INVERSE TRIDIAGONAL Z–MATRICES

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in memory of his great contributions to linear algebra

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Abstract

In this paper, we consider matrices whose inverses are tridiagonal Z–matrices. Based on a characterization of symmetric tridiagonal matrices by Gantmacher and Krein, we show that a matrix is the inverse of a tridiagonal Z–matrix if and only if, up to a positive scaling of the rows, it is the Hadamard product of a so called weak type D matrix and a flipped weak type D matrix whose parameters satisfy certain quadratic conditions. We predict from these parameters to which class of Z–matrices the inverse belongs to. In particular, we give a characterization of inverse tridiagonal M–matrices. Moreover, we characterize inverses of tridiagonal M–matrices that satisfy certain row sum criteria. This leads to the cyclopses that are matrices constructed from type D and flipped type D matrices. We establish some properties of the cyclopses and provide explicit formulae for the entries of the inverse of a nonsingular cyclops. We also show that the cyclopses are the only generalized ultrametric matrices whose inverses are tridiagonal.

1 Introduction

In many mathematical problems, Z–matrices and M–matrices play an important role. It is often useful to know the properties of their inverses, in particular when the Z–matrices and the M–matrices have a special combinatorial structure. In this paper, we investigate the properties of inverse tridiagonal Z–matrices and M–matrices, i.e., matrices whose inverses are tridiagonal Z–matrices or M–matrices. We also highlight some connections between weak type D matrices (a class that generalizes type D matrices as defined by Markham [8]) and inverse tridiagonal Z–matrices.

First, under the assumption of irreducibility, we show that a matrix is the inverse of a tridiagonal Z–matrix if and only if, up to a positive scaling of the rows, it is the Hadamard product of a weak type D matrix and a flipped weak type D matrix whose parameters satisfy certain quadratic conditions (Theorem 3.3). This characterization parallels (and is based on) the characterization of (symmetric) Green matrices by Gantmacher and Krein [6]. Further, recalling the classification of Z–matrices by Fiedler and Markham [4], we predict the class L_s of a tridiagonal Z–matrix based on the parameters of the associated weak type D matrices (Theorem 3.4). In particular, we find conditions on the parameters so that the inverse is a tridiagonal M–matrix (Corollary 3.6).

Next, we associate type D matrices with tridiagonal Z–matrices via the so called cyclopses. These are matrices that admit a block partition comprising two diagonal blocks that are of flipped type D and of type D, respectively, and two off-diagonal blocks that have constant entries. We find conditions on the parameters of the associated type D matrices and the constant off–diagonal entries so that the inverse of a cyclops exists and is a tridiagonal Z–matrix; its nonzero entries are also found explicitly in terms of the parameters (Theorem 4.6). When a cyclops is a priori nonsingular, we provide necessary and sufficient conditions so that its inverse is a tridiagonal Z–matrix (Corollary 4.7); as before we can predict the class L_s of the tridiagonal Z–matrix (Theorem 4.8).

Cyclopses (with nonnegative entries) were encountered by the authors as a special case of the generalized ultrametric matrices (see [13] and [9]), which is a class of inverse (row and column diagonally dominant) M–matrices. We conclude by finding necessary and sufficient conditions so that a cyclops is the inverse of a (row and column) diagonally dominant tridiagonal M–matrix (or equivalently a totally nonnegative generalized ultrametric matrix) (see Theorems 4.10, 4.12, and 4.14). These results amount to a characterization of the generalized ultrametric matrices whose inverses are tridiagonal.

We continue with the precise definitions of the terms mentioned above and the notational conventions.

2 Preliminaries

We let e denote the all ones vector and e_j the j–th standard basis vector in $\mathbb{R}^n$. Given a positive integer n we let $\langle n \rangle = \{1, 2, \ldots, n\}$. Let $\circ$ denote the Hadamard (i.e., entrywise) product of matrices. For $A = [a_{ij}] \in \mathbb{R}^{n,n}$, by $A(i|j)$ we denote the submatrix of A obtained by deleting the i–th row and the j–th column. Given $R, S \subseteq \langle n \rangle$ we write A_{RS} for the submatrix of A whose rows and columns are indexed by R and S, respectively. If $S = \langle n \rangle \setminus R$ and if A_{RR} is nonsingular, then the *Schur complement* of A_{RR} in A is defined and denoted by

$$A/A_{RR} = A_{SS} - A_{SR}(A_{RR})^{-1}A_{RS}.$$

It is well known that $\det A = \det A_{RR} \det(A/A_{RR})$.

We call $A = [a_{ij}] \in \mathbb{R}^{n,n}$ a Z–matrix if $a_{ij} \leq 0$ for all $i \neq j$. For any nonnegative integer $s \leq n$ we denote by L_s the set of all matrices $A = tI - B \in \mathbb{R}^{n,n}$, where B is an entrywise nonnegative matrix and where $\rho_s(B) \leq t < \rho_{s+1}(B)$. Here $\rho_s(B)$ denotes the maximum among the spectral radii of all the $s \times s$ principal submatrices of B (we take $\rho_0 = -\infty$ and $\rho_{n+1} = \infty$). In particular, A is an *M–matrix* if it can be written as $A = tI - B$, where B is an entrywise nonnegative matrix and $\rho(B) := \rho_n(B) \leq t$.

The next theorem, found in [11] and [16], is a characterization of the nonsingular Z–matrices in L_s.

Theorem 2.1 *Let $A \in \mathbb{R}^{n,n}$ be nonsingular Z–matrix. Then $A \in L_s$ if and only if one of the following alternative cases a) or b) holds:*

a) **(i)** $\det A < 0$,

(ii) *all principal minors of A^{-1} of order greater than or equal to $n - s$ are nonpositive, and*

(iii) *there exists a positive principal minor of A^{-1} of order $n - s - 1$.*

b) **(i)** $\det A > 0$,

(ii) *all principal minors of A^{-1} of order greater than or equal to $n - s$ are nonnegative, and*

(iii) *there exists a negative principal minor of A^{-1} of order $n - s - 1$.*

Markham defined in [8] type D matrices as follows: $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is of *type* D (or *a type* D *matrix*) if

$$a_{ij} = \begin{cases} a_i, & i \leq j, \\ a_j, & i > j, \end{cases} \qquad \text{where } a_n > a_{n-1} > \ldots > a_1.$$

We refer to the a_i $(i = 1, 2, \ldots, n)$ as the *parameters* of A. We also consider similarly constructed matrices, without constraints on the parameters a_i, to which we refer as of *weak type*

D. Moreover, we call A a *flipped type* D *matrix* (resp., a *flipped weak type* D *matrix*) if PAP^T is a type D matrix (resp., a weak type D matrix), where P is the permutation that reverses the order of the indices $1, 2, \ldots, n$. We enumerate the parameters of a weak type D matrix, as well as the parameters of a flipped weak type D matrix in a way such that the i–th parameter is equal to the i–th diagonal entry of the matrix. To illustrate these definitions and the relevant notation, let

$$A = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then A is of type D with parameters a_i given by $(-1, 2, 3)$ and B is of flipped weak type D with parameters b_i given by $(-3, 2, 1)$.

