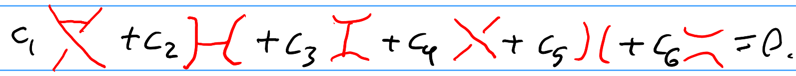
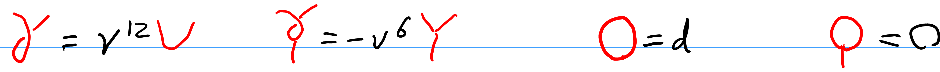
The quantum exceptional series

Suppose you have an invariant of framed trivalent graphs that satisfies a skein identity of the form

(A0) 

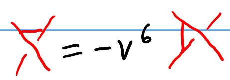
(For instance, perhaps there is only a 5-dimensional space of diagrams with 4 external vertices.)

Suppose also that the framing changes by a scalar multiple, that a circle acts by a scalar, and that the a self-loop is zero.

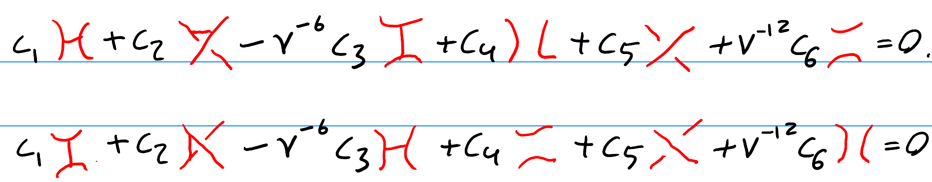


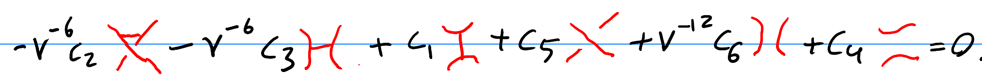
(Right now only v^6 is fixed; we will pick a root later.)
Now it seems that everything is determined with just one free parameter, at least for generic values of v.

For reference, we can change the framing of an X like so:

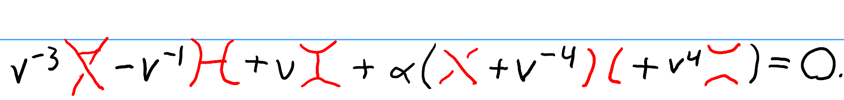


Now take (A0) and braid the bottom two outputs, then rotate 90 degrees:



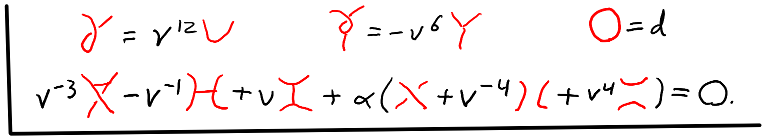
(A1) 

This relation has the same form as (A0). If we repeat this operation, we get another relation (A2) and (A3), with (A3) = v^{-12}(A0). Thus, by averaging, we may assume that (A1) = v^{-4}(A0) (possibly after changing the value of v). The relation then becomes the fundamental relation

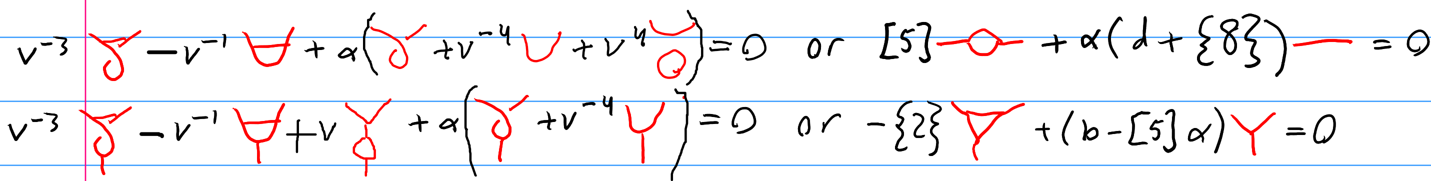
(IHX) 

(Here, we assumed that c1 != 0; otherwise, the relation reduces to the Jones skein relation, and I believe the trivalent vertex is forced to be 0.)

The right way to think about this is that our initial relation (A0) has the symmetries of a tetrahedron. The braiding operation to get from (A0) to (A1) is rotating that tetrahedron around one vertex by 1/3 of a full rotation. If we do that 3 times, we get back where we started, but with an extra twist in the strand leaving at the vertex.

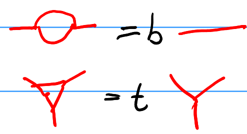


From IHX, we immediately get bigon and triangle reduction:



using the notation [n] = v^n - v^{-n} and {n} = v^n + v^{-n} = [2n]/[n].

