq m-1, \end{cases} \quad m = 6n+2. \end{aligned} \tag{48}$$

Thus, the period matrices A and B constructed by

$$A_{jk} = \int_{\mathfrak{a}_k} \widetilde{\omega}_j, \qquad B_{jk} = \int_{\mathfrak{b}_k} \widetilde{\omega}_j, \tag{49}$$

are invertible. Defining the matrix $C = A^{-1}, \tau = CB$, the Riemannian bilinear relation makes it possible to verify that the matrix τ is symmetric $(\tau_{jk} = \tau_{kj})$ and has a positive definite imaginary part (Im $\tau > 0$) [32,33]. If we normalize $\widetilde{\omega} = (\widetilde{\omega}_1, \cdots, \widetilde{\omega}_{m-1})$ into new basis $\omega = (\omega_1, \cdots, \omega_{m-1})$

$$\omega_j = \sum_{l=1}^{m-1} C_{jl} \widetilde{\omega}_l, \tag{50}$$

then, we have

$$\int_{\mathfrak{a}_k} \omega_j = \delta_{jk}, \quad \int_{\mathfrak{b}_k} \omega_j = \tau_{jk}, \quad j,k = 1, \cdots, m-1.$$

The Riemann theta function $\theta(\underline{z})$ [32,33] on $\mathcal{K}_{m-1}$ is defined as

$$\theta(\underline{z}) = \sum_{\underline{N} \in \mathbb{Z}^{m-1}} \exp(2\pi i < \underline{z}, \underline{N} > + \pi i < \tau \underline{N}, \underline{N} >), \quad \underline{z} \in \mathbb{C}^{m-1},$$

where $< \cdot, \cdot >$ stands for the Euclidean scalar product.

Let $\omega^{(2)}_{P_\infty,2}(P)$ denote the normalized Abelian differential of the second kind, which is holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty\}$ with a pole of order 2 at P_∞ and satisfies

$$\int_{\mathfrak{a}_j} \omega^{(2)}_{P_\infty,2}(P) = 0, \quad j = 1, \cdots, m-1, \tag{51}$$

$$\omega^{(2)}_{P_\infty,2}(P) = (\zeta^{-2} + O(1)) d\zeta, \quad P \to P_\infty, \tag{52}$$

where $\zeta = \lambda^{-\frac{1}{3}}$ is a local coordinate near P_∞. The b-periods of the differential $\omega^{(2)}_{P_\infty,2}$ are denoted by

$$U_2^{(2)} = (U^{(2)}_{2,1}, \cdots, U^{(2)}_{2,m-1}), \quad U^{(2)}_{2,j} = \frac{1}{2\pi i} \int_{\mathfrak{b}_j} \omega^{(2)}_{P_\infty,2}(P). \tag{53}$$

Furthermore, let $\omega^{(3)}_{P_\infty,\hat{v}_0(x,t_r)}(P)$ denote the normalized Abelian differential of the third kind defined by

$$\omega^{(3)}_{P_\infty,\hat{v}_0(x,t_r)}(P) = -\frac{y^2(P) + 2y^2(\hat{v}_0(x,t_r)) + S_m(v_0(x,t_r))}{\lambda - v_0(x,t_r)} \frac{d\lambda}{3y^2(P) + S_m(\lambda)} + \sum_{j=1}^{m-1} \gamma_j \omega_j, \tag{54}$$

which is holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty, \hat{\nu}_0(x,t_r)\}$ and has simple poles at P_∞ and $\hat{\nu}_0(x,t_r)$, with corresponding residues $+1$ and -1. The constants $\{\gamma_j\}_{j=1}^{m-1}$ are determined by the normalization condition

$$\int_{\mathfrak{a}_j} \omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P) = 0, \quad j = 1, \cdots, m-1. \tag{55}$$

