Algebraic structures on modules of diagrams

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Abstract.

There exists a graded $\mathbf{Z}$-algebra Λ acting in a natural way on many modules of 3-valent diagrams. Every simple Lie superalgebra with a nontrivial invariant bilinear form induces a character on Λ. Classical and exceptional Lie algebras and the Lie superalgebra $\mathrm{D}(2,1,\alpha)$ produce eight distinct characters on Λ and eight distinct families of weight functions on chord diagrams. As a consequence we prove that weight functions coming from semisimple Lie superalgebras do not detect every element in the module $\mathcal{A}$ of chord diagrams.

Introduction.

V. Vassiliev [Va] has recently defined a new family of knot invariants. Actually every knot invariant with values in an abelian group may be seen as a linear map from the free $\mathbf{Z}$-module $\mathbf{Z}[\mathcal{K}]$ generated by isomorphism classes of knots. This module is a Hopf algebra and has a natural filtration $\mathbf{Z}[\mathcal{K}] = I_0 \supset I_1 \supset \dots$ defined in terms of singular knots, and a Vassiliev invariant of order n is an invariant which is trivial on I_{n+1}. Coefficients of Jones [J], HOMFLY [H], Kauffman [Ka] polynomials are Vassiliev invariants.

The associated graded Hopf algebra $\mathrm{Gr}\mathbf{Z}[\mathcal{K}] = \bigoplus_n I_n/I_{n+1}$ is finitely generated over $\mathbf{Z}$ in each degree but its rank is completely unknown. Actually $\mathrm{Gr}\mathbf{Z}[\mathcal{K}]$ is a certain quotient of the graded Hopf algebra $\mathcal{A}$ of chord diagrams [BN]. Every Vassiliev invariants of order n induces a weight function of degree n, (i.e. a linear form of degree n on $\mathcal{A}$). Conversely, every weight function can be integrated (via the Kontsevich integral) to a knot invariant. Very few things are known about the algebra $\mathcal{A}$. Rationally, $\mathcal{A}$ is the symmetric algebra on a graded module $\mathcal{P}$, and so-called Adams operations split $\mathcal{A}$ and $\mathcal{P}$ in a direct sum of modules defined in terms of 3-valent diagrams called chinese characters in [BN]. The rank of $\mathcal{P}$ is known in degree < 10.

Every Lie algebra equipped with a nonsingular invariant bilinear form and a finite

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dimensional representation induces a weight function on $\mathcal{A}$. It was conjectured in [BN] that weight functions corresponding to classical simple Lie algebras detect every nontrivial element in $\mathcal{A}$.

In this paper, we define a graded algebra Λ acting on many modules of diagrams like $\mathcal{P}$ and every Lie algebra equipped with a nonsingular invariant bilinear form induces a character on Λ. With this procedure, we construct eight characters from Λ to a polynomial algebra of two variables for three of them, and to $\mathbf{Q}[t]$ for the others. These eight characters are algebraically independent. As a consequence, we construct a primitive element in $\mathcal{A}$ which is rationally nontrivial and killed by every semisimple Lie algebra and Lie superalgebra equipped with a nonsingular invariant bilinear form and a finite-dimensional representation.

1. Modules of diagrams

Let Γ be a curve, i.e. a compact 1-dimensional manifold. Let X be a finite set. A (Γ, X)-*diagram* is a finite 3-valent graph K (i.e. a graph where every vertex is univalent or 3-valent) equipped with the following data:

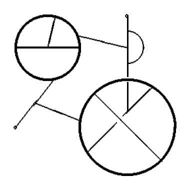
— an isomorphism from the disjoint union of Γ and X to a subgraph of K sending $\partial\Gamma \cup X$ to the set of univalent vertices of K

— for every 3-valent vertex x of K, a cyclic order of the set of oriented edges arriving at x.

The class of (Γ, X)-diagrams will be denoted by $\mathcal{D}(\Gamma, X)$.

Usually, a (Γ, X)-diagram will be represented by a 3-valent graph immersed in the plane in such a way that, at every 3-valent vertex, the cyclic order is given by the orientation of the plane.

Example of a (Γ, X)-diagram where Γ has two closed components and X has two elements:

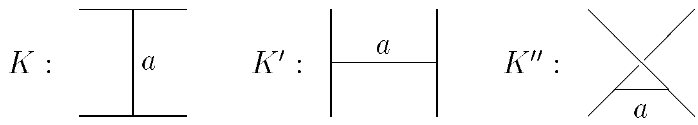


Let $\mathcal{C}$ be a sub-class of $\mathcal{D}(\Gamma, X)$ which is closed under changing cyclic order. Denote by $\mathcal{A}(\mathcal{C})$ the quotient of the free $\mathbf{Z}$-module generated by isomorphism classes of (Γ, X)-diagrams in $\mathcal{C}$ by the following relations:

— if K is a (Γ, X)-diagram in $\mathcal{C}$, and K' is obtained from K by changing the cyclic order at one vertex, we have:

$$\text{(AS)} \qquad K' \equiv -K$$

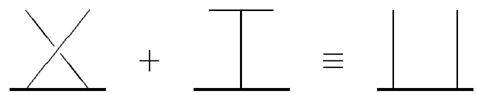
— if K, K', K'' are three (Γ, X)-diagrams in $\mathcal{C}$ which differ only near an edge a in the following way:



we have:

(IHX) $$K \equiv K' - K''.$$

Remark: If the edge meets the curve Γ the relation (IHX) is called (STU) in [BN]:



The module $\mathcal{A}(\mathcal{C})$ is a graded $\mathbf{Z}$-module. The degree $\partial^\circ K$ of a (Γ, X)-diagram K is the opposite of the Euler characteristic of the pair (K, Γ).

If X is empty and K and Γ are connected, the degree of K is half the number of vertices of K.

By considering different classes of diagrams, we get the following examples of graded modules:

— the module $\mathcal{A}(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}'(\Gamma, X)$ of (Γ, X)-diagrams K such that every connected component of K meets Γ or X

— the module $\mathcal{A}_c(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}_c(\Gamma, X)$ of (Γ, X)-diagrams K such that $K - \Gamma$ is connected and non empty(*connected case*)

— the module $\mathcal{A}_s(\Gamma, X)$, if $\mathcal{C}$ is the class $\mathcal{D}_s(\Gamma, X)$ of (Γ, X)-diagrams K such that $K - \Gamma$ is connected and has at least one 3-valent vertex (*special case*)

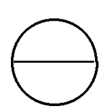
— the module $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma, \emptyset)$

— the module $\mathcal{A}_c(\Gamma) = \mathcal{A}_c(\Gamma, \emptyset)$

— the module $F(X) = \mathcal{A}_c(\emptyset, X)$. If X is the set $[n] = \{1, \dots, n\}$, the module $F(X)$ will be denoted by $F(n)$

— the module ${}_X\Delta_Y$, where X and Y are finite sets and $\mathcal{C}$ is the class of all $(\emptyset, X \coprod Y)$-diagrams.

The module $\mathcal{A}(\Gamma)$ is strongly related to the theory of links. In the case of knots, the Kontsevich integral provides a universal Vassiliev invariant with values in the module $\mathcal{A} = \mathcal{A}(S^1)$ quotiented by a certain sub-module I and completed [BN]. The module $\mathcal{A}$ is actually a commutative and cocommutative Hopf algebra (the product corresponds to the connected sum of knots) and I is the ideal generated by the following diagram of degree 1:



Remark: The definition of the module $\mathcal{A}(\Gamma)$ is slightly different as the classical one. The classical definition needs an orientation of Γ, but cyclic orders near vertices in Γ are not part of the data. The relationship between this two definitions come from the fact that, if Γ is oriented, there is a canonical choice for the cyclic order of edges arriving at each vertex in Γ.

Let $\mathcal{P} = \mathcal{A}_c(S^1)$. The inclusion $\mathcal{D}_c(S^1, \emptyset) \subset \mathcal{D}(S^1, \emptyset)$ induces a linear map from $\mathcal{P}$ to $\mathcal{A}$ and a morphism of Hopf algebras from $S(\mathcal{P})$ to $\mathcal{A}$.

1.1 Proposition: *The morphism $S(\mathcal{P}) \to \mathcal{A}$ is surjective with finite kernel in each degree.*

Proof: For $n > 0$, denote by $\mathcal{E}_n$ the submodule of $\mathcal{A}$ generated by diagrams K such that $K - \Gamma$ has at most n components. Because of relation STU, it is easy to see that, mod $\mathcal{E}_n$, $\mathcal{E}_{n+1}$ is generated by connected sums $K_1 \sharp K_2 \ldots \sharp K_{n+1}$ where $K_i - \Gamma$ are connected. That proves, by induction, that the canonical map from $S(\mathcal{P})$ to $\mathcal{A}$ is surjective. Because $S(\mathcal{P})$ and $\mathcal{A}$ are finitely generated over $\mathbf{Z}$ in each degree, it's enough to prove that the map from $S(\mathcal{P})$ to $\mathcal{A}$ is a rational isomorphism, and because $S(\mathcal{P})$ and $\mathcal{A}$ are commutative and cocommutative Hopf algebras, it is enough to prove that the map from $\mathcal{P}$ to $\mathcal{A}$ is rationally an isomorphism from $\mathcal{P}$ to the module of primitives of $\mathcal{A}$.

Consider the module $\mathcal{CD}_p$ of Chinese diagrams with p free vertices and the module $\mathcal{CD}_p^c$ of connected Chinese diagrams with p free vertices. In [BN] Bar Natan constructs a rational isomorphism from $\mathcal{A}$ to the direct sum $\bigoplus_{p>0} \mathcal{CD}_p$ which respects the comultiplication. In the same way we have a rational isomorphism from $\mathcal{P}$ to $\bigoplus_{p>0} \mathcal{CD}_p^c$.

Therefore $\mathcal{P}$ is rationally isomorphic to the module of primitives of $\mathcal{A}$. $\square$

Very few things are known about $\mathcal{A}$ and $\mathcal{P}$. They are finitely generated modules in each degree. The rank is known in degree ≤ 9. For $\mathcal{P}$, this rank is: 1, 1, 1, 2, 3, 5, 8, 12, 18 [BN]. Certain linear forms on $\mathcal{A}$ (coming from Lie algebras) are known. Rationally the module $\mathcal{P}$ splits in a direct sum of modules of connected Chinese diagrams $\mathcal{CD}_n^c$ [BN]. Actually the module $\mathcal{CD}_n^c$ is defined in the same way that $F(n)$ except that the bijection from $[n]$ to the set of monovalent vertices is forgotten. Hence this splitting may be written in the following manner:

1.2 Proposition: *There is a rational isomorphism:*

$$\bigoplus_{n>0} \mathrm{H}_0(\mathcal{S}_n, F(n) \otimes \mathbf{Q}) \longrightarrow \mathcal{P} \otimes \mathbf{Q}.$$

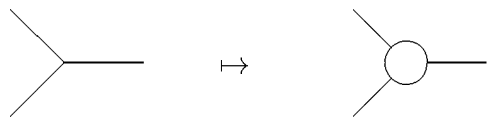
The last module ${}_X\Delta_Y$ defined above will be used later. Actually these modules define a monoidal category Δ. Objects of Δ are finite sets, and the set of morphisms $\mathrm{Hom}_\Delta(X, Y)$ is the module ${}_Y\Delta_X$. The composition of morphisms is obtained by gluing. It is a morphism from ${}_X\Delta_Y \otimes {}_Y\Delta_Z$ to ${}_X\Delta_Z$. In particular, for every finite set X, ${}_X\Delta_X$ is a $\mathbf{Z}$-algebra.

The monoidal structure is the disjoint union of finite sets or diagrams.

2. The transformation t

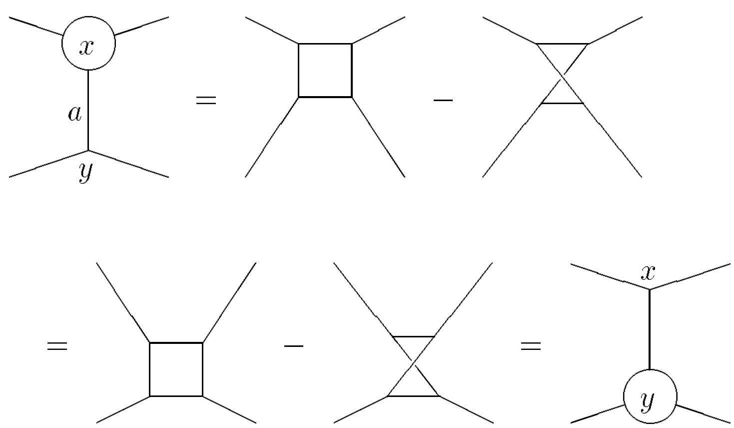
Let Γ be a curve and X be a finite set. We have three graded modules $\mathcal{A}(\Gamma, X)$, $\mathcal{A}_c(\Gamma, X)$ and $\mathcal{A}_s(\Gamma, X)$, and $\mathcal{A}_c(\Gamma, X)$ is isomorphic to $\mathcal{A}_s(\Gamma, X)$ except maybe in small degrees.