Gantmacher and Krein defined in [6] a *Green matrix* to be a matrix $G \in \mathbb{R}^{n,n}$ such that $G = A \circ B$, where A is a weak type D matrix, B is a flipped weak type D matrix. The name Green matrix is not the only name for these matrices. Originally Gantmacher and Krein called such matrices *einpaarig* or *matrix of a couple*. Moreover, Markham defined the type D matrices as a special case of Green matrices.

In our discussion, we shall also refer to the following matrices that were introduced in [13] and [9]. We say $C = [c_{ij}] \in \mathbb{R}^{n,n}$ is a *generalized ultrametric matrix* if

(i) C is entrywise nonnegative,

(ii) $c_{ii} \geq \max\{c_{ij}, c_{ji}\}$ for all $i, j \in \langle n \rangle$,

(iii) every subset of $\langle n \rangle$ with three distinct elements has a labeling $\{i, j, k\}$ such that

- **(a)** $c_{ij} = c_{ik}$,
- **(b)** $c_{ji} = c_{ki}$,
- **(c)** $\min\{c_{jk}, c_{kj}\} \geq \min\{c_{ji}, c_{ij}\}$,
- **(d)** $\max\{c_{jk}, c_{kj}\} \geq \max\{c_{ji}, c_{ij}\}$.

In the aforementioned papers, it is shown that if a generalized ultrametric matrix is nonsingular then its inverse is a row and column diagonally dominant M–matrix.

Next, we introduce a class of matrices constructed from type D matrices; as we show in Section 4, it contains matrices that are under certain additional conditions are inverse tridiagonal Z–matrices. Let $C \in \mathbb{R}^{n,n}$ and let $m \leq n$ be a nonnegative integer. We call C a *cyclops with eye* $m+$ if

$$C = \begin{bmatrix} C_{11} & b_1 E_{12} \\ b_2 E_{21} & C_{22} \end{bmatrix}, \tag{2.1}$$

where C_{11} is a $m \times m$ flipped type D matrix and C_{22} is a $(n-m)\times(n-m)$ type D matrix, viz.,

$$C_{11} = \begin{bmatrix} a_1 & a_2 & \dots & a_{m-1} & a_m \\ a_2 & a_2 & \dots & a_{m-1} & a_m \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m-1} & a_{m-1} & \dots & a_{m-1} & a_m \\ a_m & a_m & \dots & a_m & a_m \end{bmatrix}, \quad C_{22} = \begin{bmatrix} a_{m+1} & a_{m+1} & \dots & a_{m+1} & a_{m+1} \\ a_{m+1} & a_{m+2} & \dots & a_{m+2} & a_{m+2} \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m+1} & a_{m+2} & \dots & a_{n-1} & a_{n-1} \\ a_{m+1} & a_{m+2} & \dots & a_{n-1} & a_n \end{bmatrix}$$

with

$$a_1 > a_2 > \ldots > a_m \quad \text{and} \quad a_n > a_{n-1} > \ldots > a_{m+1}, \tag{2.2}$$

and where E_{12} and E_{21} are all ones matrices of appropriate sizes. We refer to the a_i ($i = 1, 2, \ldots, n$) and b_1, b_2 as the *parameters* of the cyclops C.

In the remainder of this paper, when we refer to a type D matrix, a weak type D matrix, a Green matrix, or a cyclops, we assume that the reader recalls the notation and the associated parameters indicated in this section.

3 Hadamard Products of weak type D Matrices

Gantmacher and Krein proved the following results.

Theorem 3.1 (Gantmacher and Krein [6]) *Let $G \in \mathbb{R}^{n,n}$ be symmetric. Then the following are equivalent:*

(i) *G is a nonsingular Green matrix.*

(ii) *G^{-1} is an irreducible tridiagonal matrix.*

Lemma 3.2 (Gantmacher and Krein [6]) *Let $G \in \mathbb{R}^{n,n}$ be a Green matrix with associated parameters a_i, b_i. Let $h_i := a_i b_{i-1} - a_{i-1} b_i$ for $i = 2, 3, \ldots, n$. Then*

$$\det G = a_1 b_n \prod_{i=2}^{n} h_i.$$

Moreover,

$$\det G(i|j) = \begin{cases} \det G / h_{i+1} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1. \end{cases}$$

We proceed by characterizing inverse tridiagonal Z–matrices in the spirit of Theorem 3.1.

Theorem 3.3 *Let $C \in \mathbb{R}^{n,n}$ be nonsingular and irreducible. Then the following are equivalent:*

(i) *C^{-1} is a tridiagonal Z–matrix.*

(ii) *There exists a positive diagonal matrix $D \in \mathbb{R}^{n,n}$ such that $DC = A \circ B$, where A is of weak type* D *with parameters a_i, and B is of flipped weak type* D *with parameters b_i, such that $a_i b_{i-1} - a_{i-1} b_i > 0$ for all $i = 2, 3, \ldots, n$.*

Proof: Let $h_i := a_i b_{i-1} - a_{i-1} b_i$ for $i = 2, 3, \ldots, n$ and suppose that (i) holds. As is well known (see [3] and [14]), there exists a positive diagonal matrix D^{-1} such that $C^{-1}D^{-1}$ is symmetric. Thus, by Theorem 3.1, DC is a Green matrix. We also have that $C^{-1}D^{-1} = [\gamma_{ij}]$, where

$$\gamma_{ij} = (-1)^{i+j} \det((DC)(j|i))/\det(DC).$$

Hence, by Lemma 3.2, the superdiagonal entries are

$$\gamma_{i,i+1} = \frac{-1}{h_{i+1}} \quad (i = 1, 2, \ldots, n). \tag{3.3}$$

Thus all h_i are positive. Conversely, if (ii) holds, (3.3) and Theorem 3.1 imply that $C^{-1}D^{-1}$ is a tridiagonal Z–matrix. Hence C^{-1} is a tridiagonal Z–matrix. ■

In the following theorem we determine the class L_s to which an inverse tridiagonal Z–matrix belongs.