In terms of the local coordinate $\zeta = \lambda^{-\frac{1}{3}}$ near P_∞ and $\zeta = \lambda - \nu_0(x,t_r)$ near $\hat{\nu}_0(x,t_r)$, respectively, a direct calculation shows that

$$\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P) \underset{\zeta\to 0}{=} \begin{cases} (\zeta^{-1} - \delta(m) + O(\zeta))d\zeta, & P \to P_\infty, \\ (-\zeta^{-1} + O(1))d\zeta, & P \to \hat{\nu}_0(x,t_r), \end{cases} \tag{56}$$

with

$$\delta(m) = \begin{cases} \frac{1}{\beta_0}\sum\limits_{j=1}^{m-1} \gamma_j C_{j,m-1}, & m = 6n+4, \\ \frac{1}{4\alpha_0^2}\sum\limits_{j=1}^{m-1} \gamma_j C_{j,4n+1}, & m = 6n+2. \end{cases} \tag{57}$$

Then,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P) \underset{\zeta\to 0}{=} \begin{cases} \ln\zeta + e^{(3)}_\infty(Q_0) - \delta(m)\zeta + O(\zeta^2), & P \to P_\infty, \\ -\ln\zeta + e^{(3)}_0(Q_0) + O(\zeta), & P \to \hat{\nu}_0(x,t_r), \end{cases} \tag{58}$$

with Q_0 as a chosen base point on $\mathcal{K}_{m-1} \setminus \{P_\infty, \hat{\nu}_0(x,t_r)\}$ and $e^{(3)}_\infty(Q_0), e^{(3)}_0(Q_0)$ two integration constants.

Let $\mathcal{T}_{m-1} = \{\underline{N} + \tau\underline{L}, \underline{N}, \underline{L} \in \mathbb{Z}^{m-1}\}$ be a period lattice. The complex torus $\mathcal{J}_{m-1} = \mathbb{C}^{m-1}/\mathcal{T}_{m-1}$ is called a Jacobian variety of $\mathcal{K}_{m-1}$. The Abelian mapping $\mathcal{A} : \mathcal{K}_{m-1} \to \mathcal{J}_{m-1}$ is defined as

$$\mathcal{A}(P) = \Big(\mathcal{A}_1(P), \cdots, \mathcal{A}_{m-1}(P)\Big) = \Big(\int_{Q_0}^{P} \omega_1, \cdots, \int_{Q_0}^{P} \omega_{m-1}\Big) \pmod{\mathcal{T}_{m-1}},$$

and is extended linearly to the divisor group $\mathrm{Div}(\mathcal{K}_{m-1})$

$$\mathcal{A}(\sum_k n_k P_k) = \sum_k n_k \mathcal{A}(P_k),$$

which enables us to give

$$\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\mu}_k(x,t_r)} \omega, \;\; \mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \sum_{k=1}^{m-1} \int_{Q_0}^{\hat{\nu}_k(x,t_r)} \omega, \tag{59}$$

where $\underline{\hat{\mu}}(x,t_r) = (\hat{\mu}_1(x,t_r), \cdots, \hat{\mu}_{m-1}(x,t_r)), \underline{\hat{\nu}}(x,t_r) = (\hat{\nu}_1(x,t_r), \cdots, \hat{\nu}_{m-1}(x,t_r)), \mathcal{D}_{\underline{\hat{\mu}}(x,t_r)} = \sum\limits_{k=1}^{m-1} \hat{\mu}_k(x,t_r)$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)} = \sum\limits_{k=1}^{m-1} \hat{\nu}_k(x,t_r)$.