Let K be a (Γ, X)-diagram in the class $\mathcal{D}_s(\Gamma, X)$. Take a 3-valent vertex outside of Γ. Then it is possible to modify K near this vertex in the following way:



2.1 Theorem: *This transformation induces a well defined endomorphism t of the module $\mathcal{A}_s(\Gamma, X)$.*

Proof: Let K be a diagram in the class $\mathcal{D}_s(\Gamma, X)$. Let a be an edge of K disjoint from the curve Γ. Denote vertices of a by x and y. Relations IHX imply the following:



Then transformations of K at x and y produce the same element in the module $\mathcal{A}_s(\Gamma, X)$. Since the complement of Γ in a diagram in $\mathcal{D}_s(\Gamma, X)$ is connected, the transformation t is well defined from the class $\mathcal{D}_s(\Gamma, X)$ to $\mathcal{A}_s(\Gamma, X)$.

It is easy to see that t is compatible with the relation AS. Consider a relation IHX:

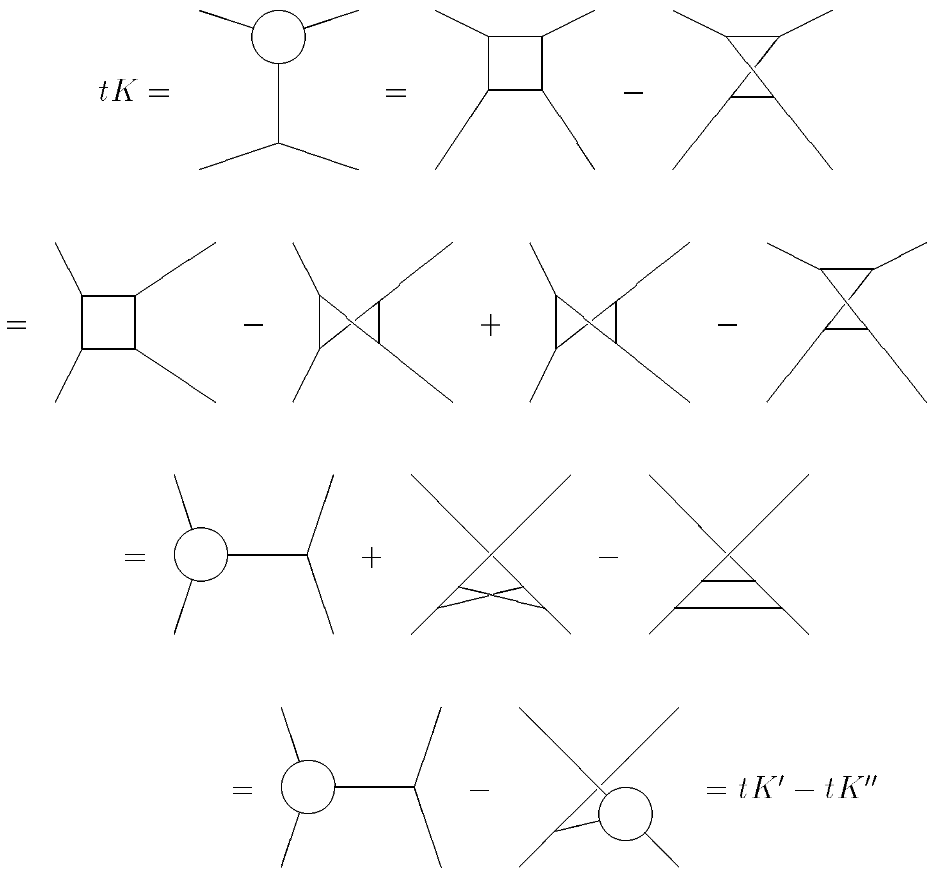
$$K \equiv K' - K''$$

where K, K' and K'' differ only near an edge a. If there is a vertex in K which is not in a and not in the curve Γ, it is possible to define tK, tK' and tK'' by using this vertex, and the relation

$$tK \equiv tK' - tK''$$

becomes obvious.

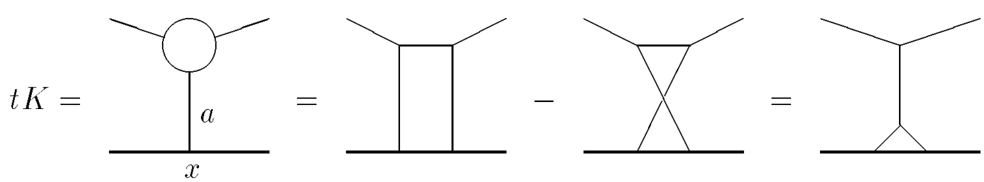
Suppose now that $\Gamma \cup a$ contains every vertex in K. Then the edge a is not contained in Γ, and that is true also for K' and K''. Therefore a doesn't meet Γ, and we have:



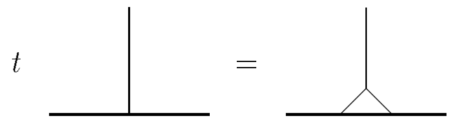
$\square$

2.2 Proposition: *The transformation t extends in a natural way to the module $\mathcal{A}_c(\Gamma, X)$.*

Proof: Let K be a diagram in the class $\mathcal{D}_c(\Gamma, X)$. Let x be a vertex of K contained in Γ. This vertex in contained in an edge a in $K - \Gamma$. If the diagram K lies in the class $\mathcal{D}_s(\Gamma, X)$, K has a vertex which is not in Γ. Therefore a has a vertex outside of Γ and we have:

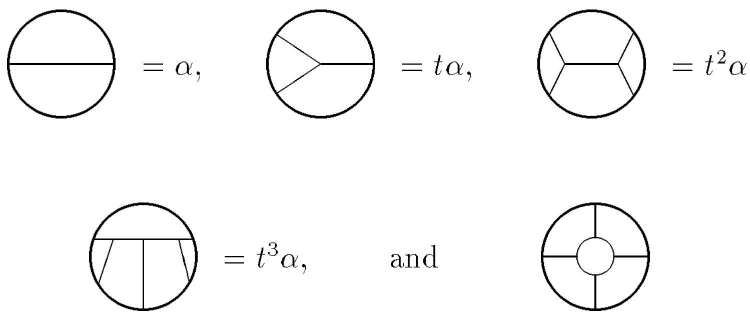


Hence t extends to the module $\mathcal{A}_c(\Gamma, X)$ by setting:



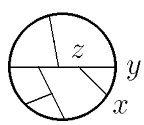
□

Example: The module $\mathcal{P} = \mathcal{A}_c(S^1) = \mathcal{A}_c(S^1, \emptyset)$ which is the module of primitives of the algebra of diagrams $\mathcal{A}$ has, in degree ≤ 4 the following basis:



Application: *Let K be a planar $(S^1, \emptyset)$-diagram of degree n such that the complement of S^1 in K is a tree. Then the class of K in the module $\mathcal{A}_c(S^1)$ is exactly $t^{n-1}\alpha$.*

Proof: Conditions satisfied by K imply that K contains a triangle xyz with an edge xy in the circle. By taking off the edge xz we get a new diagram K' such that the complement of the circle in K' is still a planar tree. By induction, the class of K' in $\mathcal{A}_c(S^1)$ is $t^{n-2}\alpha$ and the result follows.



□

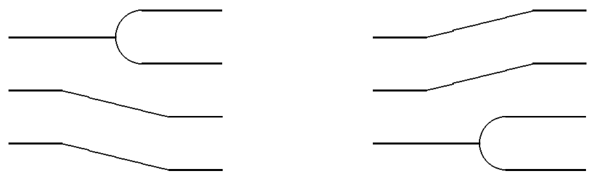
3. The algebra Λ

In this section we construct an algebra of diagrams acting on many modules of diagrams. In particular this algebra acts in a natural way on the modules $\mathcal{A}_s(\Gamma, X)$. Actually the element t is a particular element of Λ of degree 1.

The module $F(X)$ is endowed with an action of the symmetric group $\mathcal{S}(X)$. But we can also define natural maps from $F(X)$ to $F(Y)$ in the following way:

Let K be a $(\emptyset, X \coprod Y)$-diagram such that every connected component of K intersects X and Y. Then the gluing map along X induces a graded linear map φ_K from $F(X)$ to $F(Y)$. Actually the class $\mathcal{C}$ of $(\emptyset, X \coprod Y)$-diagrams satisfying this property induces a graded module ${}_X\Delta^c_Y = \mathcal{A}(\mathcal{C})$ and these modules give rise to a monoidal category Δ^c included in the category Δ. For every finite set X and Y the gluing map is a map from $F(X) \otimes {}_X\Delta^c_Y$ to $F(Y)$.

In particular we have two maps φ and φ' from $F(3)$ to $F(4)$ induced by the following diagrams:



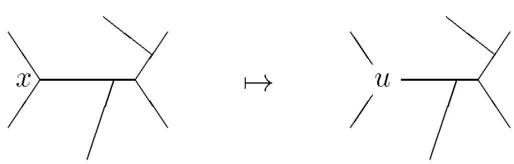
3.1 Definition: *Λ is the sub-module of $F(3)$ defined by the following conditions:*

$$\forall u \in F(3) \qquad u \in \Lambda \quad \iff \quad \forall \sigma \in \mathcal{S}_3 \quad \sigma(u) = \varepsilon(\sigma)u \quad \text{and} \quad \varphi(u) = \varphi'(u)$$

where ε is the signature homomorphism.

3.2 Proposition: *The module Λ is a graded* $\mathbf{Z}$*-algebra acting on each module $\mathcal{A}_s(\Gamma, X)$.*

Proof: Let Γ be a curve and X be a finite set. Let K be a (Γ, X)-diagram such that $K - \Gamma$ is connected and has a 3-valent vertex. Let x be such a vertex. If u is an element of Λ, we can insert u in K near x and we get a linear combination of diagrams and therefore an element uK in $\mathcal{A}_s(\Gamma, X)$.

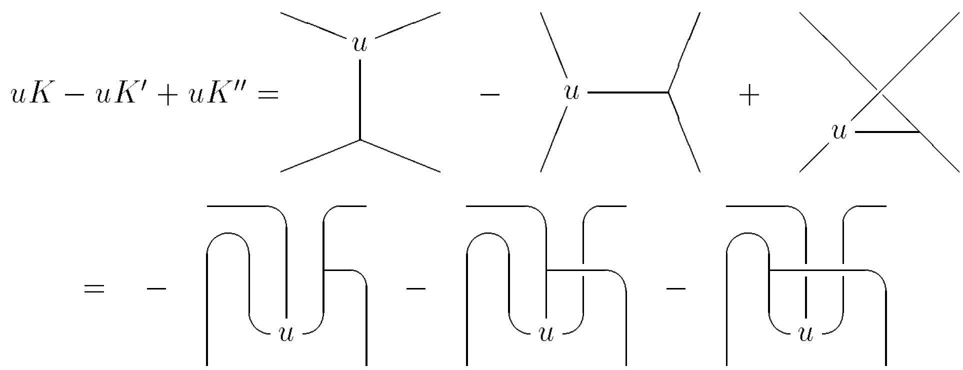


Since u is completely antisymmetric with respect to the $\mathcal{S}_3$-action, uK doesn't depend on the given bijection from $[3] = \{1, 2, 3\}$ to the set of edges arriving at x, but only on the cyclic order. Moreover, if this cyclic order is changed, uK is multiplied by -1. The second condition satisfied by u implies that the elements uK constructed by two consecutive vertices are the same. Since the complement of Γ in K is connected, uK doesn't depend on the choice of the vertex x, and uK is well-defined.

By construction, the rule $u \mapsto uK$ is a linear map from Λ to $\mathcal{A}_s(\Gamma, X)$ of degree $\partial^\circ K$. Since the transformation $K \mapsto uK$ is compatible with the relation AS, the only thing to check is to prove that this transformation is compatible with the relation IHX.

Consider a relation IHX: $K \equiv K' - K''$ corresponding to an edge a in K. If $K - \Gamma$ has at least three 3-valent vertices, it is possible to make the transformation $\Gamma \mapsto u\Gamma$ by using a vertex which is not in a, and we get the equality: $uK = uK' - uK''$.

If $K - \Gamma$ has only one 3-valent vertex, there is no relation IHX to check. In the last case, $K - \Gamma$ has exactly two 3-valent vertices and we have:

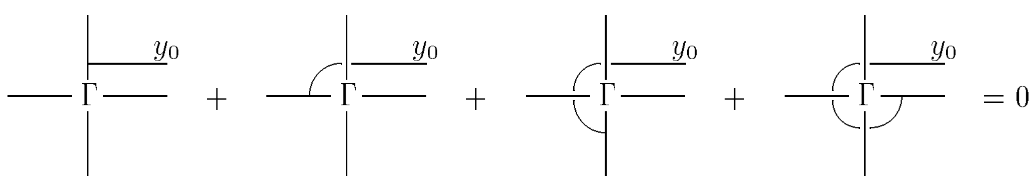


This last expression is trivial. It is not difficult to prove that directly, but it is also a special case of lemma 3.3.