Theorem 3.4 *Let $C \in \mathbb{R}^{n,n}$ be an irreducible inverse tridiagonal Z–matrix. Let D, A, and B be as in condition (ii) of Theorem 3.3. Then the following hold:*

(i) *If $\det C < 0$ then $C^{-1} \in L_s$ with $s = \min\{t-2, n-r-2\}$;*

(ii) *if $\det C > 0$ then $C^{-1} \in L_s$ with $s = \min\{t-2, n-q-2\}$,*

where

$$t = \begin{cases} n+2 & \text{if } a_i b_{i+j} - a_{i+j} b_i > 0 \text{ for all } i, j \\ \min\{j \in \langle n \rangle | \text{ there exists an } i \in \langle n \rangle \text{ with } a_i b_{i+j} - a_{i+j} b_i < 0\} & \text{otherwise,} \end{cases}$$

$$r = \begin{cases} -1 & \text{if } a_i b_{i+j} \le 0 \text{ for all } i, j \\ \max\{j \in \langle n \rangle | \text{ there exists an } i \in \langle n \rangle \text{ with } a_i b_{i+j} > 0\} & \text{otherwise,} \end{cases}$$

$$q = \begin{cases} -2, & \text{if } a_i b_{i+j} > 0 \text{ for all } i, j \\ \max\{j \in \langle n \rangle | \text{ there exists an } i \in \langle n \rangle \text{ with } a_i b_{i+j} < 0\}, & \text{otherwise.} \end{cases}$$

Proof: For $i, j \in \langle n \rangle$ with $i > j$, define $h_{ij} = a_i b_j - a_j b_i$. Since C is an inverse tridiagonal Z–matrix we have, by Theorem 3.3, that $h_{i,i-1} > 0$ for all $i = 2, 3, \ldots, n$. Moreover, by Lemma 3.2, we have that

$$\det(DC) = a_1 b_n \prod_{i=2}^{n} h_{i,i-1}, \tag{3.4}$$

where D is the positive diagonal matrix chosen in the proof of Theorem 3.3. We will proceed by considering the signs of the principal minors of DC and by applying Theorem 2.1. Since principal submatrices of Green matrices are also Green matrices, the principal minors of DC are given by formulae similar to (3.4), and their signs are determined by the corresponding quantities h_{ij} and a_i and b_i.

First, suppose $\det C < 0$, i.e., $a_1 b_n < 0$. Without loss of generality, we can assume that $a_1 > 0$ and $b_n < 0$. When $t \neq n+2$, the definition of t and (3.4) imply that there exists a principal submatrix of order $n-(t-1)$ of DC with positive determinant. (This principal submatrix is obtained by deleting rows and columns $i+1, \ldots, i+j-1$, where i, j are the minimal indices in the definition of t.) For all principal submatrices of order greater than $n-(t-1)$, the relevant $h_{i,j}$ appearing in the determinantal formula of Lemma 3.2 are positive. It is also clear from the definition of r that there exists an $(r+1) \times (r+1)$ principal submatrix with positive determinant. Moreover, all principal submatrices of order $\tilde{n}$ with $\tilde{n} > r+1$ satisfy $\tilde{a}_1 \tilde{b}_{\tilde{n}} \leq 0$, where $\tilde{a}_1$ and $\tilde{b}_{\tilde{n}}$ are the corresponding parameters. From these cases, we obtain the following: if $s = \min\{t-2, n-r-2\}$, then there exists a principal minor of order $n-s-1$ that is positive. Also, all principal minors of order greater than $n-s-1$ are nonpositive. Thus, by Theorem 2.1, $C^{-1} \in L_s$, showing (i). Similarly we obtain (ii). ■

It is shown in [11] that if $C^{-1} \in L_s$ and $\lfloor \frac{n}{2} \rfloor \leq s < n$, then $\det C < 0$. For inverse tridiagonal Z–matrices, this result can be established by considering the changes of the signs in the sequences of the parameter a_i and b_i. With q as in Theorem 3.4, if $\det C > 0$ and C is not entrywise nonnegative, one obtains that $q + 1 \geq \lfloor \frac{n}{2} \rfloor$.

Example 3.5 In the following examples we apply Theorem 3.4.
(i) Consider the matrix

$$C = \begin{bmatrix} -2 & -2 & -2 & -2 \\ -2 & -1 & -1 & -1 \\ -2 & -1 & 2 & 2 \\ -2 & -1 & 2 & 3 \end{bmatrix}$$

for which we can write $DC = A \circ B$, where $D = I$, A is of type D with parameters a_i given by $(-2, -1, 2, 3)$, and B is of (flipped) weak type D with parameters b_i given by $(1, 1, 1, 1)$. Notice that $a_i b_{i-1} - a_{i-1} b_i > 0$ for all $i = 2, 3, 4$. Moreover, $\det C < 0$, $t = 6$ and $r = 1$. Thus $C^{-1} \in L_1$.
(ii) Let

$$C = \begin{bmatrix} -24 & -20 & -16 & -4 \\ -20 & -10 & -8 & -2 \\ -16 & -8 & 16 & 4 \\ -4 & -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -4 & -4 & -4 & -4 \\ -4 & -2 & -2 & -2 \\ -4 & -2 & 4 & 4 \\ -4 & -2 & 4 & 5 \end{bmatrix} \circ \begin{bmatrix} 6 & 5 & 4 & 1 \\ 5 & 5 & 4 & 1 \\ 4 & 4 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

C is the Hadamard product of a weak type D and a flipped weak type D matrix with parameters $(-4, -2, 4, 5)$ and $(6, 5, 4, 1)$, respectively. By Theorem 3.3, C^{-1} is a tridiagonal Z–matrix and $\det C < 0$. Moreover, $r = 1$ and $t = 2$ and hence $C^{-1} \in L_0$.

The results above yield the following characterization of inverse tridiagonal M–matrices.

Corollary 3.6 *Let $C \in \mathbb{R}^{n,n}$ be nonsingular and irreducible. Then the following are equivalent:*

(i) *C^{-1} is a tridiagonal M–matrix*

(ii) *There exists a positive diagonal matrix $D \in \mathbb{R}^{n,n}$ such that $DC = A \circ B$, where A is of a weak type* D *with parameters a_i, and B is of flipped weak type* D *with parameters b_i, such that all the parameters have the same sign and*

$$0 < \frac{a_1}{b_1} < \frac{a_2}{b_2} < \ldots < \frac{a_n}{b_n}.$$

Proof:

(i) implies (ii): As $C^{-1} \in L_n$, we have that $\det C > 0$ and that C^{-1} is a tridiagonal Z–matrix. The implication now follows from Theorems 3.3 and 3.4.