Taking the local coordinate $\zeta = \lambda^{-\frac{1}{3}}$ near $P_\infty \in \mathcal{K}_{m-1}$ in (31), the Laurent series of $\phi(P,x,t_r)$ can be explicitly expressed as

$$\phi(P,x,t_r) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty} \kappa_j(x,t_r)\zeta^j, \qquad P \to P_\infty, \tag{60}$$

where

$$\kappa_0=1,\quad \kappa_1=(v^{-\frac{1}{3}})_x,\quad \kappa_2=\tfrac{2}{9}v^{-\frac{5}{3}}v_{xx}-\tfrac{7}{27}v^{-\frac{8}{3}}v_x^2,\quad \kappa_3=\tfrac{1}{3}v^{-\frac{2}{3}}(v^{-\frac{1}{3}})_{xxx},$$
$$\kappa_j=-\frac{1}{3}[v^{-1}(v^{\frac{1}{3}}\kappa_{j-2})_{xx}+3v^{-\frac{2}{3}}\sum_{i=0}^{j-1}\kappa_{j-1-i}(v^{\frac{1}{3}}\kappa_i)_x+\sum_{i=1}^{j-1}\kappa_i\kappa_{j-i}$$
$$+\sum_{i=1}^{j-1}\sum_{l=0}^{j-i}\kappa_i\kappa_l\kappa_{j-i-l}],\quad (j\geq 2).$$

Taken together with (25), Equation (60) shows that the divisor $(\phi(P,x,t_r))$ of $\phi(P,x,t_r)$ is

$$(\phi(P,x,t_r))=\mathcal{D}_{\hat{v}_0(x,t_r),\hat{v}_1(x,t_r),\cdots,\hat{v}_{m-1}(x,t_r)}(P)-\mathcal{D}_{P_\infty,\hat{\mu}_1(x,t_r),\cdots,\hat{\mu}_{m-1}(x,t_r)}(P), \tag{61}$$

which implies that $\hat{v}_0(x,t_r),\hat{v}_1(x,t_r),\cdots,\hat{v}_{m-1}(x,t_r)$ are m zeros and $P_\infty,\hat{\mu}_1(x,t_r),\cdots,\hat{\mu}_{m-1}(x,t_r)$ are m poles of $\phi(P,x,t_r)$. In view of (58) and (60), the Riemann and Riemann–Roch theorems ensure that the following theorem holds.

Theorem 1. *Let the curve $\mathcal{K}_{m-1}$ be nonsingular, $P=(\lambda,y)\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$, and (x,t_r), $(x_0,t_{0,r})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{v}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then, $\phi(P,x,t_r)$ may be explicitly constructed by the formula*

$$\phi(P,x,t_r)=\frac{\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}\exp\left(e_\infty^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,\hat{v}_0(x,t_r)}\right), \tag{62}$$

where M is the Riemann constant vector.

Based on the above results, we will obtain the Riemann theta function representations of solutions for the entire Harry–Dym-type hierarchy immediately.

Theorem 2. *Assume that the curve $\mathcal{K}_{m-1}$ is nonsingular and let $(x,t_r)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{v}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then, the Harry–Dym-type hierarchy admits quasi-periodic solutions*

$$(v^{-\frac{1}{3}})_x=\partial_{U_2^{(2)}}\ln\frac{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))}+\delta(m), \tag{63}$$

where $\partial_{U_2^{(2)}}=\sum_{j=1}^{m-1}U_{2,j}^{(2)}\frac{\partial}{\partial z_j}$, and $\delta(m)$ are defined in (57).

Proof. A direct calculation gives the following asymptotic expressions near P_∞ $(P\to P_\infty)$

$$\frac{\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}\underset{\zeta\to0}{=}1-[\partial_{U_2^{(2)}}\ln\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))]\zeta+O(\zeta^2),$$
$$\frac{\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))}\underset{\zeta\to0}{=}1-[\partial_{U_2^{(2)}}\ln\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))]\zeta+O(\zeta^2),$$

which, together with (58) and (62), imply that

$$\phi(P,x,t_r)\underset{\zeta\to0}{=}[\zeta^{-1}+\partial_{U_2^{(2)}}\ln\frac{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{v}}(x,t_r)}))}+\delta(m)+O(\zeta)],\quad P\to P_\infty. \tag{64}$$

Comparing (60) with (64), we arrive at (63). □

Next, we shall give 1-genus and 3-genus quasi-periodic solutions of the Equation (16) for $n=0$.

