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Chapter 6

On a Tensor Category for the Exceptional Lie Groups

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6.1 Introduction

The construction of representations of Lie groups is intertwined with combinatorics. For instance the combinatorial Littlewood-Richardson rule tells us how the tensor product of two irreducible GL_n-representations W_1 and W_2 decomposes into irreducibles, provided n is big with respect to the size of the Young diagrams corresponding to W_1 and W_2. Moreover, each irreducible GL_n-representation occurs in a tensor power $V^{\otimes d}$ of the natural representation V and can be isolated by use of combinatorics of the symmetric group S_d acting on $V^{\otimes d}$ by permuting the d factors. The intriguing part of the latter two combinatorial involvements is that they hardly depend on the series parameter n. For instance, once the primitive idempotents of the enveloping algebra of S_d inside $\mathrm{End}(V^{\otimes d})$ are figured out, the irreducible components of $V^{\otimes d}$ for GL_n follow (again, at least for n sufficiently large).

The tensor category captures the essence of the combinatorics of these decompositions, which enables a simultaneous treatment of representations for a series of Lie groups parameterized by a parameter such as n above. In Section 6.2 of this paper we introduce tensor categories in a somewhat informal way, giving the basic machinery and terminology for this paper. In Section 6.3 we treat an example based on GL_n, in which both V and its dual are taken as starting point (rather than just V), using diagrams as building blocks. We describe a diagrammatic notation which was introduced by Penrose (see [8]) as a substitute for overly indexed tensor notation. This notation is well known to physicists (cf. Cvitanović [2]). But the main goal of this paper is to discuss Deligne's conjecture ([3]) about the existence of a tensor category for the exceptional Lie groups. We describe the conjecture in Section 6.4. Prior to Deligne, work on these representation categories was conducted by El Houari ([6]) and Vogel ([9]). In Section 6.5, a generic tensor category for automorphism groups of complex finite-dimensional Lie algebras is introduced. Finally, in Section 6.6, we show some implications of Deligne's conjecture for low degree representations and some of the problems arising in an attempt to understand the structure of the conjectured tensor category.

The authors would like to thank Deligne for his valuable suggestions for improvement of this paper.

6.2 Tensor Categories

This section will give the necessary definitions and properties of tensor categories. The main purpose is to show how concepts like duality are paraphrased in categorical language. To prevent it from growing too big, we sometimes omit the details. These can be found in for example the first section of [5]. We assume the reader is familiar with the notions of category and functor, as can be found in [7].

In this paper, a *tensor category* $\mathcal{C}$ will be a category $\mathcal{C}$ together with a tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying certain associativity and commutativity conditions, and an identity object 1 satisfying $1 \otimes X = X \otimes 1 = X$ for all objects X. One important consequence of the associativity and commutativity conditions is that they permit us to define, for any finite set I, a functor

$$\bigotimes_{i \in I} : \mathcal{C}^I \to \mathcal{C}$$

in a 'consistent' (functorial) way. A *tensor functor* $F : \mathcal{C} \to \mathcal{D}$ will be a functor preserving the tensor product with its associativity and commutativity conditions. In this vein, there is the notion of *tensor equivalence* for tensor categories.

In the categories that we consider, the Hom-sets are abelian groups with morphism composition being bilinear (in other words, they are Ab-categories in the terminology of [7]); moreover all functors are supposed to respect addition. Then $\mathrm{End}_{\mathcal{C}}(X)$ is a ring with multiplication coming from morphism composition. The ring $\mathrm{End}_{\mathcal{C}}(1)$ can be seen to be commutative and its multiplication can be seen to coincide with the tensor product of morphisms. All Hom-sets are modules over this ring with the action given by $f \cdot g = f \otimes g$ for $f : 1 \to 1$, $g : X \to Y$ where we use the identifications $1 \otimes X = X$ and $1 \otimes Y = Y$. We speak of a tensor category *over the ring R* if $\mathrm{End}_{\mathcal{C}}(1) \cong R$.

The fundamental example of a tensor category is the category of finite-dimensional vector spaces over a field k equipped with the usual tensor product $\otimes$. Another example is the category having finite sets as objects, bijections as morphisms, and disjoint union as tensor product.

Let G be a group and k a field. Taking the finite-dimensional representations of G over k as objects and equivariant maps as morphisms gives what we will call the *representation category* of G (over k), denoted $\mathbf{Rep}_G$. We can make this into a tensor category by adding the usual tensor product $\otimes$ of representations. An important property of this category is that, for objects X and Y, the vector space $\mathrm{Hom}_k(X, Y)$ with the usual G-action is again an object of the category. In categorical sense $\mathrm{Hom}_k(X, Y)$ distinguishes itself as the object that gives natural isomorphisms

$$\mathrm{Hom}_G(T, \mathrm{Hom}_k(X,Y)) \to \mathrm{Hom}_G(T \otimes X, Y)$$

for all objects T (by mapping $f : T \to \mathrm{Hom}_k(X,Y)$ to the map $t \otimes x \mapsto f(t)(x)$). In other words, $\mathrm{Hom}_k(X,Y)$ represents the functor $T \mapsto \mathrm{Hom}_G(T \otimes X, Y)$.

Now we consider an arbitrary tensor category $\mathcal{C}$. Let X, Y be two objects. If the functor $T \mapsto \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y)$ is representable, we let $\underline{\mathrm{Hom}}(X,Y)$ denote the representing object. Taking $T = \underline{\mathrm{Hom}}(X,Y)$, the natural isomorphism gives

$$\mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(X,Y), \underline{\mathrm{Hom}}(X,Y)) \cong \mathrm{Hom}_{\mathcal{C}}(\underline{\mathrm{Hom}}(X,Y) \otimes X, Y).$$

We let $\mathrm{ev}_{X,Y} : \underline{\mathrm{Hom}}(X,Y) \otimes X \to Y$ denote the morphism corresponding to $\mathrm{id}_{\underline{\mathrm{Hom}}(X,Y)}$. The isomorphisms $\eta_T : \mathrm{Hom}_{\mathcal{C}}(T, \underline{\mathrm{Hom}}(X,Y)) \to \mathrm{Hom}_{\mathcal{C}}(T \otimes X, Y)$ are then given by

$$\eta_T(f) = \mathrm{ev}_{X,Y} \circ (f \otimes \mathrm{id}_X).$$

The *dual* $X^\vee$ of an object X is defined to be $\underline{\mathrm{Hom}}(X, 1)$. We let $\mathrm{ev}_X : X^\vee \otimes X \to 1$ denote the map $\mathrm{ev}_{X,1}$. Let $f : X \to Y$ be a morphism and suppose both X and Y have duals. The dual morphism ${}^{\top}\! f : Y^\vee \to X^\vee$ is defined to be $\eta_{Y^\vee}^{-1}(\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f))$: the unique morphism satisfying $\mathrm{ev}_X \circ ({}^{\top}\! f \otimes \mathrm{id}_X) = \mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f)$.

