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Algèbres de Lie/*Lie Algebras*

Computational evidence for Deligne's conjecture regarding exceptional Lie groups

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Abstract. For $j \leq 4$, we obtain, for all exceptional groups, a uniform decomposition of the jth tensor power of the adjoint representation, in agreement with the conjectures of Deligne [1].

Évidence pour une conjecture de Deligne sur les groupes de Lie exceptionnels

Résumé. *Pour $j \leq 4$, on obtient pour tous les groupes exceptionnels une décomposition uniforme de la puissance tensorielle j-ième de la représentation adjointe, en accord avec les conjectures de Deligne* [1].

Version française abrégée

Soit $\mathcal{G}^0$ le groupe déployé adjoint de type A_1, A_2, G_2, D_4, F_4, E_6, E_7, ou E_8 et soit Γ le groupe des automorphismes de son diagramme de Dynkin. Le groupe Γ est cyclique d'ordre 2 pour A_2 et E_6, isomorphe à S_3 pour D_4 et trivial pour les autres types. Soit $\mathcal{G}$ le produit semi-direct de Γ par $\mathcal{G}^0$ (vu comme groupe algébrique sur $\mathbb{Q}$) et soit $\mathfrak{g}$ son algèbre de Lie.

Pour les 5 types exceptionnels, Vogel a observé que $\overset{2}{\wedge}\mathfrak{g}$ se décompose en la somme de $\mathfrak{g}$ et d'une représentation irréductible X, tandis que $S^2\mathfrak{g}$ se décompose en $1 + Y + Z$ avec Y et Z irréductibles. Utilisant que les seuls éléments invariants de $S^4\mathfrak{g}$ sont les multiples du carré du Casimir, il montre que les dimensions de $\mathfrak{g}$, X, Y, Z et les scalaires par lesquels agit l'opérateur de Casimir sont donnés par des fonctions rationnelles d'un paramètre a. Cette structure persiste pour tous les groupes considérés ci-dessus, avec $a = 1/2$, $1/3$, $1/4$, $1/6$, $1/9$, $1/12$, $1/18$, $1/30$.

Deligne a demandé (pour tous ces groupes sauf D_4) si un tel phénomène persistait pour les puissances tensorielles supérieures de la représentation adjointe. Dans cette Note, nous donnons les résultats de calculs fait à l'aide du programme LiE [2]. Nous trouvons que pour chaque groupe $\mathcal{G}$, il existe des représentations virtuelles $X_i\,(2 \leq i \leq 4)$, $Y_i\,(2 \leq i \leq 4)$, $Y_i^*\,(2 \leq i \leq 4)$, A, C, C^*,

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D, D^*, E, F, F^*, G, G^*, H, H^*, I, I^*, et J, irréductibles, nulles ou l'opposée d'une représentation irréductible, en termes desquelles (et des représentations triviales et adjointes) les représentations $\mathfrak{g}^{\otimes 3}$ et $\mathfrak{g}^{\otimes 4}$, et plus précisément les pléthismes de $\mathfrak{g}$ de degré ≤ 4 se décomposent de façon uniforme (en « constituants »). Avec les notations précédentes, on a $X_2 = X$, $Y_2 = Y$, $Y_2^* = Z$. La décomposition des carrés extérieurs et symétriques des constituants X_2, Y_2 et Y_2^* de $\mathfrak{g}^{\otimes 2}$, celle des produits tensoriels (constituant de $\mathfrak{g}^{\otimes 2}$) $\otimes$ (constituant de $\mathfrak{g}^{\otimes 2}$) et celle des produits tensoriels $\mathfrak{g}\otimes$ (constituant de $\mathfrak{g}^{\otimes 3}$) est également uniforme. Si on regarde * comme une involution laissant fixe 1, $\mathfrak{g}$, X_i, A, E et J, toutes ces formules de décomposition sont stables par l'involution *.

Sur chacun des modules considérés, l'opérateur de Casimir agit par un scalaire, donné par une forme linéaire en a dépendant du module. L'involution * échange a et $a^* := -1/6 - a$.

