Examples of Adjunctions

Mitchell Rowett

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Present:

- Yiming
- Mitchell
- Keeley
- Josh
- Chris
- Likun
- Adwait
- Yossi
- Roger

Material Covered

After Yiming had defined adjunctions (via hom-sets), units, and counits, we discussed how duality was used in **Basic Category Theory** during the definition of units and counits on page 51. Specifically, it took us a while to determine how reversing the arrows for the construction of the unit gives the construction of the counit, but we eventually figured it out.

We then discussed (very informally) the correspondence between adjunctions $F \dashv G$ and pairs $(1_{\mathcal{A}} \to GF,\ FG \to 1_{\mathcal{B}})$ of natural transformations satisfying the triangle identities, though we didn't discuss the proof of this correspondence. This is an area that certainly needs more technical detail to describe properly, but we didn't have enough time to go into detail on Friday.

We then covered how to construct the following adjunctions, and their units and counits:

Free $\dashv$ Forgetful

In each of these examples, U is the forgetful functor, and F is the free functor.

$$\begin{tikzcd} \mathbf{Grp} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "F"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Vect}_{\mathbf{k}} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "F"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{Ab} \arrow[r, bend left, "U"] \arrow[r, phantom, "\top"] & \mathbf{Grp} \arrow[l, bend left, "F"] \end{tikzcd}$$

Set

Fix a set B. We can define functors $(-) \times B : A \mapsto A \times B$ and $(-)^B : C \mapsto C^B$. There is a canonical bijection

$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$

given by currying. This defines an adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "(-)^B"] \arrow[r, phantom, "\top"] & \mathbf{Set} \arrow[l, bend left, "- \times B"] \end{tikzcd}$$

Initial Objects

Let $\mathcal{A}$ be a category. A functor $1 \to \mathcal{A}$ can be thought of as an object of $\mathcal{A}$ (specifically, the object which 1 is mapped to). Hence a left adjoint to the unique functor $\mathcal{A} \to 1$ (if it exists) is an initial object of $\mathcal{A}$.

Difficulty

None of these examples caused any particular difficulty in constructing the adjunction or finding the unit/counit. We went into detail with the **Grp/Set** adjunction, and then went over the other two forgetful/free adjunctions more quickly before going into the last two adjunctions in detail.

This was the right amount of technical detail for these adjunctions, as it made sure we could construct adjunctions but allowed us to skip doing the near-identical forgetful/free adjunctions.

After the meeting

After the meeting, a couple of us (Yiming, Chris, Josh and I) discussed the string diagrams on page 56 of Leinster. This was *very* handwavy, but gave an intuitive explanation of why the triangle identities might be plausible. Again, the definition of the triangle identities and the correspondence between adjunctions $F \dashv G$ and pairs $(1_{\mathcal{A}} \to GF,\ FG \to 1_{\mathcal{B}})$ of natural transformations satisfying the triangle identities is definitely something which was not covered in sufficient technical detail.