Tree Recursion

Class outline:

- Order of recursive calls
- Tree recursion
- Counting partitions

递归调用的顺序

树递归

对分区进行计数

Order of recursive calls

The cascade function

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

What would this display?

```
cascade(123)
```

The cascade function

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

What would this display?

```
cascade(123)
```

```
1
12
123
12
1
```

```
123
12
1
12
123
```

级联

Cascade environment diagram

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)

cascade(123)
```

View in PythonTutor

- Each cascade frame is from a different call to cascade.
- Until the Return value appears, that call has not completed.
- Any statement can appear before or after the recursive call.

Global frame

cascade	→ func cascade(n)[parent=Global]

f1: cascade[parent=Global]

n	123
Return value	None

f2: cascade[parent=Global]

n	12
Return value	None

f3: cascade[parent=Global]

n	1
Return value	None

Print output:

```
123
12
1
12
123
```

Two definitions of cascade

```
def cascade(n):
    if n < 10:
        print(n)
    else:
        print(n)
        cascade(n//10)
        print(n)
```

```
def cascade(n):
    print(n)
    if n >= 10:
        cascade(n//10)
        print(n)
```

简单但可读性不强

- If two implementations are equally clear, then the shorter one is usually better
- When learning to write recursive functions, put the base cases first
- Both are recursive functions, even though only the first has typical structure

Inverse cascade

How can we output this cascade instead?

```
1
12
123
12
1
```

Inverse cascade solution

```
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)
```

```
grow = lambda n: f_then_g(                  )
shrink = lambda n: f_then_g(                  )
```

View in PythonTutor

Inverse cascade solution

```
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)
```

```
grow = lambda n: f_then_g(                  )
shrink = lambda n: f_then_g(                  )
```

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Inverse cascade solution

```
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)
```

```
grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g(                    )
```

View in PythonTutor

Inverse cascade solution

```
def inverse_cascade(n):
    grow(n)
    print(n)
    shrink(n)

def f_then_g(f, g, n):
    if n:
        f(n)
        g(n)
```

妙啊

```
grow = lambda n: f_then_g(grow, print, n//10)
shrink = lambda n: f_then_g(print, shrink, n//10)
```

缩小后打印

打印后缩小

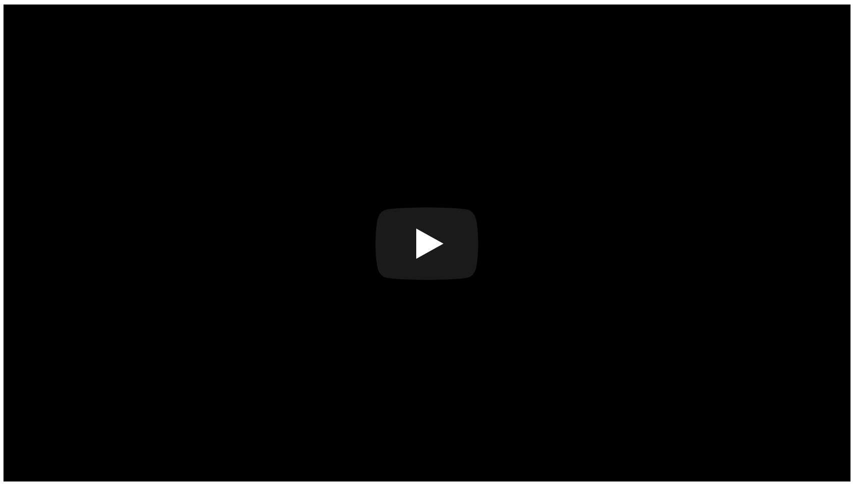
View in PythonTutor

Tree recursion
(Multiple recursion)

树形递归

Tree Recursion

Tree-shaped processes arise whenever a recursive function makes more than one recursive call.



Sierpinski curve

Recursive Virahanka-Fibonacci

The nth number is defined as:

$$\text{virfib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \text{virfib}(n-1) + \text{virfib}(n-2) & \text{otherwise} \end{cases}$$

