PROBLEMS FOR THE CATEGORY THEORY READING COURSE, 2018

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1. Assignment 1, due end of Week 3

1.1. Functors.

(1) Leinster 1.2.21 (functors preserve isomorphisms)
(2) Leinster 1.2.27, 1.2.28b (full, faithful)

1.2. Natural transformations.

(1) In fdVec, show that the functors id and $**$ are naturally isomorphic.
(2) Show the the vertical composition of two natural transformations is in fact a natural transformation.
(3) Prove *carefully* that the horizontal composition of two natural transformations is again a natural transformation.
(4) Show that a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories if and only if it is *fully faithful* and *essentially surjective*. (Hint: faithul is easier than full; then use that the inverse of an equivalence is an equivalence.) [1] [2]

1.3. Universal properties.

(1) Prove that two initial objects in a category are isomorphic.
(2) For each of the following categories, decide whether there is an initial, final, and/or zero object, and if so, describe them: FinSet, fdVec, Top, Top_* (pointed topological spaces), Semigroups, Groups.
(3) Describe the product of two objects as the terminal object in some category.
(4) Describe the tensor product of two vectors spaces as the initial object in some category.
(5) Describe both the (binary) product and coproduct in the following categories: FinSet, Top, Top_*, AbGroup, Group.

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[1]Hint: there's a nice proof in Maclane, or you can try reading https://raw.githubusercontent.com/semorrison/lean-category-theory/master/src/categories/equivalence/characterisation.lean

[2]Did you notice using the axiom of choice?

2. Assignment 2, due at the end of Week 6

2.1. **Equivalences.**

(1) Choose one of the following pairs of categories, and briefly describe the equivalence (but don't show me all the details – ideally just talk about the most difficult part):
   - Compact Hausdorff topological spaces, and commutative C^*-algebras.
   - Subgroups of $\pi_1(X)$, and covers of X. (A condition is needed on X to make this work. What is it?)
   - Subgroups of $\mathrm{Gal}(E \subset F)$, and intermediate field extensions. (A condition is needed on $E \subset F$ to make this work. What is it?)

2.2. **Adjunctions.**

(1) Consider the forgetful functor from abelian groups to groups. What is its left adjoint?
(2) In the category of finite dimensional vector spaces, show that $- \otimes V$ is biadjoint to $- \otimes V^*$.
(3) Prove that the 'hom-set isomorphism' and 'unit/counit' definitions of an adjunction are equivalent.

2.3. **Idempotent completion.**

The 'idempotent completion' $\mathrm{Kar}(C)$ (also called the 'Karoubi envelope') is defined as follows:

$$\mathrm{Obj}\,\mathrm{Kar}(C) = \left\{(X \in \mathrm{Obj}(C), p : X \to X) \mid |p^2 = p\right\}$$
$$\mathrm{Kar}(C)((X,p) \to (X',p')) = \{f \in C(X \to X') \mid |fp = f = p'f\}.$$

(1) Let primeVec denote the full subcategory of fdVec (the finite dimensional vector spaces) consisting of vector spaces with prime dimensions. Show that $\mathrm{Kar}(\mathrm{primeVec}) \cong \mathrm{Vec}$.
(2) Construct an equivalence $\iota_C : \mathrm{Kar}(\mathrm{Kar}(C)) \cong \mathrm{Kar}(C)$.
(3) Show that there is a fully faithful functor $C \to \mathrm{Kar}(C)$ given by $X \mapsto (X, 1_X)$.

2.4. **The Yoneda lemma.**

(1) Are there set-theoretic difficulties hiding in the Yoneda lemma?
(2) Explain why $\mathrm{Fun}(C^{\mathrm{op}} \to \mathcal{D}) = \mathrm{Fun}(C \to \mathcal{D}^{\mathrm{op}})$. Prove that $\mathrm{Set}^{\mathrm{op}} \not\cong \mathrm{Set}$, but that $\mathrm{fdVec}^{\mathrm{op}} \cong \mathrm{fdVec}$.

3. Assignment 3, due at the end of Week 9

(1) Find an example of a monoidal functor which is not naturally isomorphic to any strict monoidal functor.