Soit $j = 2$, 3 ou 4. Chacune des formules de décomposition relative à $\mathfrak{g}^{\otimes j}$ fournit une relation linéaire entre dimensions de modules. Des relations additionnelles sont obtenues en considérant la trace de l'opérateur de Casimir : utiliser (1) et (2) de [1]. Supposant connues les dimensions des constituants de $\mathfrak{g}^{\otimes i}$ pour $i < j$, on obtient ainsi suffisamment de relations $\mathbb{Q}(a)$-linéaires pour déterminer uniquement des fonctions rationnelles en a donnant les dimensions des modules pour $a = 1/2$, $1/3$, $1/4$, $1/6$, $1/9$, $1/12$, $1/18$, $1/30$, selon le type. Par construction, les formules obtenues sont stables par l'involution *, agissant sur le module et sur a. Miracle : les numérateurs et dénominateurs des fonctions rationnelles obtenues sont des produits de facteurs linéaires rationnels.

Let $\mathcal{G}^0$ denote the split adjoint connected Lie group of type A_1, A_2, G_2, D_4, F_4, E_6, E_7 or E_8, and let Γ be the automorphism group of its Dynkin diagram. Then Γ is cyclic of order 2 for E_6 and A_2, $\Gamma \cong S_3$ or D_4, and Γ is trivial for the other types. We extend $\mathcal{G}^0$ by Γ to form the algebraic group $\mathcal{G}$ over Q and write $\mathfrak{g}$ for its Lie algebra.

It was observed by Vogel (at least for the 5 exceptional groups $\mathcal{G}$) that the $\mathcal{G}$-modules $\overset{2}{\wedge}\,\mathfrak{g}$ and $\mathrm{Sym}^2\mathfrak{g}$ can be decomposed into irreducibles in a uniform way, and that the dimension of the constituents and the values by which the Casimir operator is acting on them are given by rational functions of a parameter a. For the types A_1, A_2, G_2, D_4, F_4, E_6, E_7, E_8, we have $a = 1/2$, $1/3$, $1/4$, $1/6$, $1/9$, $1/12$, $1/18$, $1/30$.

Deligne asked (for all 8 groups except D_4) whether this could be extended to higher tensor powers of $\mathfrak{g}$. In this Note we give the results of computations done using the program LiE [2]. It turns out that for each group we can find virtual representations $X_i\,(2 \leq i \leq 4)$, $Y_i\,(2 \leq i \leq 4)$, $Y_i^*\,(2 \leq i \leq 4)$, A, C, C^*, D, D^*, E, F, F^*, G, G^*, H, H^*, I, I^*, and J such that the pattern extends in a nice way to $\mathfrak{g}^{\otimes 3}$ and $\mathfrak{g}^{\otimes 4}$. These representations are either irreducible, 0, or the negative of an irreducible. Here * can be thought of as an involution keeping X_i, A, E, and J fixed.

We now give the decomposition formulae. Using notation of [1] for plethysms we find:

$$[(2)]\,\mathfrak{g} = \mathrm{Sym}^2\,\mathfrak{g} = 1 + Y_2 + Y_2^*,$$
$$[(1,\,1)]\,\mathfrak{g} = \mathfrak{g} + X_2.$$

$$[(3)]\,\mathfrak{g} = \mathfrak{g} + X_2 + A + Y_3 + Y_3^*,$$
$$[(2,\,1)]\,\mathfrak{g} = 2\mathfrak{g} + X_2 + Y_2 + Y_2^* + A + C + C^*,$$
$$[(1,\,1,\,1)]\mathfrak{g} = 1 + X_2 + Y_2 + Y_2^* + X_3.$$

$$[(4)]\mathfrak{g} = 1 + 2Y_2 + 2Y_2^* + C + C^* + X_3 + D + D^* + J + Y_4 + Y_4^*,$$

$$[(3,\ 1)]\mathfrak{g} = 2\mathfrak{g} + 3X_2 + 2Y_2 + 2Y_2^* + 3A + 2C + 2C^* + X_3 + Y_3 + Y_3^* + D + D^* + E + F + F^* + G + G^*,$$

$$[(2,\ 2)]\mathfrak{g} = 2 \cdot 1 + X_2 + 3Y_2 + 3Y_2^* + A + C + C^* + 2X_3 + D + D^* + E + H + H^* + J,$$

$$[(2,\ 1,\ 1)]\mathfrak{g} = 3\mathfrak{g} + 4X_2 + Y_2 + Y_2^* + 3A + 2C + 2C^* + X_3 + Y_3 + Y_3^* + E + F + F^* + I + I^*,$$

$$[(1,\ 1,\ 1,\ 1)]\mathfrak{g} = \mathfrak{g} + Y_2 + Y_2^* + A + C + C^* + X_3 + X_4.$$