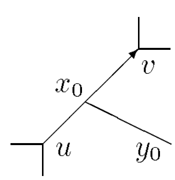
Therefore the transformation $\Gamma \mapsto u\Gamma$ is compatible with the relation IHX and induces a well-defined transformation from $\mathcal{A}_s(\Gamma, X)$ to itself. In particular, Λ acts on itself and this module is an algebra and $\mathcal{A}_s(\Gamma, X)$ is a Λ-module. $\square$

3.3 Lemma: *Let X be a finite set and Y be the set X with one extra point y_0 added. Let K be a connected $(\emptyset, X)$-diagram. For every $x \in X$ denote by K_x the $(\emptyset, Y)$-diagram obtained by adding to K an extra edge from y_0 to a point in K near x, the cyclic order near the new vertex being given by taking the edge arriving at y_0 first, the edge arriving at x after and the last edge at the end.*

Then the element $\sum\limits_x K_x$ is trivial in the module $F(Y)$.

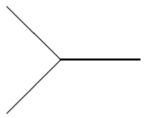
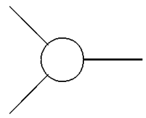
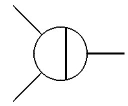


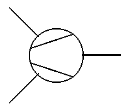
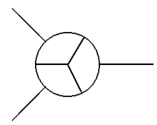
Proof: For every oriented edge a of K from a vertex u to a vertex v, we can connect y_0 to K by adding an extra edge from y_0 to a new vertex x_0 in a and we get a $(\emptyset, Y)$-diagram K_a where the cyclic order between edges arriving at x_0 is (x_0u, x_0y_0, x_0v).



It is clear that the expression $K_a + K_b$ is trivial if b is the edge a with the opposite orientation. Moreover if a, b and c are the three edges starting from a 3-valent vertex of K, the sum $K_a + K_b + K_c$ is also trivial. Therefore the sum ΣK_a for all oriented edge a of K is trivial and is equal to the sum ΣK_a for all oriented edge a starting from a vertex in X. That proves the lemma. $\square$

In degree less to 4, the module Λ is generated (over $\mathbf{Z}$) by the following diagrams:

$1 \quad =$  $t \quad =$  $t^2 \quad =$ 

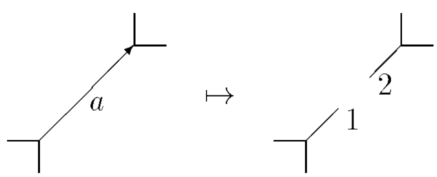
$t^3 \quad =$  

4. Structure of modules $F(n)$ for small values of n

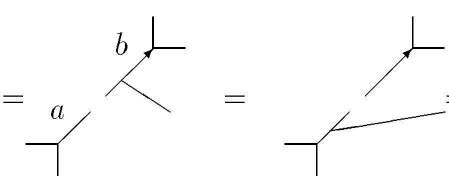
The module $F(n)$ is a Λ-module except for $n = 0, 2$. But the sub-module $F'(n) = \mathcal{A}_s(\emptyset, X)$ of $F(n)$ generated by diagrams having at least one 3-valent vertex is a Λ-module. For $n \neq 0, 2$, $F'(n)$ is equal to $F(n)$ and for $n = 0, 2$, $F(n)$ is isomorphic to $\mathbf{Z} \oplus F'(n)$.

4.1 Proposition: *Adding an edge between the elements of* $[2]$ *induces an isomorphism from* $F(2)$ *to* $F(0)$.

Proof: This map is clearly surjective.
Let K be a connected $(\emptyset, [0])$-diagram. Let a be an oriented edge of K. We can cut off a part of a and we get a $(\emptyset, [2])$-diagram $\varphi(K, a)$.



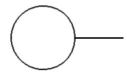
Let a and b be consecutive edges in K. Because of lemma 3.3, we have:

$\varphi(K, a) =$  $= \varphi(K, b)$

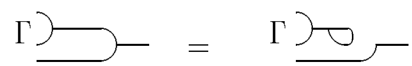
Therefore $\varphi(K, a)$ is independent of the choice of a and induces a well defined map from $F(0)$ to $F(2)$ which is obviously the inverse of the map above. $\square$

4.2 Corollary: *The action of the symmetric group $\mathcal{S}(2)$ on $F(2)$ is trivial.*

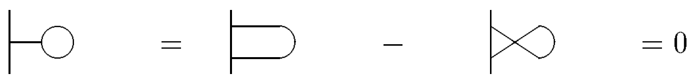
4.3 Proposition: *The module $F(1)$ is isomorphic to $\mathbf{Z}/2$ and generated by the following diagram:*



Proof: The diagram above is clearly a generator of $F(1)$ in degree 1, and the antisymmetric relation implies that this element is of order 2. Let K be a $(\emptyset,[1])$-diagram of degree > 1. We have:

$K =$ 

and this last diagram contains the following diagram:

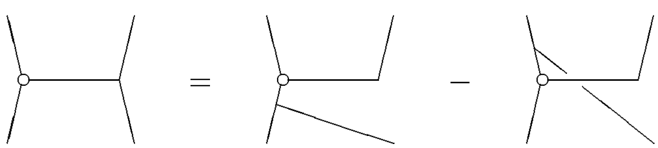


□

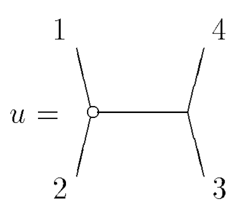
4.4 Proposition: *The quotient map from* $[3]$ *to a point induces a surjective map from* $\mathrm{F}(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^-$ *to* $\mathrm{F}'(0)$ *and its kernel is a* $\mathbf{F}_2$*-vector space.*

Proof: Here the group $\mathcal{S}(3)$ acts on $\mathbf{Z} = \mathbf{Z}^-$ via the signature. Actually, the module $F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^-$ is isomorphic to the module $\mathcal{M}$ generated by connected 3-valent diagrams without univalent vertex, pointed by a vertex and equipped with cyclic order near every vertex and where the relations are the antisymmetric relation AS everywhere and the relation IHX outside of the special vertex.

Because ot the lemma 3.3, we have in $\mathcal{M}$:



Set:



This element u lies in the module $\widetilde{\mathrm{F}}(4)$ where $\widetilde{\mathrm{F}}(n)$ is the module generated by connected 3-valent diagrams having $[n]$ as set of monovalent vertices and pointed by a 3-valent vertex and equipped with cyclic order near every 3-valent vertex and where the relations are the antisymmetric relation AS everywhere and the relation IHX outside of the special vertex and relation above. The module $\mathcal{M}$ is actually the module $\widetilde{\mathrm{F}}(0)$ and we have by gluing a map from $F(4) \underset{\mathcal{S}_4}{\otimes} \widetilde{F}(4)$ to $\mathcal{M}$.

The relation above is the following:

$$u = \sigma_2\sigma_3\sigma_2 u + \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 u \tag{1}$$

On the other hand, the antisymmetric relation implies:

$$\sigma_1 u = \sigma_3 u = -u \tag{2}$$

If we multiply (1) by σ_3, we obtain:

$$\sigma_3 u = -u = \sigma_3\sigma_2\sigma_3\sigma_2 u + \sigma_3\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 u$$

$$= \sigma_2\sigma_3 u + \sigma_2\sigma_1\sigma_2\sigma_3 u = -\sigma_2 u - \sigma_1\sigma_2\sigma_1 u = -\sigma_2 u + \sigma_1\sigma_2 u.$$

And then:

$$\sigma_1\sigma_2 u = (\sigma_2 - 1)u. \tag{3}$$

Let's multiply this formula by $\sigma_1\sigma_2\sigma_3$:

$$\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2 u = \sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 u = -\sigma_2\sigma_3\sigma_1\sigma_2 u = \sigma_1\sigma_2\sigma_3(\sigma_2 - 1)u$$

$$= (\sigma_3 - 1)\sigma_1\sigma_2\sigma_3 u = -(\sigma_3 - 1)\sigma_1\sigma_2 u$$

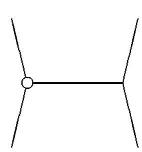
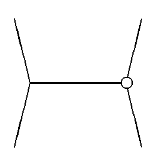
$$\Rightarrow (\sigma_2\sigma_3 + 1 - \sigma_3)\sigma_1\sigma_2 u = (\sigma_2\sigma_3 + 1 - \sigma_3)(\sigma_2 - 1)u = 0,$$

and we get:

$$(\sigma_2\sigma_3\sigma_2 - \sigma_2\sigma_3 + \sigma_2 - 1 - \sigma_3\sigma_2 + \sigma_3)u = -2(1 + \sigma_3\sigma_2 - \sigma_2)u = 0.$$

Let v be the element $(1 + \sigma_3\sigma_2 - \sigma_2)u = (1 + \sigma_3\sigma_2 + \sigma_2\sigma_3)u$. This element is of order 2 and:

$$v = u + \sigma_2\sigma_3(1 + \sigma_2\sigma_3)u = u + \sigma_2\sigma_3(1 - \sigma_2)u = u - \sigma_2\sigma_3\sigma_1\sigma_2 u$$

$=$  $-$ 

Let α be an element in $F'(0)$ represented by a 3-valent diagram K. Take a vertex x_0 in K. The pair (K, x_0) represents a well-defined element β in the module $\mathcal{M} \simeq F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^-$ and 2β doesn't depend on the choice of the vertex x_0. Hence the rule $\alpha \mapsto 2\beta$ is a well-defined map ψ from $F'(0)$ to $F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^-$. Denote by φ the canonical map from $F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^-$ to $F'(0)$. We have:

$$\varphi\psi = 2 \quad \text{and} \quad \psi\varphi = 2$$

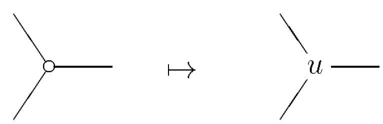
and proposition 4.4 follows. $\square$

4.5 Proposition: *Let $F(3)^-$ be the submodule of $F(3)$ defined by:*

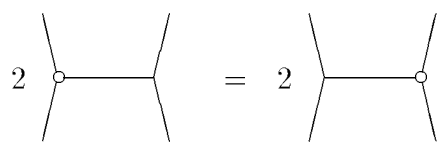
$$\forall u \in \mathrm{F}(3) \qquad u \in \mathrm{F}(3)^- \Leftrightarrow (\forall \sigma \in \mathcal{S}_3 \quad \sigma(u) = \varepsilon(\sigma)u)$$

where ε is the signature homomorphism. Then Λ is a submodule of $F(3)^-$ and the quotient $F(3)^-/\Lambda$ is a $\mathbf{F}_2$-vector space.

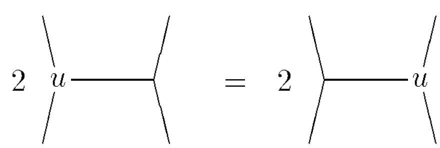
Proof: Let u be an element of $F(3)^-$. If v is an element of $\widetilde{F}(4)$ represented by a diagram K equipped with a special vertex x_0, we can insert u in K near x_0 and we get a well-defined element $f(v)$ in the module $F(4)$.



But we have in $\widetilde{F}(4)$:

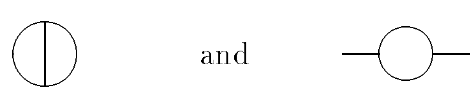


and that implies in $F(4)$:



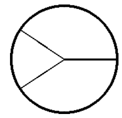
Therefore $2u$ lies in Λ. $\square$

4.6 Corollary: *The modules $F'(0)\otimes\mathbf{Z}[1/6]$ and $F'(2)\otimes\mathbf{Z}[1/6]$ are free $\Lambda\otimes\mathbf{Z}[1/6]$-modules of rank one generated by:*



Proof: Since $\mathcal{S}_3$ is a group of order 6, the identity induces an isomorphism from $F(3)^- \otimes \mathbf{Z}[1/6]$ to $F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^- \otimes \mathbf{Z}[1/6]$ and the corollary follows easily. $\square$

4.7 Corollary: *Let u be the primitive element of $\mathcal{A} = \mathcal{A}(S^1, \emptyset)$ represented by the diagram:*



Then the map $\lambda \mapsto \lambda u$ from Λ to the module $\mathcal{P}$ of primitives of $\mathcal{A}$ is rationally injective.