Composing $\mathrm{ev}_X : X^\vee \otimes X \to 1$ with the natural isomorphism $\psi_X : X \otimes X^\vee \to X^\vee \otimes X$ (obtained from the axioms) we find a morphism $X \otimes X^\vee \to 1$ that, using $\mathrm{Hom}_{\mathcal{C}}(T, X^{\vee\vee}) \cong \mathrm{Hom}_{\mathcal{C}}(T \otimes X^\vee, 1)$, corresponds to a morphism $X \to X^{\vee\vee}$. Objects X for which this morphism is an isomorphism are called *reflexive*.

We say that a tensor category $\mathcal{C}$ is *rigid* if $\underline{\mathrm{Hom}}(X,Y)$ exists for any two objects of $\mathcal{C}$, all objects are reflexive, and certain naturally defined morphisms are isomorphisms (among which are morphisms $Y \otimes X^\vee \to \underline{\mathrm{Hom}}(X,Y)$). For such categories we define the *trace* $\mathrm{Tr}(f)$ of a morphism $f : X \to X$ as follows. Dualizing ev_X gives a morphism $\delta_X : 1 \to X \otimes X^\vee$. We set

$$\mathrm{Tr}(f) = \mathrm{ev}_X \circ \psi_X \circ (f \otimes \mathrm{id}_{X^\vee}) \circ \delta_X.$$

Note that $\mathrm{Tr}(f)$ is an element of $\mathrm{End}_{\mathcal{C}}(1)$.

These definitions coincide with the usual ones for $\mathbf{Rep}_G$ via the isomorphism $\mathrm{End}_{\mathcal{C}}(1) \cong k$. For an object X of $\mathbf{Rep}_G$, the evaluation map ev_X assigns to $f \otimes v$ the value $f(v)$. Given a basis $\{e_i\}$ of X and dual basis $\{e^i\}$ in $X^\vee$, we have $\delta_X(1) = \sum_i e_i \otimes e^i$. The *dimension* $\dim(X)$ of an object X is defined to be $\mathrm{Tr}(\mathrm{id}_X)$. For $\mathbf{Rep}_G$ this is the usual dimension of X modulo the characteristic of k.

Finally, suppose $\mathcal{C}$ is an abelian tensor category over a field k (that is, the underlying category is abelian in the sense of [7]). Then $\mathcal{C}$ is called *semisimple* if, for each object X, the k-algebra $\mathrm{End}_{\mathcal{C}}(X)$ is semisimple.

6.3 A Representation Category for the General Linear Groups

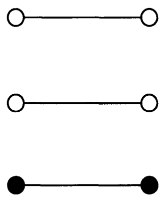
The groups under consideration in this section are the complex general linear groups $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$ $(n \in \mathbb{N})$. Let t be an indeterminate. We define a tensor category $\mathcal{C}$ over $\mathbb{Z}[t]$ that essentially describes GL_n-representations. For the objects we take finite sets A endowed with an orientation $\epsilon_A : A \to \{+,-\}$. We define an *admissible diagram* from A to B to be a set of arrows going from the positive points of A and negative points of B to the negative points of A and positive points of B, such that every point of $A \cup B$ is a begin point or end point of exactly one arrow. We also allow the diagram to have a finite number of closed (oriented) circles.

Note that we can compose such diagrams by putting them next to each other and connecting arrows. For $\mathrm{Hom}_{\mathcal{C}}(A,B)$ we take the free $\mathbb{Z}[t]$-module on the admissible diagrams, modulo the relation 'adding a circle' = 'multiplication by t'. Composition of these morphisms is induced by the composition of diagrams. For the tensor product of A and B we take the disjoint union of A and B.

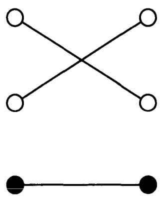
The rigidity of this category is easy to verify: the dual of an object is obtained by changing the orientation of its points.

In drawings, we denote positive points by a white dot and negative points by a black dot. Connecting line segments denote the arrows. The direction of the arrow is clear once A and B are known, and is hence omitted. With composition in mind, we will depict a diagram $A \to B$ with A on the right side and B on the left side (to make composition easier).

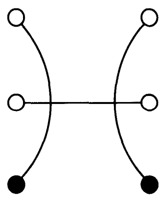
Example 6.1. The ring $\mathrm{End}_{\mathcal{C}}(\{+,+,-\})$ is a free $\mathbb{Z}[t]$-module with basis



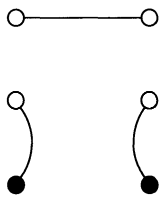
1



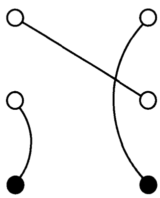
σ



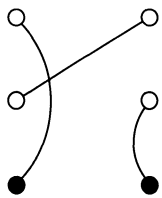
τ



$\sigma\tau\sigma$



$\sigma\tau$

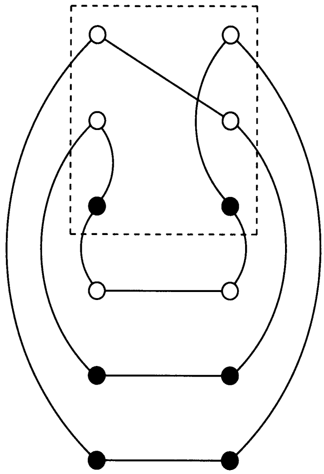


$\tau\sigma$

Multiplication is given by

	1	σ	τ	$\sigma\tau\sigma$	$\sigma\tau$	$\tau\sigma$
1	1	σ	τ	$\sigma\tau\sigma$	$\sigma\tau$	$\tau\sigma$
σ	σ	1	$\sigma\tau$	$\tau\sigma$	τ	$\sigma\tau\sigma$
τ	τ	$\tau\sigma$	$t\tau$	$\tau\sigma$	τ	$t\tau\sigma$
$\sigma\tau\sigma$	$\sigma\tau\sigma$	$\sigma\tau$	$\sigma\tau$	$t\sigma\tau\sigma$	$t\sigma\tau$	$\sigma\tau\sigma$
$\sigma\tau$	$\sigma\tau$	$\sigma\tau\sigma$	$t\sigma\tau$	$\sigma\tau\sigma$	$\sigma\tau$	$t\sigma\tau\sigma$
$\tau\sigma$	$\tau\sigma$	τ	τ	$t\tau\sigma$	$t\tau$	$\tau\sigma$