$$\mathfrak{g} \otimes X_2 = \mathfrak{g} + X_2 + Y_2 + Y_2^* + A + C + C^* + X_3,$$

$$\mathfrak{g} \otimes Y_2 = \mathfrak{g} + X_2 + Y_2 + A + C + Y_3.$$

$$\mathfrak{g} \otimes A = X_2 + Y_2 + Y_2^* + 2A + C + C^* + X_3 + D + D^* + E + F + F^* + J,$$

$$\mathfrak{g} \otimes C = X_2 + Y_2 + A + 2C + X_3 + Y_3 + D + E + F + G + H + I,$$

$$\mathfrak{g} \otimes X_3 = X_2 + A + C + C^* + X_3 + E + F + F^* + I + I^* + X_4,$$

$$\mathfrak{g} \otimes Y_3 = Y_2 + C + Y_3 + D + G + Y_4.$$

$$[(2)]X_2 = 1 + 2Y_2 + 2Y_2^* + A + C + C^* + 2X_3 + D + D^* + E + H + H^* + J + X_4,$$

$$[(1,\ 1)]X_2 = \mathfrak{g} + 2X_2 + 2A + C + C^* + Y_3 + Y_3^* + E + F + F^* + I + I^*,$$

$$X_2 \otimes Y_2 = \mathfrak{g} + 2X_2 + Y_2 + Y_2^* + 2A + 2C + C^* + X_3 + Y_3 + D + E + F + F^* + G + I,$$

$$[(2)]Y_2 = 1 + 2Y_2 + Y_2^* + C + X_3 + D + H + J + Y_4,$$

$$[(1,\ 1)]Y_2 = \mathfrak{g} + X_2 + A + C + Y_3 + F + G,$$

$$Y_2 \otimes Y_2^* = X_2 + Y_2 + Y_2^* + A + C + C^* + X_3 + D + D^* + E.$$

The starred versions of these formulae hold was well. Warning: the decomposition formulae for the plethysms in $\mathfrak{g}^{\otimes 4}$ have only been checked on $\mathcal{G}^0$.

The following two tables describe the irreducible (virtual) constituents, in terms of their highest weights. For $A_2.2$ and $E_6.2$, an irreducible representation V is either induced by an irreducible representation W of $\mathcal{G}^0$ (non-isomorphic to its transformed by the non-trivial element σ of Γ), or remains irreducible when restricted to $\mathcal{G}^0$. In the latter case, it is determined by its restriction to $\mathcal{G}^0$ and by the sign $\pm$ by which σ acts on the highest weight subspace. Notation: λ^0 (resp. $\lambda^{\pm}$), for λ the highest weight of W (resp. V). For $D_4 \cdot S_3$, the possibilities are: (*a*) The restriction to $\mathcal{G}^0$ is twice an irreducible W. On the highest weight subspace, S_3 acts by its 2-dimensional representation. (*b*) The restriction to $\mathcal{G}^0$ is irreducible; S_3 acts trivially, or by the sign character, on the highest weight subspace. (*c*) The representation is induced from an irreducible representation W of a subgroup $D_4.2$. The representation W has an irreducible restriction to $\mathcal{G}^0$ and is described as for $A_2.2$ and $E_6.2$. (*d*) The representation is induced from an irreducible representation of $\mathcal{G}^0$. Notation: 2λ, $\lambda^{\pm}$, $\lambda^{0\pm}$, λ^0.