```python
def virfib(n):
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
```

Recursive Virahanka-Fibonacci

The nth number is defined as:

$$\text{virfib}(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ \text{virfib}(n-1) + \text{virfib}(n-2) & \text{otherwise} \end{cases}$$

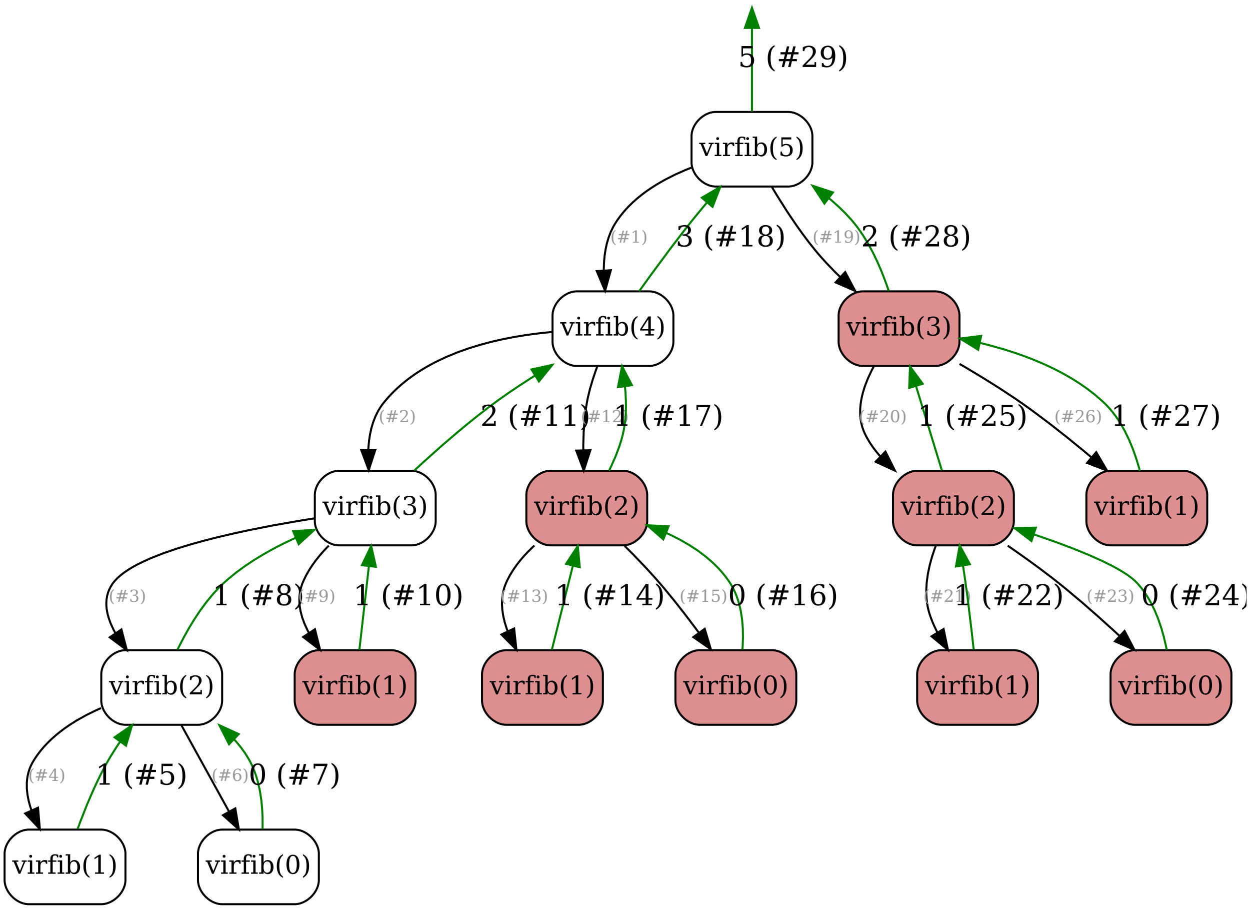
```python
def virfib(n):
    """Compute the nth Virahanka-Fibonacci number, for N >= 1.
    >>> virfib(2)
    1
    >>> virfib(6)
    8
    """
    if n == 0:
        return 0
    elif n == 1:
        return 1
    else:
        return virfib(n-1) + virfib(n-2)
```

A tree-recursive call graph

遍历树形结构

Redundant computations

The function is called on the same number multiple times. 🙀



(We will speed up this computation dramatically in a few weeks by remembering results)

Counting partitions

计算分区

Counting partitions problem

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

2 + 4 = 6

1 + 1 + 4 = 6

3 + 3 = 6

1 + 2 + 3 = 6

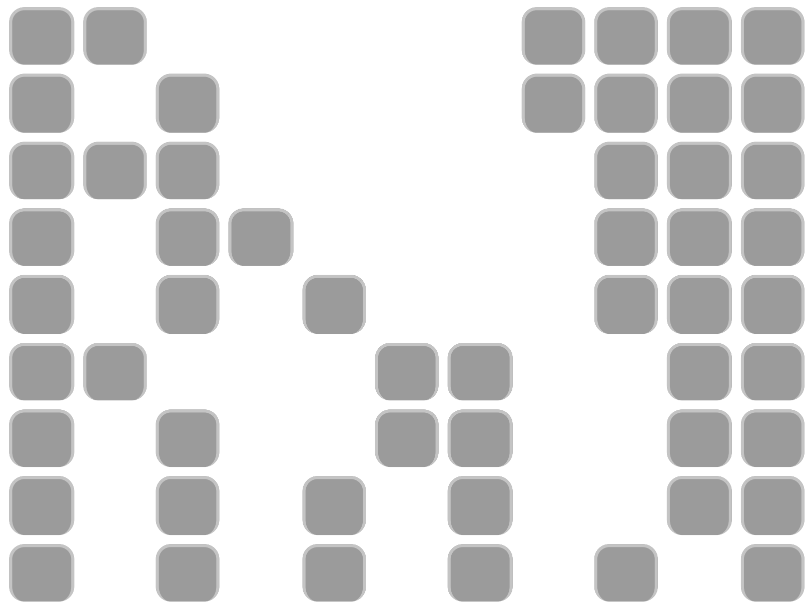
1 + 1 + 1 + 3 = 6

2 + 2 + 2 = 6

1 + 1 + 2 + 2 = 6

1 + 1 + 1 + 1 + 2 = 6

1 + 1 + 1 + 1 + 1 + 1 = 6



Counting partitions approach

