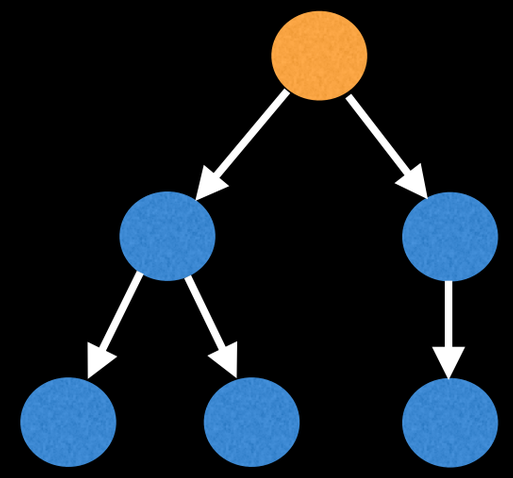
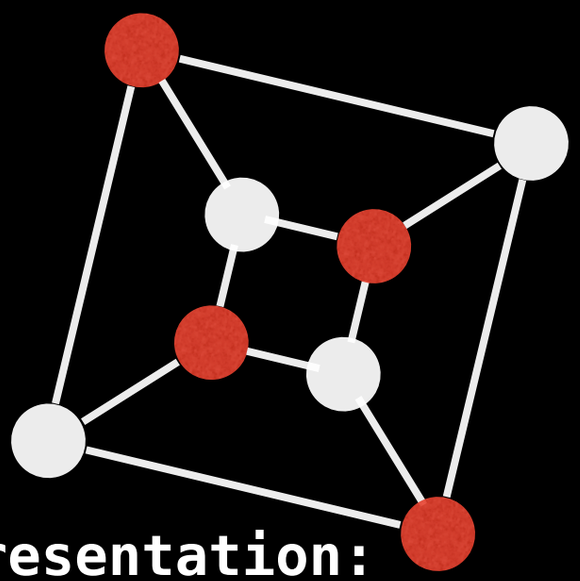


Graph Theory Intro & Overview



Sorokin Semen

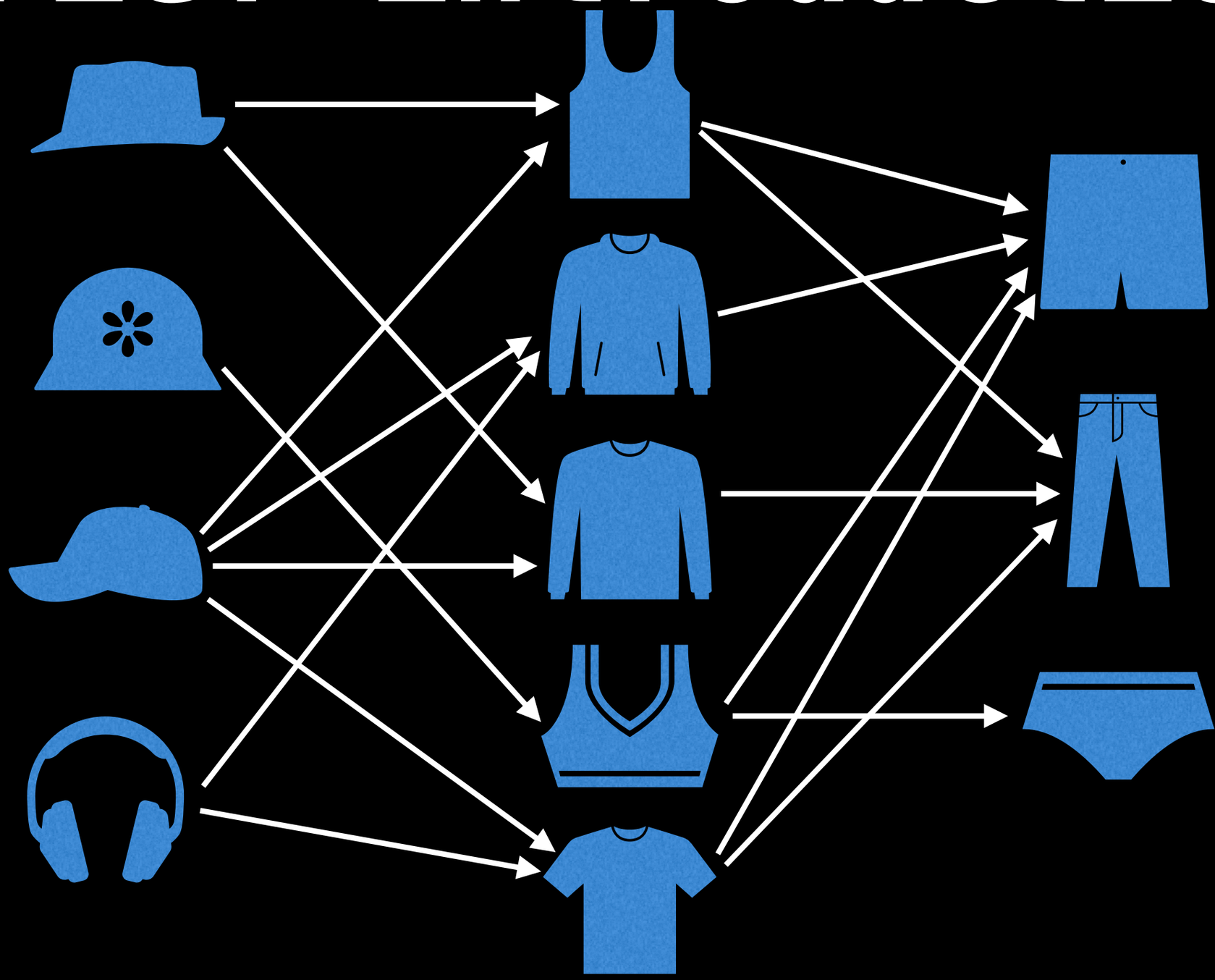
Presentation:
https://github.com/williamfiset/Algorithms/tree/master/slides/graphtheory

Brief introduction

Graph theory is the mathematical theory of the properties and applications of graphs (networks).

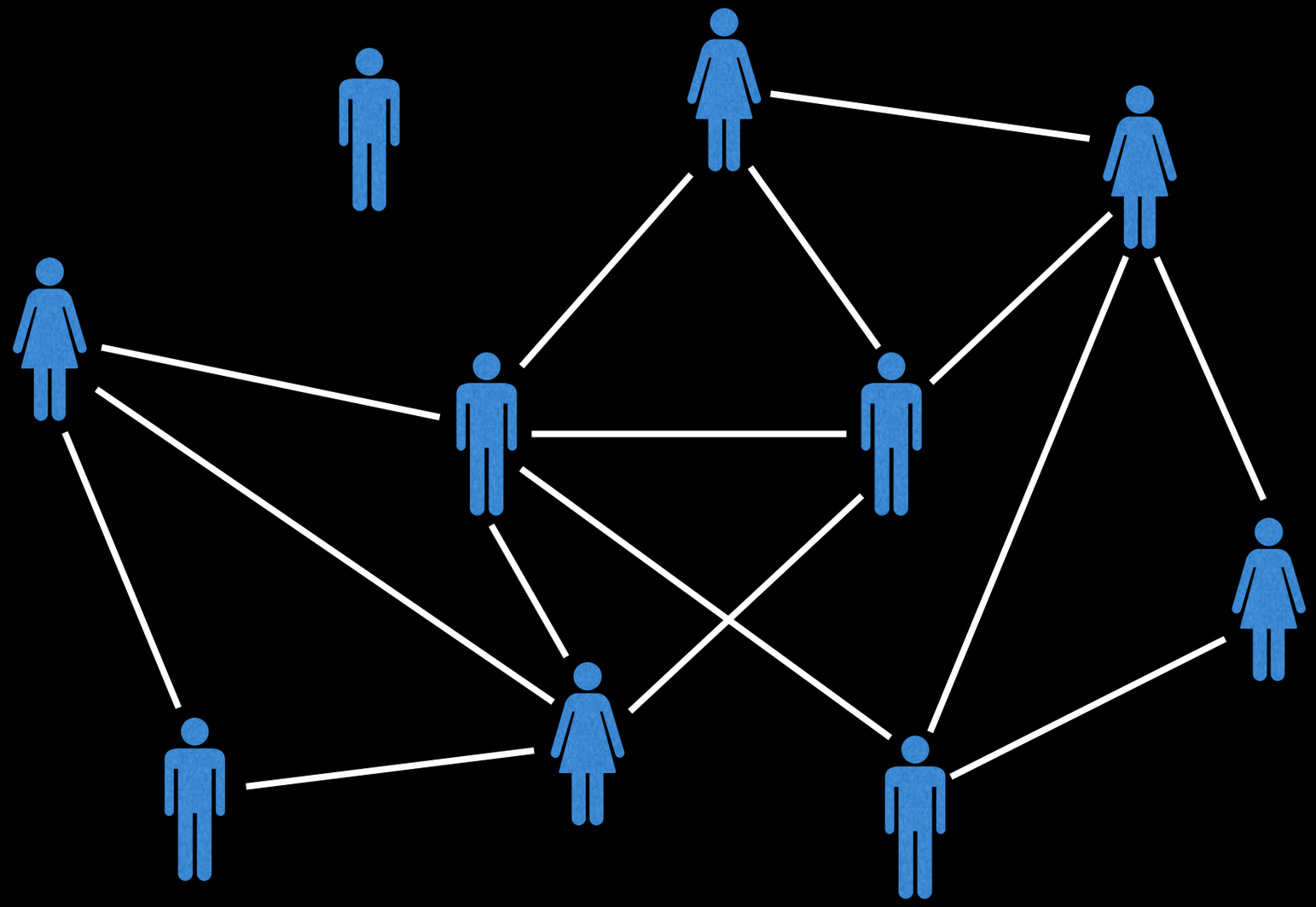
The goal of this series is to gain an understanding of how to apply graph theory to real world applications.

Brief introduction



A graph theory problem might be:
Given the constraints above, how many different sets of clothing can I make by choosing an article from each category?

Brief introduction



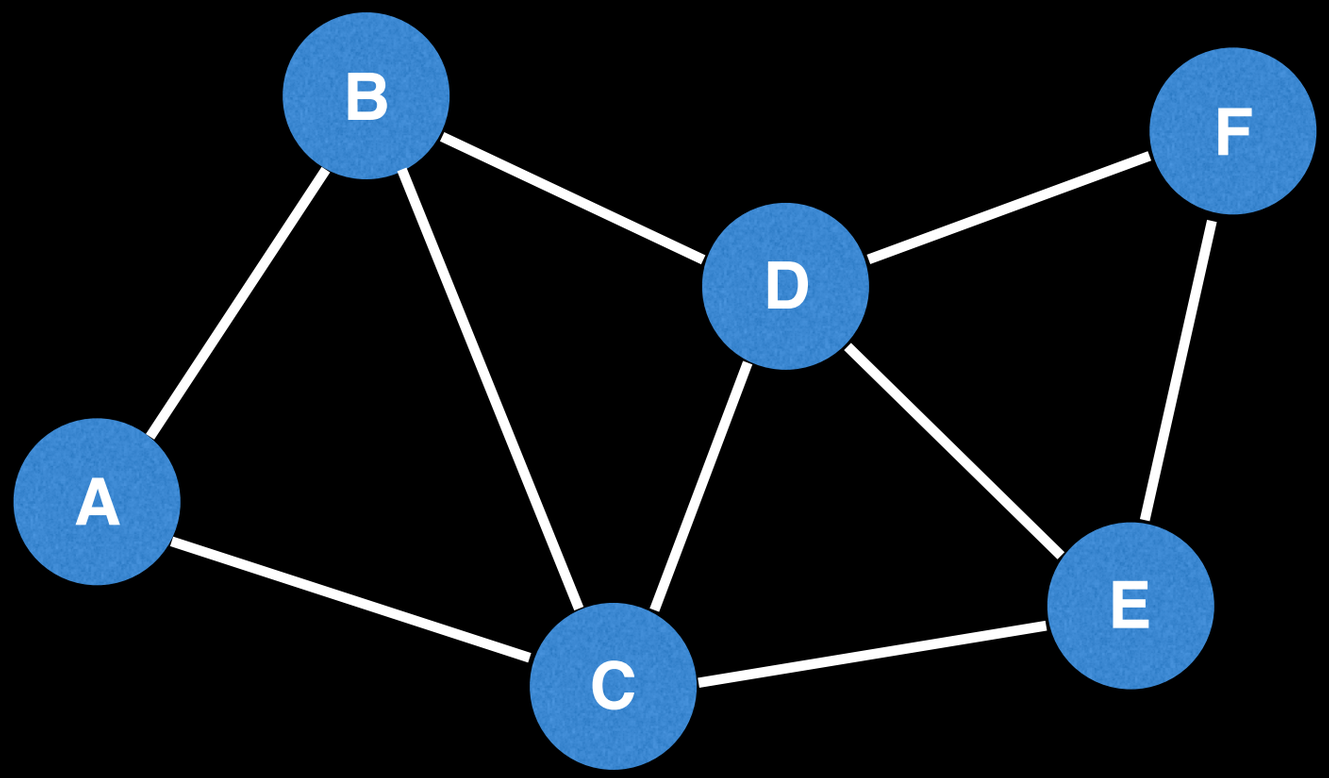
The canonical graph theory example is
a social network of friends.

This enables interesting questions such as: how many friends does person X have? Or how many degrees of separation are there between person X and person Y?