Traces are computed by the formula of Section 6.2. Graphically, this gives:

$\mathrm{Tr}(\sigma\tau) =$  $= t$

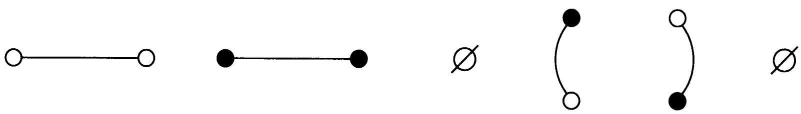
Similarly, $\mathrm{Tr}(1) = t^3$, $\mathrm{Tr}(\sigma) = \mathrm{Tr}(\tau) = \mathrm{Tr}(\sigma\tau\sigma) = t^2$ and $\mathrm{Tr}(\tau\sigma) = t$.

We now show how to obtain $\mathbf{Rep}_{\mathrm{GL}_n}$ from $\mathcal{C}$. The first step is to substitute t by the integer n. So we consider the category $\mathcal{C}_n$ having the same objects as $\mathcal{C}$ and with morphisms

$$\mathrm{Hom}_{\mathcal{C}_n}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_{\mathbb{Z}[t]} \mathbb{C},$$

where $\mathbb{Z}[t] \to \mathbb{C}$ is determined by the specialization $t \mapsto n$.

We define a functor $F : \mathcal{C}_n \to \mathbf{Rep}_{\mathrm{GL}_n}$ as follows. Let V denote the natural representation of GL_n. For S an object of $\mathcal{C}_n$, we set $F(S) = \bigotimes_{i \in S} V^{\epsilon(i)}$, where $V^+ = V$ and $V^- = V^\vee$. Note that any diagram without circles can be obtained by forming tensor products of copies of the four morphisms



id$_+$ id$_-$ ev δ

Therefore we set

$$\begin{aligned} F(\mathrm{id}_+) &= \mathrm{id}_V : V \to V, \\ F(\mathrm{id}_-) &= \mathrm{id}_{V^\vee} : V^\vee \to V^\vee, \\ F(\mathrm{ev}) &= \mathrm{ev}_V : V^\vee \otimes V \to 1, \\ F(\delta) &= \delta_V : 1 \to V \otimes V^\vee, \end{aligned}$$

and extend this first to all diagrams by forming tensor products, and then to all morphisms by linearity. This is clearly well defined, but we need to check that it commutes with composition. This follows from standard duality arguments. For example, the relation 'adding a circle' = 'multiplication by n' is respected since F maps circles to the morphism in $\mathbf{Rep}_{\mathrm{GL}_n}$ representing the dimension of V, which is n.

Now by the first fundamental theorem for GL_n (see [10]), the map from $\mathrm{Hom}_{\mathcal{C}_n}(X,Y)$ to $\mathrm{Hom}_{\mathrm{GL}_n}(F(X),F(Y))$ induced by F is surjective for all objects X,Y. To see this, consider the case where $X=\emptyset$ and $Y=\{1,2,\dots,2k\}$ with $\epsilon_Y(i)=+$ for $1\le i\le k$ and $\epsilon_Y(i)=-$ for $k+1\le i\le 2k$. Let $\{e_i\}$ be a basis of V and let $\{e^j\}$ denote its dual basis (satisfying $e^j(e_i)=\delta_{ij}$). Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}_n}(F(X),F(Y)) &= \mathrm{Hom}_{\mathrm{GL}_n}(\mathbb{C},V^{\otimes k}\otimes V^{\vee\otimes k})\\ &\cong \left(V^{\otimes k}\otimes V^{\vee\otimes k}\right)^{\mathrm{GL}_n},\end{aligned}$$

and a spanning set for this space is given by the $k!$ elements

$$\sum_{i_1,\dots,i_k} e_{i_1}\otimes\cdots\otimes e_{i_k}\otimes e^{i_{\sigma(1)}}\otimes\cdots\otimes e^{i_{\sigma(k)}},$$

with σ running over S_k. These elements clearly correspond to the $k!$ elements formed by tensoring k copies of δ_V in all possible ways.

For arbitrary X and Y we either can reduce to the case above by dualizing, or both $\mathrm{Hom}_{\mathcal{C}_n}(X,Y)$ and $\mathrm{Hom}_{\mathrm{GL}_n}(F(X),F(Y))$ are 0 (this happens when the total number of positive points in $X\otimes Y^\vee$ does not equal the number of negative points).

In general the map $\mathrm{Hom}_{\mathcal{C}_n}(X,Y)\to\mathrm{Hom}_{\mathrm{GL}_n}(F(X),F(Y))$ need not be injective. Let $I_n(X,Y)$ denote its kernel. The elements of $I_n(X,Y)$ can be identified without the help of $\mathbf{Rep}_{\mathrm{GL}_n}$:

Lemma 6.1. *The elements of $I_n(X,Y)$ are exactly the morphisms $f:X\to Y$ having $\mathrm{Tr}_{\mathcal{C}_n}(g\circ f)=0$ for all $g:Y\to X$.*

Proof. Consider morphisms $f:X\to Y$ and $g:Y\to X$ in $\mathcal{C}_n$. Then

$$F(\mathrm{Tr}_{\mathcal{C}_n}(g\circ f))=\mathrm{Tr}_{\mathrm{GL}_n}(F(g)\circ F(f)).$$