	A_1	$A_2.2$	G_2	$D_4.S_3$	F_4
$\mathfrak{g}$	[2]	$[1, 1]^-$	[0, 1]	$[0, 1, 0, 0]^+$	[1, 0, 0, 0]
X_2	0	$[0, 3]^0$	[3, 0]	$[1, 0, 1, 1]^+$	[0, 1, 0, 0]
Y_2	[4]	$[2, 2]^+$	[0, 2]	$[0, 2, 0, 0]^+$	[2, 0, 0, 0]
Y_2^*	0	$[1, 1]^+$	[2, 0]	$[0, 0, 0, 2]^{0+}$	[0, 0, 0, 2]
A	0	$[2, 2]^-$	[2, 1]	$[0, 1, 0, 2]^{0+}$	[1, 0, 0, 2]
C	0	$[1, 4]^0$	[3, 1]	$[1, 1, 1, 1]^+$	[1, 1, 0, 0]
C^*	− [2]	0	[1, 1]	2[1, 0, 1, 1]	[0, 0, 1, 1]
X_3	− [4]	0	[4, 0]	$[0, 0, 2, 2]^{0+}$	[0, 0, 2, 0]
Y_3	[6]	$[3, 3]^-$	[0, 3]	$[0, 3, 0, 0]^+$	[3, 0, 0, 0]
Y_3^*	0	$[0, 0]^-$	[1, 0]	2[0, 1, 0, 0]	[0, 0, 1, 0]
D	0	$[3, 3]^+$	[2, 2]	$[0, 2, 0, 2]^{0+}$	[2, 0, 0, 2]
D^*	[2]	0	0	2[0, 2, 0, 0]	[1, 0, 1, 0]
E	0	0	[1, 2]	2[1, 1, 1, 1]	[1, 0, 1, 1]
F	0	0	[5, 0]	$[1, 0, 1, 3]^{0+}$	[0, 1, 0, 2]
F^*	0	$-[2, 2]^-$	0	$[0, 0, 2, 2]^{0-}$	[0, 0, 1, 2]
G	0	$[2, 5]^0$	[3, 2]	$[1, 2, 1, 1]^+$	[2, 1, 0, 0]
G^*	0	$-[0, 0]^-$	0	$[0, 0, 0, 2]^{0-}$	[1, 0, 0, 1]
H	0	$[0, 6]^0$	[6, 0]	$[2, 0, 2, 2]^+$	[0, 2, 0, 0]
H^*	− [0]	$-[1, 1]^+$	0	$[1, 0, 1, 1]^-$	[0, 0, 0, 3]
I	− [6]	0	[4, 1]	$[0, 1, 2, 2]^{0+}$	[1, 0, 2, 0]
I^*	0	$-[0, 3]^0$	0	$[0, 1, 0, 2]^{0-}$	[0, 1, 0, 1]
J	0	0	0	$[0, 0, 0, 4]^{0+}$	[0, 0, 0, 4]
X_4	0	$-[1, 4]^0$	0	$[1, 0, 1, 3]^{0-}$	[0, 0, 2, 1]
Y_4	[8]	$[4, 4]^+$	[0, 4]	[0, 4, 0, 0]	[4, 0, 0, 0]
Y_4^*	0	0	0	2[0, 0, 0, 0]	[0, 0, 0, 1]

The weight λ is a linear combination $\sum a_i\omega_i$ of fundamental weights. The fundamental weights are numbered as in the tables of Bourbaki, and λ is described as the sequence of the coefficients a_i.

For V an irreducible $\mathcal{G}$-module, let $\gamma(V)$ denote the scalar by which the Casimir operator acts on V. For each of the 25 modules above, this value has been computed. It turns out that for each module this scalar is a linear expression in a. If we scale the Casimir operator such that $\gamma(\mathfrak{g}) = 1$, we find:

V	$\gamma(V)$	V	$\gamma(V)$
$\mathfrak{g}$	1	E	$21/6$
X_n	n	F	$22/6+2a$
Y_n	$n+(n^2-n)\,a$	G	$4+8a$
A	$16/6$	H	$4+6a$
C	$3+3a$	I	$4+4a$
D	$22/6+4a$	J	$20/6$

With $a^* = -1/6 - a$ these expressions remain valid after applying the involution *. This determines the value $\gamma(V)$ for the starred modules V missing from the above table.

Let $j \in \{2, 3, 4\}$. Each of the decomposition formulae belonging to $\mathfrak{g}^{\otimes j}$ gives a linear relation between the dimensions of the modules, provided the dimensions of the modules occurring in $\mathfrak{g}^{\otimes i}$ with $i < j$ are given. Additional relations were obtained by computing the trace of the Casimir operator, where we used (1) and (2) of [1]. In this way we obtain sufficiently many linear equations over $\mathbf{Q}(a)$ to solve for the dimensions of the constituents occurring in $\mathfrak{g}^{\otimes j}$ but not before. This procedure leads to a unique solution. Thus we find rational functions in a giving the dimensions of our modules for the special values $a = 1/2,\ 1/3,\ 1/4,\ 1/6,\ 1/9,\ 1/12,\ 1/18,\ 1/30$. Miraculously, for all these rational functions both numerator and denominator factor in $\mathbf{Q}[a]$ as a product of linear