Types of Graphs

Undirected Graph

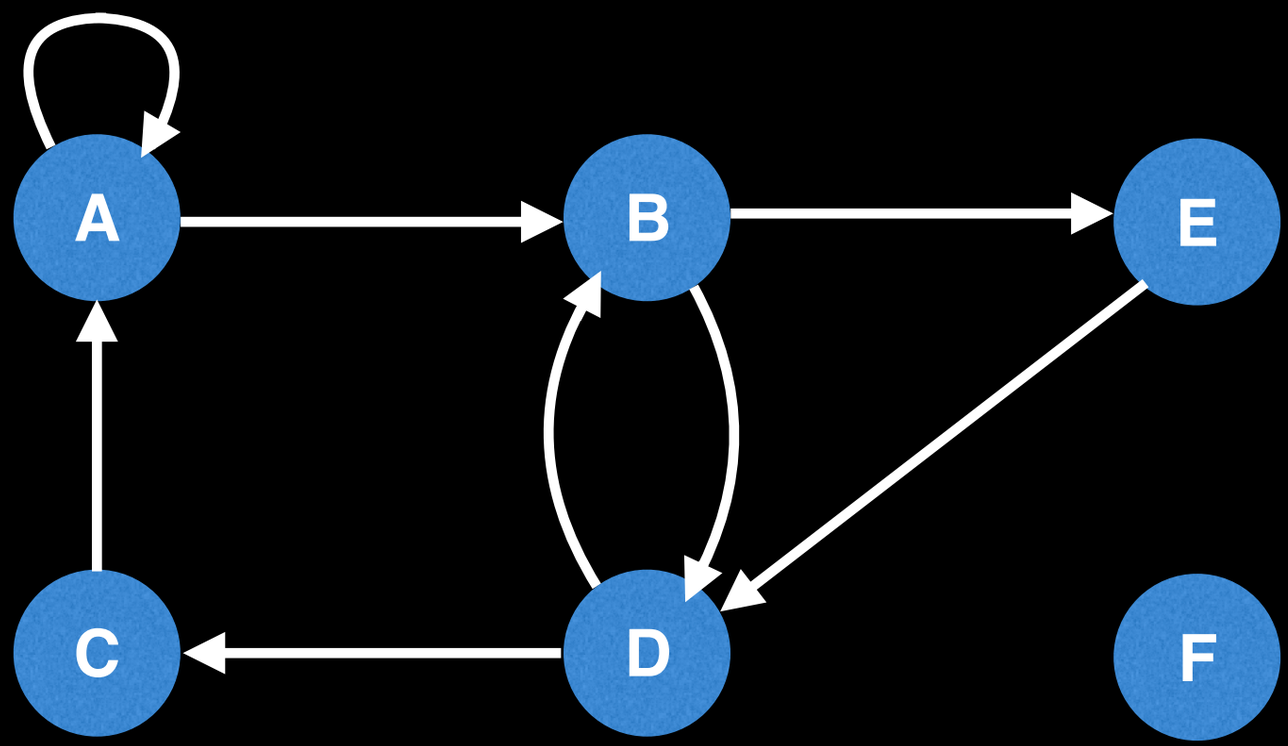
An **undirected graph** is a graph in which edges have no orientation. The edge (*u*, *v*) is identical to the edge (*v*, *u*). – Wiki



In the graph above, the nodes could represent cities and an edge could represent a bidirectional road.

Directed Graph (Digraph)

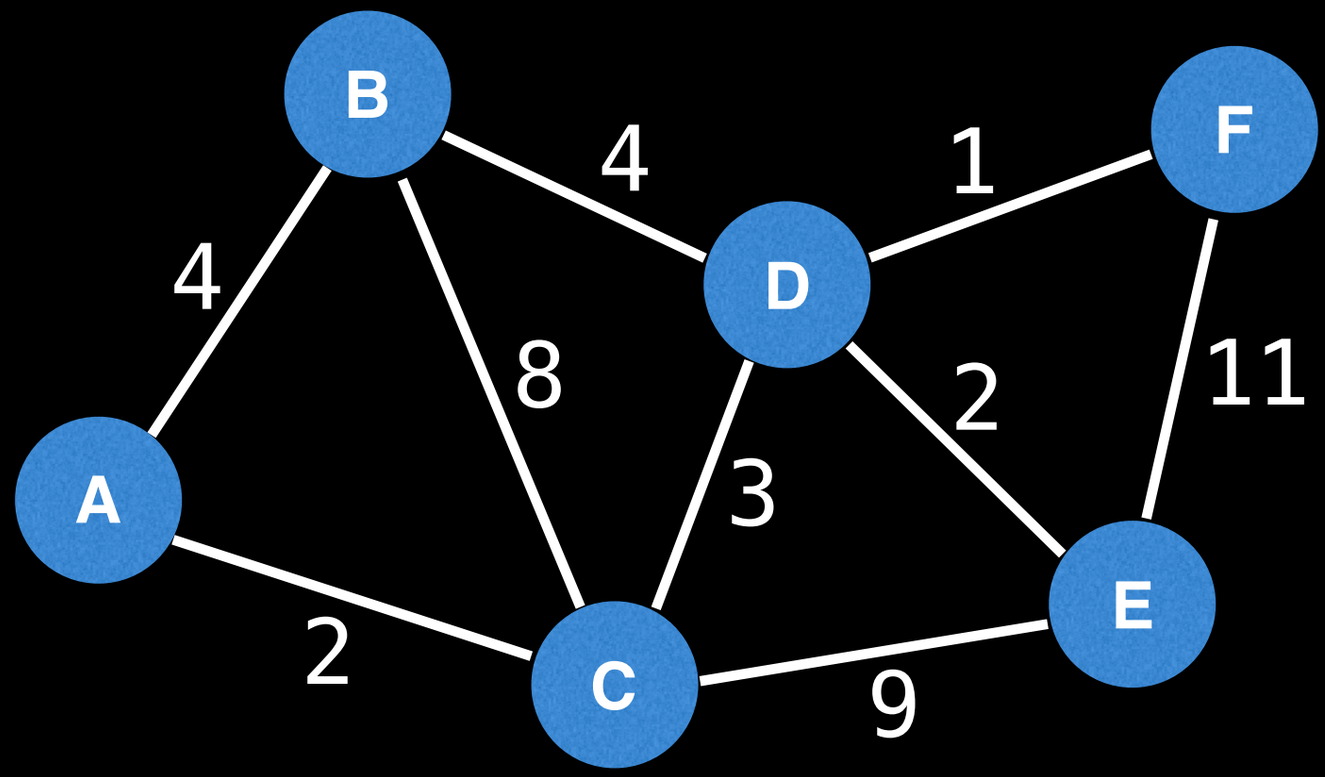
A ***directed graph*** or ***digraph*** is a graph in which edges have orientations. For example, the edge (u, v) is the edge *from* node u *to* node v.



In the graph above, the nodes could represent people and an edge (u, v) could represent that person u bought person v a gift.

Weighted Graphs

Many graphs can have edges that contain a certain weight to represent an arbitrary value such as cost, distance, quantity, etc…

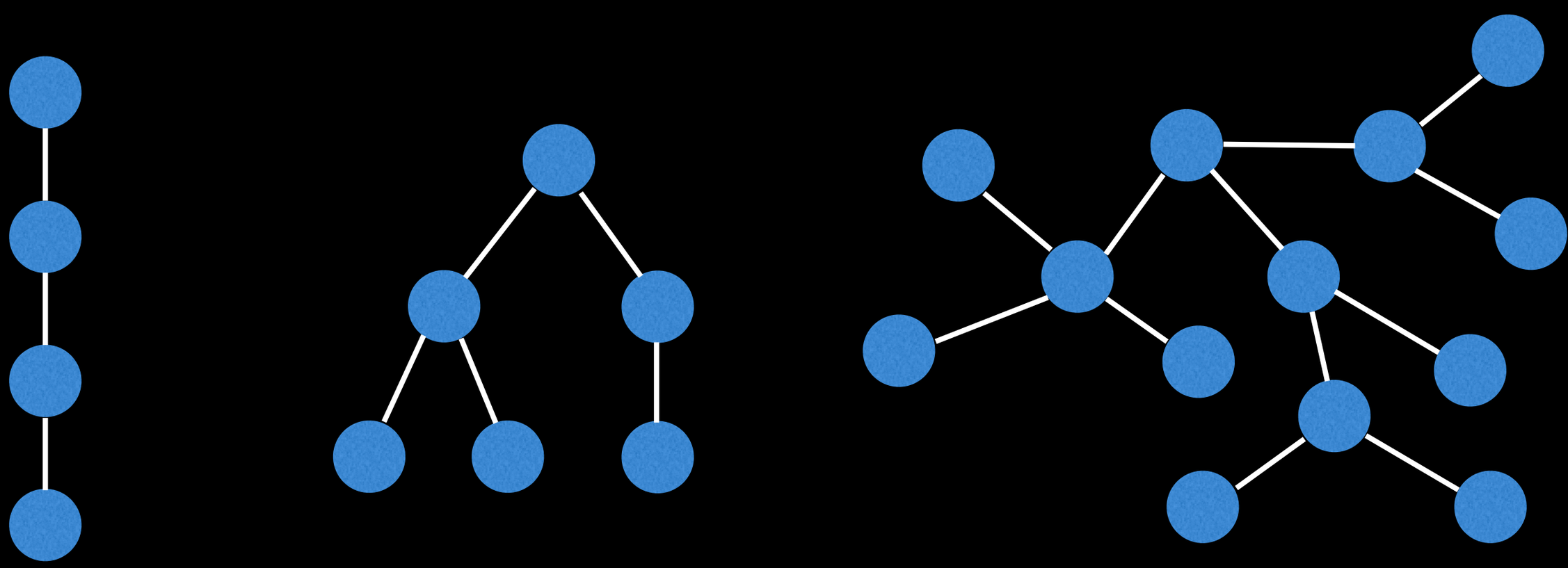


NOTE: I will usually denote an edge of such a graph as a triplet (u, v, w) and specify whether the graph is directed or undirected.

Special Graphs

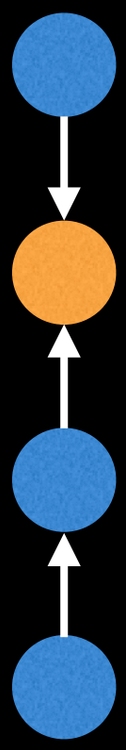
Trees!

A **tree** is an **undirected graph with no cycles**. Equivalently, it is a connected graph with N nodes and N-1 edges.

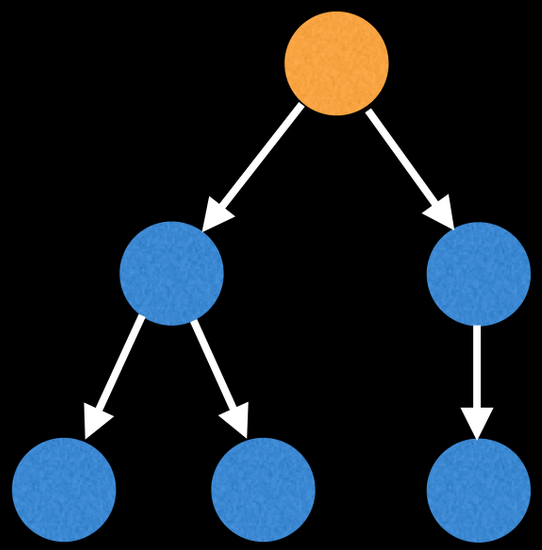


Rooted Trees!

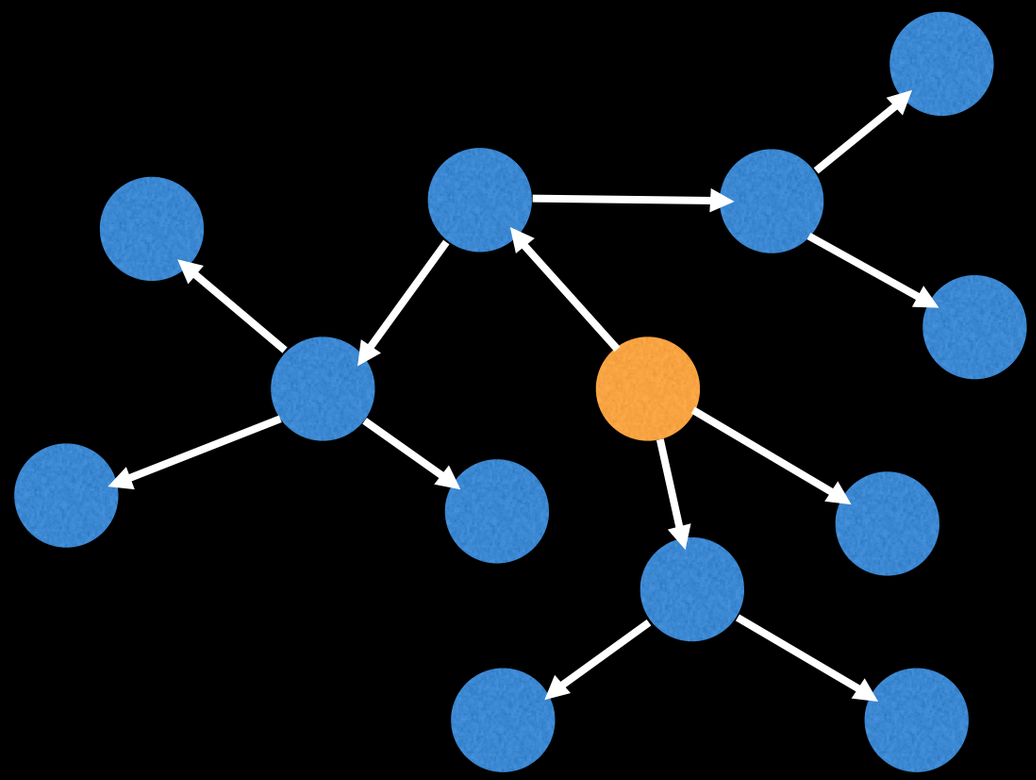
A **rooted tree** is a tree with **a designated root node** where every edge either points away from or towards the root node. When edges point away from the root the graph is called an **arborescence (out-tree)** and anti-arborescence (in-tree) otherwise.