Since the map $F:\mathrm{End}_{\mathcal{C}_n}(1)\to\mathrm{End}_{\mathrm{GL}_n}(1)$ corresponds to the identity map $\mathbb{C}\to\mathbb{C}$, it follows that $\mathrm{Tr}_{\mathcal{C}_n}(g\circ f)=0$ if and only if $\mathrm{Tr}_{\mathrm{GL}_n}(F(g)\circ F(f))=0$. Therefore it suffices to see that a morphism $f:X\to Y$ in $\mathbf{Rep}_{\mathrm{GL}_n}$ is the zero map if and only if $\mathrm{Tr}_{\mathrm{GL}_n}(g\circ f)=0$ for all $g:Y\to X$. This follows easily from the reductivity of GL_n. □

The collection of $I_n(X,Y)$ can be considered as an ideal of $\mathcal{C}_n$: if $f:X\to Y$ and $g:Y\to Z$, then $g\circ f\in I_n(X,Z)$ whenever $f\in I_n(X,Y)$ or $g\in I_n(Y,Z)$, and if $f:X_1\to Y_1$ and $g:X_2\to Y_2$ then $f\otimes g\in I_n(X_1\otimes X_2,Y_1\otimes Y_2)$ whenever $f\in I_n(X_1,Y_1)$ or $g\in I_n(X_2,Y_2)$. This observation allows us to define a category $\mathcal{C}_n/I_n$ having the same objects as $\mathcal{C}_n$, with morphisms

$$\mathrm{Hom}_{\mathcal{C}_n/I_n}(X,Y)=\mathrm{Hom}_{\mathcal{C}_n}(X,Y)/I_n(X,Y),$$

and composition $\circ$ and tensor $\otimes$ induced by those of $\mathcal{C}_n$.

Now the functor F induces a functor $\overline{F} : \mathcal{C}_n/I_n \to \mathbf{Rep}_{\mathrm{GL}_n}$ defining a full embedding of $\mathcal{C}_n/I_n$ in $\mathbf{Rep}_{\mathrm{GL}_n}$. Of course, $\mathbf{Rep}_{\mathrm{GL}_n}$ has more objects. Since every GL_n-module is isomorphic to a submodule of some $V^{\otimes n} \otimes V^{\vee \otimes m}$, we can extend $\mathcal{C}_n/I_n$ to a category tensor equivalent to $\mathbf{Rep}_{\mathrm{GL}_n}$ by formally adding direct sums and images of idempotents. The result of this process is the *Karoubian envelope* $(\mathcal{C}_n/I_n)^{\mathrm{kar}}$.

Theorem 6.1. *For each $n \in \mathbb{N}$, the functor $\overline{F}$ extends to a tensor equivalence between the categories $(\mathcal{C}_n/I_n)^{\mathrm{kar}}$ and $\mathbf{Rep}_{\mathrm{GL}_n}$.*

Before we turn our attention to $\mathcal{C}$ again, we note that a consequence of the second fundamental theorem for GL_n (cf. [10]) is that for X, Y fixed, the map

$$\mathrm{Hom}_{\mathcal{C}_n}(X,Y) \to \mathrm{Hom}_{\mathrm{GL}_n}(F(X), F(Y))$$

is actually an isomorphism for n sufficiently large. So

$$\mathrm{Hom}_{\mathcal{C}}(X,X) \otimes_{\mathbb{Z}[t]} \mathbb{C} = \mathrm{End}_{\mathcal{C}}(X) \otimes_{\mathbb{Z}[t]} \mathbb{C}$$

with $t \mapsto n$ is a semisimple finite-dimensional $\mathbb{C}$-algebra for $n \gg 0$. This implies that

$$\mathrm{End}_{\mathcal{C}}(X) \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t)$$

is a semisimple finite-dimensional $\mathbb{Q}(t)$-algebra for all objects X. Therefore we define a tensor category $\mathcal{C}_t$ over $\mathbb{Q}(t)$ by taking the objects of $\mathcal{C}$, with morphisms

$$\mathrm{Hom}_{\mathcal{C}_t}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes_{\mathbb{Z}[t]} \mathbb{Q}(t).$$

Since $\mathrm{End}_{\mathcal{C}_t}(X)$ is now a semisimple algebra for all objects X, the Karoubian envelope $\mathcal{C}_t^{\mathrm{kar}}$ is a semisimple category.

Computations performed in $\mathcal{C}_t$, such as decomposition of a tensor product, will have significance in $\mathbf{Rep}_{\mathrm{GL}_n}$ as long as n is sufficiently large. So $\mathcal{C}_t$ basically describes $\mathbf{Rep}_{\mathrm{GL}_n}$ for large n.

Example 6.2. The isotypic components of $\{+,+,-\}$ in $\mathcal{C}_t^{\mathrm{kar}}$ can be found by computing the primitive central idempotents of the endomorphism algebra $A = \mathrm{End}_{\mathcal{C}_t}(\{+,+,-\})$ of which the multiplication table was given in Example 6.1. These turn out to be

$$\begin{aligned} e_1 &= \frac{1}{t^2-1}(t(\tau + \sigma\tau\sigma) - (\sigma\tau + \tau\sigma)), \\ e_2 &= \frac{1}{2}(1+\sigma) + \frac{1}{2(t+1)}(\sigma\tau + \tau\sigma + \tau + \sigma\tau\sigma), \\ e_3 &= \frac{1}{2}(1-\sigma) + \frac{1}{2(t-1)}(\sigma\tau + \tau\sigma - \tau - \sigma\tau\sigma). \end{aligned}$$

Since $e_1 A$ is a 4-dimensional subspace, the isotypic component $\mathrm{Im}(e_1)$ is the direct sum of two isomorphic irreducible representations. Likewise, we see that $\mathrm{Im}(e_2)$ and $\mathrm{Im}(e_3)$ are both irreducible. The dimensions of the components are given as traces of the idempotents, which can be computed by linearity from the traces of the basis elements in Example 6.1:

$$\begin{aligned}
\mathrm{Tr}(e_1) &= 2t,\\
\mathrm{Tr}(e_2) &= t(t+2)(t-1)/2,\\
\mathrm{Tr}(e_3) &= t(t+1)(t-2)/2.
\end{aligned}$$

For GL_n ($n \geq 3$), this reflects the fact that $V \otimes V \otimes V^\vee$ decomposes into three isotypic components of dimensions $2n$, $n(n+2)(n-1)/2$, $n(n+1)(n-2)/2$, with the first component being a sum of two irreducibles.