(ii) implies (i): By Theorem 3.3, C^{-1} is a tridiagonal Z–matrix. Since C is entrywise positive and since every inverse positive Z–matrix is an M–matrix, (i) holds. ■

4 Cyclopses

In this section, we consider inverse tridiagonal Z–matrices that satisfy certain row sum and column sum criteria. This leads to a new class of matrices that we have defined as cyclopses in Section 2. We begin with some auxiliary results.

Lemma 4.1 ([11, Observation 3.8]) *Let $A \in \mathbb{R}^{n,n}$ be of type D with parameters a_i and such that $a_1 \neq 0$. Then*

$$A^{-1}e = \left[\frac{1}{a_1}, 0, 0, \ldots, 0\right]^T.$$

From Lemma 3.2, for a type D matrix with parameters a_i, we obtain that

$$\det C = a_1 \prod_{j=2}^{n} (a_j - a_{j-1}). \tag{4.5}$$

Lemma 4.2 *Let $C \in \mathbb{R}^{n,n}$ be a cyclops with eye $m+$ parameters a_i, b_1, b_2. Suppose that $a_m \neq 0$ and $a_{m+1} \neq 0$. Then*

$$C/C_{11} = C_{22} - \frac{b_1 b_2}{a_m} E_1, \quad C/C_{22} = C_{11} - \frac{b_1 b_2}{a_{m+1}} E_2,$$

where E_1, E_2 are all ones matrices of appropriate sizes.

Proof: Follows from Lemma 4.1 and the fact that $C/C_{11} = C_{22} - b_1 b_2 E_{21} C_{11}^{-1} E_{12}$. ■

Theorem 4.3 *Let $C \in \mathbb{R}^{n,n}$ be a cyclops with eye $m+$ and parameters a_i, b_1, b_2. Then*

$$\det C = (a_m a_{m+1} - b_1 b_2) \prod_{j=1}^{m-1} (a_{m-j} - a_{m-j+1}) \prod_{j=m+2}^{n} (a_j - a_{j-1}).$$

Proof:

Case I ($a_m \neq 0$ or $a_{m+1} \neq 0$): if $a_m \neq 0$, then by Lemma 4.5, C_{11} is nonsingular and

$$\det C = \det C_{11} \det(C/C_{11})$$

$$= a_m \prod_{j=1}^{m-1} (a_{m-j} - a_{m-j+1}) \det(C_{22} - b_1 b_2 / a_m E_1)$$

$$= a_m \prod_{j=1}^{m-1} (a_{m-j} - a_{m-j+1})(a_{m+1} - \frac{b_1 b_2}{a_m}) \prod_{j=m+2}^{n} ((a_j - \frac{b_1 b_2}{a_m}) - (a_{j-1} - \frac{b_1 b_2}{a_m}))$$

$$= (a_m a_{m+1} - b_1 b_2) \prod_{j=1}^{m-1} (a_{m-j} - a_{m-j+1}) \prod_{j=m+2}^{n} (a_j - a_{j-1}).$$

If $a_{m+1} \neq 0$ the result follows in a similar manner.

Case II ($a_m = a_{m+1} = 0$): if $a_m = a_{m+1} = 0$ and either $b_1 = 0$ or $b_2 = 0$, then C has a row of zeros (and thus zero determinant) and the result follows. Assume that $b_1 \neq 0$ and $b_2 \neq 0$. Let $R = \{m, m+1\}$, $S = \{1, 2, \ldots, m-1\}$, $T = \{m+2, m+3, \ldots, n\}$ and $U = S \cup T$. Then C_{RR} is nonsingular since $a_{m-1} > a_m = 0$. Hence

$$\det C = \det C_{RR} \det(C/C_{RR}) = -b_1 b_2 \det \begin{bmatrix} C_{SS} & 0 \\ 0 & C_{TT} \end{bmatrix}$$

$$= -b_1 b_2 (a_{m-1} a_{m+2}) \prod_{j=2}^{m-1} (a_{m-j} - a_{m-j+1}) \prod_{j=m+3}^{n} (a_j - a_{j-1})$$

$$= (a_m a_{m+1} - b_1 b_2) \prod_{j=1}^{m-1} (a_{m-j} - a_{m-j+1}) \prod_{j=m+2}^{n} (a_j - a_{j-1}).$$

■

The following is an immediate consequence of the above theorem.

Corollary 4.4 *Let $C \in \mathbb{R}^{n,n}$ be a cyclops with eye $m+$ and parameters a_i, b_1, b_2. Then C is nonsingular if and only if $a_m a_{m+1} - b_1 b_2 \neq 0$. Moreover,* $\operatorname{sgn}(\det C) = \operatorname{sgn}(a_m a_{m+1} - b_1 b_2)$.

Next, we shall give explicit formulae for the entries of the inverse of a nonsingular cyclops. We first need another result on type D matrices proved in [12]. We denote by $\otimes$ the Kronecker product of matrices.

Theorem 4.5 *Let A be a nonsingular matrix of type* D *with parameters a_i. Then the inverse of A is given by*

$$A^{-1} = \sum_{i=1}^{n} v^{(i)}(v^{(i)})^T \otimes (a_i - a_{i-1})^{-1}, \tag{4.6}$$

with $a_0 \equiv 0$. Here the vectors $v^{(i)} = [v_j^{(i)}]$ are defined as

$$v_j^{(i)} = \begin{cases} -1 & \textit{if } \ j = i-1 \\ 1 & \textit{if } \ j = i \\ 0 & \textit{otherwise.} \end{cases} \tag{4.7}$$

It follows that the entries α_{ij} of the inverse of a type D matrix $A \in \mathbb{R}^{n,n}$ are zero except for

$$\alpha_{11} = \frac{1}{a_1} + \frac{1}{a_2 - a_1}, \quad \alpha_{nn} = \frac{1}{a_n - a_{n-1}},$$

$$\alpha_{ii} = \frac{1}{a_i - a_{i-1}} + \frac{1}{a_{i+1} - a_i} \quad (i = 2, 3, \ldots, n-1),$$

$$\alpha_{i,i+1} = \alpha_{i+1,i} = -\frac{1}{a_{i+1} - a_i}.$$

Similarly, for a flipped type D matrix A we have

$$\alpha_{11} = \frac{1}{a_1 - a_2}, \quad \alpha_{nn} = \frac{1}{a_n} + \frac{1}{a_{n-1} - a_n},$$

$$\alpha_{ii} = \frac{1}{a_i - a_{i+1}} + \frac{1}{a_{i-1} - a_i} \quad (i = 2, 3, \ldots, n-1),$$