	$E_6.2$	E_7	E_8
$\mathfrak{g}$	$[0, 1, 0, 0, 0, 0]^+$	$[1, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 1]$
X_2	$[0, 0, 0, 1, 0, 0]^+$	$[0, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 0]$
Y_2	$[0, 2, 0, 0, 0, 0]^+$	$[2, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 2]$
Y_2^*	$[1, 0, 0, 0, 0, 1]^+$	$[0, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 0]$
A	$[1, 1, 0, 0, 0, 1]^+$	$[1, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 1]$
C	$[0, 1, 0, 1, 0, 0]^+$	$[1, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 1]$
C^*	$[0, 0, 0, 0, 1, 1]^0$	$[0, 1, 0, 0, 0, 0, 1]$	$[0, 1, 0, 0, 0, 0, 0, 0]$
X_3	$[0, 0, 1, 0, 1, 0]^+$	$[0, 0, 0, 1, 0, 0, 0]$	$[0, 0, 0, 0, 0, 1, 0, 0]$
Y_3	$[0, 3, 0, 0, 0, 0]^+$	$[3, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 3]$
Y_3^*	$[1, 0, 0, 0, 0, 1]^-$	$[0, 0, 0, 0, 0, 0, 2]$	0
D	$[1, 2, 0, 0, 0, 1]^+$	$[2, 0, 0, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 0, 2]$
D^*	$[1, 1, 0, 0, 0, 1]^-$	$[1, 0, 0, 0, 0, 0, 2]$	0
E	$[0, 1, 0, 0, 1, 1]^0$	$[1, 1, 0, 0, 0, 0, 1]$	$[0, 1, 0, 0, 0, 0, 0, 1]$
F	$[1, 0, 0, 1, 0, 1]^+$	$[0, 0, 1, 0, 0, 1, 0]$	$[1, 0, 0, 0, 0, 0, 1, 0]$
F^*	$[0, 0, 1, 0, 0, 2]^0$	$[0, 0, 0, 0, 1, 0, 1]$	$[0, 0, 1, 0, 0, 0, 0, 0]$
G	$[0, 2, 0, 1, 0, 0]^+$	$[2, 0, 1, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 1, 2]$
G^*	$[0, 0, 0, 1, 0, 0]^-$	0	$-[0, 1, 0, 0, 0, 0, 0, 0]$
H	$[0, 0, 0, 2, 0, 0]^+$	$[0, 0, 2, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 2, 0]$
H^*	$[0, 0, 0, 0, 0, 3]^0$	0	0
I	$[0, 1, 1, 0, 1, 0]^+$	$[1, 0, 0, 1, 0, 0, 0]$	$[0, 0, 0, 0, 0, 1, 0, 1]$
I^*	$[0, 0, 1, 0, 1, 0]^-$	$[0, 2, 0, 0, 0, 0, 0]$	0
J	$[2, 0, 0, 0, 0, 2]^+$	$[0, 0, 0, 0, 0, 2, 0]$	$[2, 0, 0, 0, 0, 0, 0, 0]$
X_4	$[0, 0, 2, 0, 0, 1]^0$	$[0, 1, 0, 0, 1, 0, 0]$	$[0, 0, 0, 0, 1, 0, 0, 0]$
Y_4	$[0, 4, 0, 0, 0, 0]^+$	$[4, 0, 0, 0, 0, 0, 0]$	$[0, 0, 0, 0, 0, 0, 0, 4]$
Y_4^*	$[0, 1, 0, 0, 0, 0]^-$	0	$-[1, 0, 0, 0, 0, 0, 0, 0]$

factors. The formulae are given in the following table, where $\lambda = -6a$. As all formulae are stable under $*$ we omit the starred modules (observe that $\lambda^* = 1 - \lambda$).

$$\dim \mathfrak{g} = -2\frac{(\lambda+5)(\lambda-6)}{\lambda(\lambda-1)},$$

$$\dim X_2 = 5\frac{(\lambda+5)(\lambda-6)(\lambda+3)(\lambda-4)}{\lambda^2(\lambda-1)^2},$$

$$\dim Y_2 = -90\frac{(\lambda-4)(\lambda+5)}{\lambda^2(\lambda-1)(2\lambda-1)}.$$

$$\dim A = -27\frac{(\lambda-4)(\lambda-5)(\lambda-6)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^2(3\lambda-1)(3\lambda-2)(\lambda-1)^2},$$

