PROPOSAL TO MSRI:
<PROGRAM TITLE>

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Date: _______________

1. ORGANIZERS

Our research program will be organized by <List of Organizers with Affiliations – Note that 1 or 2 of them are prominent women>. We propose to have the program <in Spring /Fall Year>. All five members of the organizing committee have committed to spending the entire semester at MSRI. We have received commitments from several key participants, listed in §8 below, to participate if the program is approved. A number of them have committed to spending the entire semester at MSRI and have expressed the desire to arrange their sabbaticals in order to be able to do this.

2. SUMMARY

Our program will focus on the deformation theory of geometric structures on manifolds, and the resulting geometry and dynamics. Formally a subfield of differential geometry and topology, with a heavy infusion of Lie theory, its richness stems from close relations to dynamical systems, algebraic geometry, representation theory, Lie theory, partial differential equations, number theory, and complex analysis.

Hyperbolic structures on surfaces provide the first nontrivial examples, and the classical Teichmüller space is the prototype of a deformation space of locally homogeneous structures. More general deformation spaces arise from the space of representations of the fundamental group of a manifold in a Lie group, which appears also as the moduli space of flat connections on the manifold. These "character varieties" have played an important role in developing topological invariants of manifolds, particularly in dimensions 3 and 4.

Teichmüller space can be realized as subset of the space of representations of a surface group into PSL(2, R). What has recently been called "higher Teichmüller theory" by Fock and Goncharov concerns certain deformation spaces arising from subsets of the space of representations of a surface groups into Lie groups of higher rank, e.g. PSL(n, R), which share some of the properties of classical Teichmüller space.

Recent interest in this subject has also come from mathematical physics, through Witten's suggestion relating representations in the Hitchin components, which furnish examples of higher Teichmüller spaces, to W_n-algebras, and applications of Hitchin representations to the geometric Langlands program. These unexpected inter-relationships underscore this subject's richness, timeliness and diversity. A central goal of the program will be to bring together researchers who work in the more fully developed areas of Teichmüller geometry and deformation spaces of hyperbolic structures in low dimensions with researchers studying more general deformation spaces in order to explore these new connections.

The organizers will place an emphasis on encouraging and supporting young mathematicians, as well as under-represented minorities, as they embark on careers in mathematics. All of us have been successful in supervising Ph.D. students and/or mentoring postdocs. In this proposal we detail some ways we plan to carry out this mission as part of the program.

3. PROPOSED RESEARCH DIRECTIONS

This section addresses some potential research directions to be explored during the program.

3.1. Examples of higher deformation spaces. The study of deformation spaces of hyperbolic structures on surfaces (sometimes referred to as "Teichmüller theory") concerns (discrete) subgroups of PSL(2, R). Through the uniformization theorem, the Teichmüller space of (marked) Riemann surfaces homeomorphic to a fixed topological surface Σ is identified with the Teichmüller space of hyperbolic-geometry structures on Σ. (Sometimes the space is called Fricke- Teichmüller space). Consequently methods of hyperbolic geometry became a useful tool in studying Riemann surfaces. The Lie groups PSL(2, C) and O(1,n) act by isometries on hyperbolic 3¬space and hyperbolic n-real space. In particular these are simple Lie groups of rank one.

Although not as extensively studied among topologists, the theory of complex hyperbolic manifolds provides a rich set of examples, although basic questions remain unknown. The structure group is U(n, 1), also of rank one. The deformation theory of complex hyperbolic manifolds is another generalization of Teichmüller theory, which is only beginning to be understood. Many of these Kähler manifolds have been intensively studied by algebraic geometers. The activity in this subject is underscored by the recent discovery of new commensurability classes of non-arithmetic lattices in U(2, 1) (Deraux-Parker-Paupert).Other rank one geometries including quaternionic hyperbolic spaces and the Cayley hyperbolic plane are also intriguing and even more mysterious.