6.4 The Exceptional Series

Let $\mathfrak{g}$ denote one of the exceptional Lie algebras G_2, F_4, E_6, E_7, E_8, and let $G = \mathrm{Aut}(\mathfrak{g})$. Then G is an exceptional group of adjoint type, and can be written as $G \rtimes \Gamma$, the semidirect product of the group G and the group Γ of automorphisms of the corresponding Dynkin diagram.

It turns out that we can decompose $\mathfrak{g}^{\otimes 2}$ in a 'uniform' way:

$$\begin{aligned}
\mathrm{Sym}^2 \mathfrak{g} &= 1 + Y_2 + Y_2^*,\\
\bigwedge^2 \mathfrak{g} &= \mathfrak{g} + X_2,
\end{aligned}$$

with

$$\begin{aligned}
\dim \mathfrak{g} &= -2\frac{(\mu-5)(\mu+6)}{\mu(\mu+1)}\\
\dim X_2 &= 5\frac{(\mu-5)(\mu+6)(\mu-3)(\mu+4)}{\mu^2(\mu+1)^2}\\
\dim Y_2 &= -90\frac{(\mu+4)(\mu-5)}{\mu^2(\mu+1)(2\mu+1)}\\
\dim Y_2^* &= -90\frac{(\mu-3)(\mu+6)}{\mu(\mu+1)^2(2\mu+1)}
\end{aligned}$$

and $\mu = 3/2, 2/3, 1/2, 1/3, 1/5$, for $G = G_2, F_4, E_6, E_7, E_8$, respectively. Note that $\dim Y_2$ and $\dim Y_2^*$ are related by the involution $*$ sending μ to $-1-\mu$, and that both $\dim \mathfrak{g}$ and $\dim X_2$ stay invariant under $*$.

This was observed in [9] where it was deduced from the fact that $(\mathrm{Sym}^4 \mathfrak{g})^G$ is one-dimensional for these Lie algebras. We remark that similar observations can be found in [2] and [6].

It soon turned out that the trivial Lie algebra (i.e., the one-dimensional one), those of types A_1, A_2, D_4, and also the super Lie algebra $SOSp(1,2)$ can be included in the list (corresponding to $\mu = 5, 3, 2, 1, 4$, respectively). Computations [1] gave similar results for $\mathfrak{g}^{\otimes 3}$ and $\mathfrak{g}^{\otimes 4}$. This led Deligne [3] to conjecture the existence of a semisimple tensor category $\mathcal{E}_t$ having specializations to the various $\mathbf{Rep}_G$ by $t \mapsto \mu$.

In the next sections we will investigate a possible combinatorial structure behind $\mathcal{E}_t$. First, we consider an arbitrary simple Lie algebra $\mathfrak{g}$ and construct a kind of universal category $\mathcal{L}$. Then we specialize to the exceptional algebras and try to find relations describing the exceptional series.

6.5 The Category $\mathcal{L}$

Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and set $G = \mathrm{Aut}(\mathfrak{g})$. We denote the Killing form of $\mathfrak{g}$ by $(\cdot,\cdot) : \mathfrak{g}\otimes\mathfrak{g} \to \mathbb{C}$. Since this form is non-degenerate, we can use it to identify $\mathfrak{g}$ with its dual. Hence

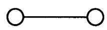
$$\mathrm{Hom}_G(\mathfrak{g}^{\otimes n}, \mathfrak{g}^{\otimes m}) \;\cong\; \left(\mathfrak{g}^{\otimes(n+m)}\right)^G .$$

Therefore we know all morphisms once we know the invariants of the spaces $\mathfrak{g}^{\otimes n}$. Independently of the type of $\mathfrak{g}$, we can already give two of these invariants: those induced by $\mathrm{id}_\mathfrak{g} : \mathfrak{g} \to \mathfrak{g}$ and $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$. To be more precise, $\mathrm{id}_\mathfrak{g}$ corresponds to the element $\sum_i e_i \otimes e^i \in \mathfrak{g}^{\otimes 2}$, with $\{e_i\}$ a basis of $\mathfrak{g}$ and $\{e^j\}$ its dual basis. Likewise, $[\cdot,\cdot]$ gives the element

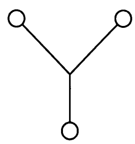
$$\sum_{i,j,k} c^{ijk} e_i \otimes e_j \otimes e_k$$

of $\mathfrak{g}^{\otimes 3}$, where $c^{ijk} = (e^k, [e^i, e^j])$, so that they are the coefficients defined by $[e^i, e^j] = \sum_k c^{ijk} e_k$.

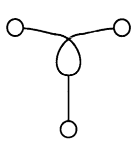
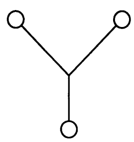
As in the previous sections, we would like a graphical representation of these elements. Letting dots correspond to $\mathfrak{g}$, we can denote the tensor element $\sum_i e_i \otimes e^i$ by the diagram



and the element $\sum c^{ijk} e_i \otimes e_j \otimes e_k$ by



Note that, since this element is alternating, we have

 $= -$ 

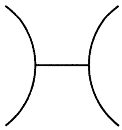
so points of valency 3 should have a fixed cyclic orientation (cyclic order of the three edges) and changing orientation corresponds to multiplication by -1.

With these, we build more invariants and diagrams. First, an invariant formed by tensoring copies of $\sum_i e_i \otimes e^i$ and $\sum c^{ijk} e_i \otimes e_j \otimes e_k$ is denoted by a disjoint union of copies of the two diagrams above. Applying contraction maps $\mathfrak{g}^{\otimes n} \to \mathfrak{g}^{\otimes(n-2)}$, that is, replacing two tensor factors by their value under the Killing form, we obtain more invariants. Appropriate diagrams are formed by connecting the points corresponding to the contracted components.

The diagrams obtained this way are finite graphs for which each vertex has valency 1, 2, or 3, and where any trivalent vertex has a fixed cyclic orientation assigned to it. Any such diagram can be dissected into copies of the two basic diagrams in an essentially unique way (up to order of contraction).