In-tree



Out-tree

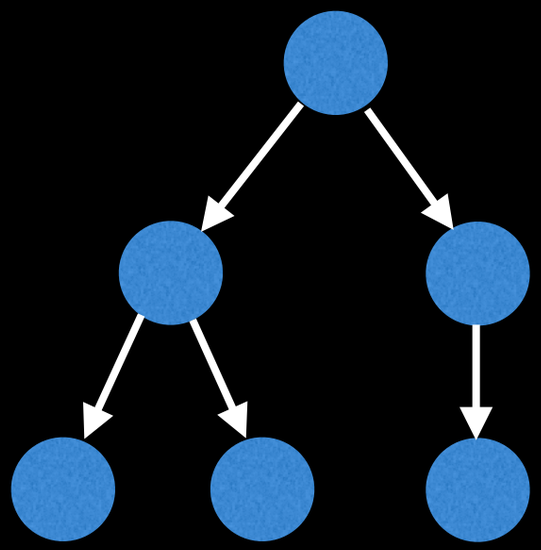


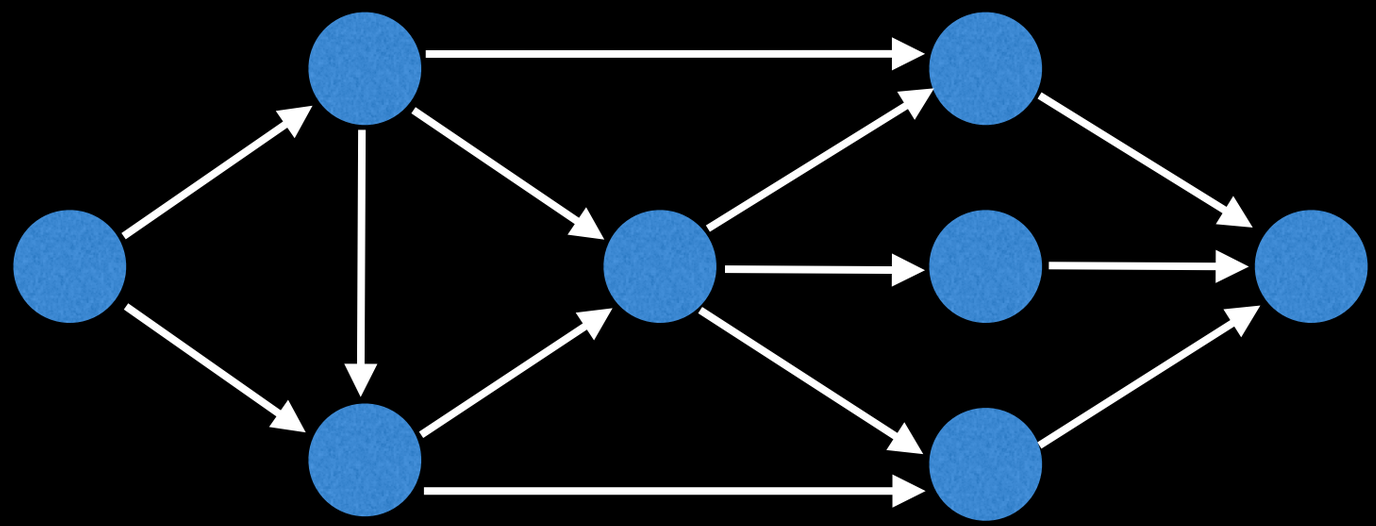
Out-tree

Directed Acyclic Graphs (DAGs)

DAGs are **directed graphs with no cycles**.
These graphs play an important role in representing structures with dependencies.
Several efficient algorithms exist to operates on DAGs.

Cool fact: All out-trees are DAGs but **not all DAGs are out-trees.**

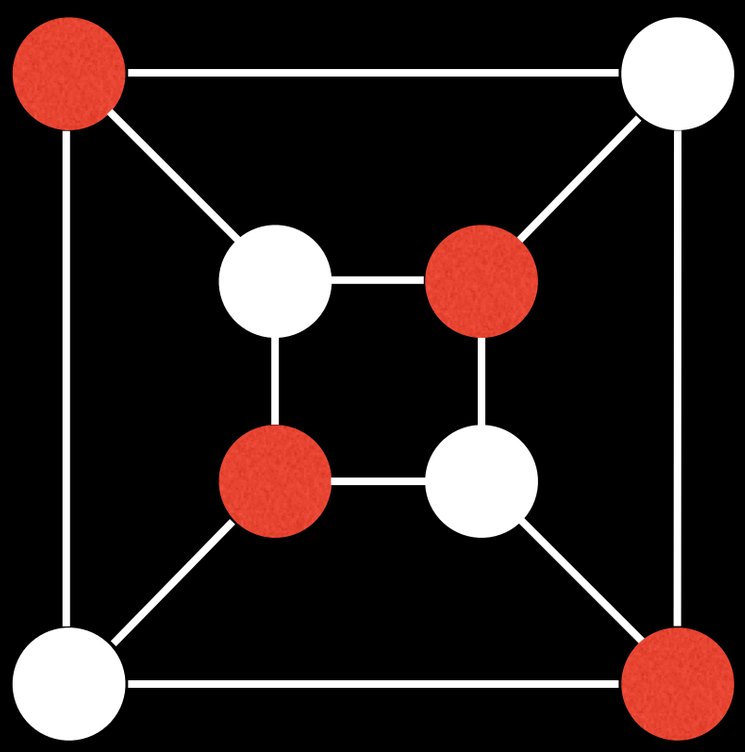


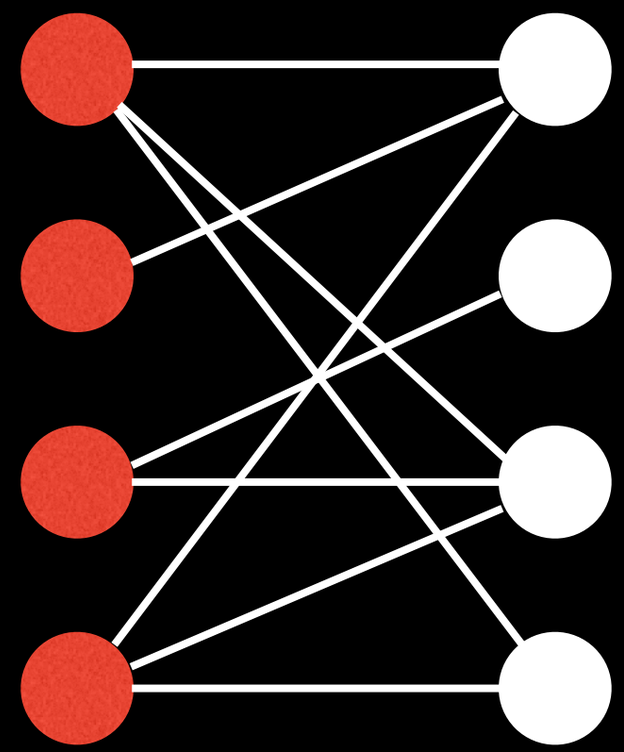


Bipartite Graph

A **bipartite graph** is one whose *vertices* can be split into two independent groups U, V such that every edge connects betweens U and V.

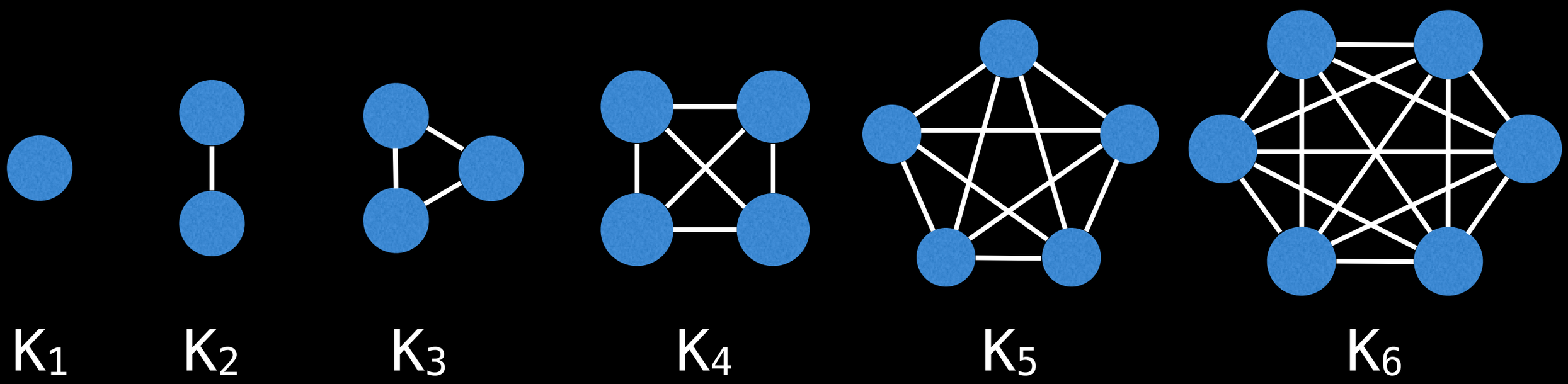
Other definitions exist such as: The graph is two colourable or there is no odd length cycle.





Complete Graphs

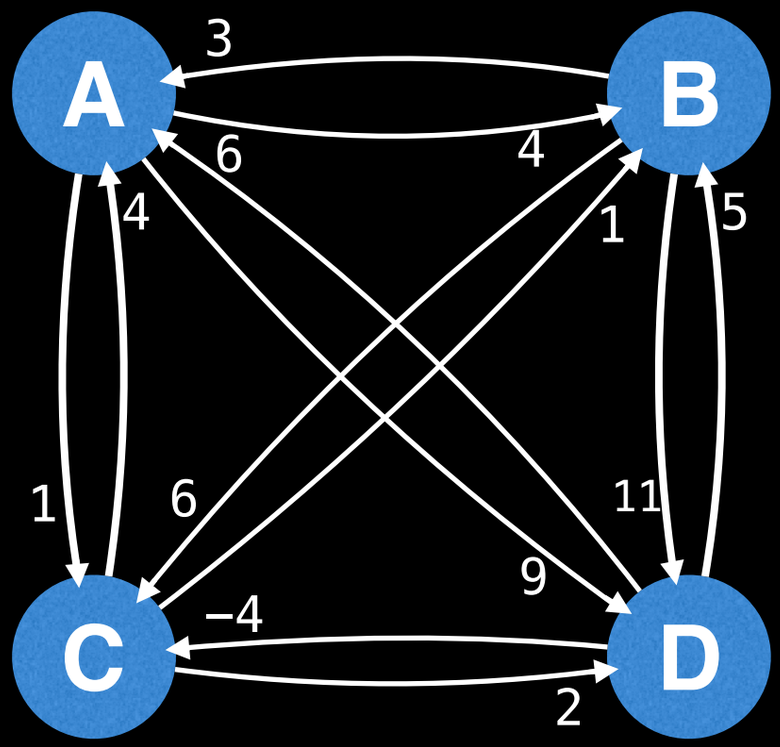
A **complete graph** is one where there is a unique edge between every pair of nodes. A complete graph with n vertices is denoted as the graph K_n.



Representing Graphs

Adjacency Matrix

A **adjacency matrix** *m* is a very simple way to represent a graph. The idea is that the cell m[i][j] represents the edge weight of going from node i to node j.



	A	B	C	D
A	0	4	1	9
B	3	0	6	11
C	4	1	0	2
D	6	5	-4	0

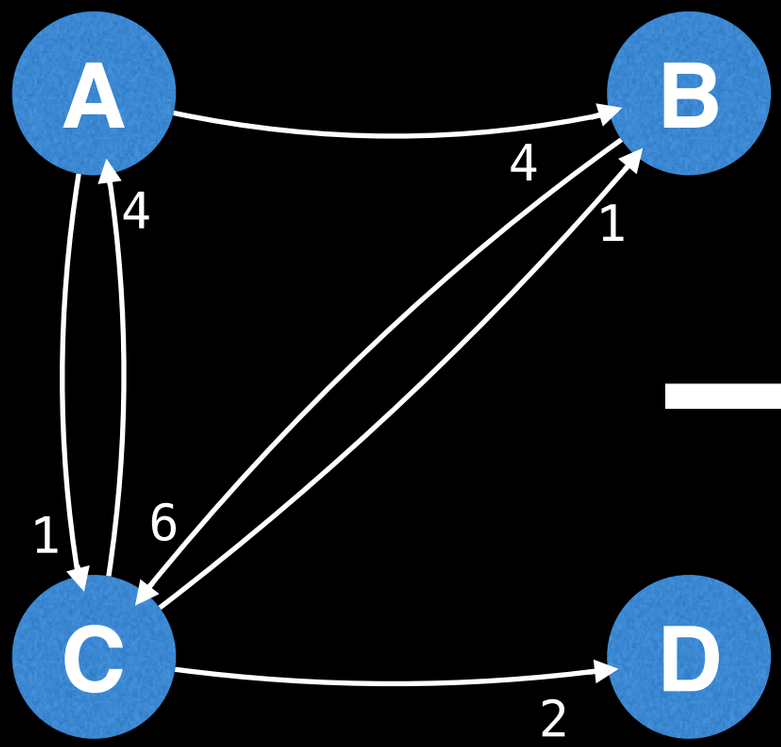
NOTE: It is often assumed that the edge of going from a node to itself has a cost of zero.