(i) $\alpha_0\neq0,\beta_0=0$, through direct calculation, we have

$$
\begin{aligned}
&m=2,\ \mathcal{K}_1:\mathcal{F}_1=y^3+\eta_1 y+8\alpha_0^3\lambda^2+\eta_2=0,\ g=1,\\
&E_1=-8\alpha_0^3 v^{-1}(x,t_r)(\lambda-\mu_1(x,t_r)),\ F_2=8\alpha_0^3(\lambda-\nu_0(x,t_r))(\lambda-\nu_1(x,t_r)),\\
&\widetilde{\omega}_1=\tfrac{d\lambda}{3y^2(P)+\eta_1},\ \omega_1=C_{11}\widetilde{\omega}_1,\ \mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)})=\int_{Q_0}^{\hat{\nu}_1(x,t_r)}\omega_1,\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})=\int_{Q_0}^{\hat{\mu}_1(x,t_r)}\omega_1,\\
&\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P)=-\tfrac{y^2(P)+2y^2(\hat{\nu}_0(x,t_r))+\eta_1}{\lambda-\nu_0(x,t_r)}\tfrac{d\lambda}{3y^2(P)+\eta_1}+\gamma_1\omega_1,\\
&\phi(P,x,t_r)=\tfrac{\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}\exp\left(e_\infty^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}\right),\\
&[v^{-\frac{1}{3}}(x,t_r)]_x=\partial_{U_2^{(2)}}\ln\tfrac{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))}+\tfrac{\gamma_1 C_{11}}{4\alpha_0^2},
\end{aligned}
$$

where η_1,η_2 are arbitrary constants and γ_1 is determined by $\int_{\mathfrak{a}_1}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P)=0$.

(ii) $\alpha_0\in\mathbb{R},\beta_0\neq 0$, tedious computing indicates that

$$
\begin{aligned}
&m=4,\ \mathcal{K}_4:\mathcal{F}_4=y^3+(6\alpha_0\beta_0\lambda^2+\eta_1)y-\beta_0\lambda^4+8\alpha_0^3\lambda^2+\eta_2=0,\ g=3,\\
&E_3=-\beta_0^3 v^{-1}(x,t_r)\prod_{j=1}^{3}(\lambda-\mu_j(x,t_r)),\ F_4=\beta_0^3\prod_{j=1}^{4}(\lambda-\nu_j(x,t_r)),\\
&\widetilde{\omega}_1=\tfrac{d\lambda}{3y^2(P)+6\alpha_0\beta_0\lambda^2+\eta_1},\ \widetilde{\omega}_2=\tfrac{\lambda d\lambda}{3y^2(P)+6\alpha_0\beta_0\lambda^2+\eta_1},\ \widetilde{\omega}_3=\tfrac{y(P)d\lambda}{3y^2(P)+6\alpha_0\beta_0\lambda^2+\eta_1},\\
&\omega_j=\sum_{l=1}^{3}C_{jl}\widetilde{\omega}_l,\ j=1,2,3,\ \mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)})=\sum_{k=1}^{3}\int_{Q_0}^{\hat{\nu}_k(x,t_r)}\omega,\ \mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})=\sum_{k=1}^{3}\int_{Q_0}^{\hat{\mu}_1(x,t_r)}\omega,\\
&\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P)=-\tfrac{y^2(P)+2y^2(\hat{\nu}_0(x,t_r))+6\alpha_0\beta_0\nu_0^2(x,t_r)+\eta_1}{\lambda-\nu_0(x,t_r)}\tfrac{d\lambda}{3y^2(P)+6\alpha_0\beta_0\lambda^2+\eta_1}+\sum_{j=1}^{3}\gamma_j\omega_j,\\
&\phi(P,x,t_r)=\tfrac{\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))\theta(M-\mathcal{A}(P)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}\exp\left(e_\infty^{(3)}(Q_0)-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}\right),\\
&[v^{-\frac{1}{3}}(x,t_r)]_x=\partial_{U_2^{(2)}}\ln\tfrac{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(M-\mathcal{A}(P_\infty)+\mathcal{A}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}))}+\tfrac{1}{\beta_0}\sum_{j=1}^{3}\gamma_j C_{j3},
\end{aligned}
$$