$$\alpha_{i,i+1} = \alpha_{i+1,i} = -\frac{1}{a_i - a_{i+1}}.$$

Theorem 4.6 *Let $C \in \mathbb{R}^{n,n}$ be a cyclops with eye $m+$ and parameters a_i, b_1, b_2. Suppose that $a_m a_{m+1} - b_1 b_2 \neq 0$. Then $A = C^{-1} = [\alpha_{ij}]$ exists and is a tridiagonal matrix with entries given by*

$$\alpha_{11} = \frac{1}{a_1 - a_2},$$

$$\alpha_{ii} = \frac{1}{a_i - a_{i+1}} + \frac{1}{a_{i-1} - a_i} \quad (i = 2, 3, \ldots, m-1),$$

$$\alpha_{mm} = \frac{a_{m+1}}{a_m a_{m+1} - b_1 b_2} + \frac{1}{a_{m-1} - a_m},$$

$$\alpha_{i,i+1} = \alpha_{i+1,i} = -\frac{1}{a_i - a_{i+1}} \quad (i = 1, 2, \ldots, m-1),$$

$$\alpha_{m+1,m+1} = \frac{a_m}{a_m a_{m+1} - b_1 b_2} + \frac{1}{a_{m+2} - a_{m+1}},$$

$$\alpha_{ii} = \frac{1}{a_i - a_{i-1}} + \frac{1}{a_{i+1} - a_i} \quad (i = m+2, m+3, \ldots, n-1)$$

$$\alpha_{nn} = \frac{1}{a_n - a_{n-1}},$$

$$\alpha_{i,i+1} = \alpha_{i+1,i} = -\frac{1}{a_{i+1} - a_i} \quad (i = m+1, m+2, \ldots, n-1).$$

Moreover,

$$\alpha_{m,m+1} = -\frac{b_1}{a_m a_{m+1} - b_1 b_2}, \quad \alpha_{m+1,m} = -\frac{b_2}{a_m a_{m+1} - b_1 b_2}.$$

Proof: Recall that our assumption that $a_m a_{m+1} - b_1 b_2 \neq 0$ is equivalent to C being invertible.

Case I ($a_m \neq 0$ and $a_{m+1} \neq 0$): if $a_m \neq 0$ then C_{11} is a nonsingular flipped type D matrix and hence it is the inverse of a tridiagonal Z–matrix (by the results in [11]). If $a_{m+1} \neq 0$, then C_{22} is a nonsingular type D matrix and hence it is also the inverse of a tridiagonal Z–matrix. By Lemma 4.2, C/C_{11} is of type D and C/C_{22} is of flipped type D. Moreover, by (4.5) applied to C/C_{11} and C/C_{22} and since $a_m a_{m+1} - b_1 b_2 \neq 0$, both Schur complements are nonsingular and thus (using formulas from [2, (10), p. 773])

$$A = \begin{bmatrix} (C/C_{22})^{-1} & -b_1 C_{11}^{-1} E_{12} (C/C_{11})^{-1} \\ -b_2 C_{22}^{-1} E_{21} (C/C_{22})^{-1} & (C/C_{11})^{-1} \end{bmatrix}.$$

Since C/C_{22} is of flipped type D, its inverse is a tridiagonal Z–matrix with all row sums zero except the m–th (last). Since C/C_{11} is of type D, its inverse is a tridiagonal Z–matrix with all row sums zero except the first, which corresponds to the $(m+1)$–st row of A.

Now one can easily get the entries of $(C/C_{22})^{-1}$ and $(C/C_{11})^{-1}$ using Theorem 4.5. Furthermore, it follows that

$$\begin{aligned} -b_1 C_{11}^{-1} E_{12} (C/C_{11})^{-1} &= -\frac{b_1}{a_m} e_m e_1^T (C/C_{11})^{-1} \\ &= -\frac{b_1}{a_m (a_{m+1} - \frac{b_1 b_2}{a_m})} e_m e_1^T \\ &= -\frac{b_1}{a_m a_{m+1} - b_1 b_2} e_m e_1^T. \end{aligned}$$

Similarly,

$$-b_2 C_{22}^{-1} E_{21} (C/C_{22})^{-1} = -\frac{b_2}{a_m a_{m+1} - b_1 b_2} e_1 e_{n-m}^T.$$

This establishes the result in Case I.

Case II ($a_m = 0$ or $a_{m+1} = 0$): if $a_m = 0$ or $a_{m+1} = 0$ (or both), form a new cyclops from C by replacing the m–th or the $(m+1)$–st parameter (or both) by real numbers that approach zero. The result then follows from Case I and continuity. ■

Notice that if C is as in the previous theorem, then all row sums and column sums of $A = C^{-1}$ are zero, except at least one of the m–th or the $(m+1)$–st (for otherwise C would be singular).

Corollary 4.7 *Let $C \in \mathbb{R}^{n,n}$ be a nonsingular cyclops with eye $m+$ and parameters a_i, b_1, b_2. Then C^{-1} is a Z–matrix if and only if the following two conditions hold:*

(i) $b_1 = 0$ *or* $\operatorname{sgn}(b_1) = \operatorname{sgn}(a_m a_{m+1} - b_1 b_2)$,

(ii) $b_2 = 0$ *or* $\operatorname{sgn}(b_2) = \operatorname{sgn}(a_m a_{m+1} - b_1 b_2)$.

Theorem 4.8 *Let $C \in \mathbb{R}^{n,n}$ be a nonsingular cyclops with eye $m+$ and parameters a_i, b_1, b_2, whose inverse is a Z–matrix. Let*

$$\chi = \{\ k - j \mid a_j a_k - b_1 b_2 < 0,\ j \le m,\ k \ge m+1\ \},$$

$$\Upsilon = \{\ k - j \mid a_j a_k - b_1 b_2 > 0,\ j \le m,\ k \ge m+1\ \},$$

and define

$$x = \begin{cases} \min(\chi) & \textit{if } \chi \neq \emptyset \\ n+1 & \textit{otherwise,} \end{cases}$$

$$y = \begin{cases} \min(\Upsilon) & \textit{if } \chi \neq \emptyset \\ n+1 & \textit{otherwise,} \end{cases}$$

$$r = \textit{number of positive } a_j \textit{ with } j \le m,$$

$$t = \textit{number of positive } a_j \textit{ with } j > m.$$

Then the following hold:

(i) *If* $\det C > 0$ *and* $a_m > 0$*, then* $C \in L_n$ *(i.e., C is an inverse M–matrix.)*

(ii) *If* $\det C > 0$ *and* $a_m \le 0$*, then* $C \in L_s$*, where*

$$s = n - 1 - \max\{m, n-m, n-x+1\}.$$

(iii) *If* $\det C < 0$*, then* $C \in L_s$*, where*

$$s = n - 1 - \max\{r, t, n-y+1\}.$$

Proof: Let B be any principal submatrix of C, partitioned as in (2.1). Then B is one of three types:

1. B is a principal submatrix of C_{11}, in which case, by (4.5), $\det B$ has the same sign as the parameter a_j with the largest index contained in B.