Adjacency Matrix

Pros	Cons
Space efficient for representing dense graphs	Requires **O(V²)** space
Edge weight lookup is **O(1)**	Iterating over all edges takes **O(V²)** time
Simplest graph representation	

Adjacency List

An **adjacency list** is a way to represent a graph as a map from nodes to lists of edges.




```
A -> [(B,4),(C,1)]
B -> [(C,6)]
C -> [(A,4),(B,1),(D,2)]
D -> []
```

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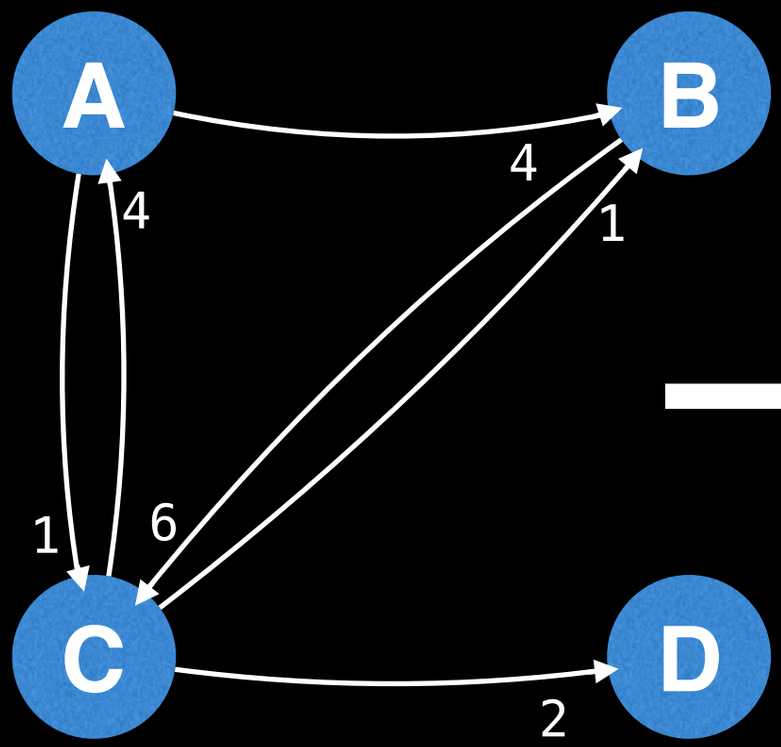



```
A -> [(B,4),(C,1)]
B -> [(C,6)]
C -> [(A,4),(B,1),(D,2)]
D -> []
```

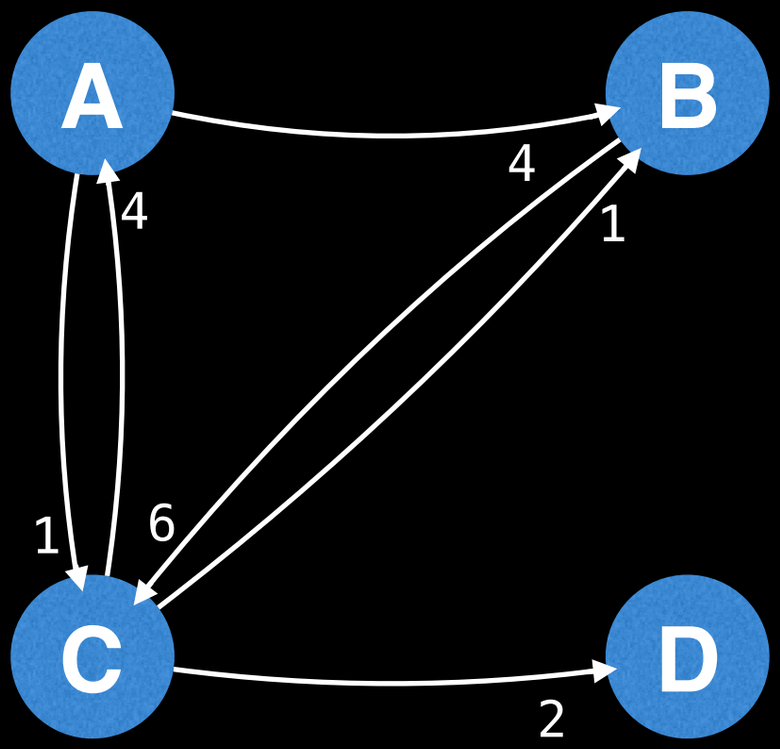
Node C can reach
- Node A with cost 4
- Node B with cost 1
- Node D with cost 2

Adjacency List

Pros	Cons
Space efficient for representing sparse graphs	Less space efficient for denser graphs.
Iterating over all edges is efficient	Edge weight lookup is **O(E)**
	Slightly more complex graph representation

Edge List

An **edge list** is a way to represent a graph simply as an unordered list of edges. Assume the notation for any triplet (u,v,w) means: “the cost from node u to node v is w”



[(C,A,4), (A,C,1),
(B,C,6), (A,B,4),
(C,B,1), (C,D,2)]

This representation is seldomly used because of its lack of structure. However, it is conceptually simple and practical in a handful of algorithms.

Edge List

Pros	Cons
Space efficient for representing sparse graphs	Less space efficient for denser graphs.
Iterating over all edges is efficient	Edge weight lookup is **O(E)**
Very simple structure	

Common Graph Theory Problems

For the upcoming problems ask yourself:

Is the graph directed or undirected?

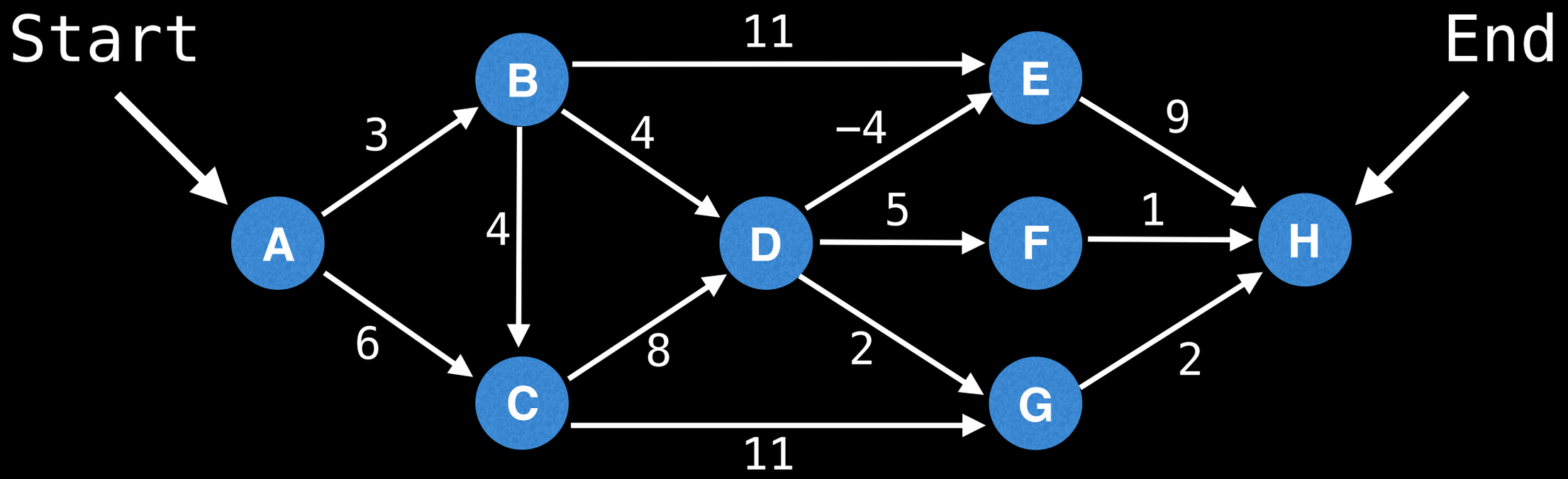
Are the edges of the graph weighted?

Is the graph I will encounter likely to be sparse or dense with edges?

Should I use an adjacency matrix, adjacency list, an edge list or other structure to represent the graph efficiently?

Shortest path problem

Given a weighted graph, find the shortest path of edges from node A to node B.



Algorithms: BFS (unweighted graph), Dijkstra's, Bellman–Ford, Floyd–Warshall, A*, and many more.

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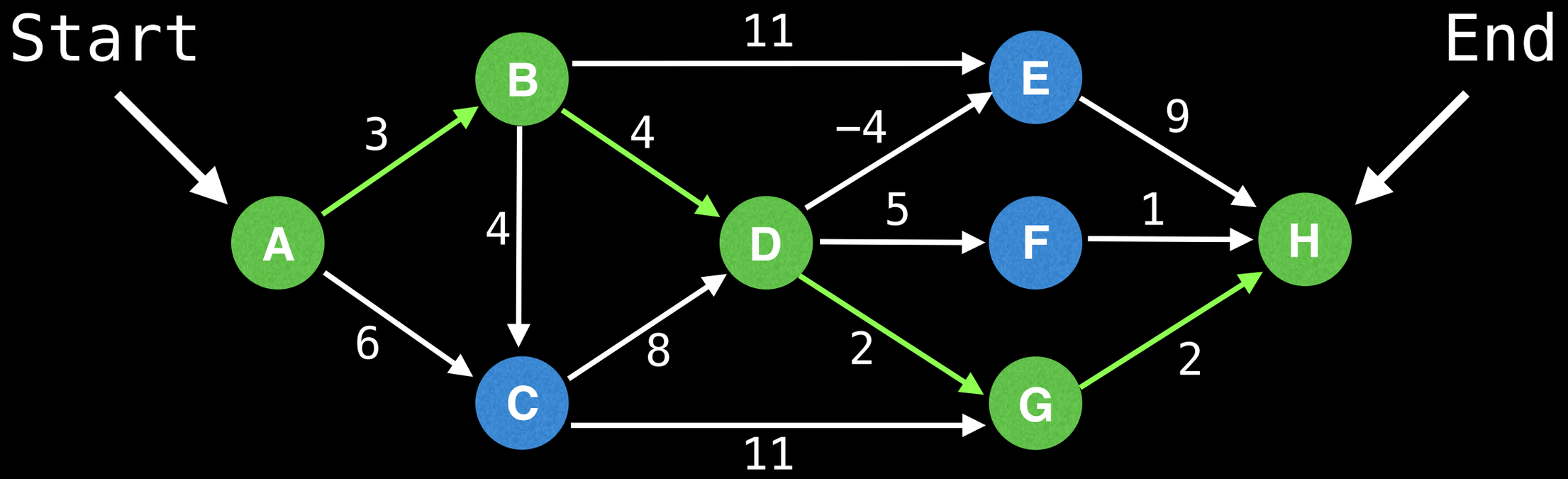
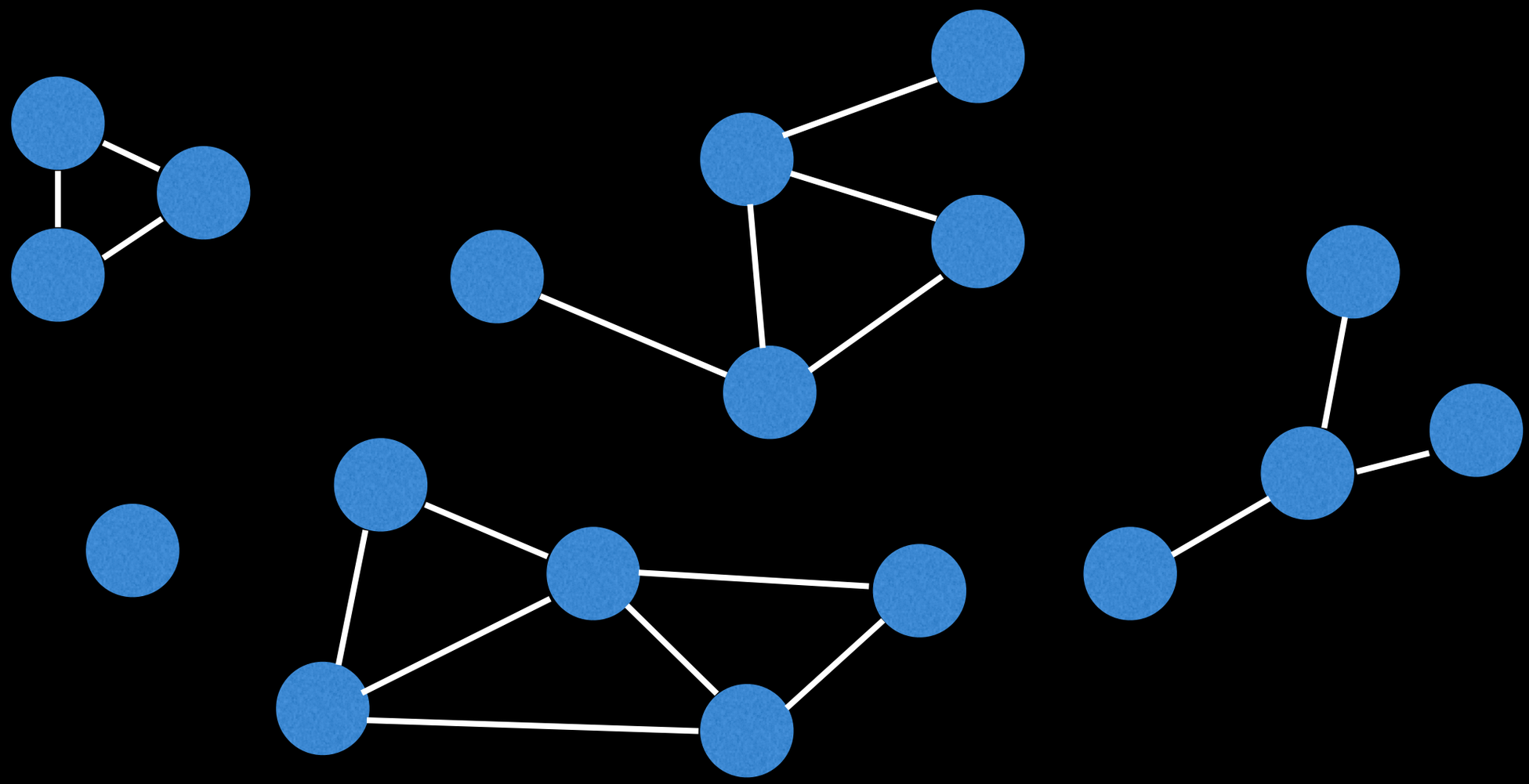


Algorithms: BFS (unweighted graph), Dijkstra's, Bellman–Ford, Floyd–Warshall, A*, and many more.