Proof: That's a consequence of the fact that $\mathcal{P}$ contains rationally the module $F'(2) \underset{\mathcal{S}(2)}{\otimes} \mathbf{Z} = F'(2)$. $\square$

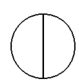
It is not clear that Λ is commutative, but it's almost the case. If α and β are elements in Λ, and u an element of a module $\mathcal{A}_s(\Gamma, X)$ represented by a diagram with at least two 3-valent vertices outside of Γ, we may construct $\alpha\beta u$ by using α and β modifications near two different vertices. Therefore: $\alpha\beta u = \beta\alpha u$.

4.8 Proposition: *The algebra Λ has the following properties:*

$$\forall \alpha, \beta, \gamma \in \Lambda \qquad \partial^\circ\gamma > 0 \Rightarrow \alpha\beta\gamma = \beta\alpha\gamma,$$

$$\forall \alpha, \beta \in \Lambda \qquad 12\alpha\beta = 12\beta\alpha.$$

Proof: The first formula is a special case of the property explained above. For the second one, just use that property where u is represented by the diagram



in $F'(0)$ and remark that the composite: $\Lambda \to F(3) \underset{\mathcal{S}(3)}{\otimes} \mathbf{Z}^- \to F'(0)$ has a kernel annihilated by $6 \times 2 = 12$. $\square$

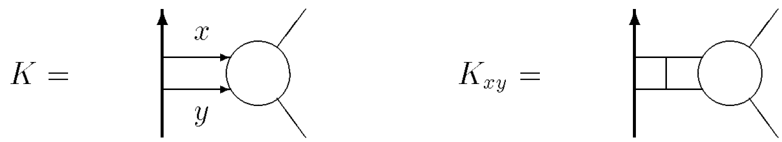
Remark: At this time there is no counterexample to the following conjecture:

Conjecture: *The kernel of any epimorphism from $\mathcal{S}_4$ to $\mathcal{S}_3$ acts trivially on the module $F(4)$.*

If this conjecture is true it is not difficult to deduce that Λ is commutative and equal to $F(3)$ and that $F'(2)$ and $F'(0)$ are free Λ-modules of rank 1.

5. Constructing elements in Λ

Let Γ be a curve and X be a finite set. Let K be a (Γ, X)-diagram in the class $\mathcal{D}_s(\Gamma, X)$. Let Y be a finite set in $K - \Gamma$ outside of the 0-skeleton of K. Consider an orientation of K near each point of Y. For each subset $\{x, y\}$ of Y, $x \neq y$, denote by K_{xy} the diagram obtained from K by adding an edge u joining x and y, the cyclic order near x (resp. y) being the part of the edge in K arriving at x first (resp. starting from y first) and the other part after.



The sum in $\mathcal{A}_s(\Gamma, X)$ of classes of K_{xy}'s for all subsets $\{x, y\} \subset Y$ will be denoted by $[K]_Y$.

5.1 Lemma: *Let X, Y, Z be finite sets and Γ be a curve. Let K be a $(\emptyset, X \cup Y)$-diagram and K' be a $(\Gamma, X \cup Y \cup Z)$-diagram such that Γ in $K \underset{X \cup Y}{\cup} K'$ lies in the class $\mathcal{D}_s(\Gamma, Z)$. Direction toward K (resp. toward K') induces an orientation of $K \cup K'$ near every point of X (resp. Y).*

Then we have the following formula:

$$[K \cup K']_X - pt[K \cup K'] = [K \cup K']_Y - qt[K \cup K']$$

where $p = \#X$, $q = \#Y$ and $[K \cup K']$ is the class of $K \cup K'$ in the module $\mathcal{A}_s(\Gamma, Z)$.

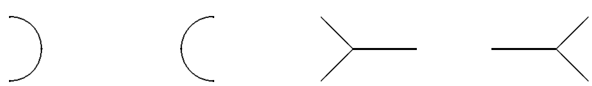
Proof: Consider a map f from $K \cup K'$ to $[0, 1]$ such that:

— f is smooth and generic on every edge of $K \cup K'$

— $f^{-1}(0) = X \qquad f^{-1}(1) = Y$.

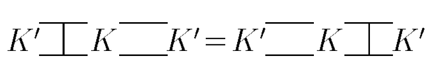
For every regular value $\gamma \in [0, 1]$, denote by X_γ the set $f^{-1}(\gamma)$. The map f induces an orientation of $K \cup K'$ near each point of X_γ, and $G(\gamma) = [K \cup K']_{X_\gamma} - \#(X_\gamma)t[K \cup K']$ is well-defined in $\mathcal{A}_s(\Gamma, Z)$. We have to prove that G is constant in the set on regular values in $[0, 1]$.

Since f is chosen to be generic, it is possible to cut $[\alpha, \beta]$ in elementary intervals J_λ such that $f^{-1}(J_\lambda)$ are of the form $U \cup V$ where f is a trivial fibration on U and V is one of the following possibilities:



where the restriction $f|V$ is a vertical projection in each of these pictures. And it is easy to check that G has the same value at the endpoints of J_λ. Therefore G is constant and the lemma is proven. $\square$

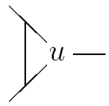
A special case of this lemma is the following:



in $\mathcal{A}_s(\Gamma, Z)$.

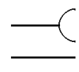
5.2 Corollary: *The element t is central in Λ.*

Proof: For every $u \in \Lambda$, we have:

$$ut =$$  $$= tu$$

$\square$

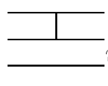
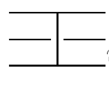
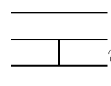
Let Γ_4 be the kernel of any epimorphism from the symmetric group $\mathcal{S}_4$ to $\mathcal{S}_3$. Consider the element $\omega \in {}_3\Delta_4$ represented by the following diagram:



By gluing from the left or the right, we get a map $u \mapsto u\omega$ from $F(3)$ to $F(4)$ or a map $u \mapsto \omega u$ from $F(4)$ to $F(3)$. Denote by E the submodule of $F(4)$ of all elements $u \in F(4)$ satisfying the following conditions:

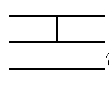
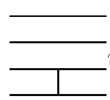
$$\forall \sigma \in \mathcal{S}_4 \qquad \omega\sigma u \in \Lambda \qquad \text{and} \qquad \forall \sigma \in \Gamma_4 \qquad \sigma u = u.$$

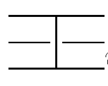
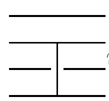
For every $u \in F(4)$, define elements xu, yu, zu by:

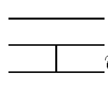
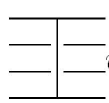
$xu =$  u $\qquad yu =$  u $\qquad zu =$  u

5.3 Proposition: *The module E is a graded $\Lambda[\mathcal{S}_4]$-submodule of $F(4)$ and for every $u \in E$, the elements xu, yu, zu belong to E, and $xu + yu + zu = 2tu$.*

Proof: The fact that E is a graded $\Lambda[\mathcal{S}_4]$-submodule of $F(4)$ is obvious. Let $u \in F(4)$. Because of lemma 5.1, we have:

$xu =$  $u =$  u

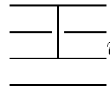
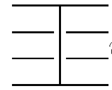
$yu =$  $u =$  u

$zu =$  $u =$  u

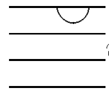
Hence, if σ is a permutation in $\mathcal{S}_4$, there exists an element $\theta \in \{x, y, z\}$ such that $\sigma xu = \theta\sigma u$. More precisely $\mathcal{S}_4$ acts on the set $\{x, y, z\}$ via an epimorphism $\sigma \mapsto \hat{\sigma}$ from $\mathcal{S}_4$ to $\mathcal{S}_3$, and we have:

$$\sigma xu = \hat{\sigma}(x)\sigma u, \qquad \sigma yu = \hat{\sigma}(y)\sigma u, \qquad \sigma zu = \hat{\sigma}(z)\sigma u.$$

We have:

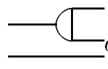
$xu + yu + zu =$  $u +$  $u +$  u

Because of lemma 3.3, we have:

$xu + yu + zu =$  $u = 2tu$

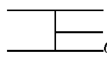
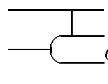
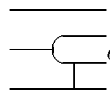
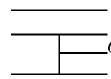
Moreover, if $u \in F(4)$ is Γ_4-invariant, xu, yu, zu are Γ_4-invariant too, and the last thing to do is to prove that, for every $u \in E$, $\omega x \sigma u, \omega y \sigma u, \omega z \sigma u$ belong to Λ.

We have:

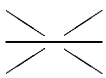
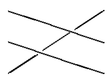
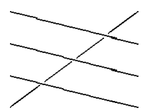
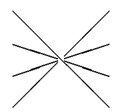
$\omega x \sigma u =$  $\sigma u = t\omega\sigma u \ \in \ \Lambda$

$$\omega z \sigma u = 2t\omega\sigma u - \omega x \sigma u - \omega y \sigma u$$

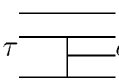
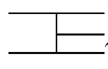
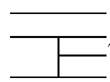
and it is enough to prove that $\omega y \sigma u$ belongs to Λ. Because of lemma 5.1, we have:

$\omega y \sigma u =$  $\sigma u =$  $\sigma u =$  $\sigma u =$  σu

Let s, τ, τ', θ be the permutations in $\mathcal{S}_4$ or $\mathcal{S}_3$ represented by the following diagrams:

$s =$  $\tau =$  $\tau' =$  $\theta =$ 

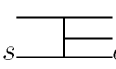
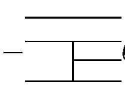
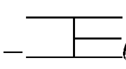
We have:

$\tau\omega y \sigma u = \tau$  $\sigma u =$  $\tau' \sigma u =$  $\tau' \sigma u$

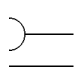
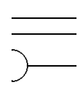
Then:

$$\tau\omega y \sigma u = \omega y \tau' \sigma u \qquad \Rightarrow \qquad \tau^2 \omega y \sigma u = \omega y \tau'^2 \sigma u$$

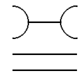
But τ'^2 lies in Γ_4 and $\tau^2 \omega y \sigma u = \omega y \sigma u$. Therefore $\omega y \sigma u$ is invariant under cyclic permutations. We have also:

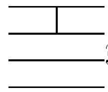
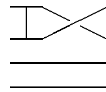
$s\omega y \sigma u = s$  $\sigma u = -$  $\theta\sigma u = -$  $\theta \sigma u$

Since θ lies in Γ_4 also, $s\omega y \sigma u = -\omega y \sigma u$ and $\omega y \sigma u$ belongs to the submodule $F(3)^-$ of $F(3)$. Consider the following diagrams:

$\omega' =$  $\qquad \omega'' =$ 

We have to prove the last equality: $\omega'\omega y\sigma u = \omega''\omega y\sigma u$. But we have:

$$\omega''\omega y\sigma u = \tau'^2\omega'\tau^2\omega y\sigma u = \tau'^2\omega'\omega y\sigma u = \tau'^2$$  $y\sigma u$

$= \tau'^2$  $y\sigma u - \tau'^2$  $y\sigma u = \tau'^2 xy\sigma u - \tau'^2 x\sigma_1 y\sigma u$

where σ_i is the i^{th} transposition. Then:

$$\omega''\omega y\sigma u = xy\tau'^2\sigma u - x\sigma_3 y\tau'^2\sigma u = xy\tau'^2\sigma u - x\sigma_1 y\sigma_1\sigma_3\tau'^2\sigma u$$

But τ'^2 and $\sigma_1\sigma_3$ belong to Γ_4 and:

$$\omega''\omega y\sigma u = xy\sigma u - x\sigma_1 y\sigma u = \omega'\omega y\sigma u$$

which finishes the proof. $\square$

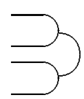
Consider now the graded Λ-algebra R generated by the group $\mathcal{S}_3$ in degree 0 and letters x, y, z in degree 1, where relations are:

$$\forall\sigma\in\mathcal{S}_3 \qquad \sigma x = \sigma(x)\sigma \qquad \sigma y = \sigma(y)\sigma \qquad \sigma z = \sigma(z)\sigma$$

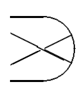
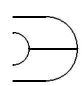
$$x + y + z = 2t$$

Because of proposition 2.1, the module E is a graded R-module.

Consider the following element of $F(4)$:

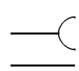
$a =$ 

It is not difficult to see that a lies in E. Let τ be a permutation in $\mathcal{S}_4$ inducing the permutation: $x \mapsto y \mapsto z \mapsto x$ via the map $\mathcal{S}_4 \to \mathcal{S}_3$, and set: $b = \tau a$, $c = \tau b$. We have:

$b = -$  $\qquad c = -$ 

and $a + b + c = 0$.