2. B is a principal submatrix of C_{22}, in which case $\det B$ has the same sign as the parameter a_j with the smallest index contained in B.

3. B is neither a principal submatrix of C_{11} nor of C_{22}; in this case $\det B$ has the same sign as $a_j a_k - b_1 b_2$, where a_j has the largest index less than m contained in B, and a_k has the smallest index greater than $m+1$ contained in B.

If $\det C > 0$, by Corollary 4.4 we have that $a_m a_{m-1} - b_1 b_2 > 0$. Since C is an inverse Z–matrix, it follows from our previous results that $b_1 \geq 0, b_2 \geq 0$. Hence a_m and a_{m+1} are nonzero and have the same sign. If $a_m > 0$, it follows that C is nonnegative and hence an inverse M–matrix, i.e., (i) holds. If $a_m \leq 0$, then $a_{m+1} < 0$ and $\det C_{11}$, $\det C_{22}$ are both negative; thus C has negative principal minors of sizes $m \times m$ and $(n-m) \times (n-m)$. We need also consider submatrices of the third type; the largest such submatrix with a negative determinant is of size $(n-x+1) \times (n-x+1)$. By Theorem 2.1 applied to $A = C^{-1}$, we have that (ii) holds.

If $\det C < 0$, then s is determined by the size of the largest submatrix of C with a positive determinant. The largest submatrix of C_{11} with a positive determinant is $r \times r$. The largest submatrix of C_{22} with a positive determinant is $t \times t$. The largest submatrix of C of the third type is of size $(n-y+1) \times (n-y+1)$, and (iii) follows. ■

Example 4.9 The following example illustrates a cyclops and its tridiagonal inverse, computed by Theorem 4.6.

$$C = \begin{bmatrix} 4 & 3 & 2 & -1 & -1 & -1 \\ 3 & 3 & 2 & -1 & -1 & -1 \\ 2 & 2 & 2 & -1 & -1 & -1 \\ -4 & -4 & -4 & 1 & 1 & 1 \\ -4 & -4 & -4 & 1 & 2 & 2 \\ -4 & -4 & -4 & 1 & 2 & 3 \end{bmatrix}, \quad C^{-1} = \begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 0.5 & -0.5 & & \\ & & -2 & 0 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}.$$

Note that, as shown in Theorem 4.6, all row sums and column sums of C^{-1} are zero expect the 3–rd and the 4–th. Moreover, using Theorem 4.3, one easily obtains that $\det C = -2$. Applying Theorem 4.8, we have $y = 3, r = 3, t = 3$, and thus $C^{-1} \in L_1$.

Next, we will characterize generalized ultrametric matrices whose inverses are tridiagonal. We begin with the irreducible case. We remind the reader that $C \in \mathbb{R}^{n,n}$ is called *irreducible* if its directed graph, $\Gamma(C)$, is strongly connected (see e.g., [1]). Also recall that C is called *totally nonnegative* if all its minors are nonnegative.

Theorem 4.10 *Let $C \in \mathbb{R}^{n,n}$ be a nonsingular matrix. Then the following are equivalent:*

(i) *There is $m \in \langle n \rangle$ such that C^{-1} is an irreducible row and column diagonally dominant tridiagonal M–matrix whose row and columns sums are all zero, except at least one of the m–th or $(m+1)$–st.*

(ii) *There is $m \in \langle n \rangle$ and $a_i, b_1, b_2 \in \mathbb{R}$ such that C is a cyclops with eye $m+$ and parameters a_i, b_1, b_2 satisfying*

$$\min\{a_m, a_{m+1}\} \geq \max\{b_1, b_2\} \quad \textit{and} \quad \min\{b_1, b_2\} > 0.$$

(iii) *C is an irreducible generalized ultrametric matrix whose inverse is tridiagonal.*

(iv) *C is a totally nonnegative irreducible generalized ultrametric matrix.*

Proof: Let $C = [c_{ij}]$ and $A = C^{-1}$.

(i) implies (ii): By [9, Theorem 3.2] applied to A and A^T, $c_{ii} = c_{ij} = c_{ji}$ for all $1 \leq j < i \leq m$ and $c_{ii} = c_{ik} = c_{ki}$ for all $m+1 \leq i < k \leq n$. Since A is a tridiagonal M–matrix, by [10, Theorem 4.1], $c_{mk} = \frac{c_{m,m+1}c_{m+1,k}}{c_{m+1,m+1}} = c_{m,m+1}$ and $c_{jk} = \frac{c_{jm}c_{mk}}{c_{mm}} = c_{mk}$ for all $j \leq m$, and $k \geq m+1$. Similarly, $c_{jk} = c_{m+1,k} = c_{m+1,m}$ for all $j \geq m+1, k \leq m$. Thus C is a cyclops with eye $m+$ and parameters $a_i = c_{ii}, b_1 = c_{m,m+1}, b_2 = c_{m+1,m}$. Since A is a nonsingular M–matrix, all principal minors of C are positive and thus $a_m a_{m+1} - b_1 b_2 > 0$. Since A is irreducible, the entries of C are all positive. Lastly, the inequality $min\{a_m, a_{m+1}\} \geq max\{b_1, b_2\}$ follows from the facts that C is an entrywise positive cyclops and that C^{-1} is a row and column diagonally dominant matrix (and thus each diagonal entry of C is greater than or equal to the other entries in the corresponding row and column, see [5, Theorem (3,5)]).

(ii) implies (iii): Notice first that the conditions on the parameters of the cyclops imply that C is an generalized ultrametric matrix. By Corollary 4.7, C is invertible, and by [10, Theorem 4.1], C^{-1} is tridiagonal.

(iii) implies (i): We only need to show that the row and column sums are as claimed.

Claim I: For all $j < i < k$, either $c_{ii} = c_{ik} = c_{ki}$ or $c_{ii} = c_{ij} = c_{ji}$.