Connectivity

Does there exist a path between node A and node B?



Typical solution: Use union find data structure or any search algorithm (e.g DFS).

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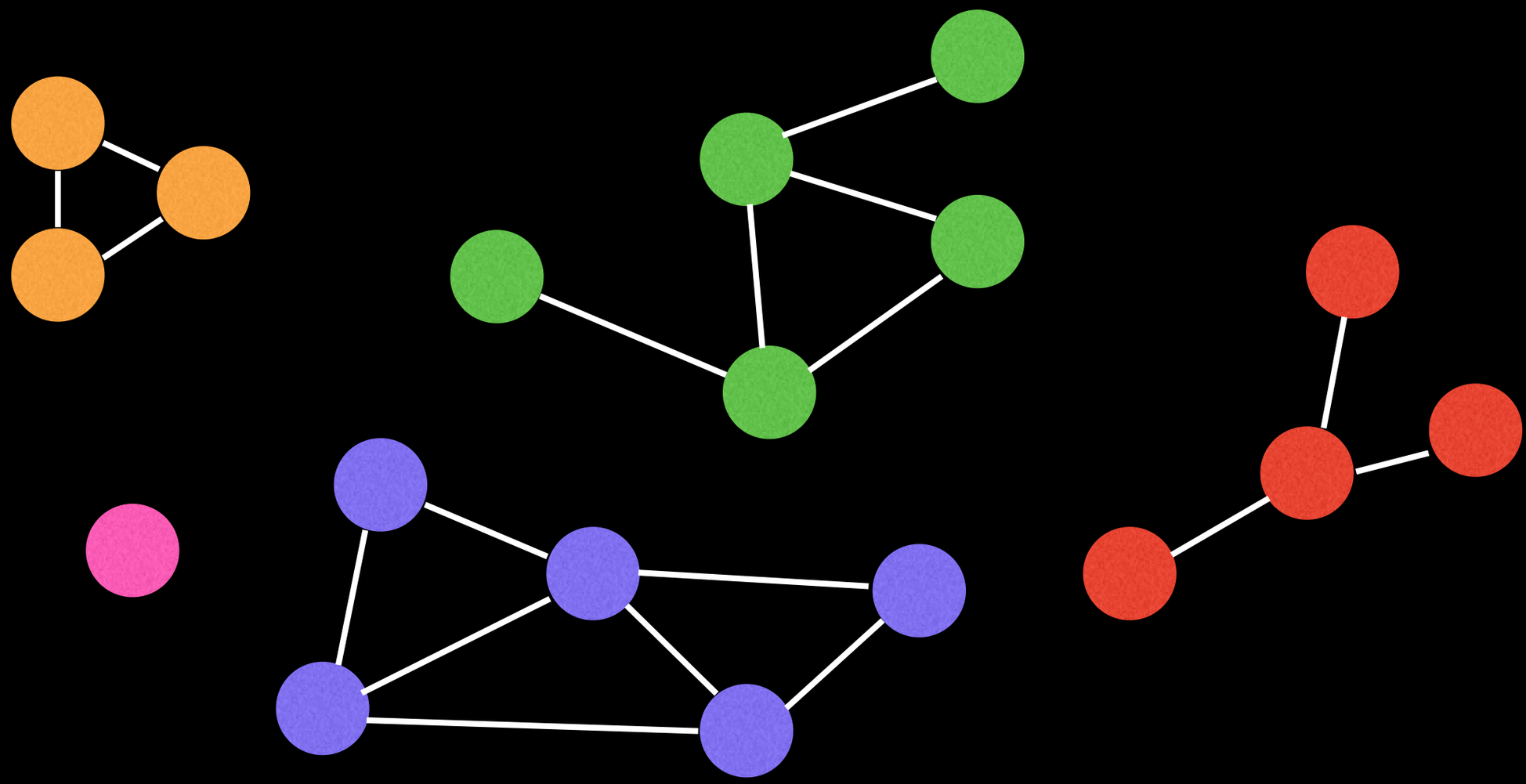
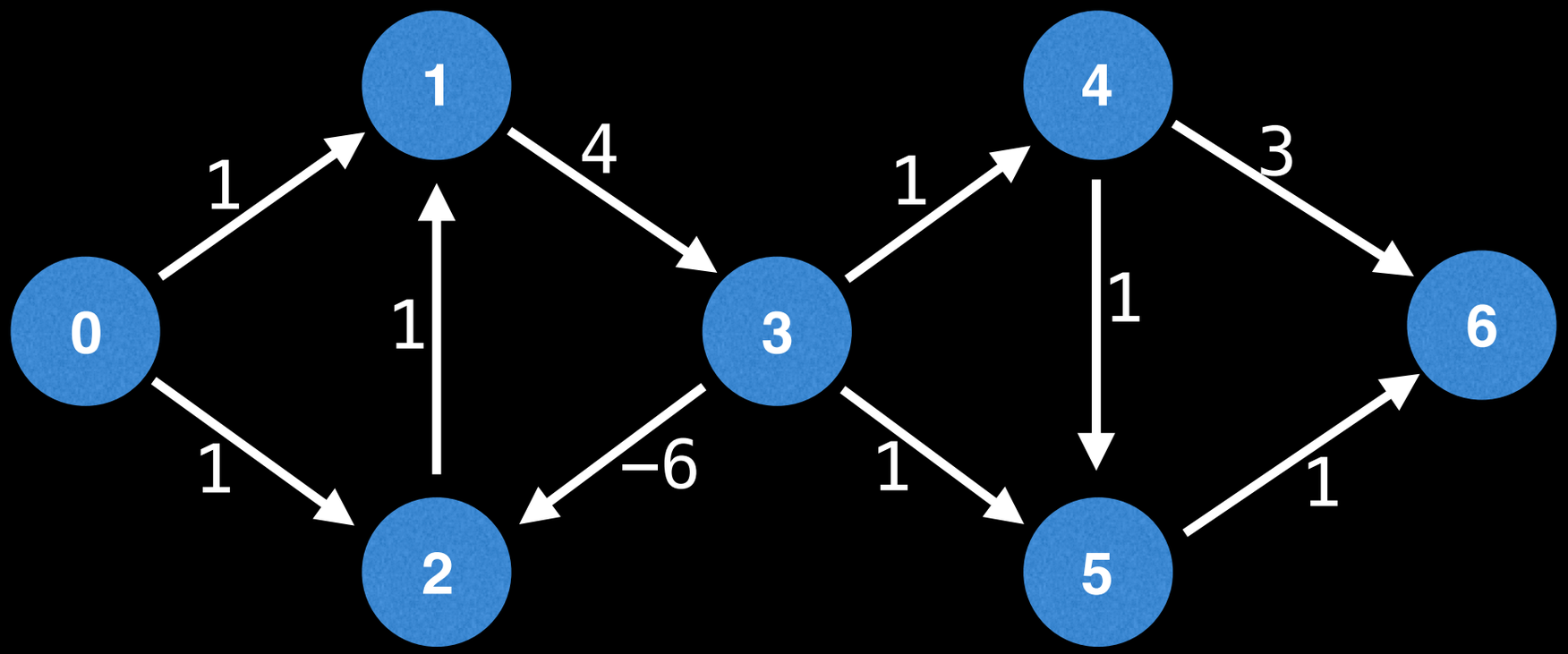


Typical solution: Use union find data structure or any search algorithm (e.g DFS).

Negative cycles

Does my weighted digraph have any negative cycles? If so, where?



Directly in negative cycle

Unaffected node

Algorithms: Bellman–Ford and Floyd–Warshall

Negative cycles

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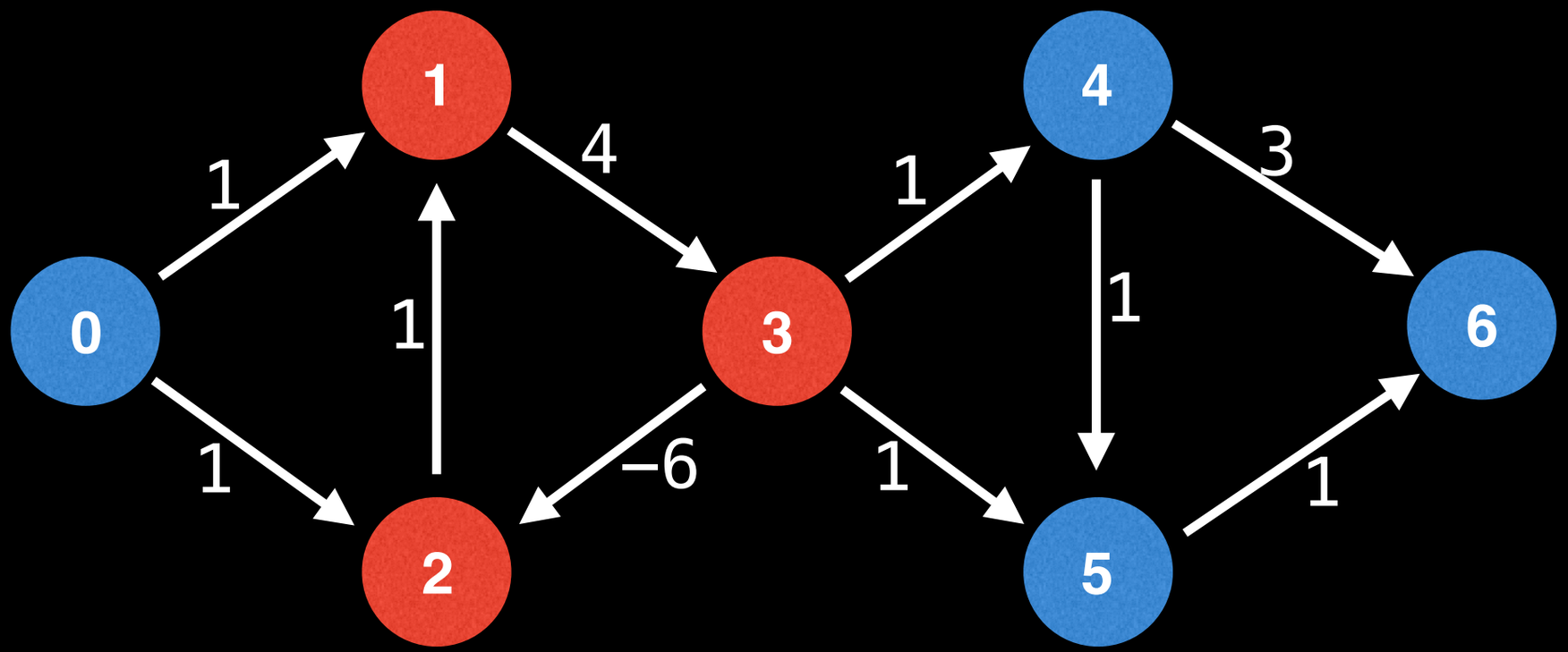
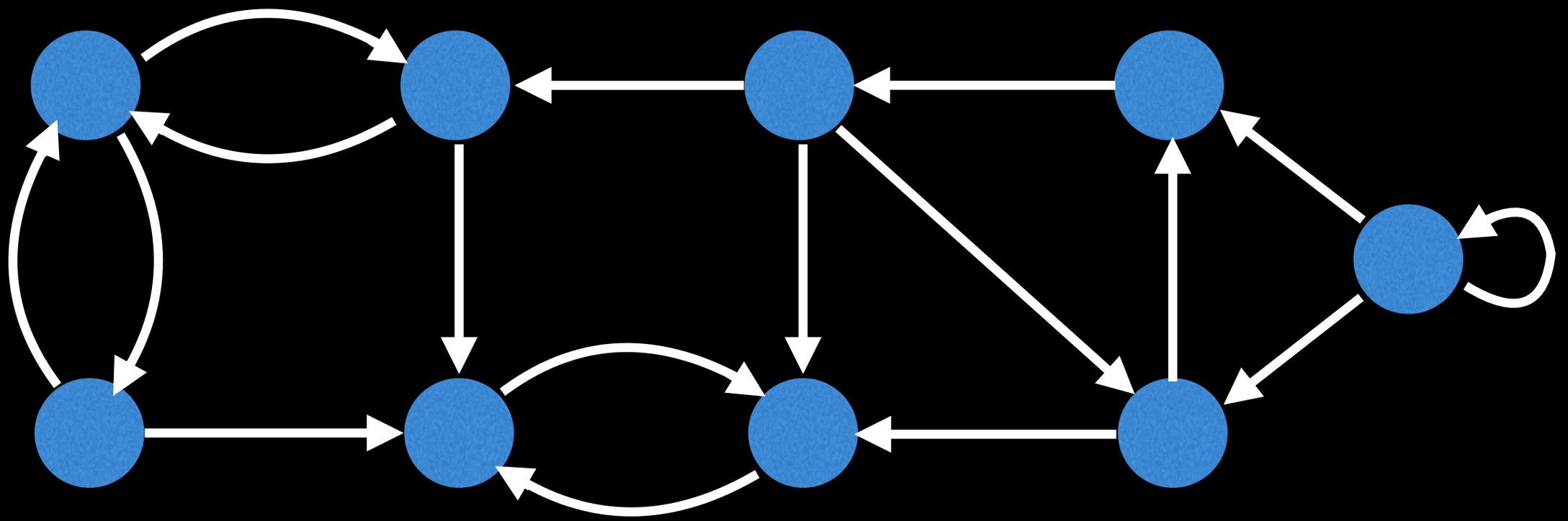


Algorithms: Bellman–Ford and Floyd–Warshall

Strongly Connected Components

Strongly Connected Components (SCCs) can be thought of as **self-contained cycles** within a **directed graph** where every vertex in a given cycle can reach every other vertex in the same cycle.