Higher rank Lie groups include SL(n, R), Sp(2n, R), O(n, m) or U(n, m). Such groups act isometrically on symmetric spaces of nonpositive curvature, but some of the sectional curvatures are zero. That leads to several rigidity results, such as Margulis super-rigidity of lattices and the rigidity results of Kleiner-Leeb and Quint concerning convex cocompact subgroups. These rigidity results force one to ask different sorts of questions and in particular force attention on discrete subgroups which are not lattices, and not even Nevertheless, in recent years meaningful generalizations of the space of Fuchsian representations of a surface groups into PSL(2, R) (Teichmüller space), of the spaces of quasi-Fuchsian representations into PSL(2, C) and of the space of convex cocompact subgroups of O(1,n) have been found in the context of higher rank Lie groups.

<Optional: Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.2. Hitchin representations. One simple extension concerns PSL(3, R), the group of projective transformations of RP2 . The higher Teichmüller space in this case consists of convex projective structures, that is, representations of a surface S of genus g> 1 as a quotient of a convex domain in RP2 by a discrete subgroup of PSL(3, R) acting properly. This space is homeomorphic to R16g−16 (Goldman) and defines a connected component in the deformation space of representations π1(Σ) −→ PSL(3, R) (Choi-Goldman). In particular, this connected component contains Fuchsian representations into O(2, 1) — hyperbolic structures on Σ regarded as projective structures using the Klein-Beltrami projective model of the hyperbolic plane.

SL(2, R) and SL(3, R) are examples of split real forms. In 1990, Hitchin showed, using gauge-theoretic techniques, that for every split real form G (such as SL(n, R) or Sp(2n, R)), the deformation space of representations π1(Σ) −→ G contains components H such that:

- H is homeomorphic to R(2g−2)dim(G);
- H contains Fuchsian representations with respect to the (Kostant) principal representation SL(2, R) −→ G;
- Every representation in H is a discrete embedding (<Names>).

Such components are called Hitchin components and the corresponding representa¬tions Hitchin representations.

<Optional: Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.3. Maximal representations. The group SL(2, R) and its symmetric space the hyperbolic plane H2 generalize in another direction. The complex structure of H2 turns H2 into a Hermitian symmetric space. Hermitian symmetric spaces of rank one are the complex

hyperbolic spaces discussed above. When G acts on a Hermitian symmetric space, this extra structure determines an integer invariant of surface group representations, called the *Toledo invariant*. For a certain integer d(G) depending solely on G, the Toledo invariant $\tau(\rho)$ of any representation $\pi_1(\Sigma_g) \to G$
−
satisfies:

(1) $|\tau(\rho)| \leq d(G)|\chi(\Sigma)|$
(2)

A representation ρ realizing equality in (1) is called maximal.

When G = SL(2, R), then $\tau(\rho)$ is just the Euler class of the oriented RP^1-bundle associated to ρ. Moreover, d(G) = 1 and (1) is just the famous Milnor-Wood inequality for flat oriented circle bundles. Furthermore ρ is maximal if and only if ρ is a discrete embedding (Goldman),that is a Fuchsian representation. Thus the set of equivalence classes of maximal representations identifies with the Teichmüller space of Σ. Using this result, Toledo showed that any maximal representation of $\pi_1(\Sigma)$ in U(n, 1) must factor through the stabilizer U(1, 1) × U(n − 1) of a complex hyperbolic line (the Poincar´e disc), a striking rigidity theorem for representations of the surface group $\pi_1(\Sigma)$ in a rank one Lie group.

Recently <Names> have proved that maximal representations into the isometry group G satisfy many basic properties shared by Fuchsian representations:

- Maximal representations are discrete embeddings;
- Maximal representations comprise a union of connected components;
- The image of a maximal representation is Zariski dense in the isometry group of a special Hermitian symmetric subspace (maximal subspace of tube type); in particular it is reductive.

<Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.4. Anosov representations. Hitchin representations and maximal representations generalize the space of Fuchsian representations, motivating the term "higher Teichmüller spaces" for their deformation spaces. Recently <name> has defined the notion of Anosov representations which include these special representation but applies in a more general context. The concept of Anosov representations is defined for representations of any finitely generated word-hyperbolic group into any real semisimple Lie group.