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

Counting partitions approach

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

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Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

Use at least one 4

Counting partitions approach

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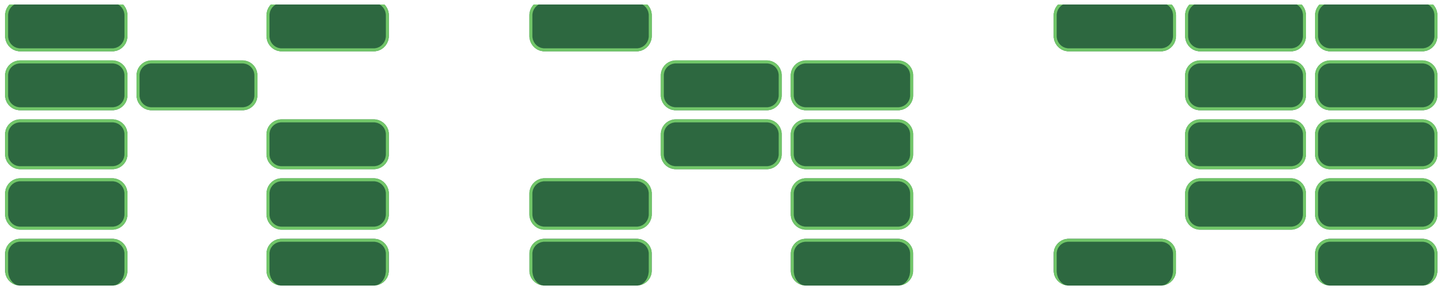
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

Use at least one 4

Don't use any 4



Counting partitions approach

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

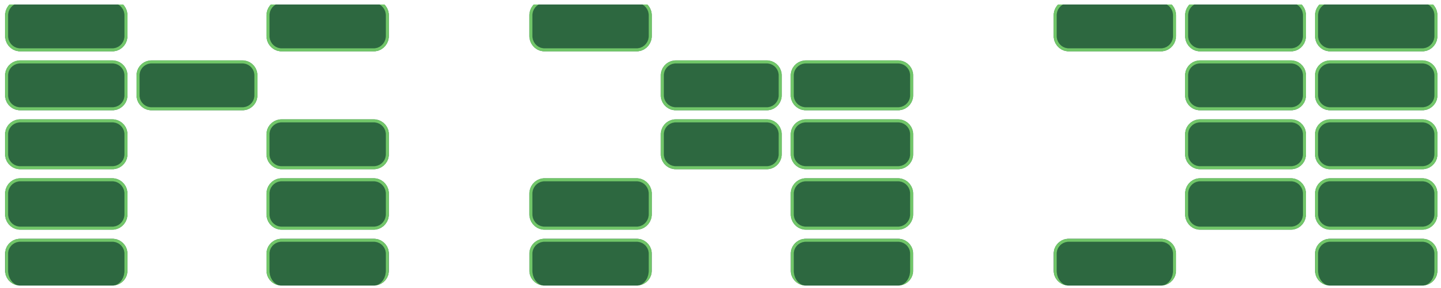
```
count_partitions(6, 4) # n, m
```

Recursive decomposition: finding simpler instances of the problem.

Explore two possibilities:

- Use at least one 4
- Don't use any 4

Tree recursion often involves exploring different choices.



Counting partitions approach

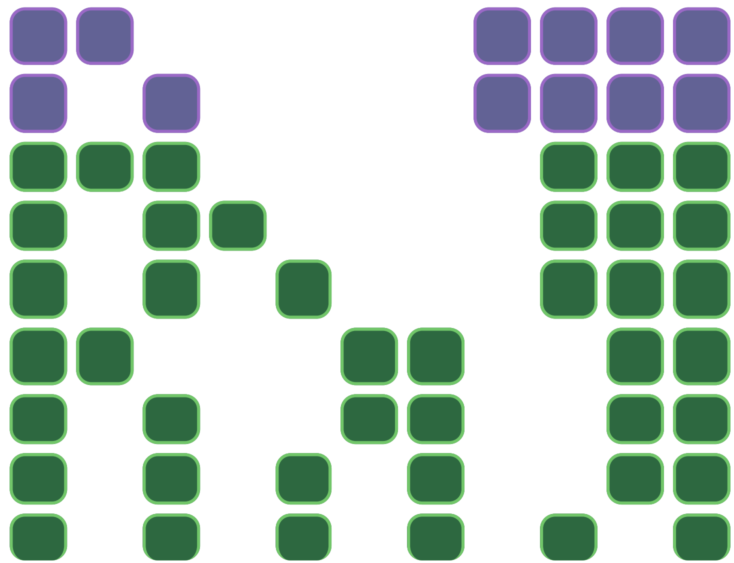
The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

`count_partitions(2, 4)`

`count_partitions(6, 3)`



Counting partitions approach

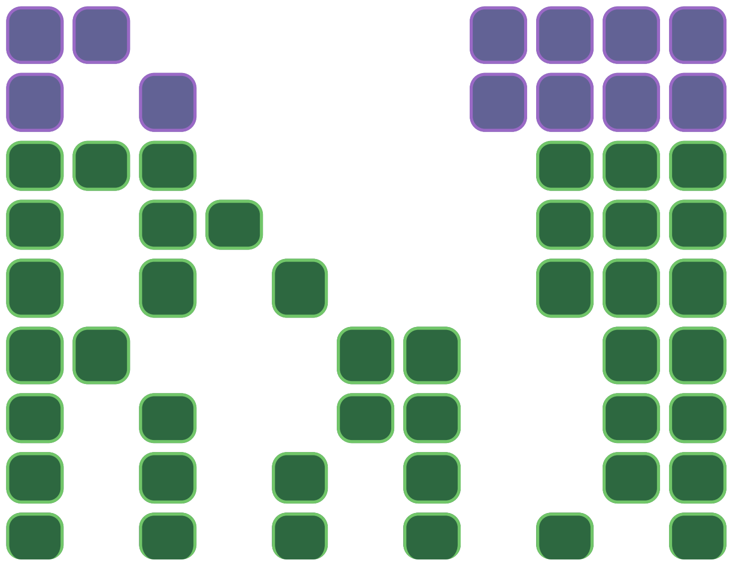
The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