Proof of Claim I: From [10, Theorem 4.1], we see that for all $j < i < k$, $c_{jk} = \frac{c_{ji}c_{ik}}{c_{ii}}$ and $c_{kj} = \frac{c_{ki}c_{ij}}{c_{ii}}$. Consider the *triangle* on vertices i, j, k (see [9, Definition 2.3]). If i is the *preferred vertex*, then $c_{ji} = c_{ki}$ and $c_{ij} = c_{ik}$, and hence $c_{jk} = c_{kj}$ and $c_{jk} \geq max\{c_{ji}, c_{ik}\}$. But since C is a generalized ultrametric matrix, $c_{ii} \geq max\{c_{ji}, c_{ik}\}$, thus $c_{ii} = c_{ij} = c_{ji} = c_{jk} = c_{kj} = c_{ik} = c_{ki}$. If j is preferred, it follows that $c_{jk} = c_{ji}$ and $c_{kj} = c_{ij}$ and thus $c_{ii} = c_{ik} = c_{ki}$. If k is preferred then $c_{ii} = c_{ij} = c_{ji}$. This establishes Claim I.

Claim II: If p is the first nonzero row sum of A, then all other row sums are zero, except possibly the $(p+1)$–st.

Proof of Claim II: For all $j < p+1$, [9, Theorem 3.2] implies $c_{p+1,p+1} \neq c_{j,p+1}$. Suppose there is $q > p+1$ such that q–th row sum is nonzero. Then for all $k > p+1$, [9, Theorem 3.2] implies $c_{p+1,p+1} \neq c_{k,p+1}$. If we apply Claim I with $i = p+1$ we have a contradiction that establishes Claim II.

Claim III: If the q–th row sum of A is nonzero, then the column sums of columns $1, 2, \ldots, q-2$ are zero.

Proof of Claim III: By [9, Theorem 3.2], for all $i < q$ and $k \geq q$, $c_{ii} \neq c_{ki}$ and hence by Claim I, for all $j < i$, $c_{ii} = c_{ij} = c_{ji}$. If we now apply [9, Theorem 3.2] to A^T, we see that columns $1, 2, \ldots, q-1$ must have zero column sums. This establishes Claim III.

Suppose now that the p–th row sum of A is the first nonzero row sum. Let P be the permutation matrix which reverses the order of the indices $1, 2, \ldots, n$.

If the $(p+1)$–st row sum of A is also nonzero, then by applying Claim III to A and PAP^T, with $q = p$ and with $q = p+1$, we see that the only possible nonzero column sums are the p–th and the $(p+1)$–st. Taking $m = p$, the implication is proven.

If the $(p+1)$–st row sum is zero, then by applying Claim III to A and PAP^T, with $q = p$, we see that the only possible nonzero column sums are the $(p-1)$–st, the p–th and the $(p+1)$–st. By Claim II applied to A^T, either the $(p-1)$–st or the $(p+1)$–st sum is zero. By choosing m appropriately to be either $p-1$ or p, the implication is proven.

(iii) if and only if (iv): Follows from the results in [9] or [13], and in [7]. ∎

Corollary 4.11 *A matrix A is of type* D *with parameter $a_1 > 0$ if and only if A^{-1} is a tridiagonal M–matrix with the only nonzero row and column sums being the first.*

We say that C is a *G–cyclops* if it is nonsingular and satisfies any of the equivalent conditions of Theorem 4.10. We also refer to a matrix all of whose entries are equal as a *flat* matrix.

Theorem 4.12 *Let C be a nonsingular matrix that is reducible but not completely reducible. Then the following are equivalent:*

(i) *Either C or C^T is of the form*

$$B := \begin{pmatrix} B_{11} & B_{12} & B_{13} & \ldots & B_{1m} \\ 0 & B_{22} & B_{23} & \ldots & B_{2m} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & B_{m-1,m-1} & B_{m-1,m} \\ 0 & \ldots & \ldots & 0 & B_{mm} \end{pmatrix},$$

where B_{11} is a G–cyclops whose last column is of constant value given by f_{11}; B_{mm} is a G–cyclops whose first row is of constant value given by f_{mm}; each B_{st}, $1 \leq s < t \leq m$, is a flat matrix whose fixed value is given by f_{st}; each B_{ss}, $2 \leq s \leq m-1$, is either a positive number f_{ss} or an entrywise positive matrix of the form

$$B_{ss} = \begin{pmatrix} f_{ss} & f_{ss} \\ g_{ss} & f_{ss} \end{pmatrix},$$

with $g_{ss} < f_{ss}$; and for some $2 \leq q \leq m-1$,

$$f_{11} \geq f_{22} \geq \ldots \geq f_{qq} > 0,$$

$$0 < f_{q+1,q+1} \leq f_{q+2,q+2} \leq \ldots \leq f_{mm},$$

$$f_{1t} = f_{2t} = \ldots = f_{t-1,t} = f_{tt} \quad \text{for} \quad 1 < t \leq q,$$

$$f_{ss} = f_{s,s+1} = \ldots = f_{s,m-1} = f_{sm} \quad \text{for} \quad q < s < m,$$

$$0 < f_{st} = f_{q,q+1} \leq \min\{f_{qq}, f_{q+1,q+1}\} \quad \text{for} \quad 1 \leq s \leq q < t \leq m.$$

(ii) *C is a generalized ultrametric matrix whose inverse is tridiagonal.*

(iii) *C is a totally nonnegative generalized ultrametric matrix.*

Proof:

(i) implies (ii): It is easy to see that the conditions on the f_{ij} guarantee that C is a generalized ultrametric matrix and that it satisfies [10, Theorem 4.1 (ii)].

(ii) implies (i): Let $C = [c_{ij}]$ and $A = C^{-1} = [a_{ij}]$. Since A is a tridiagonal M–matrix, it must satisfy [10, Theorem 4.1 (ii)].

We begin by showing that C or C^T must be block upper triangular with no zero entries in or above the diagonal blocks. Suppose that A has a zero entry on the superdiagonal and a zero entry on the subdiagonal. For simplicity, we will assume that $a_{j,j+1} = 0$ and $k \geq j$ is the smallest integer such that $a_{k+1,k} = 0$ (otherwise take $A = (C^T)^{-1}$). If $k = j$ then A is completely reducible contradicting our hypothesis, hence we will assume that $j < k$. Since j does not access $j+1$ in $\Gamma(C)$, by [15, Lemma 2.2], $c_{j,j+1} = 0$. Similarly $c_{k+1,k} = 0$. By [10, Theorem 4.1 (ii)], $c_{ji} = \frac{c_{j,j+1}c_{j+1,i}}{c_{j+1,j+1}} = 0$ for all $i \geq j+1$, and $c_{k+1,l} = \frac{c_{k+1,k}c_{kl}}{c_{kk}} = 0$ for all $l \leq k$. Consider the triangle (see [10, Definition 2.3]) on $j, k, k+1$. Either $c_{kj} = 0$ or $c_{k,k+1} = 0$. If $c_{k,k+1} = 0$ then by [15, Lemma 2.2] $a_{k,k+1} = 0$ and hence A is completely reducible, contradicting our hypothesis. If $c_{kj} = 0$, then by [15, Lemma 2.2], k does not have access to j in $\Gamma(A)$ and hence there must be an i with $j \leq i < k$ such that $a_{i+1,i} = 0$, contradicting the minimality of k. So either the superdiagonal of C^{-1} or the subdiagonal of C^{-1} contains only nonzero entries. It follows that either C or C^T is block upper triangular, as represented by B. It remains to show that the blocks of $B = [b_{ij}]$ are as claimed. Without loss of generality, assume that all the entries on the superdiagonal of A are nonzero.