A word-hyperbolic group Γ acts naturally on its boundary ∂Γ, which is a compact metric space. Let G be a semisimple Lie group and P <G a parabolic subgroup which is conjugate to its opposite group (in the case when G = SL(n, R) one could for example take P to be the stabilizer of a k-plane and an incident n − k-plane). A representation $\rho : \Gamma \to G$ is Anosov if and only if there exists a ρ-equivariant continuous map $\xi : \partial\Gamma \to G/P$ such that

- For any distinct points t
= t' ∈ ∂Γ, the pair ξ(t),ξ(t') is transverse.
- The map ξ satisfies some contraction property (it comes from some Anosov flow).

Important examples of Anosov representations are quasi-Fuchsian representations and embeddings of convex cocompact subgroups of rank one Lie groups, as well as (complex) projective Schottky groups (defined by Nori and Seade-Verjovsky) or discrete subgroups of PGL(n, R) acting cocompactly on a convex domain in projective space (studied by Benoist).

Recent work of <Names>, indicates that Anosov representations have many interesting geometric properties:

- Anosov representations form an open subset of the representation variety.
- Anosov representations have finite kernel and discrete image.
- Anosov representations are quasi-isometric embeddings.

Many of the proposed topics explore to what extent features known from the Fuchsian representations in PSL(2, R) persist for these generalizations.

<Optional: Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.5. Mapping class group dynamics. The mapping class group is a fundamental object in low-dimensional topology. Deformation spaces of geometric structures on manifolds provide examples of natural actions of the mapping class group.

Of particular interest is the action of the mapping class group of a closed ori¬entable surface on character varieties of surface groups. The mapping class group Mod(Σ) of a closed orientable surface Σ can be identified with an index two sub-group of the *outer automorphism group* Out π1(Σ) . The action of Mod(Σ) on the Teichmüller space T (S), viewed as a component of the PSL(2, R)-character variety, is proper and its quotient is the moduli space of Riemann surfaces homeomorphic to Σ. The properness of the action of the mapping class group persists in other contexts; in particular, the action on the space of Anosov representations is proper. This includes Hitchin representations in PSL(n, R) (Goldman for n = 3, Labourie for all n), maximal representations (Wienhard) and convex cocompact representa¬tions, including classical quasi-Fuchsian representations into PSL(2, C). In contrast, when the image group G of the representation is compact, Goldman and Pickrell-Xia showed that the mapping class group acts ergodically on each connected component. In the case of PSL(2, C) it is conjectured that the mapping class group acts ergodi¬cally on the complement of the space of quasifuchsian representations. One piece of evidence for this conjecture is that no point in the boundary of quasifuchsian space can lie in any domain of discontinuity for the action of the mapping class group. So, even for noncompact groups, these actions are dynamically interesting.

Given a general group Γ, one may consider the action of Out(Γ) on the G-character variety of Γ and consider questions concerning the ergodic decomposition and topological dynamics of the action. Minsky's theory of primitive stable representations of free groups into PSL(2, C) is a prime example of this developing direction, which is yet unexplored for higher rank Lie groups. The case where Γ is a 3-manifold group and G = PSL(2, C) has been studied by Canary, Lee and Storm. This work indicates that the dynamical behavior in the closed orientable surface group case is quite special.

Further interesting questions are raised in recent work by <Names>. For complex simple Lie groups, one finds both open subsets upon which the action is proper as well as subsets with chaotic dynamics.

Understanding the boundary of the space of Anosov representations and the action of the mapping class group on it – as we do for PSL(2, C)-could help in understanding an open question: can one find a discrete representation of the fundamental group of a compact hyperbolic 3-manifold in PSL(3, R)?

<Optional: Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.6. Geometric structures on the deformation space. Deformation spaces of representations of surface groups carry natural symplectic structures. On classical Teichmüller space finer and richer geometric structures exist. For example, there is the Ahlfors-Bers complex structure which may be defined by using an embedding of Teichmüller space into the the space of representations of the surface group into PSL(2, C) via quasi-Fuchsian groups. This agrees with the general Kodaira-Spencer-Kuranishi structure on the deformation space of complex structures.