Note that contracting a tensor element with $\sum_i e_i \otimes e^i$ gives the same element again. For diagrams, this has as consequence that we can divide any edge into two edges connected by a point of valency 2 without changing the corresponding invariant. Therefore we can neglect points of valency 2 and consider graphs with vertices of valency 1 and cyclically oriented vertices of valency 3 only. We call these *trivalent diagrams* and usually omit the dots at the points of valency 1.

By way of example, we indicate how to write down the tensor element corresponding to the diagram



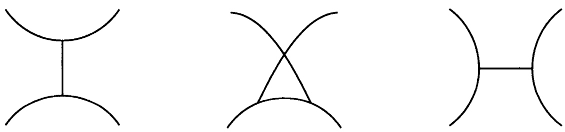
We interpret the cyclic orientation as being given by 'clockwise is positive'. It is formed by contracting two oriented trivalent nodes, say

$$\sum_{i,j,k} c^{ijk} e_i \otimes e_j \otimes e_k \quad \text{and} \quad \sum_{p,q,r} c^{pqr} e_p \otimes e_q \otimes e_r,$$

along k and p, giving

$$\sum_{i,j,k,p,q,r} c^{ijk} c^{pqr} (e_k, e_p) e_i \otimes e_j \otimes e_q \otimes e_r.$$

The Jacobi identity can easily be translated into a relation between diagrams. We consider the three elements of $\mathfrak{g}^{\otimes 4}$



α β γ

As before the orientation of the trivalent nodes is assumed to be clockwise. To compare these elements, it is easiest to look at the induced functionals $\mathfrak{g}^{\otimes 4} \to 1$.

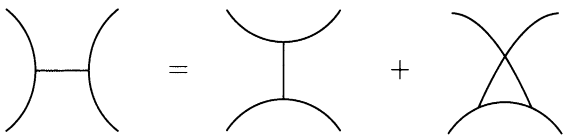
Labeling the endpoints anti-clockwise by x, y, z, w, with x denoting the lower-left endpoint, we have

$$(\alpha, x \otimes y \otimes z \otimes w) = ([x, y], [z, w]),$$
$$(\beta, x \otimes y \otimes z \otimes w) = ([z, x], [y, w]),$$

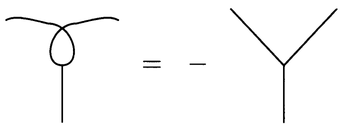
and

$$\begin{aligned}(\gamma, x \otimes y \otimes z \otimes w) &= ([w, x], [y, z]) = (w, [x, [y, z]]) \\ &= (w, -[y, [z, x]] - [z, [x, y]]) \\ &= (-([w, y], [z, x]) - ([w, z], [x, y]) \\ &= ([z, x], [y, w]) + ([x, y], [z, w]) \\ &= (\alpha + \beta, x \otimes y \otimes z \otimes w).\end{aligned}$$

Hence $\gamma = \alpha + \beta$:



This diagrammatic formula is known as the IHX relation (cf. [6, 9]). As we saw before, anti-commutativity of $[\cdot, \cdot]$ is expressed by

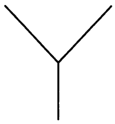


We use these diagrams to construct a category $\mathcal{L}$. In this construction, the diagrams are viewed as combinatorial objects, without reference to the Lie algebra $\mathfrak{g}$. As objects we take finite sets (without orientation). For finite sets X, Y we define $\mathrm{Hom}_{\mathcal{L}}(X, Y)$ to be the free $\mathbb{Z}$-module with basis the trivalent diagrams that have as set of monovalent points the disjoint union of X and Y, modulo the two relations above. It is clear how to define composition on diagrams. We extend this by linearity to obtain composition in $\mathcal{L}$. As usual, we take disjoint unions for tensor products.

It can easily be seen that any object in $\mathcal{L}$ is self-dual and that $\mathcal{L}$ is rigid. We summarize this section in the following theorem.

Theorem 6.2. *Let $\mathfrak{g}$ denote a finite-dimensional complex Lie algebra having a non-degenerate Killing form $(\cdot, \cdot) : \mathfrak{g}^{\otimes 2} \to \mathbb{C}$ and let $G = \mathrm{Aut}(\mathfrak{g})$. There exists a tensor functor $F : \mathcal{L} \to \mathbf{Rep}_G$, unique up to unique isomorphism, mapping singletons to $\mathfrak{g}$ and satisfying*

$$F(\text{———}) = (1 \mapsto \textstyle\sum e_i \otimes e^i),$$

$F($  $) = (1 \mapsto \sum c^{ijk} e_i \otimes e_j \otimes e_k).$

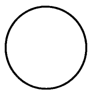
The diagrams are interpreted here as morphisms $1 \to \mathfrak{g}^{\otimes n}$; all other interpretations follow by dualizations of tensor factors.

6.6 The Exceptional Categories

Let $\mathfrak{g}$ be one of the exceptional simple Lie algebras G_2, F_4, E_6, E_7, E_8, and let d denote its dimension. Set $G = \mathrm{Aut}(\mathfrak{g})$, and let $F : \mathcal{L} \to \mathbf{Rep}_G$ be as in Theorem 6.2. In Section 6.6.1, we look for linear combinations of morphisms in the kernel of $F : \mathrm{Hom}_{\mathcal{L}}(X, Y) \to \mathrm{Hom}_G(F(X), F(Y))$, where X, Y are small tensor powers of $\mathfrak{g}$. In Section 6.6.2 this is used to compute $\mathrm{End}(\mathfrak{g}^{\otimes 2})$ and the corresponding decomposition of $\mathfrak{g}^{\otimes 2}$. Based on the relations found so far, a tensor category quotient of $\mathcal{L}$ can be found through which F factors. This can be seen as a first step towards the conjectured category $\mathcal{E}_t$. In Section 6.6.3, we observe that more relations are needed to describe the conjectured category $\mathcal{E}_t$ fully.

6.6.1 Finding Relations All of the relations we will obtain involve diagrams consisting of a circle and a number of legs. These diagrams have an easy interpretation as elements of $(\mathfrak{g}^{\otimes n})^\vee$: the circle with n legs corresponds to the linear map sending $x_1 \otimes \cdots \otimes x_n$ to $\mathrm{Tr}(\mathrm{ad}\ x_1 \cdots \mathrm{ad}\ x_n)$.