For every $u \in E$ denote by $< au >$ the element of Λ represented by the following diagram:

$< au >= \omega u =$  u

and set: $< bu >=< a\tau^{-1}u >$, $< cu >=< a\tau u >$. Now, we're able to define many elements in Λ. For simplicity, set: $z_1 = x$, $z_2 = y$, $z_3 = z$, $\alpha_1 = a$, $\alpha_2 = b$, $\alpha_3 = c$. For every sequence $(\beta_0, \beta_1, \ldots, \beta_p)$ in $[3] = \{1, 2, 3\}$, $p > 0$, denote by $< \beta_0\beta_1\ldots\beta_p >$ the element $< \alpha_{\beta_0} z_{\beta_1} \ldots z_{\beta_{p-1}}\alpha_{\beta_p} >$.

Let W be the graded monoid of words in [3]. The permutation group $\mathcal{S}_3$ acts on W in the obvious way. Moreover, for every $w \in W$ of degree $d > 1$, $< w >$ is an element of Λ of degree $d-1$. Denote by W' (resp. W'') the set of words $w \in W$ of degree ≥ 1 (resp. ≥ 2).

5.4 Proposition: *Elements $< w >$ satisfy the following properties:*

$$< 11 >= 2t, \qquad < 12 >= -t,$$

5.4.1 $$\forall \sigma \in \mathcal{S}_3 \ \ \forall w \in W'' \qquad < w >=< \sigma(w) >,$$

5.4.2 $$\forall w \in W' \quad < 1w > + < 2w > + < 3w >=< w1 > + < w2 > + < w3 >= 0,$$

5.4.3 $$\forall w \in W \ \ \forall \gamma \in [3] \quad < \gamma\gamma w >= t < \gamma w > \qquad < w\gamma\gamma >= t < w\gamma >,$$

5.4.4 $$\forall u, v \in W' \quad < u1v > + < u2v > + < u3v >= 2t < uv >,$$

5.4.5 $$\forall u, v \in W' \ \ \forall \gamma \in [3] \qquad < u\gamma v > + < u\gamma\sigma(v) >=< \gamma v >< u\gamma >$$

where σ is the transposition keeping γ fixed.

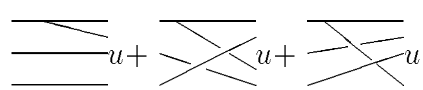
Proof: The first formula is easy to check.

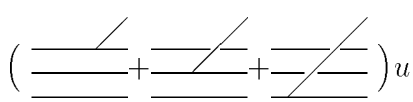
Let τ be the permutation $1 \mapsto 2 \mapsto 3 \mapsto 1$. Consider an element $\gamma w \in W''$ and a permutation $\sigma \in \mathcal{S}_3$. There exists a number p such that $\gamma = \tau^p(1)$, and we have: $< \gamma w >=< 1\tau^{-p}(w) >$. There exists also a number q such that $\sigma(\gamma) = \tau^q(1)$. Since $\tau^{-q}\sigma\tau^p$ keeps 1 fixed, $\tau^{-q}\sigma\tau^p$ is a power θ^n of the transposition θ keeping 1 fixed, and we have:

$$< \sigma(\gamma w) >=< \sigma(\gamma)\sigma(w) >=< 1\tau^{-q}\sigma(w) >=< 1\theta^{-n}\tau^{-p}(w) > .$$

But for every $u \in E$, it is easy to see that $< a\theta u >$ is equal to $- < au >$. On the other hand, if u has the form: $u = z_{j_1} \ldots z_{j_n}\alpha_k$, θu is equal to $-z_{\theta(j_1)} \ldots z_{\theta(j_n)}\alpha_{\theta(k)}$. Hence, for every word $w \in W'$, we have: $< 1\theta(w) >=< 1w >$, and that proves the relation 5.4.1.

The relation $< w1 > + < w2 > + < w3 >= 0$ is a consequence of: $a + b + c = 0$ in E. Let $w \in W'$. We have:

$< 1w > + < 2w > + < 3w >=$ 

$=$ 

and this last expression is zero because of lemma 3.3.

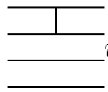
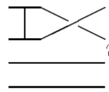
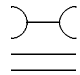
It's enough to check the formula 5.4.3 for $\gamma = 1$ and that's obvious.

The next property:

$$\forall u, v \in W' \quad < u1v > + < u2v > + < u3v >= 2t < uv >$$

is an obvious consequence of the equality: $x + y + z = 2t$ in R.

For the last property, it's enough to consider the case $\gamma = 1$. But for every u in the module E, we have:

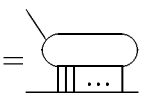
$xu - x\sigma u =$  $u -$  $u =$  $u =< au > a$

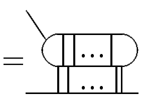
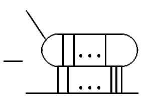
and that implies the desired formula. □

In particular, we have the following elements in Λ:

$$x_0 = 0, \quad x_1 =< 11 >, \quad x_2 =< 121 >, \quad x_3 =< 1221 >, \quad \ldots \quad x_n =< 12 \ldots 21 >,$$

$$y_1 =< 12 >, \qquad y_2 =< 121 >, \qquad y_3 =< 1212 >, \ \ldots \quad y_n =< 12121 \ldots >,$$

$\forall n > 0 \qquad x_n =$ 

$\forall p > 0 \qquad y_{2p} =$  $\qquad y_{2p+1} = -$ 

5.5 Proposition: *The subalgebra Λ_0 of Λ generated by elements $< w >$, $w \in W''$ is generated by $y_1, y_2, y_3, \ldots$. The algebra $\Lambda_0 \otimes \mathbf{Z}[1/6]$ is commutative and generated by $x_{2p+1}, p \geq 0$.*

Proof: Because of 5.4.1 and 5.4.4, Λ_0 is generated by t and elements $< 1u >$ where u is a word in letters 1 and 2. Let σ (resp. σ') be the transposition $2 \mapsto 3 \mapsto 2$ (resp $1 \mapsto 3 \mapsto 1$). If u and v are two words in W', we have:

$$< u11v >= 2t < u1v > - < u12v > - < u13v >$$

$$= 2t < u1v > - < u12v > + < u12\sigma(v) > + < 13v >< u1 >$$

$$< u22v >= 2t < u2v > - < u21v > + < u21\sigma'(v) > + < 23v >< u2 >$$

and, by induction, every element $< w >$, $w \in W''$, lies in the subalgebra generated by $< 121212 \ldots >$.

The formula above for $< u22v >$ implies:

$$< u2^n1 >\equiv< u212^{n-2}1 > + < u212^{n-2}3 >\equiv -2 < u212^{n-2}1 >$$

where $\equiv$ denotes the equality modulo decomposables. Hence we get:

$$\forall p > 0 \quad x_{2p} \equiv (-2)^{p-1} < 1212 \ldots 21 >\equiv (-2)^{p-1} y_{2p}$$

$$\forall p \quad x_{2p+1} \equiv (-2)^{p-1} < 1212\ldots 21221 > \equiv (-2)^{p-1} < \ldots 1213 > \equiv -(-2)^{p-1} y_{2p+1}.$$

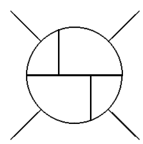
Hence $\Lambda_0 \otimes \mathbf{Z}[1/2]$ is generated by x_n's.

Moreover, we have:

$$\begin{aligned}
\forall p > 1 \quad x_{2p} &=< 12^{2p-1}1 >=< 13^{2p-1}1 > \equiv - < 123^{2p-2}1 > \equiv < 121^{2p-2}3 > \\
&\equiv - < 121^{2p-2}2 > \equiv 2 < 12121^{2p-4}2 > \equiv \ldots \equiv -(-2)^{p-2} < (12)^{p-1}112 > \\
&= (-2)^{p-2} < (12)^p 1 >= (-2)^{p-2} y_{2p} \\
x_{2p} &=< 12^{2p-1}1 > \equiv -2 < 1212^{2p-3}1 > \equiv \ldots \equiv (-2)^{p-1} y_{2p}.
\end{aligned}$$

Hence, for every $p > 1$, $3(-2)^{p-2} y_{2p} \equiv 0$. Since $y_2 = t^2 = y_1^2$, $\Lambda_0 \otimes \mathbf{Z}[1/6]$ is generated by odd y_n's (or odd x_n's). $\square$

Remark: Let E_0 be the R-submodule of E generated by a, b, c. It is not difficult to see that $F(4) = E_0$ in degree < 4, and in degree 4, the quotient $F(4)/E_0$ is a $\mathbf{Z}/2$-module generated by the following diagram ω:



It is easy to see that $< a\omega >, < b\omega >, < c\omega >$ are in Λ_0. Therefore Λ is equal to Λ_0 in degree less than 6, and generated in this range by:

$$1, \quad t, \quad t^2, \quad t^3, \quad t^4, \quad t^5, \quad y_3, \quad ty_3, \quad t^2y_3, \quad y_4, \quad ty_4, \quad y_5.$$

But y_4 is rationally a consequence of y_3. The explicit relation is:

$$3y_4 + 4ty_3 + t^4 = 0$$

It will be proved in next section that there is no algebraic relation between elements $t, x_3, x_5, \ldots$ in degree less than 10. Therefore Λ is, in degree less than 6, free over $\mathbf{Z}$ with basis:

$$1, \quad t, \quad t^2, \quad t^3, \quad t^4, \quad t^5, \quad y_3, \quad y_4 + ty_3, \quad ty_4 + t^2y_3, \quad y_5,$$

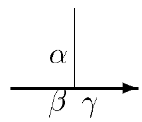
and Λ is not a polynomial algebra over $\mathbf{Z}$.

6. Detecting elements in Λ

Let L be a finite dimensional simple Lie superalgebra over a field k equipped with a nonsingular supersymmetric bilinear form $<\Gamma, \Gamma>$ invariant under the adjoint representation. Take a homogeneous basis (e_j) of L and its dual basis (e'_j). The Casimir element $\omega = \sum_j e_j \otimes e'_j \in L \otimes L$ is independent on the choice of the basis and its degree is zero.

Let Γ be an oriented curve and $X = [n]$ be a finite set. Suppose that a L-representation E_i is chosen for every component Γ_i of Γ. We will say that Γ is coloured by L-representations. Then it is possible to construct a linear map from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$ in the following way:

Let K be a (Γ, X)-diagram. Up to some changes of cyclic order we may as well suppose that, at each vertex x in Γ the cyclic order is given by (α, β, γ) where α is the edge which is not contained in Γ and β is the edge in Γ arriving at x (with the orientation of Γ).

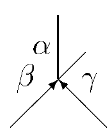


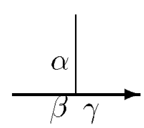
For each component Γ_i we can take a basis (e_{ij}) of E_i and its dual basis (e'_{ij}) of the dual E'_i of E_i and we get a Casimir element $\omega_i = \Sigma_j e_{ij} \otimes e'_{ij} \in E_i \otimes E'_i$. This element is of degree zero and is independent of the choice of the basis.

For each oriented edge α in K denote by $V(\alpha)$ the module L if α is not contained in Γ and E_i (resp. E'_i) if α is contained in the component Γ_i of Γ with a compatible (resp. noncompatible) orientation. If α is an oriented edge in K denote by $W(\alpha)$ the module $V(\alpha) \otimes V(-\alpha)$ where $-\alpha$ is the edge α equipped with the opposite orientation.

Let a be an edge in K. Take an orientation of a compatible with the orientation of Γ if a is contained in Γ. Denote also by $\omega(a)$ the Casimir element ω if a is not contained in Γ and the element ω_i if a is contained in Γ_i. This element belongs to the module $W(a)$ and is independent on the orientation of a. If a numbering of the set of edges of K is chosen the tensor product $W = \bigotimes_a W(a)$ is well-defined and the element $\Omega = \bigotimes_a \omega(a)$ is a well-defined element in W.

Let x be a 3-valent vertex in K. There are three oriented edges α, β and γ arriving at x (the order (α, β, γ) is chosen to be compatible with the cyclic order given at x and, if x is in Γ, α is supposed to be outside of Γ).





Then we get a module $H(x) = V(\alpha) \otimes V(\beta) \otimes V(\gamma)$. If a numbering of the set of 3-valent vertices of K is chosen, the module $\bigotimes_x H(x)$ is well-defined. We can permute (in the super sense) the big tensor product W and we get an isomorphism φ from W to the module:

$$H = L^{\otimes n} \otimes \bigotimes_x H(x)$$

and $\varphi(\Omega)$ is an element of H.

Suppose that x is not contained in Γ. Then the rule $u \otimes v \otimes w \mapsto < [u, v], w >$ induces a map f_x from $H(x)$ to k. If x is in Γ the rule $u \otimes e \otimes f \mapsto (-1)^{\partial^\circ f \partial^\circ (u \otimes e)} f(ue)$ is a map f_x from $H(x)$ to k. Hence the image of $\varphi(\Omega)$ under the tensor product of all

f_x is an element $\Phi_L(K) \in L^{\otimes n}$. Since elements w and w_i and maps f_x are of degree zero, this element $\Phi_L(K)$ doesn't depend on these numberings.