There are also various mapping class group invariant metrics. The most prominent of these are the Teichmüller and Weil-Petersson metrics. The Weil-Petersson metric is intimately connected with dynamical properties of hyperbolic structures. One such connection is via the Hausdorff dimension of the limit set of a quasi-Fuchsian representation by work of Bridgeman and McMullen. Another is with the length of random geodesics on the surface by work of Thurston and Wolpert. There are also important geometric flows associated with these metrics and there has been substantial progress understanding these flows in recent years. The Teichmüller flow is connected with the variation of flat structures on the surface and the basic quantities associated to this flow such as the Lyapunov exponents are now quite well understood due to work of <Names>. There has also been substantial progress in understanding the Weil-Petersson geometry and flow due to <Names> and recently the proof of the ergodicity of the flow due to <Names>. Also associated to the hyperbolic structure is the earthquake flow, whose ergodicity was recently proved by <name>. A central problem is to extend such structures such as a complex structures, invariant metrics or the Teichmüller, Weil-Petersson or earthquake flows to higher Teichmüller spaces.

Hitchin representations are associated to an Anosov flow. Therefore dynamical invariants of the flow give rise to natural functions on the Hitchin component. The first natural function is the entropy which is the exponential rate of growth of the length of closed orbits of the flow. Entropy should also be closely related to the dimension of the limit curves of the representation. For hyperbolic metrics the entropy is constantly equal to 1. This is no longer true for Hitchin components for PSL(n, R) with $n \geq 3$. Here one might expect a rigidity phenomenon: entropy is in general less than one and equality occurs only for Fuchsian representations. This would in spirit be close to the entropy rigidity proved by <names> in the Riemannian context.

Hitchin representations are also associated to geodesic currents. From <name's> work, two geodesic currents can be paired to produce a number called the intersection of the currents. The entropy is the square root of the self-intersection of the current. Investigating this further might lead to an analogue of Wolpert's formula which gives the second variation of this intersection in terms of the Weil-Petersson metric.

Starting points for investigations in this direction are cross ratio functions and geometric structures, which have recently been associated to higher Teichmüller spaces.

Due to the correspondence between the moduli space of flat bundles and the moduli space of Higgs bundles, these deformation spaces carry hyper-Kähler structures which are not invariant by the action of the mapping class group, but rather parametrized by Teichmüller space itself. The variation of these finer invariants as the complex structure on the surface changes is an important new direction in Teichmüller theory, a sort of continuous version of the action of the (discrete) mapping class group. Indeed, this idea plays an important role in Andersen's proof that mapping class groups do not satisfy Kazhdan's property T.

<Followed by a list of potential researchers with affiliations and identifying women and minorities.>

3.7. Uniformization and simultaneous uniformization for Hitchin compo¬nents. A crucial question is whether or not one can generalize the Weil–Petersson geometry in the context of deformation spaces. Since these deformation spaces have a symplectic nature, the question is to describe a natural complex structure of the deformation space. Thanks to the uniformization theorem, Teichmüller space is the space of complex structures and thus, by the general theory of Kodaira-Spencer-Kuranishi, inherits itself a complex structure.

For Hitchin components the conjectural picture which generalizes the uniformization theorem is the following. Let S be a compact surface, and let Q(J, n) be the space of holomorphic differentials of degree n with respect to a complex structure J on S. Fixing J, Hitchin has produced a diffeomorphism between the Hitchin component H(n) and the direct sum H(J, n) of the Q(J, p) for $n \geq p \geq 2$.

<Name> proposed the following uniformization conjecture: The quotient of the Hitchin component for SL(n, R) by the action of the mapping class group is a vector bundle over the moduli space of Riemann surface with fibre above a point (S, J) being the direct sum of the Q(J, p) for $n \geq p \geq 3$. This conjecture would provide a a Kähler structure on the Hitchin components extending the Weil-Petersson structure on Teichmüller space.
For n = 2, this conjecture reduces to the uniformization theorem. <Names> established the conjecture for n = 3, using the realization of the Hitchin component as a deformation space of convex real projective structures by Choi-Goldman.