$$\dim C = 640\frac{(\lambda-3)(\lambda-5)(\lambda+5)(\lambda+3)}{\lambda^3(\lambda-1)(2\lambda-1)(3\lambda-2)},$$

$$\dim X_3 = -10\frac{(\lambda-3)(\lambda-5)(\lambda-6)(\lambda+5)(\lambda+4)(\lambda+2)}{(\lambda-1)^3\lambda^3},$$

$$\dim Y_3 = -10\frac{(\lambda-4)(\lambda-5)(\lambda-6)(5\lambda-6)(\lambda+5)}{\lambda^3(3\lambda-1)(2\lambda-1)(\lambda-1)^2}.$$

$$\dim D = -90\frac{(\lambda-3)(\lambda-6)(5\lambda-6)(2\lambda-5)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^3(4\lambda-1)(2\lambda-1)^2(\lambda-1)^3},$$

$$\dim E = 512\frac{(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda+5)(\lambda+4)(\lambda+3)(\lambda+2)}{\lambda^2(4\lambda-1)(4\lambda-3)(2\lambda-1)^2(\lambda-1)^2},$$

$$\dim F = -810\frac{(\lambda-4)(\lambda-6)(2\lambda-5)(\lambda+5)(\lambda+3)(\lambda+2)}{\lambda^3(3\lambda-1)(2\lambda-1)^2(\lambda-1)^2},$$

$$\dim G = 45\frac{(\lambda-4)(\lambda-5)(\lambda-6)(3\lambda-4)(2\lambda-5)(\lambda+5)(\lambda+3)}{\lambda^4(3\lambda-1)(2\lambda-1)^2(\lambda-1)^2},$$

$$\dim H = -10\frac{(\lambda-5)(\lambda-6)(3\lambda-4)(5\lambda-6)(2\lambda-5)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^4(3\lambda-2)(2\lambda-1)^2(\lambda-1)^3},$$

$$\dim I = -405\frac{(\lambda-4)(\lambda-5)(\lambda+5)(\lambda+4)(2\lambda-5)(\lambda+2)(5\lambda-6)}{\lambda^4(4\lambda-3)(2\lambda-1)(3\lambda-2)(\lambda-1)^2},$$

$$\dim J = 81\frac{(\lambda-3)(\lambda-4)(\lambda-6)(2\lambda+3)(2\lambda-5)(\lambda+5)(\lambda+3)(\lambda+2)}{\lambda^2(3\lambda-1)(3\lambda-2)(2\lambda-1)^2(\lambda-1)^2},$$

$$\dim X_4 = 5\frac{(\lambda-4)(\lambda-5)(\lambda-6)(2\lambda+3)(2\lambda-5)(\lambda+5)(\lambda+4)(\lambda+3)}{\lambda^4(\lambda-1)^4},$$

$$\dim Y_4 = -5\frac{(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(3\lambda-4)(2\lambda-5)(\lambda+5)(7\lambda-6)}{\lambda^4(4\lambda-1)(3\lambda-1)(2\lambda-1)(3\lambda-2)(\lambda-1)^2}.$$

The only rational values of λ for which all those dimension formulae (including those obtained by replacing λ by $1-\lambda$) take integer values with $\dim \mathfrak{g} \geq 0$ are those corresponding to a or a^* equal to 5/6, 2/3, 1/2, 1/3, 1/4, 1/6, 1/9, 1/12, 1/18, 1/24, 1/30. The value $a = 5/6$ corresponds to the trivial group (noted by Deligne): all dimension formulae take values 1, 0, or -1 and when the dimension is ± 1, the Casimir is 0. Of the remaining values, only 2/3 and 1/24 are not accounted for by a group in our list. For $a = 2/3$, all dimension formulae take values 1, 0, or -1, with the value 1 for $\dim \mathfrak{g}$. For $a = 1/24$, the dimension formula gives $\dim \mathfrak{g} = 190$.

The rational values of λ for which $\dim \mathfrak{g}$ is negative and all dimension formulae take integer values can all be found by applying the involution $a \mapsto -a/(12a+1)$ (which corresponds to $\lambda \mapsto \lambda/(2\lambda-1)$ to the values of a above, excluding 1/12 and 1/6. This involution transforms $\dim \mathfrak{g}$ into $-244-\dim \mathfrak{g}$. The values for λ obtained in this way from 1/12 and 1/6 (namely 1/4 and 1/3) are poles of some of the dimension formulae.

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