```
count_partitions(2, 4)
count_partitions(n-m, m)
```

```
count_partitions(6, 3)
```



Counting partitions approach

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

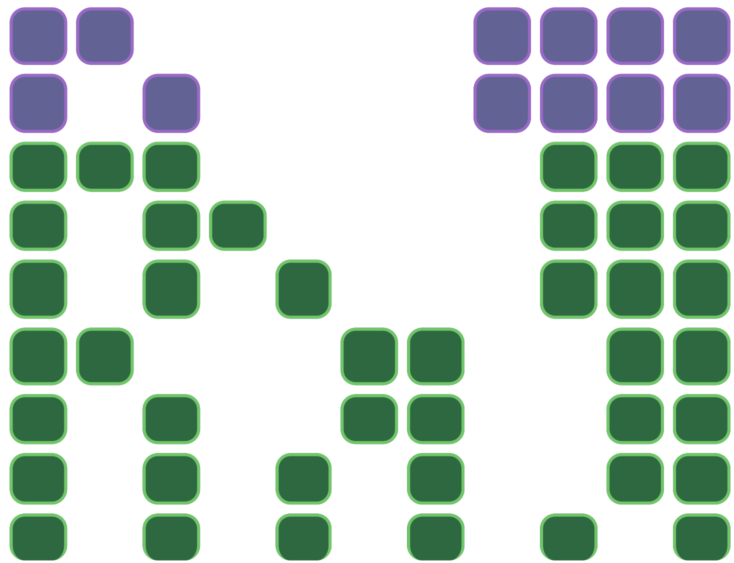
Solve two simpler problems:

`count_partitions(2, 4)`

`count_partitions(n-m, m)`

`count_partitions(6, 3)`

`count_partitions(n, m-1)`



Counting partitions code

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

with parts of size `m`:

```
count_partitions(2, 4)
count_partitions(n-m, m)
```

without parts of size `m`:

```
count_partitions(6, 3)
count_partitions(n, m-1)
```

```
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
```

```
9
"""
```

Counting partitions code

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

with parts of size `m`:

```
count_partitions(2, 4)
count_partitions(n-m, m)
```

without parts of size `m`:

```
count_partitions(6, 3)
count_partitions(n, m-1)
```

```
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
```

```
    9
    """









    else:
        with_m = count_partitions(n-m, m)
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```

Counting partitions code

The number of partitions of a positive integer `n`, using parts up to size `m`, is the number of ways in which `n` can be expressed as the sum of positive integer parts up to `m` in increasing order.

```
count_partitions(6, 4) # n, m
```