We therefore get



$$b = -\frac{\alpha(d+\{8\})}{[5]}$$

$$t = \frac{b-[5]\alpha}{\{2\}}$$

One nice solution to these equations (introducing a new free prameter w, and using the freedom to rescale vertices by a scalar):

$$d = -\frac{\{2\}[\lambda+5][\lambda-6]}{[\lambda][\lambda-1]}$$

$$b = \frac{\{\lambda+2\}\{\lambda-3\}[3]}{[1]}$$

$$\alpha = -\frac{[\lambda][\lambda-1]}{[1]}$$

$$t = \{1\}\left((v+v^{-1})\frac{w^2}{v} + (v^4-v^2-1-v^{-2}+v^{-4}) + (v+v^{-1})\frac{v}{w^2}\right)$$

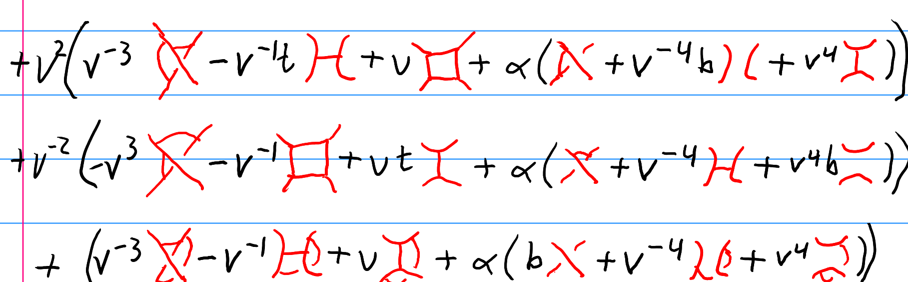
$$[k\lambda+l] := w^k v^l - w^{-k} v^{-l}$$

$$\{k\lambda+l\} := w^k v^l + w^{-k} v^{-l}$$

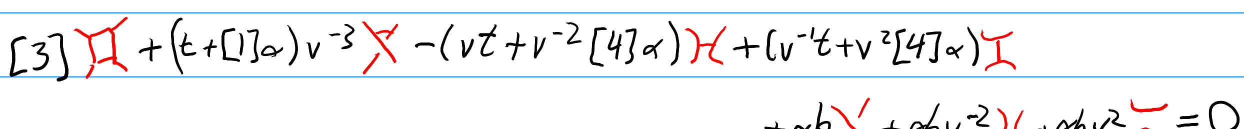
Conjecture. These relations allow us to evaluate any trivalent graph.

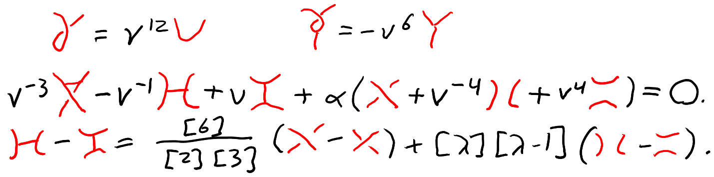
As evidence for this, let's see how to reduce a square, or change a crossing.

Take the (IHX) relation, and attach an "H" to it in the 3 natural ways respecting the symmetries of the tetrahedron:



Some more complicated diagrams (twisted squares) appear, but if we take the indicated linear combination they cancel:





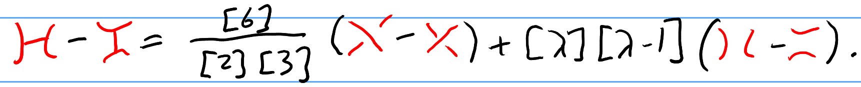
$$d = -\frac{\{2\}[\lambda+5][\lambda-6]}{[\lambda][\lambda-1]}$$

$$b = \frac{\{\lambda+2\}\{\lambda-3\}[3]}{[1]}$$

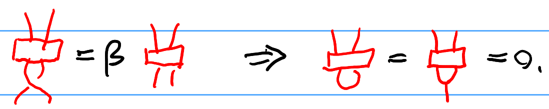
$$\alpha = -\frac{[\lambda][\lambda-1]}{[1]}$$

Not terribly beautiful yet, but it does allow us to reduce squares, as long as [3] != 1, i.e., v^6 != 1. Note that this does not apply in the classical case (naturally), although the resulting relation does have the Vogel relation as its classical limit.

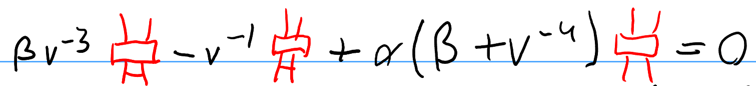
For a nicer relation, add this relation to its conjugate, obtained by reversing all crossings and sending v and w to their inverses, or alternately in this case by rotating 90 degrees and negating. Miraculously, all the complicated terms end up as nice products of monomials, and we get

(Cross) 

Let's now find more nice relations, by considering eigenvalues. Assume that we have an eigenvector acting on the 4-box space, with eigenvalue $\beta \neq v^{12}, v^{-6}$.