Since both B and A are block upper triangular, $B_{ss} = (A_{ss})^{-1}$. Since A_{ss} is irreducible, B_{ss} must be a G–cyclops by Theorem 4.10. Since the superdiagonal entries of A are nonzero, by the results in [15] each B_{st} is an entrywise positive matrix for all $s \leq t$.

Notice that if $i < j$ and i and j are in different blocks of B then $b_{ji} = 0$. This fact will be used without further remark whenever triangles are considered in the remainder of this proof. We will also write $j \in s$ to mean that b_{jj} is in the block B_{ss}.

Let $j \in 1$, $k \in 1$ and $l \notin 1$ with $j \leq k$. By considering the triangle on j, k, l, we see that $b_{jl} = b_{kl}$. But $b_{jl} = \frac{b_{jk} b_{kl}}{b_{kk}}$, hence $b_{jk} = b_{kk}$ and B_{11} is as claimed. A similar argument shows that B_{mm} is as claimed.

Consider B_{rr}, B_{ss}, and B_{tt} with $r < s < t$. Let $i \in r$, $j \in s$, $k \in s$, $l \in t$. Consider the triangle on i, j, l. Then $b_{il} = min\{b_{ij}, b_{jl}\}$. By [10, Theorem 4.1], $b_{il} = \frac{b_{ij} b_{jl}}{b_{jj}}$ and hence $b_{jj} = max\{b_{ij}, b_{jl}\}$. Similarly, $b_{kk} = max\{b_{ik}, b_{kl}\}$. From the triangle on i, j, k we see that $b_{ij} = b_{ik} \leq max\{b_{jk}, b_{kj}\}$. From the triangle on j, k, l we have that $b_{jl} = b_{kl} \leq max\{b_{jk}, b_{kj}\}$. But then $b_{jj} = max\{b_{ij}, b_{jl}\} \leq max\{b_{jk}, b_{kj}\} \leq b_{jj}$. Hence equality must hold throughout. Using the corresponding inequalities for k, we can conclude that $b_{jj} = \max\{b_{jk}, b_{kj}\} = b_{kk}$. If $j < k$, then $b_{ik} = \frac{b_{ij} b_{jk}}{b_{jj}}$ implies that $f_{ss} = b_{jk} = b_{jj} \geq b_{kj} = g_{ss}$. Since $B_{ss} = (A_{ss})^{-1}$, B_{ss} must be a nonsingular G–cyclops and hence can only be as claimed for $2 \leq s \leq m-1$.

Let now $h = min\{j \mid b_{jj} = b_{jk}$ for all $k > j\}$ (h is well defined since $b_{nn} = b_{nk}$ for $k > j$). Consider the r–th diagonal block so that $h \in r$. By the equalities in the above paragraph, if $h-1 \in r$ then $b_{h-1,h-1} = b_{h-1,h} = b_{hh} = b_{hk} = b_{h-1,k}$, contradicting the minimality of h. Hence $h-1 \notin r$. Set $q = r-1$. If $r < m$, then by the choice of h and the triangle on $h, h+1, k$, $b_{hh} = b_{h,h+1} = b_{hk} = min\{b_{h,h+1}, b_{h+1,k}\}$ for all $k > h+1$, which implies that $b_{h,h+1} \leq b_{h+1,k}$. Thus $b_{h+1,h+1} = max\{b_{h,h+1}, b_{h+1,k}\} = b_{h+1,k}$ for all $k > h+1$. We can now repeat this argument for $h+2, h+3, \ldots,$ up to largest index in the $(m-1)$–st diagonal block to conclude that for all $q < s < t$ with $j \in s$ and $k \in t$, $f_{ss} = b_{jj} = b_{jk} = f_{st}$ and thus the B_{st} are as claimed. For any $j < h$, with $j \notin 1$, by the choice of h and the inequalities in the above paragraph, we see that $b_{jj} = b_{ij}$ for all $i \leq j$. Hence B_{st} must be as claimed for all $s < t \leq q$. For $s \leq q < t$, let $j \in s$ and $k \in t$. Then $b_{jk} = \frac{b_{jh} b_{hk}}{b_{hh}} = b_{jh}$ and $b_{jh} = \frac{b_{j,h-1} b_{h-1,h}}{b_{h-1h-1}}$. By considering the triangle on $j, h-1, h$, we have that $b_{jh} = min\{b_{j,h-1}, b_{h-1,h}\} = min\{b_{h-1,h-1}, b_{h-1,h}\} = b_{h-1,h} \leq min\{b_{h-1,h-1}, b_{hh}\}$. Thus $f_{st} = b_{jk} = b_{h-1,h} = f_{q,q+1}$.

(ii) if and only if (iii): Follows from the results in [9] or [13], and in [7]. ∎

Example 4.13 The following matrix illustrates a matrix that satisfies the conditions of Theorem 4.12 (and hence it is a reducible, totally nonnegative, generalized ultrametric matrix whose inverse is tridiagonal).

$$C = \begin{bmatrix} 12 & 11 & 10 & 9 & 9 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 11 & 11 & 10 & 9 & 9 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 10 & 10 & 10 & 9 & 9 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 9 & 9 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 8 & 9 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 7 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 7 & 7 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 7 & 7 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 9 & 10 \end{bmatrix}.$$

Finally, Theorems 4.10 and 4.12 yield the following result.

Theorem 4.14 *Let $C \in \mathrm{IR}^{n,n}$ be a nonsingular matrix. Then the following are equivalent:*

(i) *C is the direct sum of matrices of the forms given in Theorem 4.10(ii) and Theorem 4.12(i).*

(ii) *C is a generalized ultrametric matrix whose inverse is tridiagonal.*

(iii) *C is a totally nonnegative generalized ultrametric matrix.*

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