Algorithms: Tarjan’s and Kosaraju's algorithm

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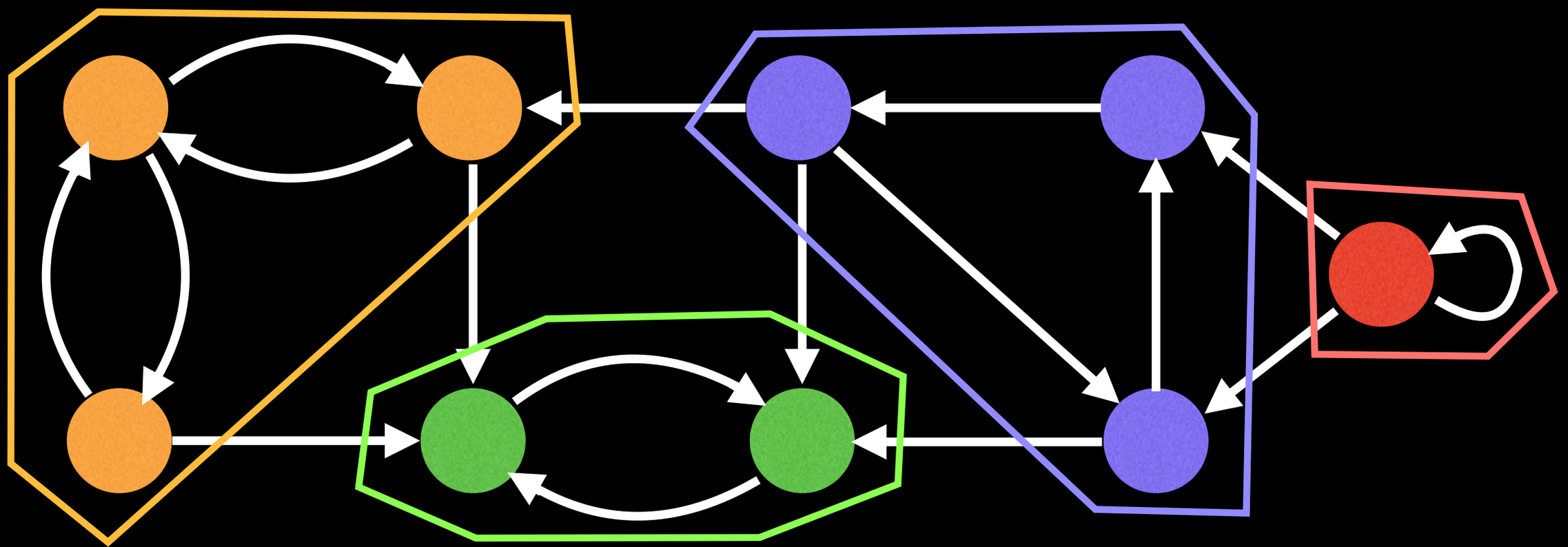
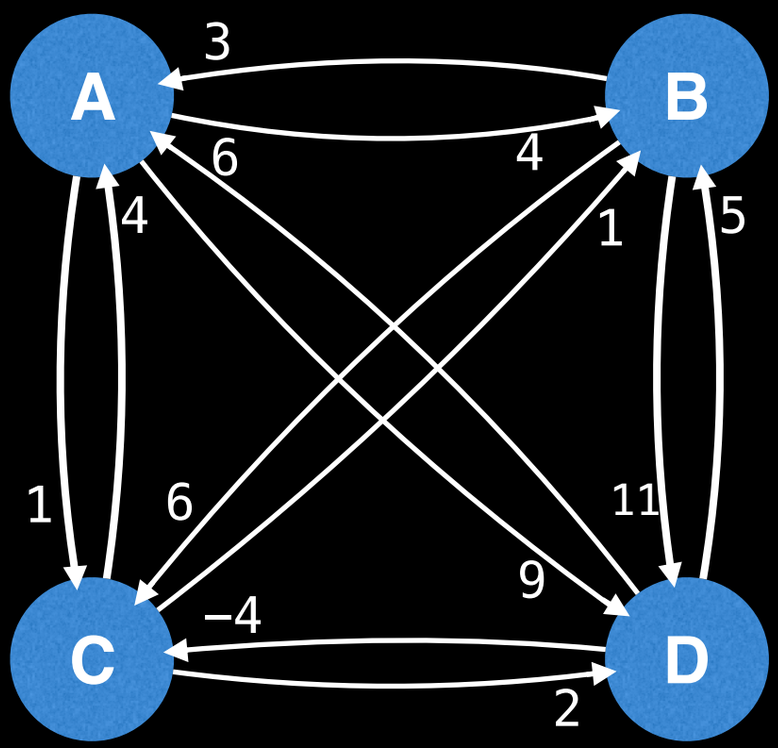
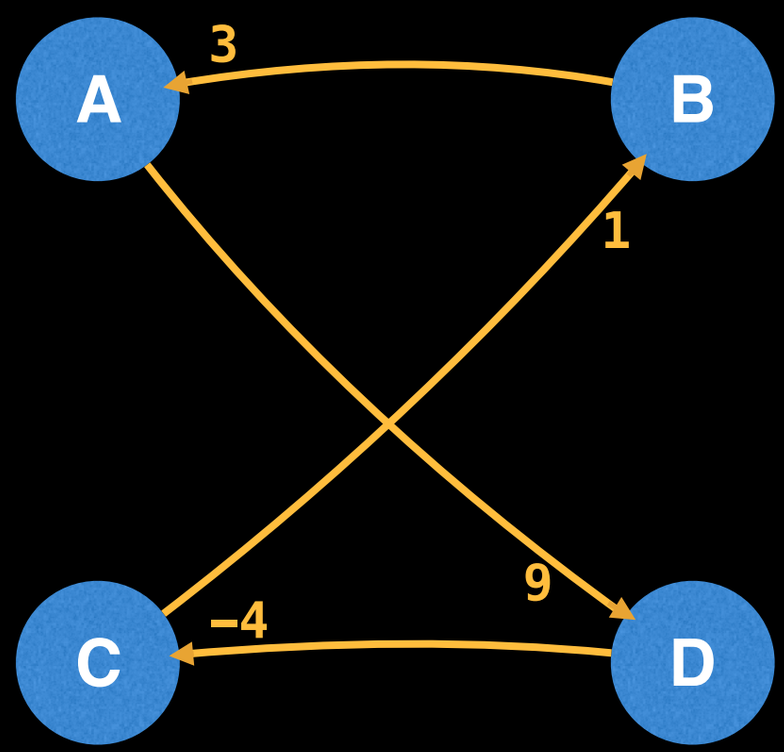


Algorithms: Tarjan’s and Kosaraju's algorithm

Traveling Salesman Problem

"Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city?" – Wiki

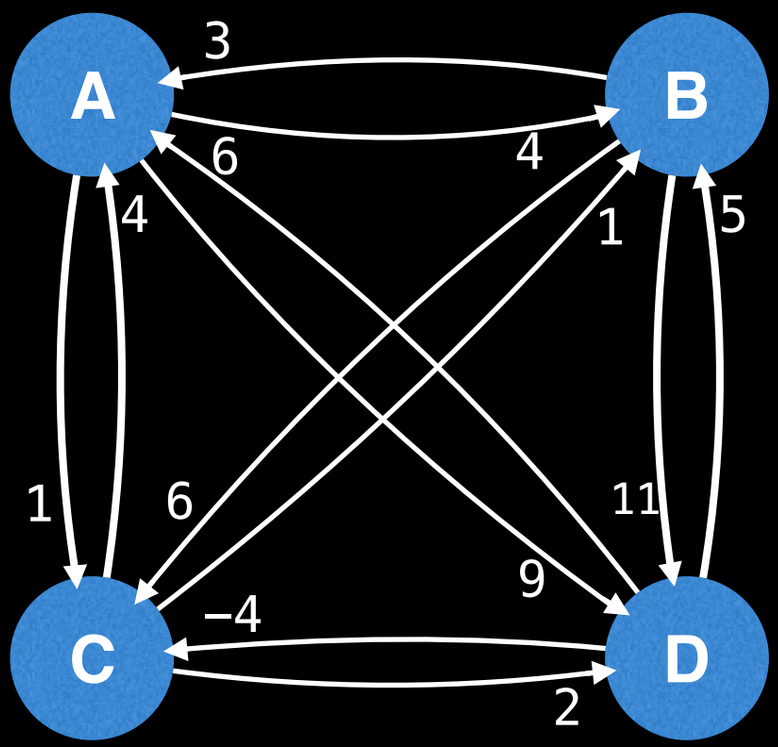


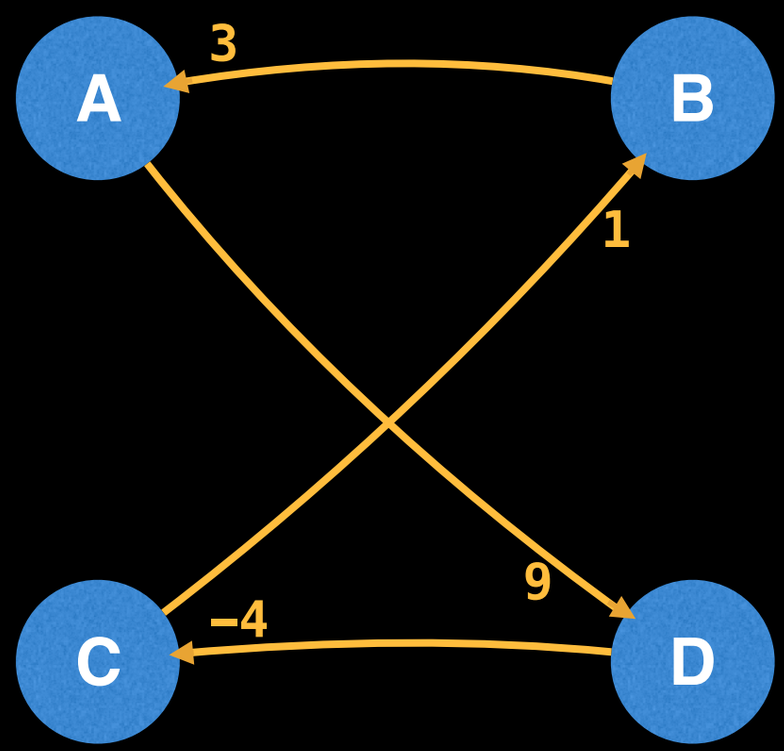


Algorithms: Held–Karp, branch and bound and many approximation algorithms

Traveling Salesman Problem

The TSP problem is NP-Hard meaning it’s a very computationally challenging problem. This is unfortunate because the TSP has several very important applications.

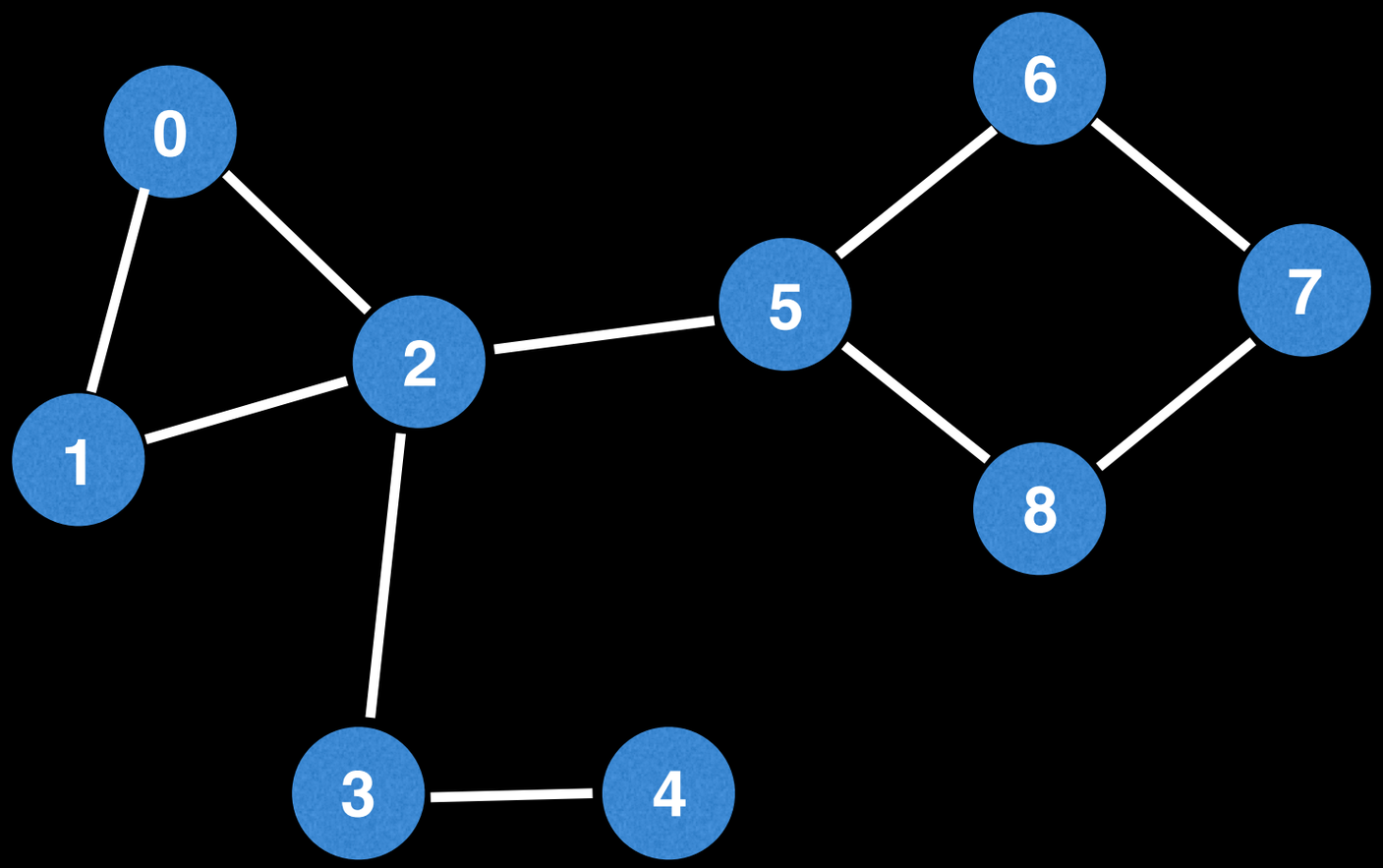




Algorithms: Held-Karp, branch and bound and many approximation algorithms

Bridges

A **bridge / cut edge** is any edge in a graph whose removal increases the number of connected components.