Multiply by (IHX) and (Cross) to see this is also an eigenvalue for the ladder operator:

IHX ⇒ 

$$\gamma(v-\beta v^{-1}) = -\frac{[\lambda][\lambda-1](\beta v^2+v^{-2})}{[1]}$$

$$\text{Cross} \Rightarrow \gamma = \frac{v^2-1+v^{-2}}{[1]}(\beta-\beta^{-1}) + [\lambda][\lambda-1]$$

These give two different equations for beta and gamma, yielding a cubic equation for beta, with solutions

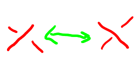
$$\beta = -1 \qquad \gamma = [\lambda][\lambda-1]$$

$$\beta = w^2 \qquad \gamma = \frac{[\lambda]\{\lambda+2\}}{[1]}$$

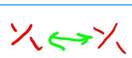
$$\beta = \frac{v^2}{w^2} \qquad \gamma = -\frac{[\lambda-1]\{\lambda-3\}}{[1]}$$

Aside: why the minus sign? The overall symmetries of the theory are

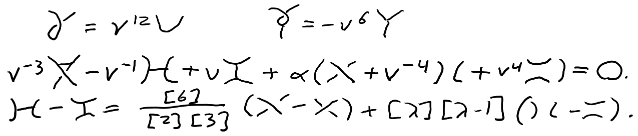
(a) $v \leftrightarrow v^{-1}$ $\quad w \leftrightarrow w$



(b) $v \leftrightarrow v$ $\quad w \leftrightarrow \frac{v}{w}$



Under (b), $[\lambda] \leftrightarrow -[\lambda-1]$

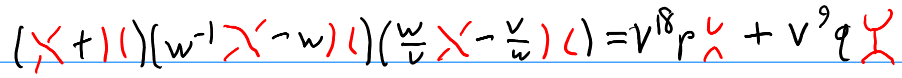


$$d = -\frac{\{2\}[\lambda+5][\lambda-6]}{[\lambda][\lambda-1]}$$

$$b = \frac{\{\lambda+2\}\{\lambda-3\}[3]}{[1]}$$

$$\alpha = -\frac{[\lambda][\lambda-1]}{[1]}$$

The charcteristic equation is then



Capping off gives

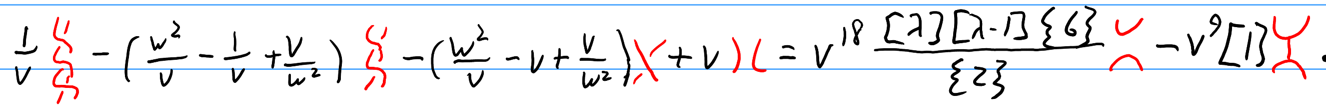
$$(v^{12}+1)(w^{-1}v^{12}-w)\left(\frac{w}{v}v^{12}-\frac{v}{w}\right) = v^{18}\, p\, d$$

$$p = \frac{\{6\}(-[\lambda-6])[\lambda+5]}{d} = \frac{[\lambda][\lambda-1]\{6\}}{\{2\}}$$

$$(-v^6+1)(-w^{-1}v^6-w)\left(-\frac{w}{v}v^6-\frac{v}{w}\right) = v^9\, q\, b$$

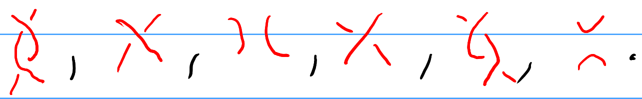
$$q = \frac{-[3]\cdot -\{\lambda-3\}\cdot -\{\lambda+2\}}{b} = -[1]$$

so we find



I have not proved this relation is correct, but there must be one of this form and it's hard to believe this is not right.

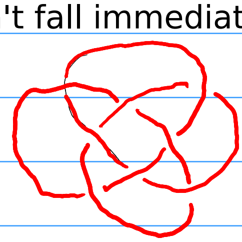
Of course, we can multiply by one twist the other direction to get a relation relating two adjacent crossings to simpler diagrams. In fact, you could get rid of graphs altogether, and get another relation relating the tangles



This allows you to turn a clasp into a clasp the other way. This suffices to compute many knots; all trefoils, but also (Noah pointed out) almost all knots with up to 8 crossings: all but three of these are alternating, and all but one of the alternating knots have a clasp region. The three non-alternating diagrams have a diagram with at least three twists ina row, so are easy to simplify.

If you apply this relation to an alternating clasp, the opposite clasp is a non-alternating diagram, which therefore simplifies.