The general conjecture is very challenging. It translates into the existence and uniqueness of a minimal surface in the quotient of SL(n, R)/SO(n, R) by the image of a Hitchin representation.

The deformation of a Fuchsian representations in PSL(2, C) leads to a quasi-Fuchsian representation. The simultaneous uniformization theorem states that the set of quasi-Fuchsian representations is parametrized by two copies of Teichmüller space. The deformation of a Hitchin representations in SL(n, R) in SL(n, C) is an Anosov representation, and one can consider the connected component of the set of Anosov representations containing this deformation as a generalization of the space of quasi-Fuchsian representations. It is intriguing to ask if there is a generalization of the simultaneous uniformization theorem in the context of higher Teichmüller spaces. One starting point for the investigation of this problem are recently defined domains of discontinuity for Anosov representations (<names>).

<Optional: Followed by a list of potential researchers with affiliations and identifying women and minorities.>

A few more sections on various topics followed.

4. DIFFERENCES WITH <YEAR> PROGRAM

We have been asked to address the differences between our proposed program and the program at MSRI on Teichmüller space and Kleinian groups in Fall 2007. One obvious difference is that among the senior participants listed below as having committed to spending a significant amount of time at MSRI, only <Names of researchers> were senior participants in <year>.
On a scientific level, the current program focuses on representations of the fundamental group of a surface into general Lie groups, and more generally on representations of the fundamental group of a manifold into a Lie group. None of the central issues of this program were part of the <year> program.

There were two primary emphases to the Teichmüller theory portion of the 2007 program. The first was the relation of rational billiards and translation surfaces to Teichmüller theory and the second is what might be called the coarse geometry of Teichmüller space and its relation to objects such as the mapping class group and the curve complex. Neither of these subjects plays an important role in our current proposal. Instead, our current proposal focusses on generalizing more analytical topics, such as the complex structure or the Weil-Petersson flow, into the setting of spaces of representations of surface groups into more general Lie groups.

The Kleinian groups portion of the previous program focussed on developments flowing from the (then) recent solutions of Thurston's Ending Lamination Conjecture and

Marden's Tameness Conjecture. The two most prominent aspects on the Kleinian groups side were the development of combinatorial models for hyperbolic 3-manifolds and the study of Cannon-Thurston maps from the Gromov boundary of a Kleinian group to its limit set. Our current proposal interacts with the theory Kleinian groups through the study of the deformation theory of Kleinian groups, which we hope will provide inspiration for the study of deformation theory of representations of surface groups into more general Lie groups.

5. Mentoring Plans

Each of our postdoctoral members will be assigned a mentor from among the senior participants. The mentors will be expected to meet weekly with their assigned postdocs to discuss research, encourage them to interact fully in the program and offer professional advice. A member of the organizing committee, Dick Canary, has agreed to oversee the mentoring process to ensure that appropriate mentoring occurs. We also expect that there will be a weekly junior research seminar which will be run for and by the postdoctoral members and visiting graduate students. A similar plan was followed quite successfully during the program on Teichmüller Theory and Kleinian Groups in Fall 2007, resulting in numerous research collaborations.

6. Related Activities

Recently many conferences and workshops have been organized on related topics. In particular the trimester at Institut Henri Poincaré beginning in <month year> concentrates on surface group representations, which is narrower in scope than this proposed activity. The Mathematical Research Community activity "The Geometry of Real Projective Structures" <month year>is aimed at young postdocs and graduate students, some of whom will be postdocs by the time of the proposed MSRI activity. There will be a <Workshop Name> workshop at the University of Michigan in <month year>aimed at graduate students and early career mathematicians on the topic of Higher Dimensional Teichmüller theory. A special semester at CRM Barcelona will occur just after the IHP activity in <year>, which will have a more algebro-geometric flavor. A workshop on "Higher Thurston-Teichmüller theory" is planned for <year>as part of the Geometry/Topology Année Thematique at CRM Montréal.