Solve two simpler problems:

with parts of size `m`:

```
count_partitions(2, 4)
count_partitions(n-m, m)
```

without parts of size `m`:

```
count_partitions(6, 3)
count_partitions(n, m-1)
```

```
def count_partitions(n, m):
    """
    >>> count_partitions(6, 4)
```

```
    9
    """
    if n == 0:
        return 1
    elif n < 0:
        return 0
    elif m == 0:
        return 0
    else:
        with_m = count_partitions(n-m, m)
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```

Count partitions call graph

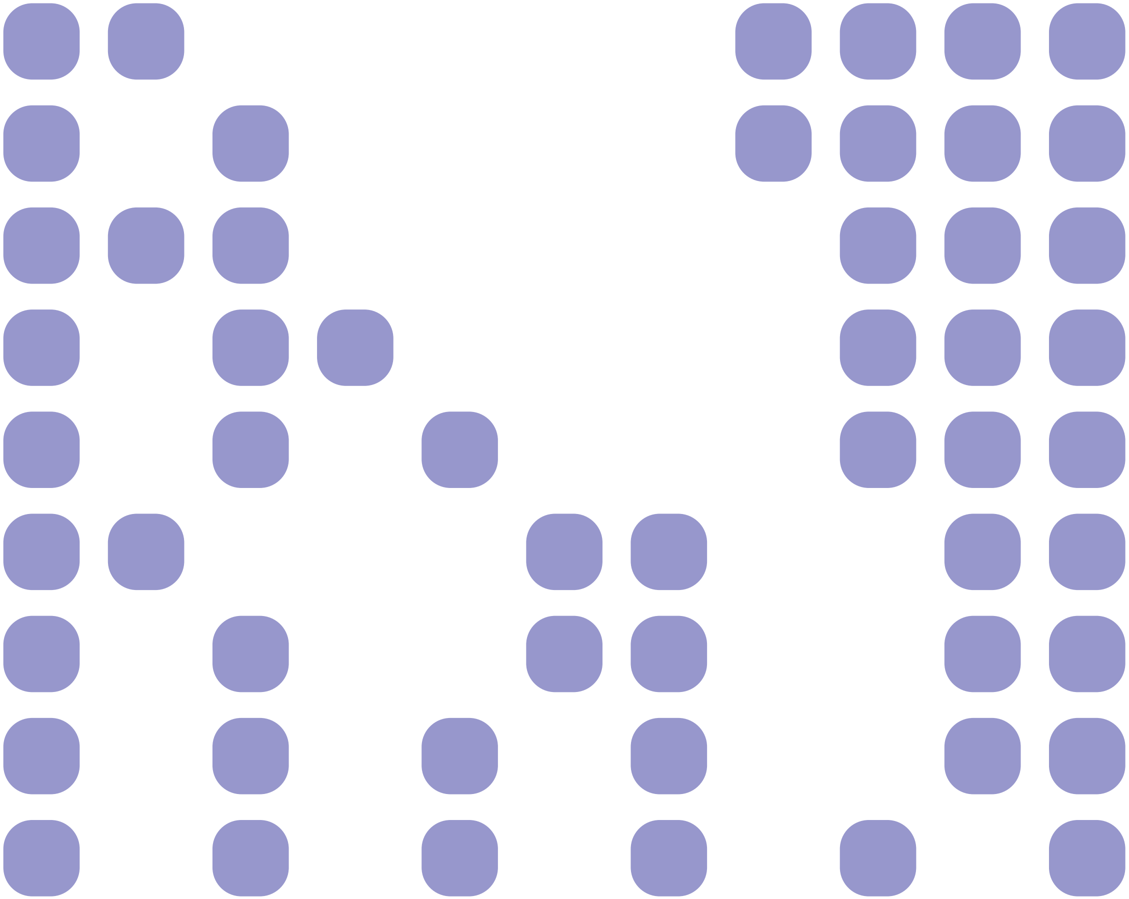
Count partitions variant

To save on unneeded calls, we can cap `m`, the maximum partition size, based on `n`. We can also add a base case for `m` of size 1.

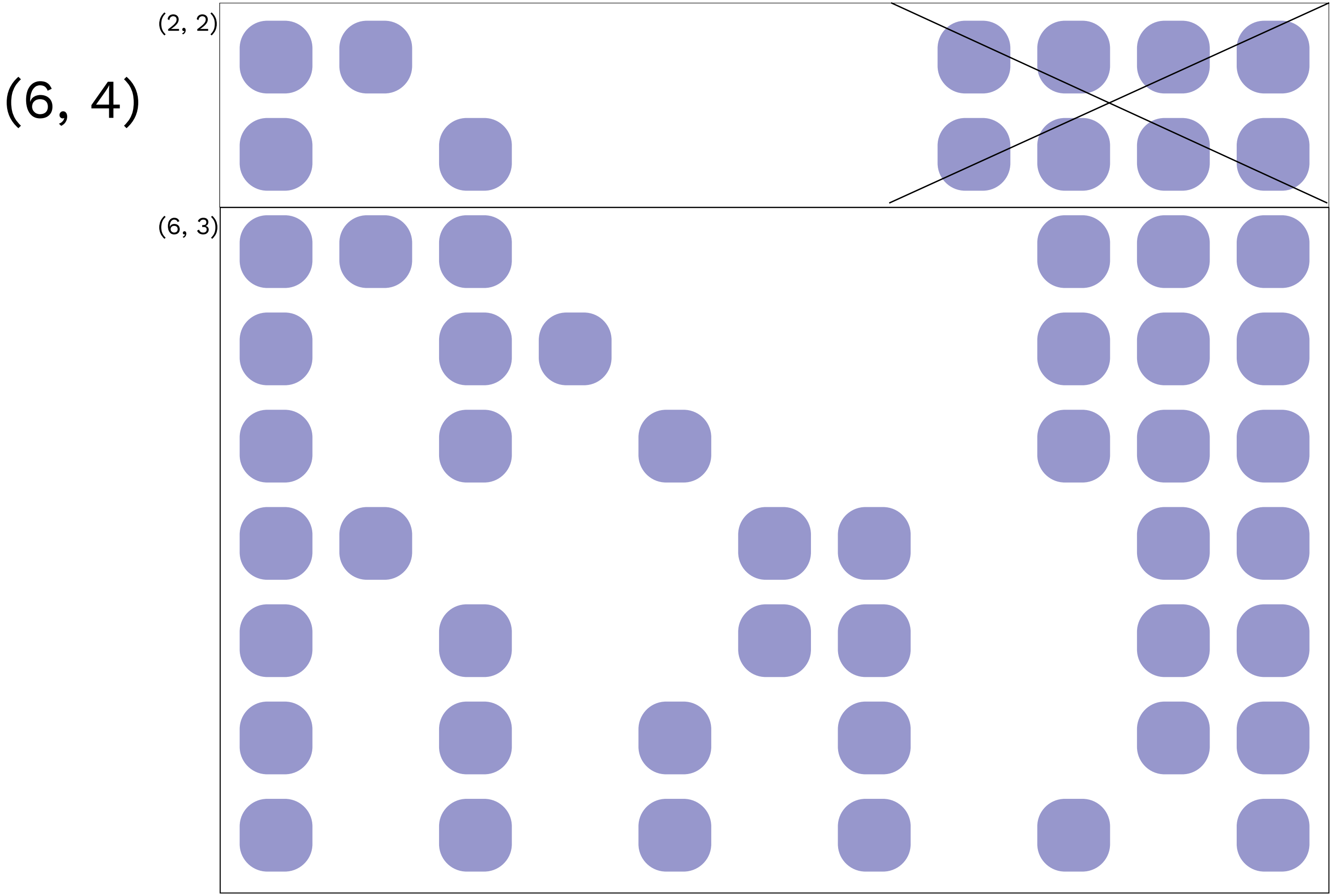
```
def count_partitions(n, m):
    if m == 1 or n == 0:
        return 1