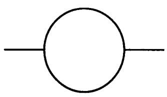
One obvious relation is given by the fact that the dimension d of $\mathfrak{g}$ is equal to the trace of $\mathrm{id}_{\mathfrak{g}}$:

 $= \quad d$

To obtain further relations, one way to proceed is to count the number of invariants in certain degrees.

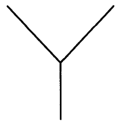
First, since $\mathfrak{g}$ is simple, there is no non-zero morphism $\mathfrak{g} \to 1$. This translates to the fact that any diagram with exactly one monovalent vertex vanishes.

Next, by the decomposition in Section 6.4 of $\mathfrak{g}^{\otimes 2}$, there is only one independent morphism $\mathfrak{g}^{\otimes 2} \to 1$. Of course, by Schur's Lemma this also follows from the simplicity of $\mathfrak{g}$. So any diagram with exactly two monovalent vertices maps to a scalar multiple of the diagram having two vertices and one connecting edge. One important example is the Casimir endomorphism of $\mathfrak{g}$:

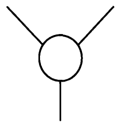
 $= \quad C$

with C a scalar. Since we can interpret the lefthand side as the morphism sending $x \otimes y$ to $\mathrm{Tr}(\mathrm{ad}\ x\ \mathrm{ad}\ y)$, we find that both sides represent the Killing form. Hence $C = 1$.

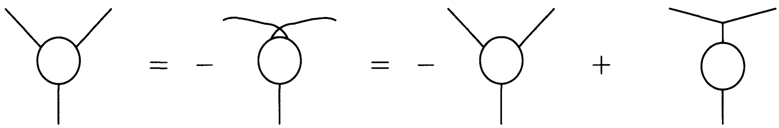
From the same decomposition of $\mathfrak{g}^{\otimes 2}$, we find that there is only one independent morphism $\mathfrak{g}^{\otimes 2} \to \mathfrak{g}$. Of course, this is the Lie bracket



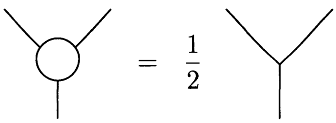
Hence any diagram with three monovalent vertices is a scalar multiple of the Lie bracket. In particular, the diagram



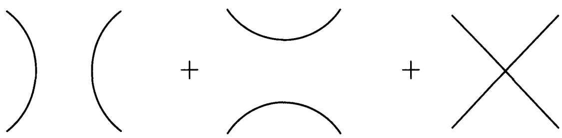
is anti-commutative:



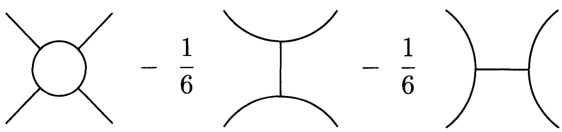
Therefore,



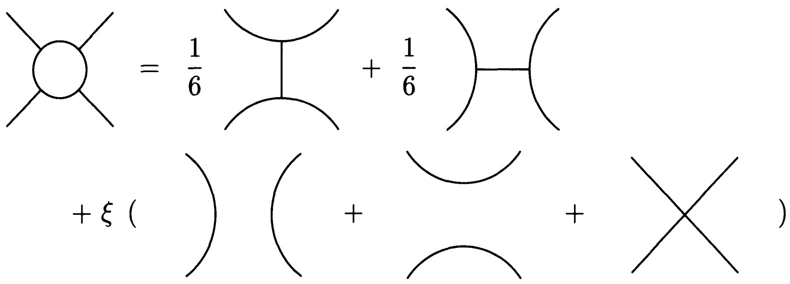
The next relation depends on the fact that $\left(\mathrm{Sym}^4\,\mathfrak{g}\right)^G$ is one-dimensional for exceptional Lie algebras. To make use of this fact, note that



is invariant under the action of S_4 (permuting vertices), and that the same holds for

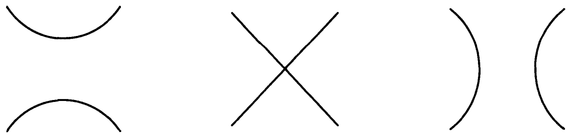


So we know that for some scalar ξ we have

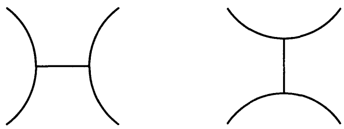


To find ξ, we simply contract two pairs of vertices to get $d = d/6 + \xi(d^2 + 2d)$, and hence $\xi = 5/(6(d+2))$.

6.6.2 Endomorphisms of the Second Tensor Power The relations found in the preceding section already are sufficient to describe the endomorphism algebra of $\mathfrak{g}^{\otimes 2}$. From the decomposition in Section 6.4, we know this will be a five-dimensional commutative algebra. As a basis we take the morphisms



$1 \quad \sigma \quad \tau$



$\alpha \quad \beta$

Using the relations it is not difficult to find the multiplication table:

	1	σ	τ	α	β
1	1	σ	τ	α	β
σ	σ	1	τ	$-\alpha$	$\beta - \alpha$
τ	τ	τ	$d\tau$	0	τ
α	α	$-\alpha$	0	α	$\alpha/2$
β	β	$\beta - \alpha$	τ	$\alpha/2$	$\frac{1}{6}(\alpha+\beta) + \frac{5}{6(d+2)}(1+\sigma+\tau)$

We can use the endomorphism algebra to verify the dimension formulae in Section 6.4. The representations occurring in the decomposition of $\mathfrak{g}^{\otimes 2}$ correspond to the primitive idempotents in the commutative algebra $\mathrm{End}(\mathfrak{g}^{\otimes 2})$. These idempotents turn out to be

$$\alpha, \frac{1}{2}(1-\sigma) - \alpha, \frac{1}{d}\tau,$$

$$B = \frac{1}{1+2\mu}\left(6\beta - 3\alpha + \frac{\mu}{2}(1+\sigma) - \frac{\mu+6}{d}\tau\right), \frac{1}{2}(1+\sigma) - \frac{1}{d}\tau - B,$$

where $\mu = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{240}{d+2}}$, so that $d = -2\frac{(\mu-5)(\mu+6)}{\mu(\mu+1)}$. The dimension of a representation is equal to the trace of the corresponding idempotent and an easy calculation gives the formulae of Section 6.4.