7. Companion Program

The organizers suggest a companion program called <Program Name>. It was felt that there would be a great deal of interaction of the proposed program with a program of this sort. Potential organizers for such a program include <List of 5 organizers with affiliation.>

8. Expected List of Key Participants

The following is a list of senior mathematicians who we have contacted to gauge their interest in this program – provided its approval. Each has expressed interest in spending a significant amount of time at MSRI – from 2 to 4 months. In addition, all five of the organizers have committed to spending the entire semester.

<Follows a list of names of researchers with their affiliations, home institutions, gender and in case of US citizen minority status if applicable. **Note, a reference to an attached spreadsheet is sufficient.>**

9. Human Resources

The organizers are committed to encouraging the participation of members of groups which are under-represented in mathematics. In this section we indicate what our efforts will be in this direction. <One or two of the program organizers> have agreed to be the contact persons for these efforts. Listed below is a collection of women and African American mathematicians already working in the fields discussed in this proposal. If the program is approved, the organizers will immediately personally contact these people to let them know of the program, so that they might plan ahead. Once the application procedures are established by MSRI for the program, we will again contact these people to let them know how to apply. We will also contact the senior participants and others so that they can make their graduate students and recent Ph.D's from underrepresented groups aware of the program. In addition, as indicated in section 6, in the several years before the commencement of our proposed MSRI program there will a number of related activities. We expect that these programs will lead to heightened interest in these relatively new fields. The second major effort will be to contact several organizations that represent under-represented minorities to see if we can identify additional individuals who would benefit from the program. X and Y will personally contact people that we have identified this way. Professor David Scott, from the HRAC Committee, has agreed to assist us in establishing contacts. The organizations that we will contact include the Association for Women in Mathematics (AWM), National Association of Mathematicians (NAM), and SACNAS which is devoted to advancing Hispanics, Chicanos, and Native Americans in science.

<Follows a list of female & under-represented minority with their affiliation. **Again a reference to a spreadsheet is sufficient**.>

We will also attempt to include as many mathematicians as possible who work at colleges and universities which do not have large resources to support research. Examples of such active mathematicians working in the field include <X, Y and Z with their affiliation>.

10. Workshops

10.1. Topical workshop. The tentative title for the topical workshop is <workshop title>. We will invite speakers whose work is at the forefront of the field at the time of this workshop. As this subject is developing so quickly, recent innovations may necessitate a shift in subject matter. The workshop's organizing committee will be drawn from amongst the organizers and the senior participants. We will make a special effort to include speakers from groups which are under-represented in mathematics.

Topics will be drawn from the main research directions of the conference, including:

- mapping class group dynamics; for example actions on spaces of Anosov representations.
- actions of automorphism groups on character varieties
- higher mapping class groups
- uniformization problems
- geometric structures and topology of deformation spaces

10.2. Connections for Women workshop. There are a number of women mathematicians who are very active in the fields of this program who we envision asking to organize the Connections for Women Workshop. They are <list of 5-6 women>. We would ask X, a senior participant, to lead the effort. The senior program organizer <name> has agreed to assist in this project. The organizers envision a series of mini-courses each consisting of a couple of introductory lectures, with discussions in small groups and question and answer sessions. In our experience this format has led to the best interaction among people learning the subjects. A panel discussion will be included as well as social activities.

10.3. Introductory workshop. The introductory workshop will be organized by the senior organizers. We envision a similar format to that of the Connections for Women Workshop. The format will consist of a series of mini-courses with discussions in small groups. Several of them will be given by the organizers and the rest by some of the key participants. Topics for the mini-courses include:

(1) Geometry of higher rank symmetric spaces compared to rank one symmetric spaces
(2) Anosov representations
(3) Affine and projective structures
(4) Mapping class group dynamics
(5) Flows on Riemann moduli space
(6) Deformation spaces of hyperbolic 3-manifolds and dynamics on character varieties