    elif n < 0:
        return 0
    else:
        # Number of partitions using a partition of size M
        leftover = n - m
        with_m = count_partitions(leftover, min(leftover, m))
        # Number of partitions using size up to M-1
        without_m = count_partitions(n, m-1)
        return with_m + without_m
```

Variant call graph

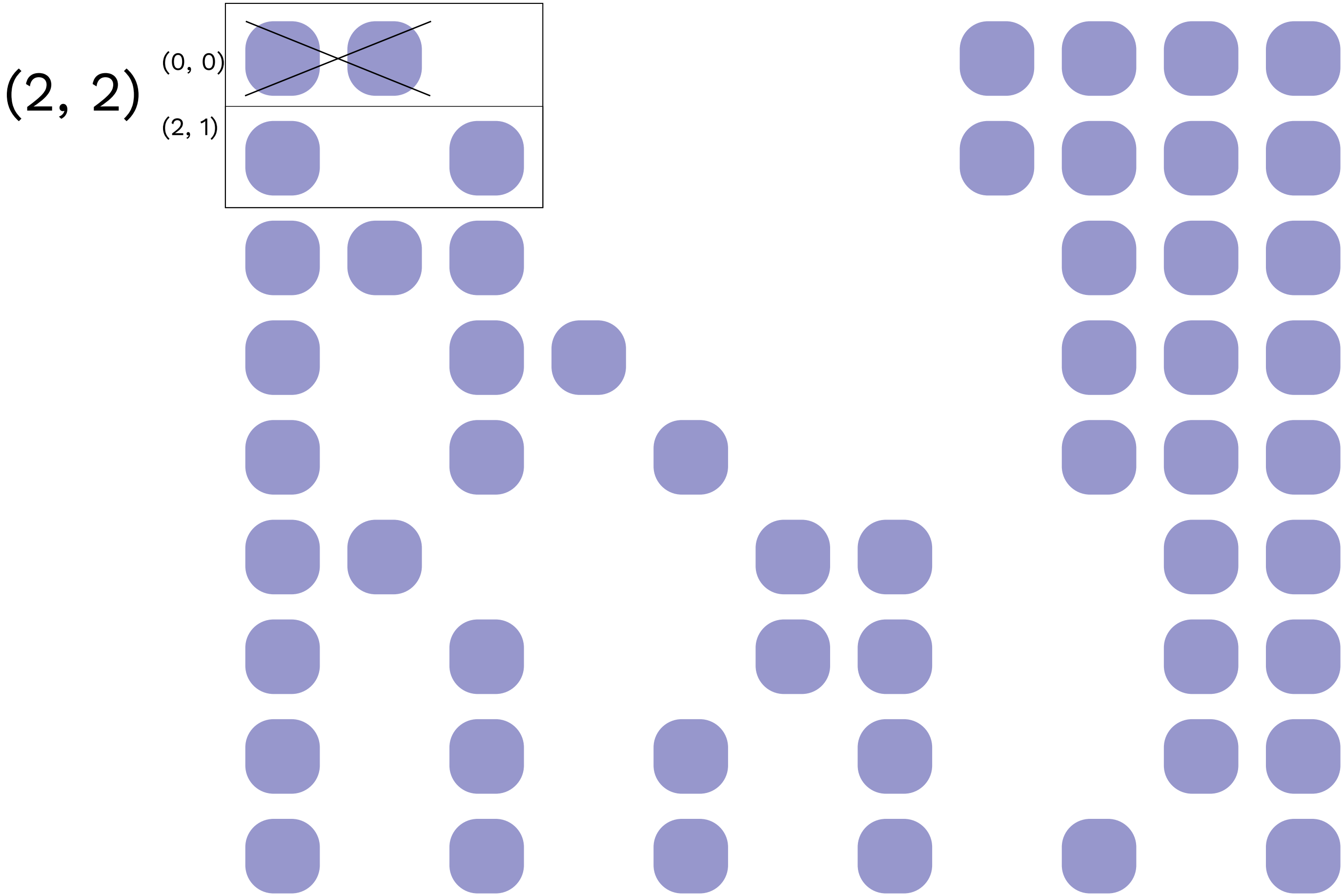
Variant visualization