Since the map $u \otimes v \otimes w \mapsto < [u, v], w >$ from $L \otimes L \otimes L$ to k is totally antisymmetric (in the super sense) the number $\Phi_L(K)$ is multiplied by -1 if one cyclic order is changed in K. Moreover, the Jacobi identity and the property of the L-action on modules E_i imply that the correspondence $K \mapsto \Phi_L(K)$ is compatible with the IHX relation. Therefore this correspondence induces a well-defined linear map Φ_L from $\mathcal{A}(\Gamma, X)$ to $L^{\otimes n}$.

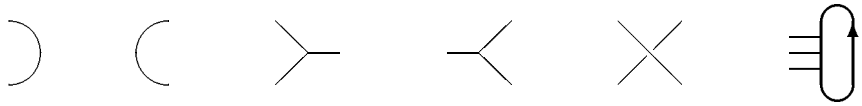
6.1 Theorem: *There is a well-defined character χ_L depending only on L and Ω, from the algebra Λ to k such that:*

For every oriented curve Γ coloured by L-representations and every finite set $X = [n]$, the map Φ_L satisfies the following property:

$$\forall \alpha \in \Lambda \quad \forall u \in \mathcal{A}_s(\Gamma, X) \quad \Phi_L(\alpha u) = \chi_L(\alpha)\Phi_L(u)$$

Proof: First of all, it is possible to extend the map Φ_L to a functor between two categories $Diag(L)$ and $\mathcal{C}(L)$. The objects of these categories are the sets $[p], p \geq 0$. For $p, q \in \mathbf{N}$ the set of morphisms in $\mathcal{C}(L)$ from $[p]$ to $[q]$ is the set of $L-$linear homomorphisms from $L^{\otimes p}$ to $L^{\otimes q}$, and the set of morphisms in $Diag(L)$ from $[p]$ to $[q]$ is the module generated by the isomorphy classes of $(\Gamma, [p] \cup [q])$-diagrams where Γ is any L-coloured oriented curve and where the relations are the antisymmetric and IHX relations.

These two categories are monoidal and $Diag(L)$ contains Δ as subcategory. Moreover, $Diag(L)$ is generated (as a monoidal category) by the following morphisms:



The last morphism depends on an integer $p \geq 0$ and a L-representation E and is a morphism in $Diag(L)$ from $[p]$ to $[0]$.

The map Φ_L associates to each L-coloured $(\Gamma, [p] \cup [q])$-diagram K an element $\Phi_L(K)$ in $L^{\otimes p} \otimes L^{\otimes q}$. But $L^{\otimes p}$ is isomorphic to its dual and $\Phi_L(K)$ may be seen as a linear map from $L^{\otimes p}$ to $L^{\otimes q}$.

It is not difficult to see that the image under Φ_L of the generators above are:

— the scalar product from $L^{\otimes 2}$ to $L^{\otimes 0} = k$,

— the Casimir from $k = L^{\otimes 0}$ to $L^{\otimes 2}$,

— the Lie bracket from $L^{\otimes 2}$ to L,

— the dual of the Lie bracket from L to $L^{\otimes 2}$,

— the map $x \otimes y \mapsto (-1)^{\partial^\circ x \partial^\circ y} y \otimes x$ from $L^{\otimes 2}$ to itself,

— the map $x_1 \otimes \ldots \otimes x_p \mapsto \tau_E(x_1 \ldots x_p)$ from $L^{\otimes p}$ to $L^{\otimes 0} = k$,

where $\tau_E(x_1 \ldots x_p)$ is the supertrace of the endomorphism $x_1 \ldots x_p$ of E.

But all of these maps are L-linear. Therefore Φ_L induces a functor still denoted by Φ_L from $Diag(L)$ to the category $\mathcal{C}(L)$.

Let Γ be a L-coloured oriented curve and $X = [n]$ be a finite set. Consider an element $\alpha \in \Lambda$ and an element $u \in \mathcal{A}_s(\Gamma, X)$ represented by a (Γ, X)-diagram K. Take a 3-valent vertex x in K and a bijection from [3] to the set of edges arriving at x. By taking off a neighborhood of x in K, we get a diagram H inducing a morphism v in $Diag(L)$ from [3] to $[n]$.

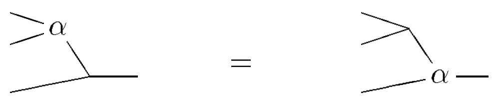
On the other hand, α induces a morphism β in $Diag(L)$ from [0] to [3], and $1 \in \Lambda$ induces an element β_0 from [0] to [3]. Let $\tilde{u}$ and $\widetilde{\alpha u}$ be the morphisms from [0] to $[n]$ induced by u and αu. We have:

$$\tilde{u} = v \circ \beta_0, \qquad \widetilde{\alpha u} = v \circ \beta.$$

Hence:

$$\Phi_L(\tilde{u}) = \Phi_L(v) \circ \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \Phi_L(v) \circ \Phi_L(\beta).$$

The elements $\alpha \in \Lambda$ and $1 \in \Lambda$ also induce morphisms γ and γ_0 from [2] to [1]. Denote by φ and φ_0 the morphisms $\Phi_L(\gamma)$ and $\Phi_L(\gamma_0)$. The morphism φ_0 is the Lie bracket and φ is L-linear and antisymmetric. Since α belongs to Λ, we have the following:



Hence for every x, y, z in L we have:

$$[\varphi(x \otimes y), z] = \varphi([x, y] \otimes z)$$

and $\varphi(x \otimes y)$ depends only on $[x, y]$. Therefore there exists a unique endomorphism f of L such that:

$$\varphi(x \otimes y) = f([x, y]).$$

Moreover, L is supposed to be simple and f is the multiplication by a scalar $\lambda \in k$. Consequently:

$$\Phi_L(\beta) = \lambda \Phi_L(\beta_0), \qquad \Phi_L(\widetilde{\alpha u}) = \lambda \Phi_L(\tilde{u}), \qquad \Phi_L(\alpha u) = \lambda \Phi_L(u).$$

Now it is easy to see that $\alpha \mapsto \lambda$ is a character depending only on the simple Lie algebra L equipped with the Casimir Ω and theorem 6.1 is proven. $\square$

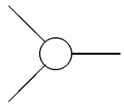
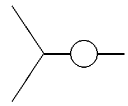
Remark: Theorem 6.1 is still true if L is a Lie superalgebra satisfying the following conditions:

— the center of L is trivial,
— $\mathrm{End}_L(L) = k$,
— L has a nonsingular invariant supersymmetric bilinear form.

From now on except in the last part of this section, L is supposed to satisfy the following conditions:

— The characteristic of the field k is different from 2,
— The superdimension d of L is non zero in k,

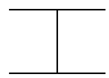
— The Casimir operator acts on L by multiplication by a nontrivial element.
Because of the formula:

2  = 

the Casimir operator $x \mapsto \sum[e_i, [e'_i, x]]$ from L to L is the multiplication by $2\chi_L(t)$. Therefore we'll denote by $t \in k$ half this number. Since the scalar product on L may be changed by multiplication by any number, Casimir operator and t are defined up to a scalar.

On the other hand, Λ is a graded algebra. Hence χ_L may be considered as a graded algebra homomorphism from Λ to $k[t]$, sending t to t.

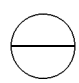
From now on, for every Lie superalgebra, we denote by Ψ the morphism from $L^{\otimes 2}$ to itself represented by the following diagram:



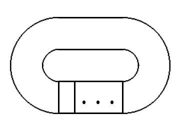
6.2 Proposition: *Let $\{\alpha_i t\}$ be the eigenvalues of the morphism Ψ, and $\{d_i\}$ be superdimensions of the corresponding characteristic spaces. Then the image of $x_n \in \Lambda$ by χ_L is:*

$$\chi_L(x_n) = \frac{\Sigma d_i \alpha_i^{n+1}}{2d} t^n.$$

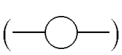
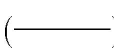
Proof: Let θ be the element of F(0) represented by the diagram



Then $x_n\theta$ is represented by the diagram



with $2n+2$ vertices. We have:

$\Phi_L($$) = 2t\Phi_L($$)$

$$\Rightarrow \qquad \Phi_L(\theta) = 2t\Phi_L(\quad \bigcirc \quad) = 2t\sum < e_i, e'_i >$$

$$= 2t\sum(-1)^{\partial^\circ e_i \partial^\circ e'_i} < e'_i, e_i >= 2t\sum(-1)^{\partial^\circ e_i} = 2td$$

where $\Sigma e_i \otimes e'_i$ is the Casimir element in $L \otimes L$. We deduce:

$$2td\chi_L(x_n) = \Phi_L(x_n\theta) = \sum < e_i \otimes e_j, \Psi^{n-1}(e'_j \otimes e'_i) > .$$

By setting: $e_\alpha = e_i \otimes e_j, e'_\alpha = e'_j \otimes e'_i, \Psi^{n+1}(e'_\alpha) = \sum A_{\alpha\beta}e'_\beta$, we get:

$$\sum < e_i \otimes e_j, \Psi^{n-1}(e'_j \otimes e'_i) >= \sum < e_\alpha, A_{\alpha\beta}e'_\beta >= \sum A_{\alpha\alpha}(-1)^{\partial^\circ e_\alpha} = \tau(\Psi^{n+1})$$

where τ is the supertrace. Then we have:

$$2td\chi_L(x_n) = \sum d_i\alpha_i^{n+1}t^{n+1}$$

and the proposition is proven. □

6.3 The sl **case.** Let E be a supermodule of superdimension $m \neq 0$. Take a homogeneous basis $\{e_i\}$ of E and denote by $\{e_{ij}\}$ the corresponding basis of End$E$. Let $L = sl(E)$ be the Lie superalgebra of endomorphisms of E with zero supertrace. The superdimension d of L is $d = m^2 - 1$ and a Casimir element of L is:

$$\Omega = \sum(-1)^{\partial^\circ e_j}e_{ij} \otimes e_{ji} - \frac{1}{m}1 \otimes 1$$

It is easy to check that Ω acts on L by multiplication by $2m$. Hence with this Casimir, we have: $t = m$.

6.4 Theorem: *There exists a graded algebra homomorphism* χ_{sl} *from* Λ *to the graded algebra* $\mathbf{Z}[t,\beta]$ *with* $\partial^\circ t = 1, \partial^\circ\beta = 2$ *such that:*

For every supermodule E *of superdimension* $m \neq 0$*, the following diagram is commutative:*

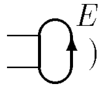
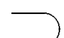
$$\begin{tikzcd} \Lambda \arrow[r, "\chi_{sl}"] \arrow[dr, "\chi_L"'] & Z[t,\beta] \arrow[d, "\gamma"] \\ & k \end{tikzcd}$$

where γ *is the ring homomorphism sending* t *to* m *and* β *to* 1*, and* L *is the Lie superalgebra* $sl(E)$*.*

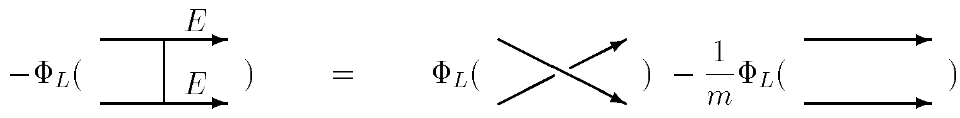
Moreover, for every $n = 2p + 1 > 0$*, we have:*

$$\chi_{sl}(x_n) = t^n + t(4\beta)^p + 2^n t\beta\frac{t^{2p} - \beta^p}{t^2 - \beta}.$$

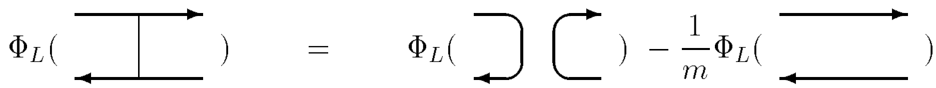
Proof: Since the scalar product of x and y in L is $< x, y >= \tau_E(xy)$ we have the following:

$\Phi_L($  $)$ $\quad = \quad$ $\Phi_L($  $)$

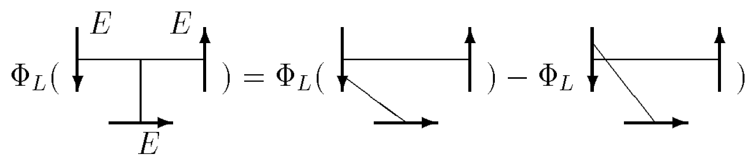
Moreover, it is not difficult to show the following:

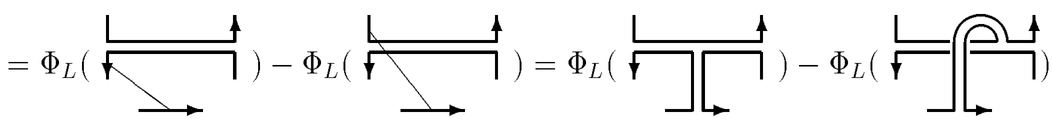


Whence:



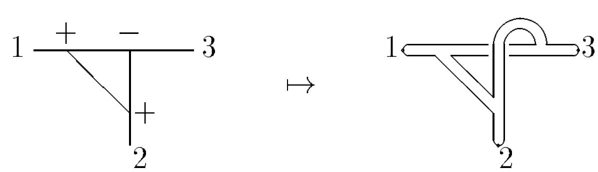
and we get:





Therefore, to compute the image by Φ_L of a $(\emptyset, [n])$-diagram K, we may proceed as follows:

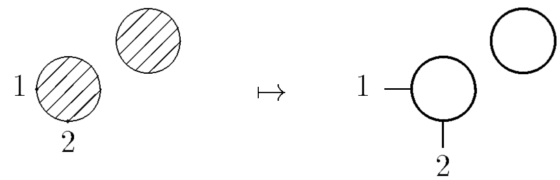
Let $S(K)$ be the set of functions α from the set of 3-valent vertices of K to ± 1. For every $\alpha \in S(K)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(x)$. If $\alpha \in S(K)$ is given we may construct a thickening of K by using the given cyclic order of edges arriving at a 3-valent vertex x if $\alpha(x) = 1$ and the other one if not, and we get an oriented surface $\Sigma_\alpha(K)$ equipped with n numbered points in its boundary.