where η_1,η_2 are arbitrary constants, $\omega=(\omega_1,\omega_2,\omega_3)$, and $\gamma_j, j=1,2,3$, are determined by $\int_{\mathfrak{a}_j}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x,t_r)}(P)=0$.

Author Contributions: Conceptualization, Q.F. and L.W.; methodology, Q.F. and L.W.; software, Q.F.; validation, Q.F. and L.W.; formal analysis, L.W.; investigation, Q.F. and L.W.; resources, Q.F. and L.W.; data curation, Q.F.; writing—original draft preparation, Q.F.; writing—review and editing, L.W.; visualization, Q.F.; supervision, L.W.; project administration, Q.F. and L.W.; funding acquisition, Q.F. and L.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by National Natural Science Foundation of China (project Nos. 11701595, 11871232), Youth Innovation Foundation of Xiamen (Project No. 3502Z20206011).

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kruskal, M.D. *Nonlinear Wave Equations, Dynomical Systems, Theory and Applications*; Moser, J., Ed.; Lecture Notes in Physics; Springer: Berlin, Germany, 1975; Volume 38, pp. 310–354.
2. Sabatier, P.C. On some spectral problems and isospectral evolutions connected with the classical string problem. *Lett. Nuovo Cim.* **1979**, *26*, 477–482. [CrossRef]
3. Li, Y.S. Evolution equations associated with the eigenvalue problem based on the equation $\phi_{xx}=[u(x)-k^2\varrho^2(x)]\phi$. *Lett. Nuovo Cim.* **1982**, *70*, 1–12.
4. Magri, F. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.* **1978**, *19*, 1156–1162. [CrossRef]
5. Wadati, M.; Ichikawa, Y.H.; Shimizu, T. Cusp soliton of a new integrable nonlinear evolution equation. *Prog. Theor. Phys.* **1980**, *64*, 1959–1967. [CrossRef]
6. Calogero, F.; Degasperis, A. *Spectral Transform and Solitons I*; Elsevier: Amsterdam, The Netherlands, 1982.
7. Konopelchenko, B.G.; Lee, J.-H. Inverse spectral transform for the Harry Dym equation on the complex plane. *Phys. D* **1995**, *81*, 32–43. [CrossRef]

