Cheat Sheet - Uncertainty calculation

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1 Definitions

Let us say you measure the length of your table to be

$$l = 50.5 \pm 0.1 \text{ cm}.$$

- The **best value** X is the value you measured and given in our example as 50.5 cm
- The **absolute error** ΔX in our example is $\Delta l = 0.1$ cm. Note that the absolute value has the same units as the best value. If the best value is scaled by a factor of a then also the absolute error is multiplied by this factor
$$\Delta(aX) = a\Delta X$$
- The **relative error** is given by the absolute error divided by the best value and has therefore no units.
$$\Delta_{rel}X = \frac{\Delta X}{X}$$
In our example $\Delta_{rel}l = \Delta l/l = 50.5 \text{ cm}/0.1 \text{ cm } = 0.002$

2 Random error

Imagine you measure the length of the table **multiple times**. Your data set (measured lengths) is given by

Length / cm
50.1
50.2
50.0
50.5

Let x_i denote the different measurement outcomes (i is just an index to label the measurements) and N the number of measurements.

- The **mean** or **average** of the data set can be calculated as
$$\overline{x} = \frac{\sum_i x_i}{N}.$$

or in our case

$$\bar{l} = \frac{50.1 + 50.2 + 50.0 + 50.5}{4} \text{ cm} = 50.2 \text{ cm}$$

TIP: In Excel you can use the AVERAGE(data) function

- The **standard error** or the error of the average is then given by

$$\Delta\overline{x} = \sqrt{\frac{\sum_i (x_i - \overline{x})^2}{N(N-1)}}$$

for our example

$$\Delta\bar{l} = \sqrt{\frac{(50.1 - 50.2)^2 + (50.2 - 50.2)^2 + ... + (50.5 - 50.2)^2}{12}} \text{ cm} = 0.11 \text{ cm}$$

The standard error gives you information about how likely it is that your measured **mean value** is close to the **true value**. For example, in our experiment the true length of the table is between 50.1 cm and 50.3 cm (i.e. the mean value ± the standard error) with a probability/certainty of about 68%

TIP: In Excel, use STDEV(data)/SQRT(COUNT(data))

3 Error propagation

Imagine you measure the width of the table to be

$$w = 30.2 \pm 0.2 \text{ cm}$$

and you want to calculate the perimeter and area of the table. How does one combine the two uncertainties. Answer: you use the **geometric mean**, which means "add the squared values and then take the square root".

- **Addition/Subtraction**

Adding/Subtracting the measured values X and Y, the combined/propagated error is given by the geometric mean of the **absolute errors** ΔX and ΔY.

$$Z = X + Y$$
$$\Delta Z = \sqrt{(\Delta X)^2 + (\Delta Y)^2}$$

So in our example, half the perimeter of the table and its error is given by

$$c = (30.2 + 50.5) \text{ cm} = 80.7 \text{ cm}$$
$$\Delta c = \sqrt{(0.1)^2 + (0.2)^2} \text{ cm} = 0.2 \text{ cm}$$

- **Multiplication/Division**

  Multiplying/Dividing X and Y, one needs to calculate the geometric mean of the **relative errors** to obtain the combined relative error

$$Z = X \cdot Y$$
$$\Delta_{rel} Z = \frac{\Delta Z}{Z} = \sqrt{\left(\frac{\Delta X}{X}\right)^2 + \left(\frac{\Delta Y}{Y}\right)^2}$$
$$\Delta Z = (\Delta_{rel} Z) \cdot Z$$

  Thus, the area of the table and its relative and absolute error are given by

$$A = 30.2 \text{ cm} \cdot 50.5 \text{ cm} = 1525.1 \text{ cm}^2$$
$$\Delta_{rel} A = \sqrt{\left(\frac{0.2}{30.2}\right)^2 + \left(\frac{0.1}{50.5}\right)^2} = 0.0069$$
$$\Delta A = 0.0069 \cdot 1525.1 \text{ cm}^2 = 10 \text{ cm}^2$$

  Therefore our final result is

$$A = (15.3 \pm 0.1) \times 10^2 \text{ cm}^2 = 15.3 \pm 0.1 \text{ dm}^2$$

- **Powers**

  If you are dealing with powers, the relative error of $Z = X^n$ is just the relative error of X multiplied by the absolute value of the exponent n (n does not have to be positive or even an integer).

$$\frac{\Delta Z}{Z} = |n| \times \frac{\Delta X}{X}$$