Denote by S_n the set of isomorphism classes of oriented connected surfaces equipped with n numbered points in its boundary. Under the connected sum, $S = S_0$ is a monoid and acts on S_n. This monoid is a graded commutative monoid freely generated by the disk δ of degree 1 and the torus β of degree 2. The set S_n is a graded S-set with dim H_1 as degree. Let $\mathbf{Z}[S_n]$ and $\mathbf{Z}[S]$ be the free modules generated by S_n and S. They are graded modules, and $\mathbf{Z}[S]$ is a polynomial algebra acting on $\mathbf{Z}[S_n]$.

If K is connected, the sum $s(K) = \Sigma\varepsilon(\alpha)\Sigma_\alpha(K)$ lies in $\mathbf{Z}[S_n]$. It is easy to check that s is compatible with antisymmetric and IHX relations and induces a well-defined graded homomorphism from $F(n)$ to $\mathbf{Z}[S_n]$. Moreover, this homomorphism is $\Lambda \times \mathbf{Z}[S]$-linear with respect to a character χ from Λ to $\mathbf{Z}[S] = \mathbf{Z}[\delta, T]$.

On the other hand, we have a map ∂ from S_n and $\mathbf{Z}[S_n]$ to $F(n)$ by sending each surface Σ with numbered points in $\partial\Sigma$ to the boundary $\partial\Sigma$ coloured by E with intervals added near each marked point:



and $\Phi_L(K)$ is equal to the sum $\Sigma\varepsilon(\alpha)\Phi_L(\partial\Sigma_\alpha(K)) = \Phi_L(\partial s(K))$. Therefore if u is an element of Λ, we have $\chi_L(u) = \chi_L(\partial\chi(u))$. Since $\chi_L \circ \partial$ is a ring homomorphism sending δ to $m = \dim E$ and β to 1, the first part of the theorem is proven (with $t = \delta$ and $\beta = T$).

In order to prove the second part of this theorem, it is enough to determine $\chi_L(x_n)$ for infinitely many values of m. Suppose now $m > 2$ and E has no odd part. Then $L = sl(E)$ is the classical Lie algebra sl_m. The morphism Ψ from $L^{\otimes 2}$ to itself is the morphism:

$$x \otimes y \mapsto \sum_{ij} [x, e_{ij}] \otimes [e_{ji}, y]$$

and because of proposition 6.2 we have to determine eigenvalues and eigenspaces of this morphism. Some of them do not depend on the Lie algebra:

6.5 Lemma: *Let L be any Lie superalgebra of superdimension $d \neq 0$. Suppose that the Casimir operator acts on L by multiplication by a nontrivial element $2t$. Then we have the following decompositions of L-modules:*

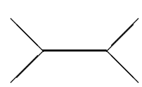
$$L^{\otimes 2} \simeq \Lambda^2(L) \oplus S^2(L)$$

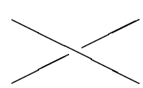
$$\Lambda^2(L) \simeq L \oplus E_1 \qquad\qquad S^2(L) \simeq k \oplus E_2$$

The morphism Ψ acts on the summands L, E_1 and k of $L^{\otimes 2}$ by multiplication by t, 0 and $2t$ respectively. Moreover, we have:

$$\dim L = d \qquad \dim E_1 = \frac{d(d-3)}{2} \qquad \dim k = 1$$

Proof: Consider the following endomorphisms of $L^{\otimes 2}$:

$\alpha = \frac{1}{2t}\Phi_L($  $)$

$\beta = \frac{1}{2}\Phi_L($ $) - \frac{1}{2}\Phi_L($  $) - \alpha$

If σ is the transposition $x \otimes y \mapsto (-1)^{\partial^\circ x \partial^\circ y} y \otimes x$ from $L^{\otimes 2}$ to itself, β is the morphism $\frac{1}{2}(1-\sigma) - \alpha$.

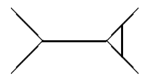
By definition of t, we have: $\alpha^2 = \alpha$. Since $\sigma\alpha = \alpha\sigma = -\alpha$, we also have:

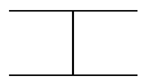
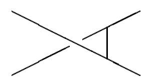
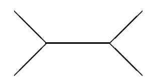
$$\alpha\beta = \beta\alpha = 0 \qquad\qquad \beta^2 = \beta \qquad\qquad \sigma\beta = \beta\sigma = -\beta$$

Hence:

$$\Lambda^2(L) = \alpha(L \otimes L) \oplus \beta(L \otimes L)$$

and $\alpha(L \otimes L)$ is isomorphic to L. Moreover:

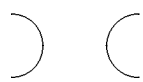
$\Psi \circ \alpha = \frac{1}{2t}\Phi_L($  $) = t\alpha$

$2\Psi \circ \beta = \Psi - \Psi \circ \sigma - 2t\alpha = \Phi_L($  $-$  $-$  $) = 0$

and Ψ acts on $\alpha(L \otimes L)$ and $\beta(L \otimes L)$ by multiplication by t and 0.

The superdimensions of $\alpha(L \otimes L)$ and $\beta(L \otimes L)$ are d and $d(d-1)/2 - d = d(d-3)/2$.

On the other hand, the element

$\gamma = \frac{1}{d}\Phi_L($  $)$

is a projector with 1-dimensional image, and Ψ acts on this image by multiplication by $2t$. $\square$

Suppose now $L = sl(E) = sl_m, m > 2$. The only thing to do is to determine the eigenvalues of Ψ acting on $E_2 \simeq S^2(L)/\Omega$. Consider the morphism $f : x \otimes y \mapsto xy + yx - \frac{2}{m}\tau(xy)$ from $L \otimes L$ to L. This homorphism is surjectif because $m > 2$. We have:

$$f\Psi(x \otimes y) = \sum((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji}) + (e_{ji}y - ye_{ji})(xe_{ij} - e_{ij}x)$$

$$-\frac{2}{m}\tau((xe_{ij} - e_{ij}x)(e_{ji}y - ye_{ji})))$$

$$= mxy + \tau(xy) + myx + \tau(yx) - \frac{2}{m}\tau(mxy + \tau(xy))$$

$$= mxy + myx - 2\tau(xy) = mf(x \otimes y).$$

The map f factorizes through E_2 and E_2 contains a summand isomorphic to L where Ψ acts by multiplication by $m = t$. Let E_3 be the kernel of $f : E_2 \to L$. By construction, the maps sending $x \otimes y$ to xy, yx, $x \otimes y - y \otimes x$ are trivial on E_3.

If z lies in L, denote by z_{ij} the entries of z. We have:

$$\Psi(x \otimes y) = \sum(x_{ki}e_{kj} - x_{jk}e_{ik}) \otimes (y_{il}e_{jl} - y_{lj}e_{li})$$

$$= \sum (xy)_{kl} e_{kj} \otimes e_{jl} + (yx)_{lk} e_{ik} \otimes e_{li} - x_{ki} e_{kj} \otimes y_{lj} e_{li} - y_{il} e_{ik} \otimes x_{jk} e_{jl}.$$

Therefore on E_3 the map Ψ is equal to Ψ':

$$\Psi'(x \otimes y) = -2 \sum x e_{ij} \otimes y e_{ji}$$

and we have:

$$\Psi'^2(x \otimes y) = 4 \sum x e_{ij} e_{kl} \otimes y e_{ji} e_{lk} = 4x \otimes y$$

Hence E_3 decomposes into two eigenspaces E_+ and E_- corresponding to the eigenvalues 2 and -2. Denote by a and b the dimensions of E_+ and E_-. Because of Proposition 6.2, $\chi_L(x_n)$ is computable if n is odd because $a+b = \dim E_3 = d(d+1)/2 - d - 1 = (d+1)(d-2)/2$. That proves theorem 6.4. $\square$

6.6 The ***osp*** **case.** Let E be a supermodule of superdimension m equipped with a supersymmetric nonsingular bilinear form $<,>$. We'll say that E is a quadratic supermodule. For every endomorphism α of E, we have a endomorphism α^* defined the formula:

$$\forall x, y \in E \qquad < \alpha^*(x), y > = (-1)^{pq} < x, \alpha(y) >$$

where p is the degree of x and q is the degree of α. An endomorphism α is *antisymmetric* if $\alpha^* = -\alpha$. Let $L = osp(E)$ be the Lie superalgebra of antisymmetric endomorphisms of E. The superdimension of L is $d = m(m-1)/2$. With the same notation as before, a Casimir of L is:

$$\Omega = \frac{1}{2} \sum_{i,j} (-1)^{\partial^\circ e_j} (e_{ij} - e_{ij}^*) \otimes (e_{ji} - e_{ji}^*)$$

and with this Casimir, $t = m - 2$. The bilinear form corresponding to Ω is half the supertrace of the product.

6.7 Theorem: *There exists a graded algebra homomorphism χ_{osp} from Λ to the graded algebra $\mathbf{Z}[t, \alpha]$ with $\partial^\circ t = \partial^\circ \alpha = 1$ such that:*

For every quadratic supermodule E of superdimension m, the following diagram is commutative:

$$\begin{tikzcd} \Lambda \arrow[r, "\chi_{osp}"] \arrow[dr, "\chi_L"'] & Z[t, \alpha] \arrow[d, "\gamma"] \\ & k \end{tikzcd}$$

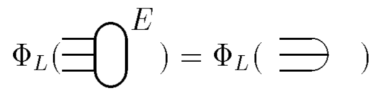
where γ is the ring homomorphism sending t to $m-2$ and α to 1.

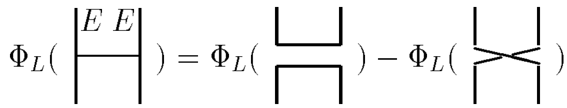
Moreover for every $n > 0$, we have:

$$t(t+\alpha)(t+2\alpha)\chi_{osp}(x_n) = \frac{(t+\alpha)(t+2\alpha)}{2} t^{n+1} + \alpha^2 (2t)^{n+1} + \frac{(t+\alpha)(t+4\alpha)}{2} (t-2\alpha)^{n+1}$$

$$+ 2 \frac{(t+2\alpha)(t+\alpha)t(t-\alpha)}{3} (4\alpha)^{n-1} + \frac{(t+4\alpha)(t+3\alpha)(t+2\alpha)(t-\alpha)}{3} (-2\alpha)^{n-1}.$$

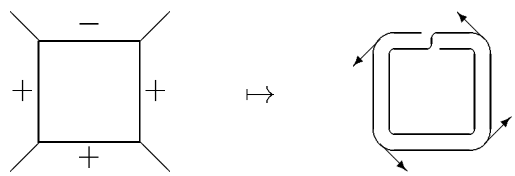
Proof: Let K be a L-coloured diagram. If we change the orientation of a component coloured by E, $\Phi_L(K)$ is unchanged. Therefore we may consider in K unoriented component coloured by E. On the other hand it is easy to see the following:





Therefore, to compute the image by Φ_L of a $(\emptyset, [n])$-diagram K, we may proceed as follows:

Let $S(K)$ be the set of functions from the set of edges of K joining two 3-valent vertices of K to ± 1. For every $\alpha \in S(K)$ denote by $\varepsilon(\alpha)$ the product of all $\alpha(a)$. If $\alpha \in S(K)$ is given we may construct a thickening of K by using the given cyclic order of edges arriving at each 3-valent vertex and making a half-twist near every edge a with negative $\alpha(a)$. So we get an unoriented surface $\Sigma_\alpha(K)$ equipped with n numbered points in its boundary and a local orientation of $\partial\Sigma_\alpha(K)$ near each of these points.