The one exception that doesn't fall immediately to this relation is 8_{18}:



Let's compute the Hopf link, just for fun. Take the 3-crossing relation, twist once the other way, and take the trace:

$$= v^{12} \qquad = -v^{6}$$

$$v^{-3} - v^{-1} + v + \alpha(+ v^{-4})(+ v^{4}) = 0.$$

$$ - = \frac{[6]}{[2][3]}(-) + [\lambda][\lambda-1](-).$$

$$d = -\frac{\{2\}[\lambda+5][\lambda-6]}{[\lambda][\lambda-1]}$$

$$b = \frac{\{2\lambda+2\}\{2\lambda-3\}[3]}{[1]}$$

$$\alpha = -\frac{[\lambda][\lambda-1]}{[1]}$$

$$\frac{1}{v}() - \left(\frac{w^2}{v} - \frac{1}{v} + \frac{v}{w^2}\right) - \left(\frac{w^2}{v} - v + \frac{v}{w^2}\right) + v = v^{6}\frac{[\lambda][\lambda-1]\{6\}}{\{2\}}d + v^{3}[1]bd$$

$$\text{Hopf} = vd\left(v^{-12}\left(\frac{w^2}{v} - \frac{1}{v} + \frac{v}{w^2}\right) + d\left(\frac{w^2}{v} - v + \frac{v}{w^2}\right) - v^{13} + v^{6}\frac{[\lambda][\lambda-1]\{6\}}{\{2\}} + v^{3}[1]b\right)$$

which is not actually a very nice expression (and doesn't factor nicely)...

```
(%i90) hopf : v*d*(v^(-12)*(w^2/v - 1/v + v/w^2) + d*(w^2/v - v + v/w^2) - v^13
+ v^6*qlam(1,0)*qlam(1,-1)*qtwo(6)/qtwo(2) + v^3*qint(1)*b);
```

(%o90)
$$\left(v\ (-v^2 - 1/(v^2))\ (w/(v^6)-(v^6)/w)\ (v^5\,w-1/(v^5\,w))\ \left(((-v^2-1/(v^2))\ (w/(v^6)-(v^6)/w)\ (v^5\,w\right.\right.$$

```
(%i92) factor(ratexpand(hopf));
```

(%o92)
$$-\left((v^4+1)\ (w-v^6)\ (w+v^6)\ (v^5\,w-1)\ (v^5\,w+1)\ (v^{22}\,w^8 - v^{12}\,w^8 - v^{10}\,w^8 + w^8 - v^{24}\,w^6 - v^{22}\,w^6 + v^{20}\,w^6 - v^{16}\,w^6 + 2\,v^{12}\,w^6 - v^8\,w^6 + v^4\,w^6 - v^2\,w^6 - w^6 + v^{26}\,w^4 + 2\,v^{24}\,w^4 - v^{20}\,w^4 + v^{18}\,w^4 + v^{16}\,w^4 - 2\,v^{14}\,w^4 - 2\,v^{12}\,w^4 + v^{10}\,w^4 + v^8\,w^4 - v^6\,w^4 + 2\,v^2\,w^4 + w^4 - v^{26}\,w^2 - v^{24}\,w^2 + v^{22}\,w^2 - v^{18}\,w^2 + 2\,v^{14}\,w^2 - v^{10}\,w^2 + v^6\,w^2 - v^4\,w^2 - v^2\,w^2 + v^{26} - v^{16} - v^{14} + v^4)\right) / \left(v^{24}\ (w-1)^2\ w^2\ (w+1)^2\ (w-v)^2\ (w+v)^2\right)$$

```
(%i93) factor(ratexpand(hopf/d));
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(%o93)
$$\left(v^{22}\,w^8 - v^{12}\,w^8 - v^{10}\,w^8 + w^8 - v^{24}\,w^6 - v^{22}\,w^6 + v^{20}\,w^6 - v^{16}\,w^6 + 2\,v^{12}\,w^6 - v^8\,w^6 + v^4\,w^6 - v^2\,w^6 - w^6 + v^{26}\,w^4 + 2\,v^{24}\,w^4 - v^{20}\,w^4 + v^{18}\,w^4 + v^{16}\,w^4 - 2\,v^{14}\,w^4 - 2\,v^{12}\,w^4 + v^{10}\,w^4 + v^8\,w^4 - v^6\,w^4 + 2\,v^2\,w^4 + w^4 - v^{26}\,w^2 - v^{24}\,w^2 + v^{22}\,w^2 - v^{18}\,w^2 + 2\,v^{14}\,w^2 - v^{10}\,w^2 + v^6\,w^2 - v^4\,w^2 - v^2\,w^2 + v^{26} - v^{16} - v^{14} + v^4\right) / \left(v^{12}\ (w-1)\ w^2\ (w+1)\ (w-v)\ (w+v)\right)$$

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(%i94)
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One more small consequence: the eigenvector with eigenvalue -1 is

$$\frac{1}{v} - \{2\lambda-1\})(+ v - \frac{[\lambda][\lambda-1]}{\{2\}} + \frac{[1]}{[3]}$$