6.6.3 The Category $\mathcal{E}_t$ We now consider the conjectural tensor category $\mathcal{E}_t$, whose 'specialization' $t \mapsto \mu$ should give the relevant data for the corresponding exceptional Lie algebra $\mathfrak{g}$. It will be a tensor category over $\mathbb{Q}(t)$, whose objects are those of $\mathcal{L}$, and whose morphisms are of the form

$$\mathrm{Hom}_{\mathcal{E}_t}(X,Y) = (\mathrm{Hom}_{\mathcal{L}}(X,Y) \otimes_{\mathbb{Z}} \mathbb{Q}(t))/I(X,Y),$$

for an ideal I containing the relations found in section 6.6.1, where we replace d by $-2\frac{(t-5)(t+6)}{t(t+1)}$. This is justified by the fact that the linear combinations found in Section 6.6.1 have coefficients depending rationally on d. Note also that the idempotents of Section 6.6.2 have meaning over $\mathbb{Q}(t)$.

Although we do not know whether it exists, we can already describe $\mathcal{E}_t$ in 'low degrees': the decomposition formulae tell us what the irreducible objects are occurring in $\mathfrak{g}^{\otimes n}$ for n small, and using diagrams we can describe the morphisms between them. So $\mathrm{End}_{\mathcal{E}_t}(\mathfrak{g}^{\otimes 2})$ is the five-dimensional algebra with multiplication table as given in Section 6.6.2.

The ad-hoc arguments of the preceding sections have been used to describe $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes n})$ for $n \leq 4$. To proceed further, we will use a more systematic approach. It depends on 'guessing' a basis for the space of morphisms $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes n})$, and then using a naturally defined bilinear form to express other elements in this basis.

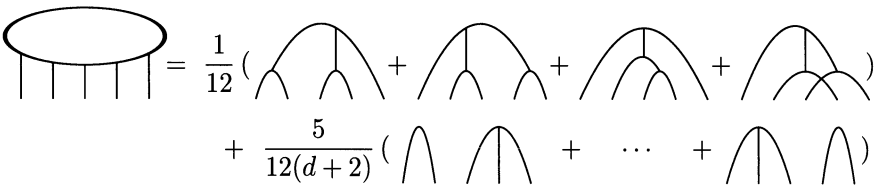
The bilinear form $\langle \cdot, \cdot \rangle : \mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes n}) \times \mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes n}) \to \mathbb{Q}(t)$ is defined as follows. Let $f, g : 1 \to \mathfrak{g}^{\otimes n}$ be morphisms. Then ${}^{\top}\!f \circ g : 1 \to 1$ corresponds to multiplication by a scalar, and we let $\langle f, g \rangle$ denote this scalar. It is easily verified that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form. If we have f and g in the form of diagrams, we can try and evaluate this diagram using the relations already found.

The Gram matrix of the $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes 4})$-basis of section 6.6.2 with respect to $\langle \cdot, \cdot \rangle$ is

$$\begin{pmatrix} d^2 & d & d & 0 & d \\ d & d^2 & d & -d & -d \\ d & d & d^2 & d & 0 \\ 0 & -d & d & d & d/2 \\ d & -d & 0 & d/2 & d \end{pmatrix}$$

The determinant is $3d^5(d+2)(d-3)^2/4$, confirming that the chosen basis elements are linearly independent.

To attack $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes 5}) = \mathrm{Hom}_{\mathcal{E}_t}(\mathfrak{g}^{\otimes 2}, \mathfrak{g}^{\otimes 3})$, we first use the decompositions of $\mathfrak{g}^{\otimes 2}$ and $\mathfrak{g}^{\otimes 3}$ to compute its dimension. This gives a number of 16 expected basis elements. As candidate basis elements we take the diagrams on 5 points having no cycles (since many diagrams having cycles can be reduced by use of the relations we have). Since this gives more than 16 elements, there will be linear dependencies. Assuming that this set of diagrams contains a basis and that $\langle \cdot, \cdot \rangle$ is nondegenerate, we can use this bilinear form to remove redundant elements. Doing this, we end up with 16 linearly independent diagrams, hence a basis. Any diagram can be expressed in this basis, and we can effectively compute the coefficients provided we know how to evaluate the relevant closed diagrams (i.e., without monovalent vertices) occurring as values of the bilinear form. For example,



In a similar fashion, a basis of $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes 6})$ can be found. The decomposition of $\mathfrak{g}^{\otimes 3}$ leads us to expect 80 elements. However, in this case diagrams having no cycles do not suffice. We explain why not. The subspace of $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes 6})$ invariant under the action of S_6 is of dimension two (which we derive from the decomposition tables, suggesting that 1 occurs with multiplicity two in $\mathrm{Sym}^6(\mathfrak{g})$). But symmetrizing (taking the sum over all permutations) any diagram with no cycles and at least one trivalent vertex, we get zero since a transposition of two endpoints meeting in the same trivalent vertex changes the sign of the diagram. Therefore the only symmetric morphism obtained from diagrams without cycles is the symmetrization of the morphism representing three tensor copies of the evaluation morphism. As the second symmetric morphism, it turns out we can take the symmetrization of the circle with six legs. This shows that the circle with six legs is linearly independent of the diagrams with no cycles. Here we have a potential obstruction for our method when applied to $\mathrm{Hom}_{\mathcal{E}_t}(1, \mathfrak{g}^{\otimes n})$ for higher n: we do not know how to rewrite closed diagrams having no cycles of length at most 5.

We conclude by making some remarks about the present state of knowledge regarding $\mathcal{E}_t$. Decomposition tables have been found for $\mathfrak{g}^{\otimes n}$, $n \leq 5$. All objects that occur have a dimension formula factoring completely in linear forms, with one exception. It is expected (or at least hoped) that this exception is in fact the sum of two or more irreducible objects of $\mathcal{E}_t$. There turns out to be an additive structure on the set of irreducible objects, much like the addition of heighest weights in the representation theory of semisimple algebraic groups. In [4], a 7-parameter family of irreducible objects is identified, including for each element a set of affine linear forms in t (each form having a multiplicity in $\mathbb{Z}$). These forms play the role of the factors occurring in the Weyl dimension formula, explaining the factorization property of the dimensions of these irreducible objects. The family should extend to infinitely many parameters. Currently work is being done to find the linear forms for larger families.

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