8. Dmitrieva, L.A. Finite-gap solutions of the Harry Dym equation. *Phys. Lett. A* **1993**, *182*, 65–70. [CrossRef]
9. Novikov, D.P. Algebraic-geometric solutions of the Harry Dym equation. *Siberian Math. J.* **1999**, *40*, 136–140. [CrossRef]
10. Li, Z. Algebro-geometric solutions of the Harry Dym hierarchy. *Int. J. Nonlinear Sci. Numer. Simul.* **2017**, *18*, 129–136. [CrossRef]
11. Qiao, Z.J. A completely integrable system assocaited with the Harry-Dym hierarchy. *J. Nonlinear Math. Phys.* **1994**, *1*, 65–74. [CrossRef]
12. Konopelchenko, B.G.; Dubrovsky, V.G. Some new integrable nonlinear evolution equations in 2+1 dimensions. *Phys. Lett. A* **1984**, *102*, 15–17. [CrossRef]
13. Antonowicz, M.; Fordy, A.P. Factorisation of energy dependent Schrödinger operators: Miura maps and modified systems. *Comm. Math. Phys.* **1989**, *124*, 465–486. [CrossRef]
14. Konopelchenko, B.G.; Oevel, W. An r-matrix approach to nonstandard classes of integrable equations. *Publ. Rims Kyoto Univ.* **1993**, *29*, 581–666. [CrossRef]
15. Antonowicz, M.; Fordy, A.P. Coupled Harry Dym equations with multi-Hamiltonian structures. *J. Phys. A Math. Gen.* **1998**, *21*, L269–L275. [CrossRef]
16. Popowicz, Z. The generalized Harry Dym equation. *Phys. Lett. A* **2003**, *317*, 260–264. [CrossRef]
17. Brunelli, J.C.; da Costa, G.A.T.F. On the nonlocal equations and nonlocal charges associated with the Harry Dym hierarchy. *J. Math. Phys.* **2002**, *43*, 6116–6128. [CrossRef]
18. Marciniak, K.; Błaszak, M. Construction of coupled Harry Dym hierarchy and its solutions from Stäckel systems. *Nonlinear Anal.* **2010**, *73*, 3004–3017. [CrossRef]
19. Błaszak, M.; Marciniak, K. Invertible coupled KdV and coupled Harry Dym hierarchies. *Stud. Appl. Math.* **2012**, *131*, 211–228. [CrossRef]
20. Tian, K.; Popowicz, Z.; Liu, Q.P. A non-standard Lax formulation of the Harry Dym hierarchy and its supersymmetric extension. *J. Phys. A Math. Theor.* **2012**, *45*, 122001. [CrossRef]
21. Ma, W.X. An extended Harry Dym hierarchy. *J. Phys. A* **2010**, *43*, 165202. [CrossRef]
22. Geng, X.G. The Hamiltonian structure and new finite-dimensional integrable system associated with Harry-Dym type equations. *Phys. Lett. A* **1994**, *194*, 44–48. [CrossRef]
23. Qiao, Z.J.; Li, S.T. A new integrable hierarchy, parametric solutions and traveling wave solutions. *Math. Phys. Anal. Geom.* **2004**, *7*, 289–308. [CrossRef]
24. Das, A.; Popowicz, Z. A nonlinearly dispersive fifth order integrable equation and its hierarchy. *J. Nonlinear Math. Phys.* **2005**, *12*, 105–117. [CrossRef]
25. Belokolos, E.D.; Bobenko, A.I.; Enol'skii, V.Z.; Its, A.R.; Matveev, V.B. *Algebro-Geometric Approach to Nonlinear Integrable Equations*; Springer: Berlin, Germany, 1994.
26. Cao, C.W.; Wu, Y.T.; Geng, X.G. Relation between the Kadomtsev-Petviashvili equation and the confocal involutive system. *J. Math. Phys.* **1999**, *40*, 3948–3970. [CrossRef]
27. Gesztesy, F.; Holden, H. *Soliton Equations and Their Algebro-Geometric Solutions*; Cambridge University Press: Cambridge, UK, 2003.
28. Dickson, R.; Gesztesy, F.; Unterkofler, K. Algebro-geometric solutions of the Boussinesq hierarchy. *Rev. Math. Phys.* **1999**, *11*, 823–879. [CrossRef]
29. Geng, X.G.; Wu, L.H.; He, G.L. Algebro-geometric constructions of the modified Boussinesq flows and quasi-periodic solutions. *Phys. D* **2011**, *240*, 1262–1288. [CrossRef]
30. Eilbeck, J.C.; Enolskii, V.Z.; Matsutani, S.; Ônishi, Y.; Previato, E. Abelian functions for trigonal curves of genus three. *Int. Math. Res. Not.* **2007**, *2007*, 38. [CrossRef]
31. Ônishi, Y. Abelian functions for trigonal curves of degree four and determinantal formulae in purely trigonal case. *Internat. J. Math.* **2009**, *20*, 427–441. [CrossRef]
32. Farkas, H.M.; Kra, I. *Riemann Surfaces*, 2nd ed.; Springer: New York, NY, USA, 1992.
33. Griffiths, P.; Harris, J. *Principles of Algebraic Geometry*; Wiley: New York, NY, USA, 1994.