(Hint: consider categories $\mathrm{Vec}^\omega G$, where G is a finite group, and $\omega \in H^3(G, k^\times)$ is a 3-cocycle. In particular, consider $G = \mathbb{Z}/2\mathbb{Z}$. The category $\mathrm{Vec}\mathbb{Z}/2\mathbb{Z}$ can be made into a monoidal category in two distinct ways: either with the 'obvious' associator, or with the associator $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ which, when Z is 1-dimensional, is either the identity if Z is in the even grading, or minus the identity if Z is in the odd grading. If you're going to use this example you should explain carefully why this associator works. You should be able to show that there are *no* strict monoidal functors between these two versions of $\mathrm{Vec}\mathbb{Z}/2\mathbb{Z}$, but nevertheless there are non-strict monoidal functors, with a non-trivial tensorator; you should write one down explicitly.)

(2) Let C be a monoidal category. We say a 'monoid object' in C (or, as we gain confidence, just a monoid in C) is a tuple $(A \in \mathrm{Obj}\, C, \iota : 1 \to A, m : A \otimes A \to A)$ satisfying some conditions. Look up, or work out, what these conditions should be. (Hint: look at chapter 7 of Etingof, Gelaki, Nikshych, and Ostrik's book Tensor Categories.) You should be able to show that a monoid object in Vec is what is usually called an associative unital algebra.

  (a) A 'module object' for a monoid object $A \in C$ is a tuple $(M, \triangleright : A \otimes M \to M)$ satisfying an appropriate condition (what is it?). A morphism f between module objects M and M' is a morphism between the underlying objects, such that $f \circ \triangleright_M = \triangleright_{M'} \circ (1_A \otimes f)$. Draw the string diagram corresponding to this axiom. Define composition of module morphisms, by imitating the definition for modules over a ring. Show that modules for a fixed monoid object form a category.

(3) Show that $\mathrm{Rep}G$, for G a finite group, forms a monoidal category.

(4) (For this part, you may assume we are looking at representations over the complex numbers.)

- If $\mathrm{Rep}G \cong \mathrm{Rep}H$, as categories, are G and H isomorphic?

  (Hint: no, give a counterexample — any pair of non-isomorphic groups with the same number of irreducible representations (equivalently, the same number of conjugacy classes) will do.)

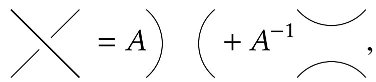
- What about if $\mathrm{Rep}G \cong \mathrm{Rep}H$ as monoidal categories, and moreover this equivalence is compatible with the forgetful functors to Vec?

  (Hint: think about the monoidal automorphisms of the forgetful functor. You should prove that (not necessarily monoidal) automorphisms of the forgetful functor $\mathrm{Rep}G \to \mathrm{Vec}$ is a group isomorphic to $\mathbb{C}[G]^\times$, and then identify the subgroup of monoidal automorphisms is G is itself. To do the first part, show that any such automorphism is determined by its component on the regular representation $\mathbb{C}[G]$; for this you may like to use that the regular representation is faithful, or (more or less equivalently) that there is a surjective G-linear map from the regular representation to any irreducible representation of G.)

4. Assignment 4, due at the end of Week 12

4.1. **Braided monoidal categories.**

(1) Show that Temperley-Lieb is a braided monoidal category, with braiding satisfying



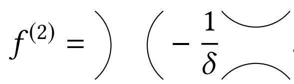
for some A so that $\delta = -A^2 - A^{-2}$.

(2) We can take a link diagram, and interpret it as a morphism $0 \to 0$ in Temperley-Lieb, by interpreting crossings according to the above formula. (Explain carefully what we're meant to do with .) The morphism space $0 \to 0$ consists of just scalar multiples of the empty diagram, so this associates a number to a link diagram. Explain why this number is a *framed* link invariant, i.e. that it doesn't change when we modify the link diagram by Reidemeister moves II and III, and a modified version of the usual Reidemeister move I.

(3) Calculate the invariant of the trefoil, according to this recipe. Show that the trefoil is not isotopic to the unknot, by showing this invariant takes different values on the two knots.

(4) What is this invariant usually called?

(5) Show that in $\mathrm{Kar}(TL)$, we have $(2, 1_2) \cong (2, f^{(2)}) \oplus (0, 1_0)$. Here $f^{(2)}$ denotes the second Jones-Wenzl idempotent:



(6) We can make another knot invariant by replacing each string of a knot diagram by two parallel strings, and somewhere on the knot inserting a copy of $f^{(2)}$.

Explain why this doesn't depend on where we insert the $f^{(2)}$. Calculate this invariant for the unknot.

(7) We say a monoid (A, m) in a braided monoidal category is commutative if $m \circ \beta = m$, where β denotes the braiding. Define a monoidal structure on the category of modules for a commutative monoid.

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