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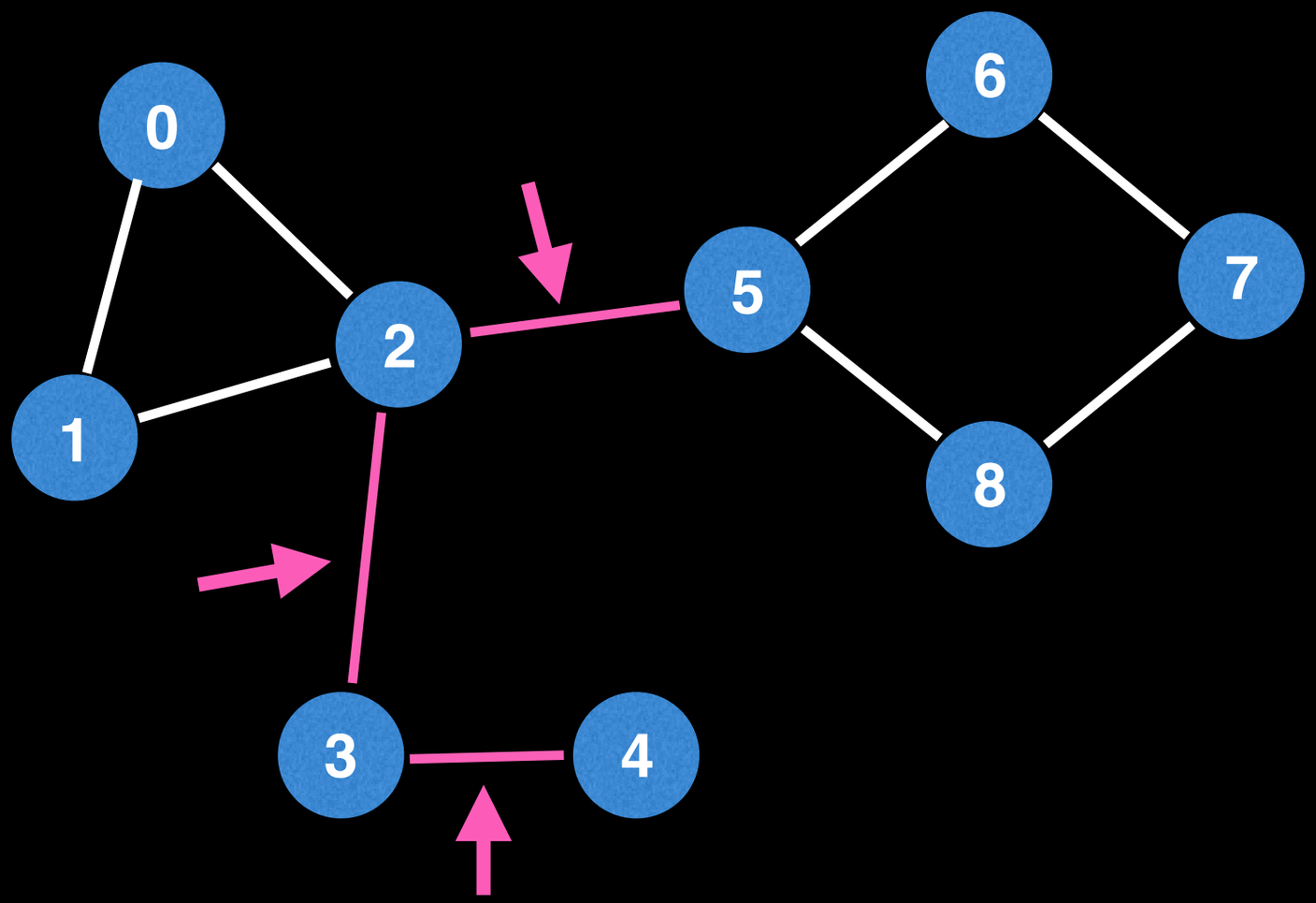
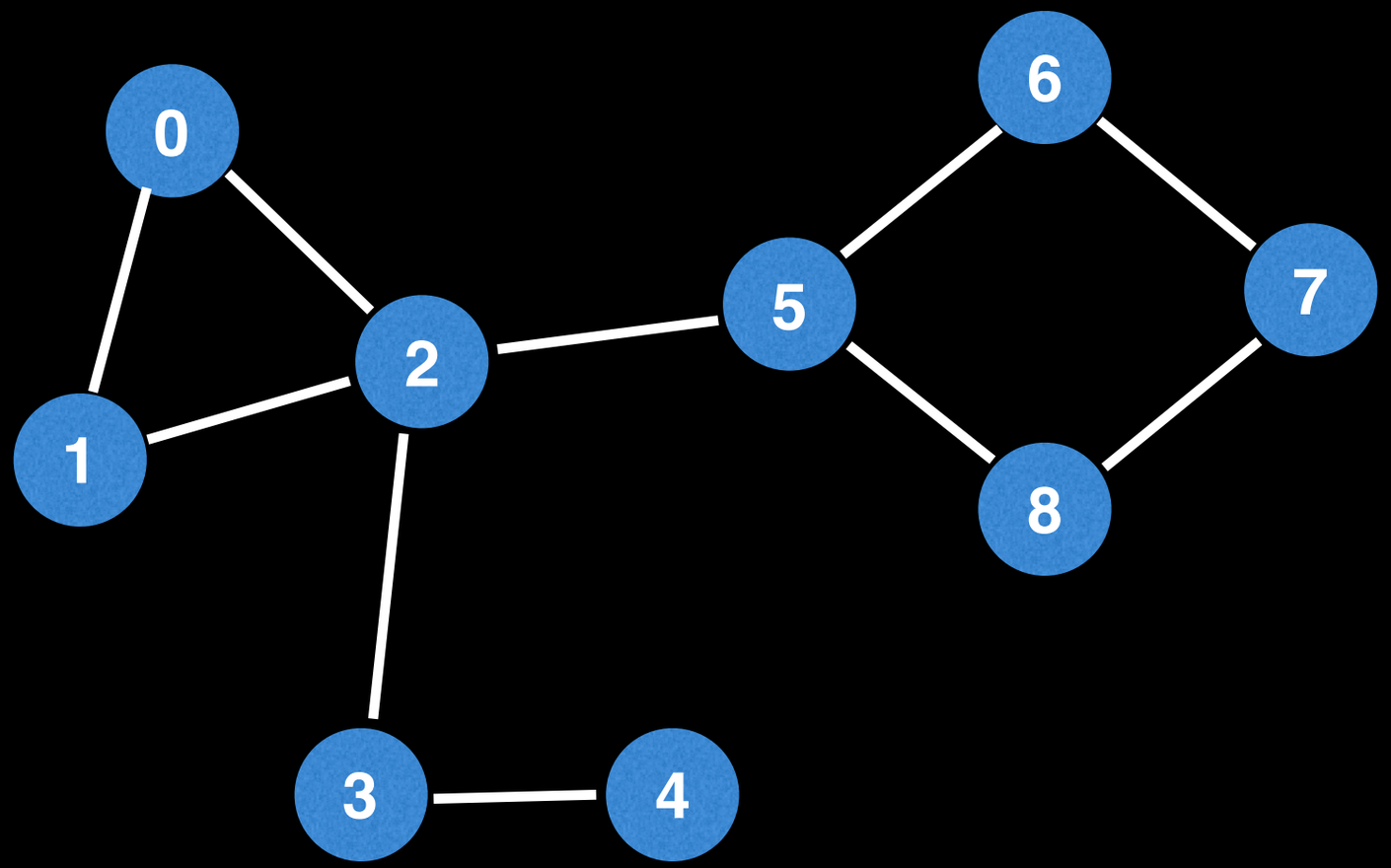


Bridges are important in graph theory because they often hint at weak points, bottlenecks or vulnerabilities in a graph.

Articulation points

An **articulation point / cut vertex** is any node in a graph whose removal increases the number of connected components.



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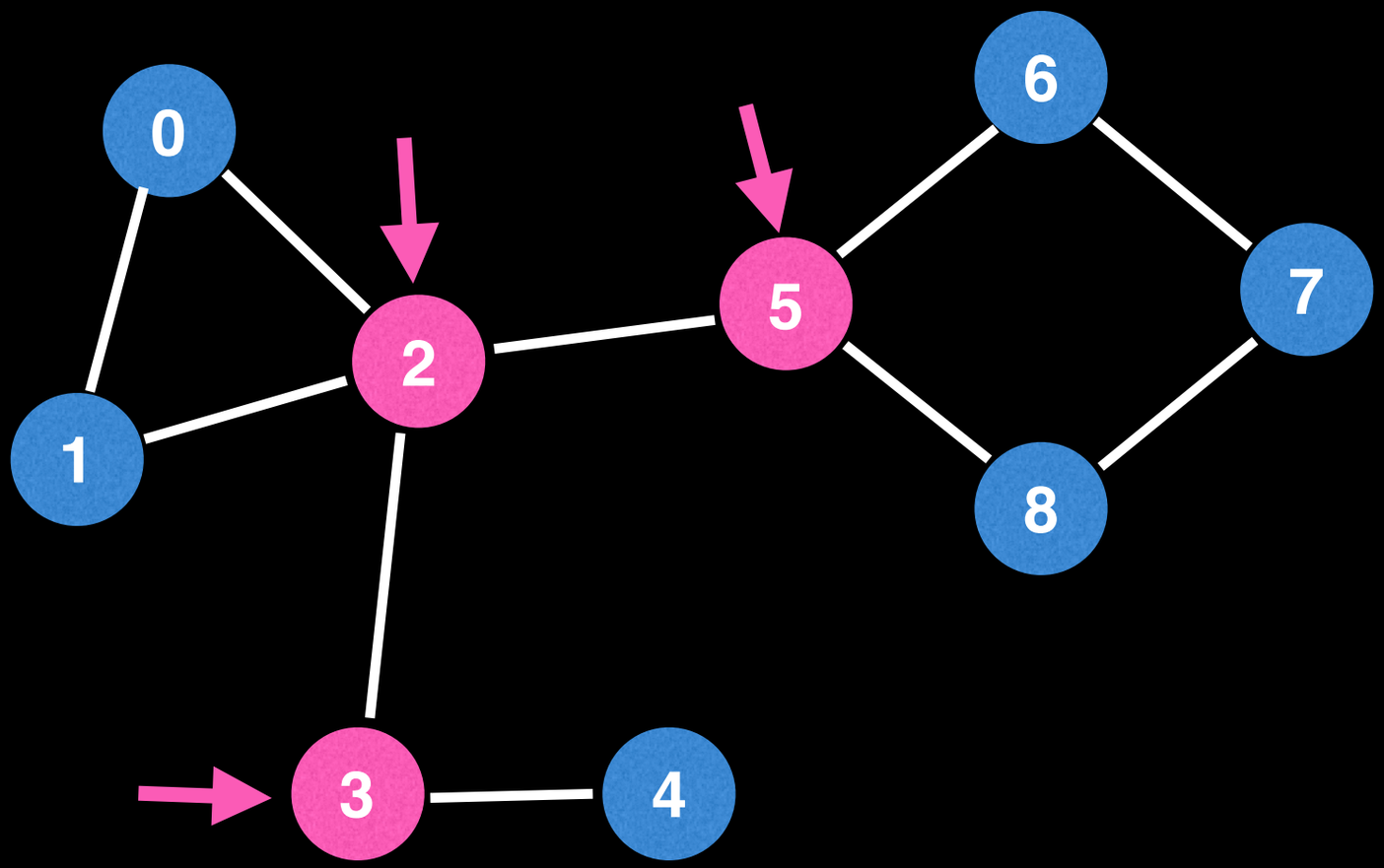
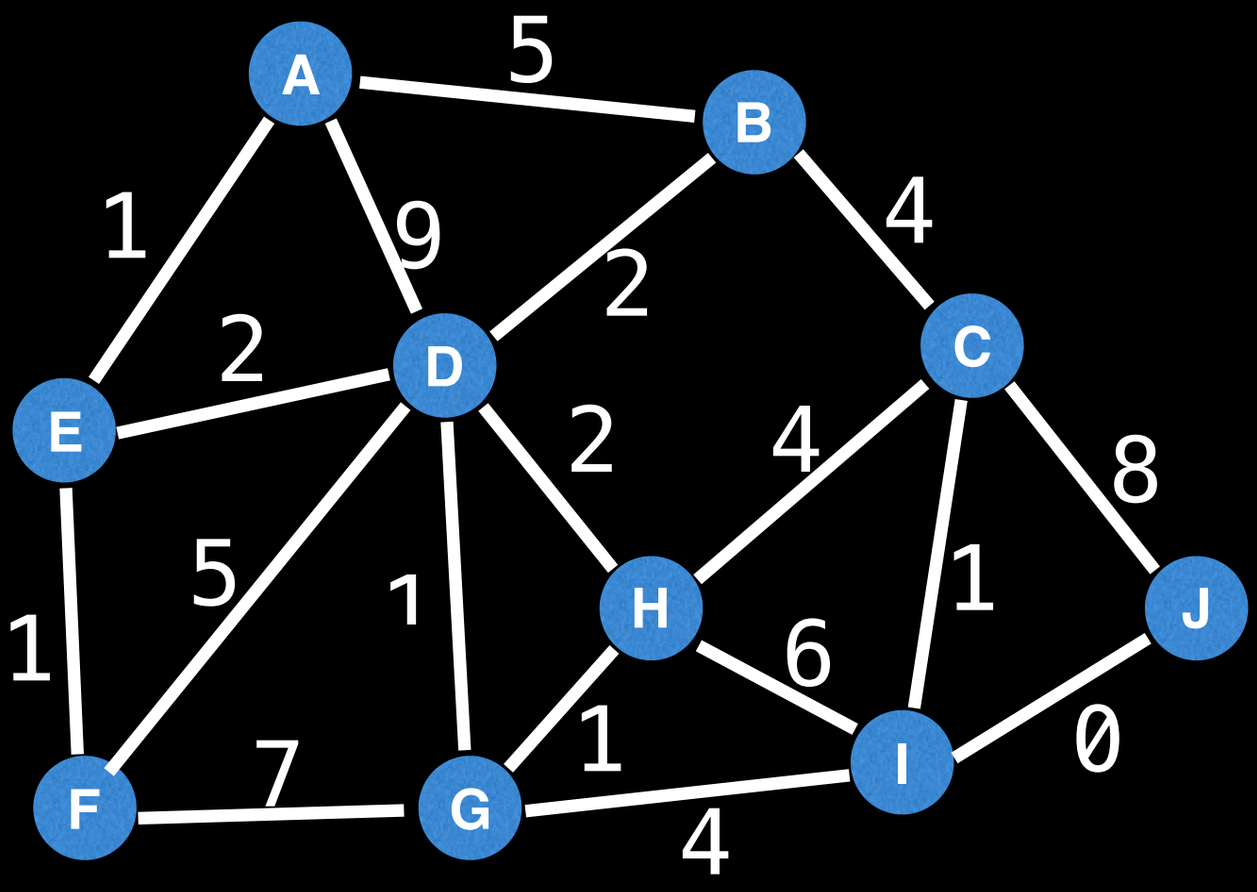