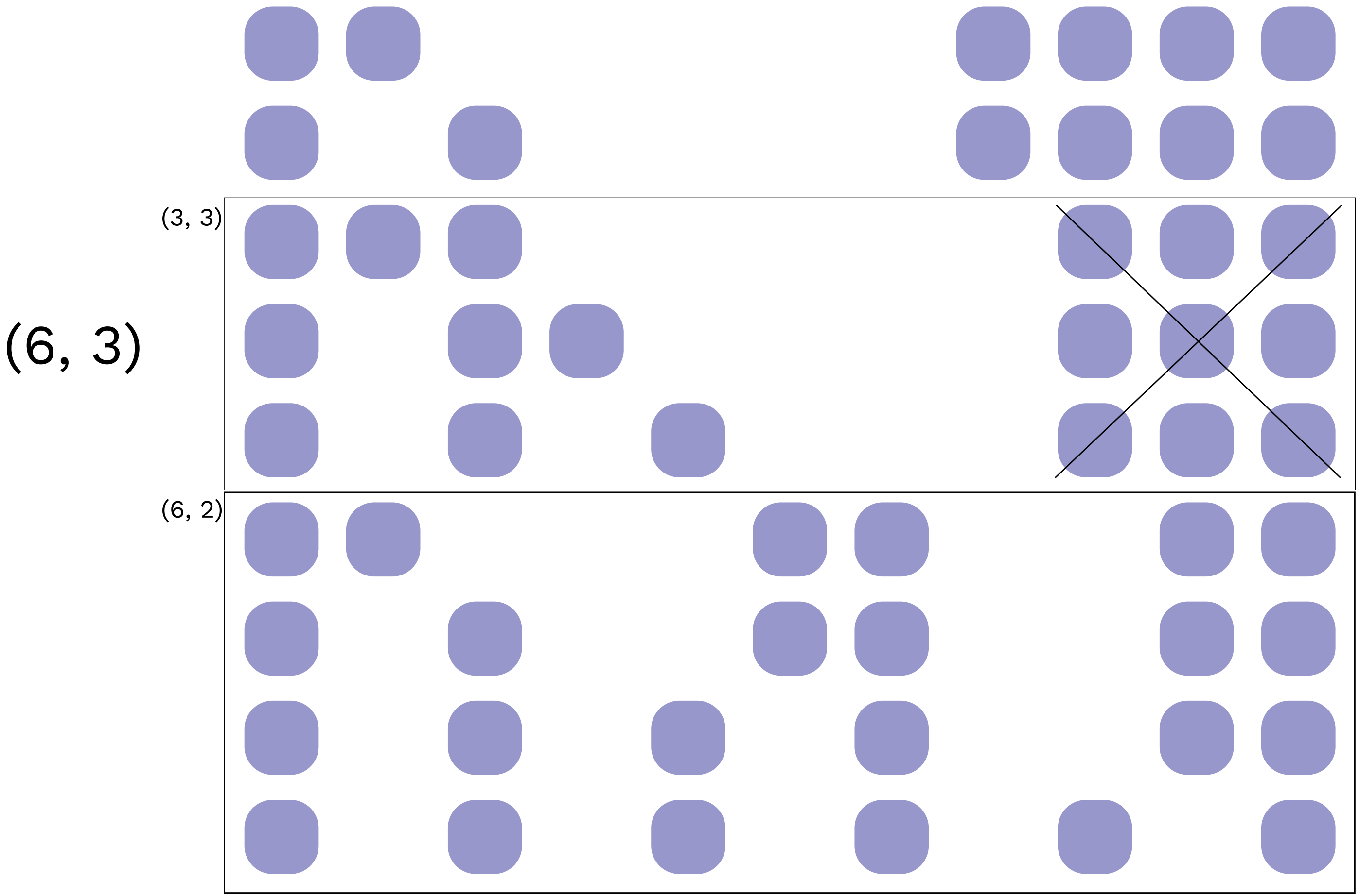
Variant visualization



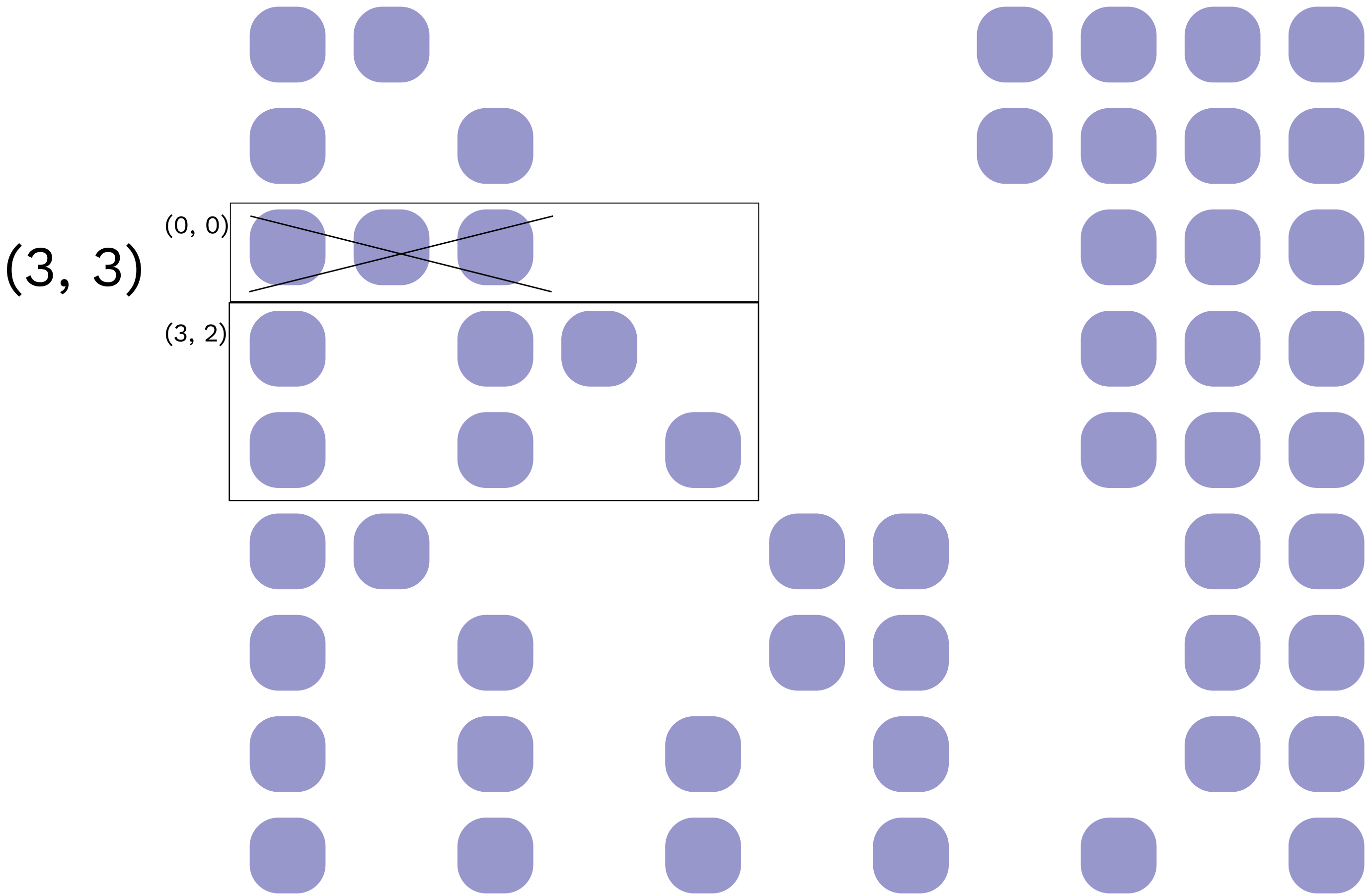
Variant visualization



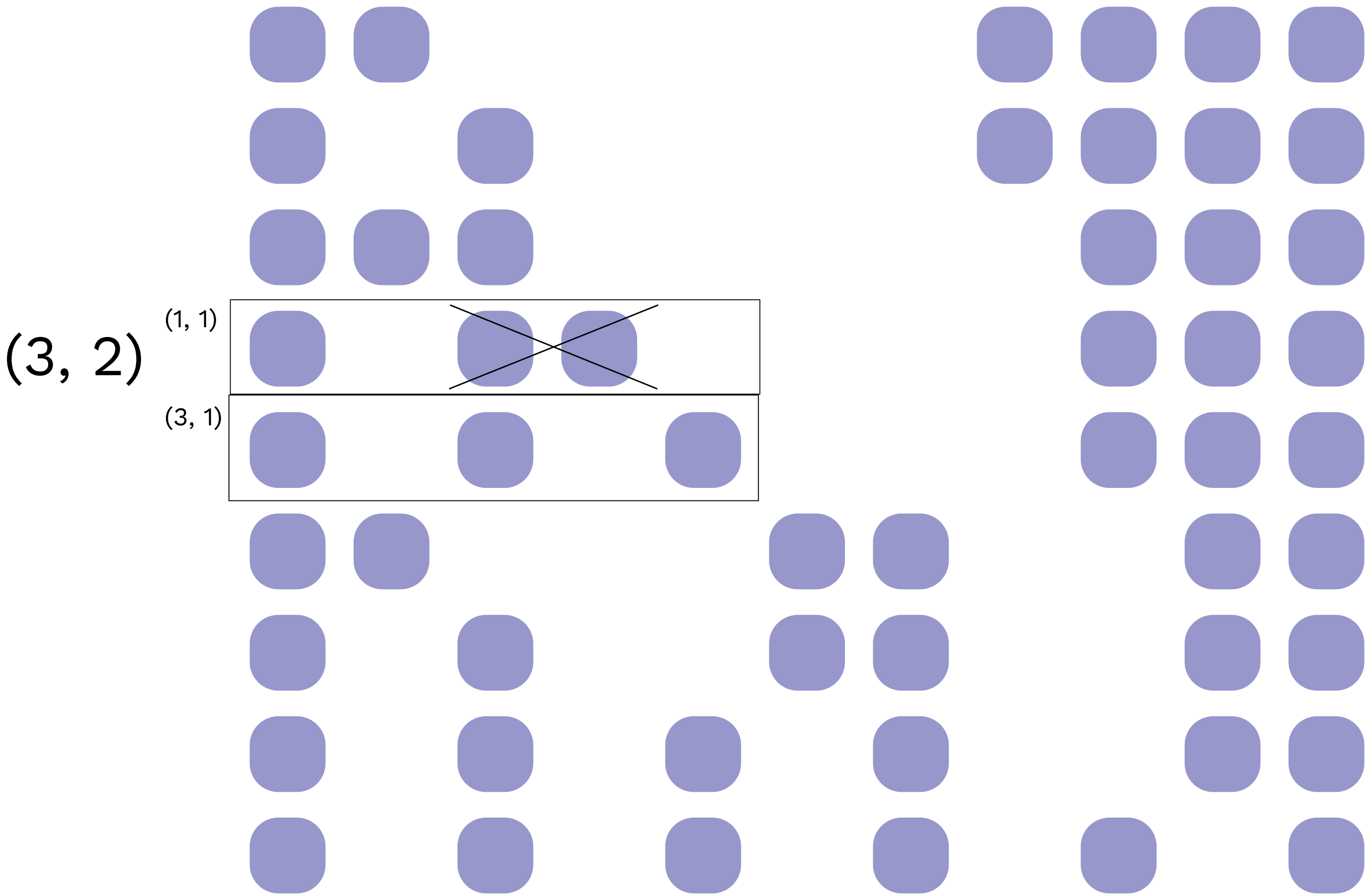
Variant visualization



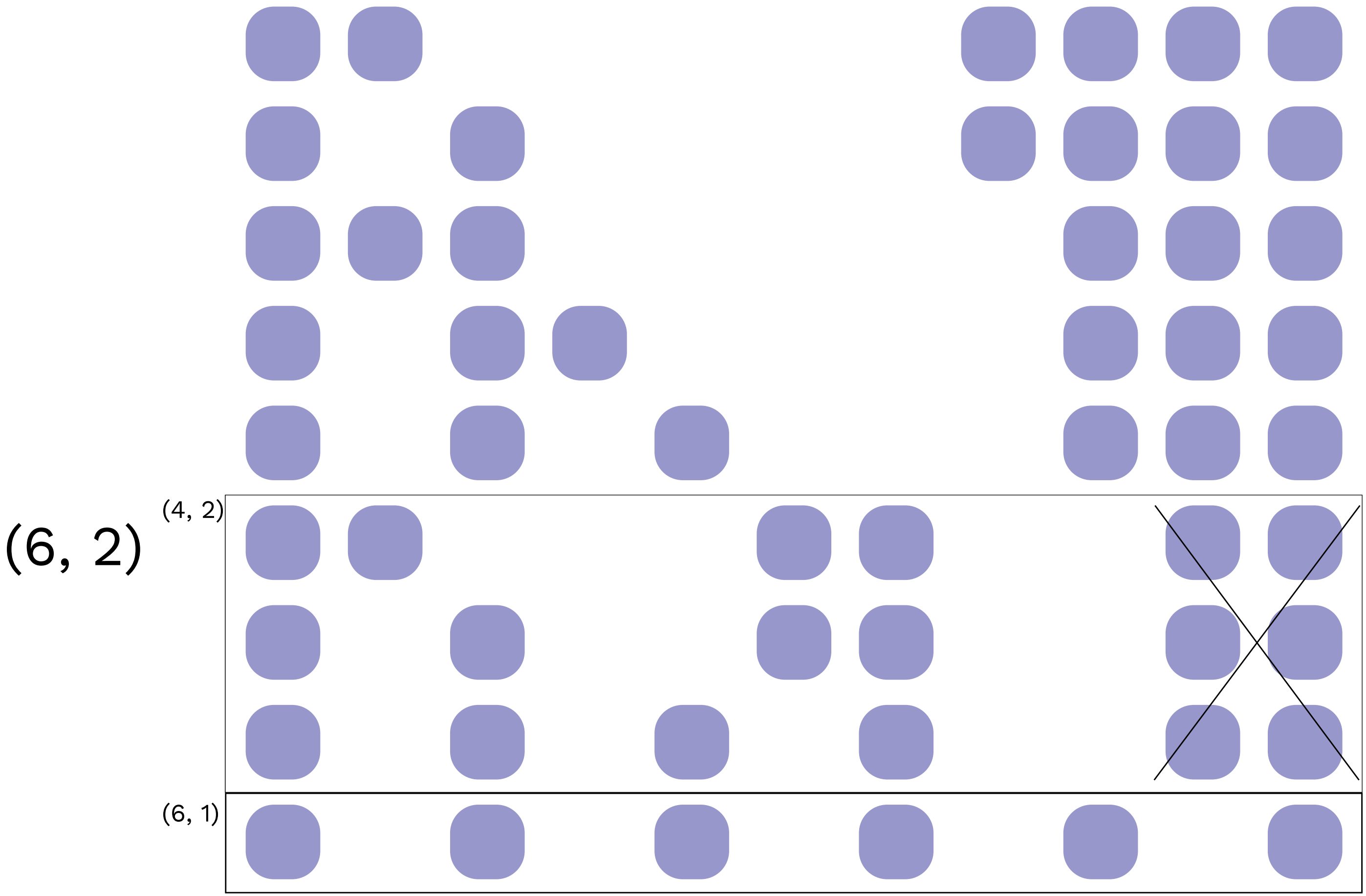
Variant visualization



Variant visualization



Variant visualization



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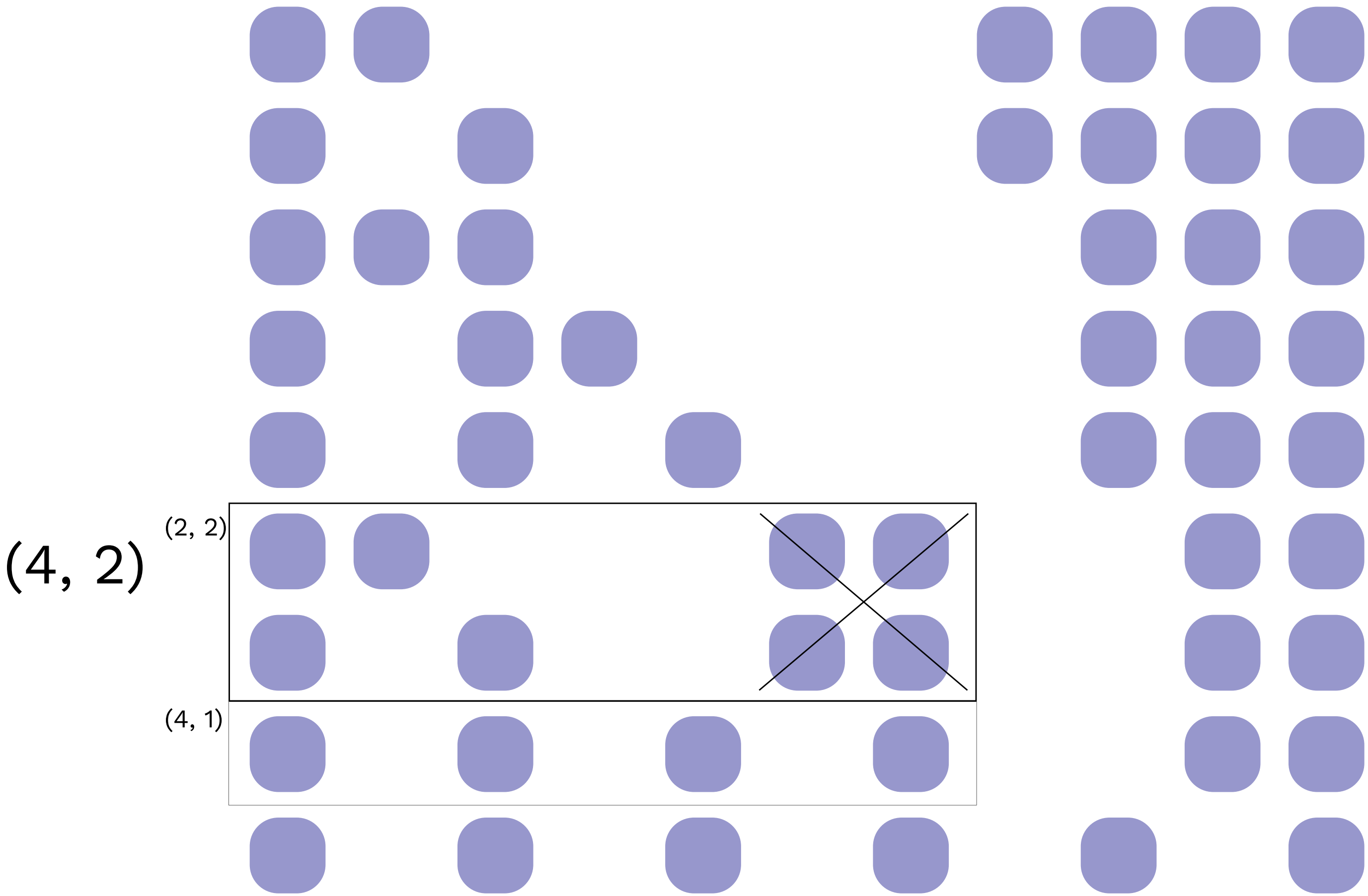

Variant visualization