Articulation points are important in graph theory because they often hint at weak points, bottlenecks or vulnerabilities in a graph.

Minimum Spanning Tree (MST)

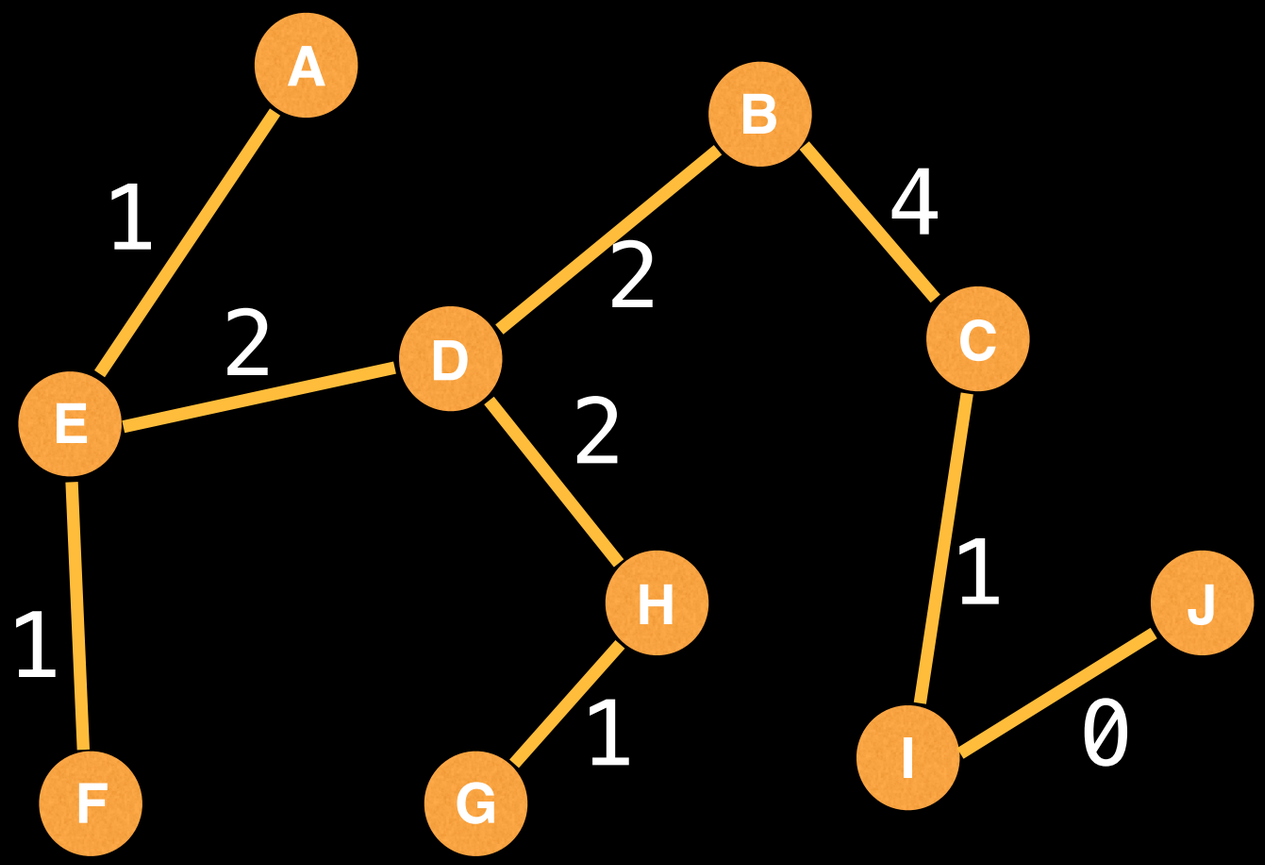
A **minimum spanning tree (MST)** is a subset of the edges of a connected, edge-weighted graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. – Wiki



Algorithms: Kruskal’s, Prim’s & Borůvka's algorithm

Minimum Spanning Tree (MST)

A **minimum spanning tree (MST)** is a subset of the edges of a connected, edge-weighted graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. – Wiki

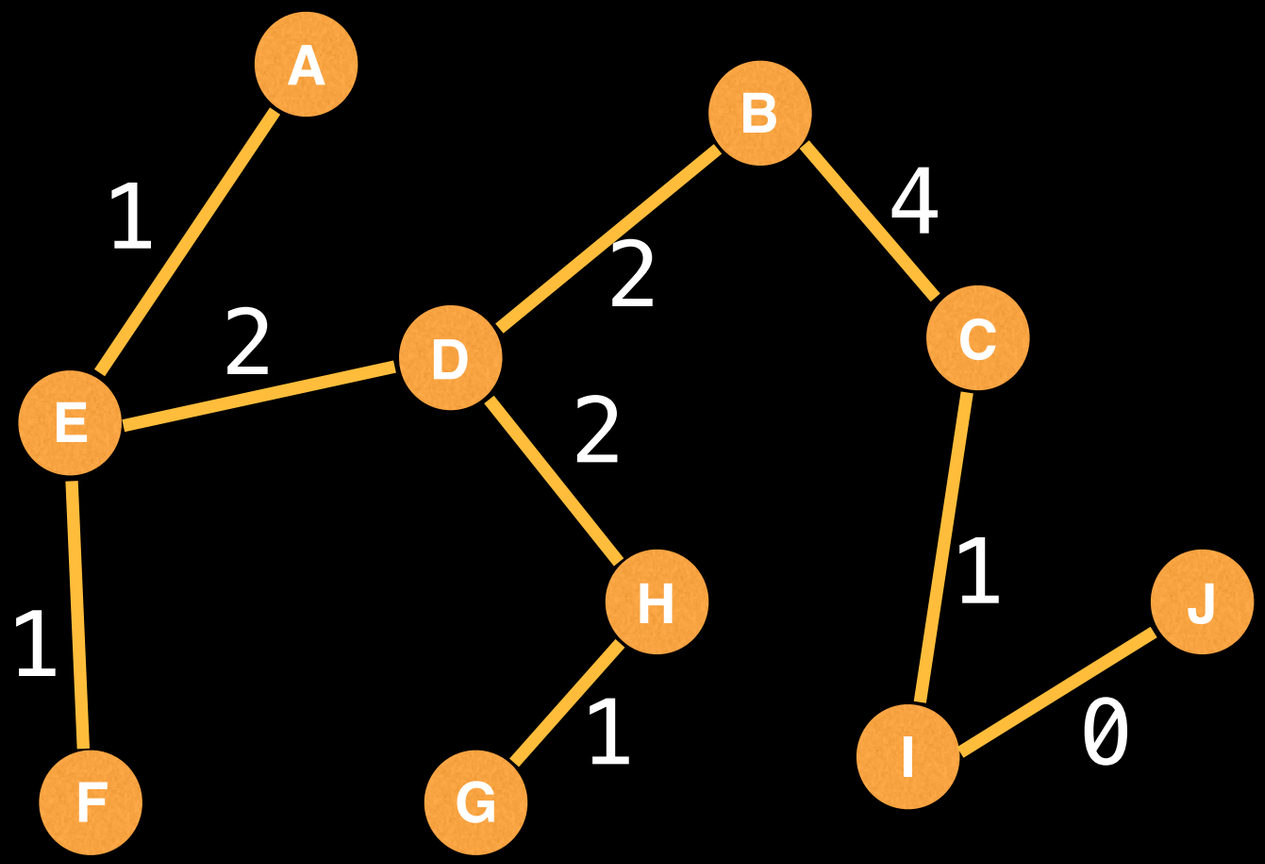


This MST has a total weight of 14. Note that MSTs on a graph are not always unique.

Algorithms: Kruskal’s, Prim’s & Borůvka's algorithm

Minimum Spanning Tree (MST)

MSTs are seen in many applications including: Designing a least cost network, circuit design, transportation networks, and etc…



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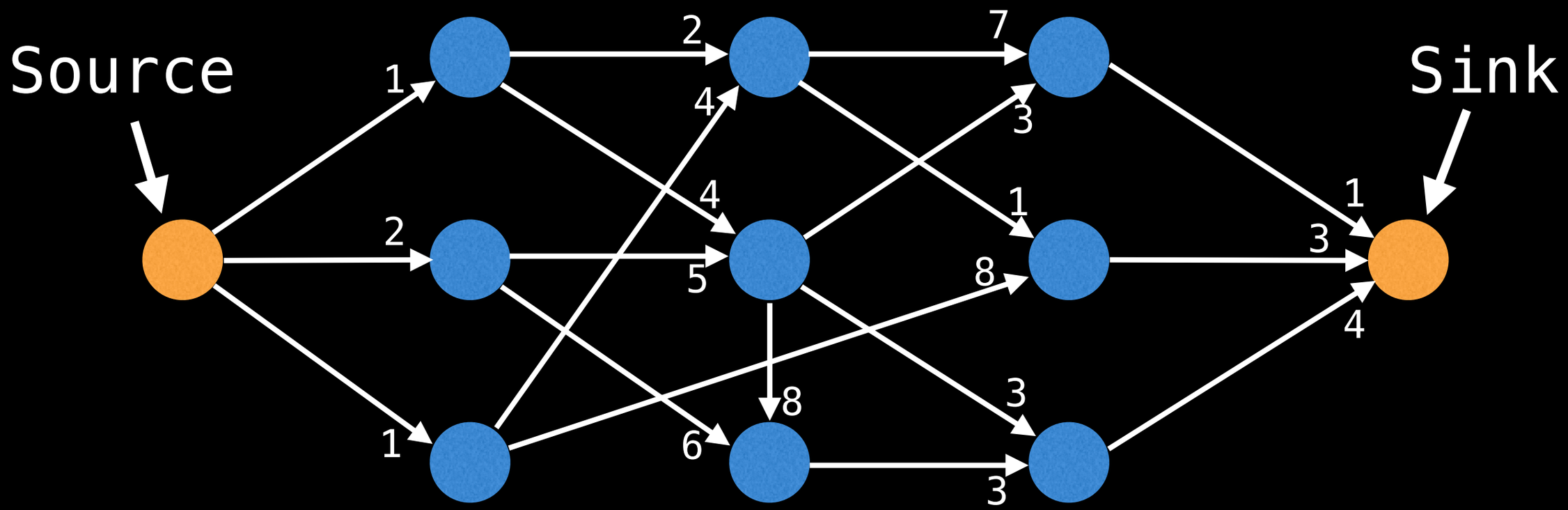
Network flow: max flow

Q: With an infinite input source how much “flow” can we push through the network?



Suppose the edges are roads with cars, pipes with water or hallways with packed with people. Flow represents the volume of water allowed to flow through the pipes, the number of cars the roads can sustain in traffic and the maximum amount of people that can navigate through the hallways.

Algorithms: Ford-Fulkerson, Edmonds-Karp & Dinic’s algorithm