Denote by US_n the set of isomorphism classes of connected surfaces Σ equipped with n numbered points in its boundary and an orientation of $\partial\Sigma$ near each of these points. Under the connected sum, $US = US_0$ is a monoid and acts on US_n. This monoid is a graded commutative monoid generated by the disk δ, the projective plane α and the torus β and the only relation is: $\alpha\beta = \alpha^3$.

Let $\mathbf{Z}(US_n)$ be the $\mathbf{Z}$-module freely generated by US_n quotiented by the following relation:

If Σ' is obtained from Σ by changing the local orientation near one point, $\Sigma + \Sigma'$ is trivial in $\mathbf{Z}(US_n)$. Then $\mathbf{Z}[US]$ is a commutative algebra and $\mathbf{Z}(US_n)$ is a graded $\mathbf{Z}[US]$-module.

If K is connected, the sum $s(K) = \sum \varepsilon(\alpha)\Sigma_\alpha(K)$ lies in $\mathbf{Z}[US_n]$. It is easy to check that s is compatible with antisymmetric and IHX relations and induces a well-defined graded homomorphism from $F(n)$ to $\mathbf{Z}[US_n]$. Moreover this homomorphism is $\Lambda \times \mathbf{Z}[US]$-linear with respect to a character χ from Λ to $\mathbf{Z}[US] = \mathbf{Z}[\delta, \alpha, \beta]/(\alpha\beta - \alpha^3)$.

On the other hand, we have a map ∂ from US_n and $\mathbf{Z}(US_n)$ to $F(n)$ by sending each surface Σ with numbered points in $\partial\Sigma$ to the boundary $\partial\Sigma$ coloured by E with intervals added near each marked point. If K is a diagram, $\Phi_L(K)$ is equal to the sum $\Sigma\varepsilon(\alpha)\Phi_L(\partial\Sigma_\alpha(K)) = \Phi_L(\partial s(K))$. Therefore if u is an element of Λ, we have $\chi_L(u) = \chi_L(\partial\chi(u))$. Since $\chi_L \circ \partial$ is a ring homomorphism sending δ to $m = \dim E$ and α and β to 1, the character χ_L factorizes through $\mathbf{Z}[\delta, \alpha] = \mathbf{Z}[US]/(\beta - \alpha^2)$ and the first part of the theorem is proven (with $t = \delta - 2\alpha$).

To prove the last part of the theorem, it is enough to consider the case where E is a classical module of large dimension m. Then the second symmetric power $S^2(L)$ decomposes into 4 simple L-modules E_0, E_1, E_2, E_3 of dimensions 1, $(m-1)(m+2)/2$, $m(m-1)(m-2)(m-3)/4!$, $m(m+1)(m+2)(m-3)/12$. The module $E_0 \oplus E_1$ is isomorphic to S^2E and E_2 is isomorphic to $\Lambda^4 E$. The L-module L is isomorphic to $\Lambda^2 E$ and the map from S^2E to S^2L is given by:

$$x.y \mapsto \sum (x \wedge e_i).(y \wedge e_i')$$

where $\{e_i'\}$ is the dual basis of the given basis $\{e_i\}$ of E. The map from $\Lambda^4 E$ to $S^2 L$ is given by:

$$x_1 \wedge x_2 \wedge x_3 \wedge x_4 \mapsto \sum_\sigma \varepsilon(\sigma)(x_{\sigma_1} \wedge x_{\sigma_2}).(x_{\sigma_3} \wedge x_{\sigma_4})$$

where ε is the signature and σ runs over all permutations. Moreover the Casimir acts on E_0, E_1, E_2, E_3 by multiplication by 0, $2m$, $4m-16$, $4m-4$.

On the other hand, the morphism Ψ is equal to $2t$ minus the morphism induced by the Casimir. Therefore Ψ acts on modules E_i by multiplication by $2m-4=2t$, $m-4$, 4, -2 and the computation of $\chi_{osp}(x_n)$ is an easy consequence of lemma 6.5.□

Remark: The use of surfaces in the *sl*- and *osp*-cases was introduced in a slightly different way by Bar-Natan to produce weight functions [BN].

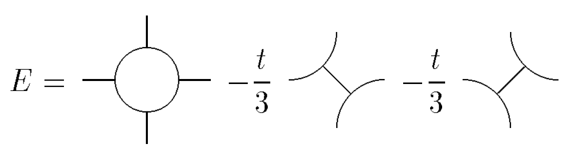
6.8 The exceptional case. Consider now a simple Lie superalgebra L over a field in characteristic 0 with a Casimir element Ω and satisfying the following conditions:

— the element t is nontrivial and

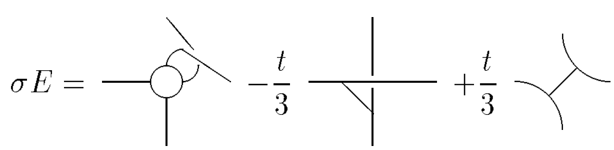
— the square of the Casimir generates the center of the enveloping algebra $\mathcal{U}$ of L in degree 4.

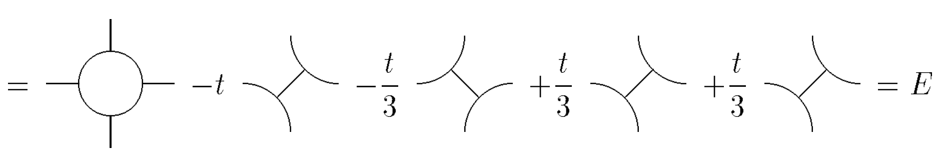
Exceptional Lie algebras E_6, E_7, E_8, F_4, G_2 satisfy these properties.

Consider the following element in $F(4) \otimes \mathbf{Q}$:



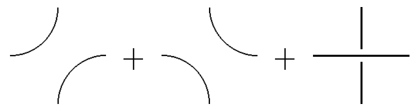
Let σ be a transposition in the symmetric group $\mathcal{S}_4$. We have:



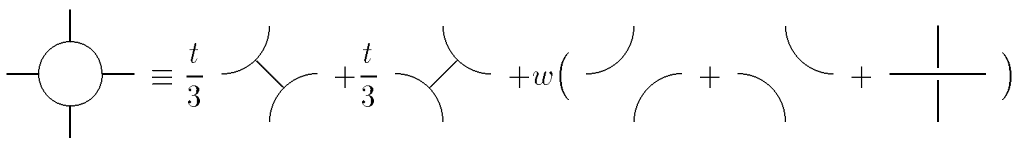


Hence E is invariant under the action of $\mathcal{S}_4$ and $\Phi_L(E) \in L^{\otimes 4}$ is L-invariant and $\mathcal{S}_4$-invariant. Because of the condition of the center of $\mathcal{U}$, the invariant submodule of $S^4 L$ is 1-dimensional and generated by the image of $\Omega \otimes \Omega$.

On the other hand the image under Φ_L of:

$E' =$ 

is also invariant under L- and $\mathcal{S}_4$-actions and goes to $3\Omega \otimes \Omega$ in $S^4 L$. Therefore $\Phi_L(E)$ is a multiple of $\Phi_L(E')$ and there exists a scalar w such that:

(*) 

where $\equiv$ represents the equality after applying Φ_L.

Let d be the superdimension of L. If we connect two consecutive free edges in diagrams above, we get the formula:

$$4t^2 = \frac{2t^2}{3} + w(d+2) \qquad \Rightarrow \qquad \frac{10t^2}{3} = w(d+2).$$

Another consequence of (*) is the following:

The morphism Ψ from $L^{\otimes 2}$ to itself satisfies the relation:

$$\Psi^2 = \frac{t}{3}\Psi + 2w$$

on the quotient $S^2 L/\Omega$.

Hence Ψ has on $S^2 L/\Omega$ only two eigenvalues λ and μ:

$$\lambda + \mu = \frac{t}{3}, \qquad \lambda\mu = -2w = -\frac{20t^2}{3(d+2)}.$$

Let a and b be the superdimensions of the corresponding eigenspaces. Because of proposition 6.2 we have:

$$0 = dt + 2t + a\lambda + b\mu$$

and $a + b$ is equal to dim $S^2 L/\Omega = d(d+1)/2 - 1 = (d-1)(d+2)/2$. Set $\alpha = \lambda t^{-1}$, $\beta = \mu t^{-1}$. Then α, β, a, b are easy to compute (if d is given) and we get:

6.9 Theorem: *Let L be one of the exceptional Lie algebras. Then:*

$$2d\chi_L(x_n) = (d + 2^{n+1} + a\alpha^{n+1} + b\beta^{n+1})t^n$$

where d, a, b, α, β are the following:

Lie algebra	d	a	α	b	β
E_6	78	650	1/2	2430	$-1/6$
E_7	133	1539	4/9	7371	$-1/9$
E_8	248	3875	2/5	27000	$-1/15$
F_4	52	324	5/9	1053	$-2/9$
G_2	14	27	5/6	77	$-1/2$

Remark: Properties above work also for $L = sl(E)$ or $L = osp(E)$ if the morphism Ψ acting on $S^2(L)/\Omega$ as only two eigenvalues, and that's the case for $sl(E)$ if $\mathrm{superdim}(E) = \pm1, \pm2, \pm3$ and for $osp(E)$ if $\mathrm{superdim}(E) = -2, -1, 0, 1, 2, 3, 8$. Moreover it's also the case for the two exceptional Lie superalgebras G(3) and F(4) with superdimensions: superdim(G(3))=3 and superdim(F(4))=8. Then the possible values of the superdimension of L are: 0,1,3,8,14,28,52,78,133,248.

6.10 The $\widetilde{\mathrm{D}}$(2,1) case. There exists an interesting Lie superalgebra depending on a parameter α called D(2,1,α). This algebra is simple and has a nonsingular bilinear supersymmetric invariant form and a Casimir element. Therefore it produces a character on Λ depending on the parameter α. Actually this algebra produces a graded character from Λ to a polynomial algebra $\mathbf{Z}[\sigma_2, \sigma_3]$.

Let A be the polynomial algebra $\mathbf{Z}[a, b, c]$ divided by the only relation $a+b+c = 0$. Consider 2-dimensional free A-modules E_1, E_2, E_3 and denote by L_i the Lie algebra $sl(E_i)$ and by X the module $E_1 \otimes E_2 \otimes E_3$. This module X is a module over the Lie algebra $L' = L_1 \oplus L_2 \oplus L_3$.

If we take isomorphisms $\Lambda^2 E_i \simeq A$, for every $x, y \in E_i$, $x \wedge y$ lies in A. On the other hand, we have a symmetric map $x \otimes y \mapsto x.y$ from $E_i \otimes E_i$ to L_i defined by:

$$\forall x, y, z \in E_i \qquad x.y(z) = x\ y \wedge z + y\ x \wedge z.$$

Then there is a morphism [,] from $(L' \oplus X)^{\otimes 2}$ to $(L' \oplus X) \otimes \mathbf{Z}[1/2]$ defined as follows: this bracket is the Lie bracket on $L_i \otimes L_i$, it is trivial on $L_i \otimes L_j$ for $i \neq j$. On $L' \otimes X$ and $X \otimes L'$ it is induced by the L'-module structure of X:

$$\forall \alpha \in L' \ \ \forall u \in X \qquad [\alpha, u] = -[u, \alpha] = \alpha u.$$

On $X \otimes X$ the bracket is defined as follows:

$$[x \otimes y \otimes y, x' \otimes y' \otimes z'] = \frac{1}{2}(a\ x.x'\ y \wedge y'\ z \wedge z' + b\ x \wedge x'\ y.y'\ z \wedge z' + c\ x \wedge x'\ y \wedge y'\ z.z').$$

With this bracket, $(L' \oplus X) \otimes \mathbf{Z}[1/2]$ is a Lie superalgebra. Define L to be the submodule of $L' \otimes \mathbf{Z}[1/2]$ generated by L and all elements of the form $\frac{1}{2}(u\alpha + v\beta + w\gamma)$ where α (resp. β, γ) is en element of L_1 (resp. L_2, L_3) congruent to the identity mod 2, and u, v, w are elements in A such that $u + v + w = 0$. It is not difficult to see that $L \oplus X$ is a Lie subalgebra of $(L' \oplus X) \otimes \mathbf{Z}[1/2]$.

The even part is L and the odd part is X. The bracket is antisymmetric (in the super sense) and the Jacobi relation holds because $a+b+c=0$. If we take a character from A to $\mathbf{C}$, we get a complex Lie superalgebra. Up to isomorphism, this algebra depends only on one parameter α and is called D(2,1,α). Here this algebra $L\oplus X$ will be denoted by $\widetilde{\mathrm{D}}(2{,}1)$.

In order to define a Casimir element in $\widetilde{\mathrm{D}}(2{,}1)$, we need some notations. Consider for each $i=1,2,3$ a basis $\{\varepsilon_{ij}\}$ of E_i and the dual basis $\{\varepsilon'_{ij}\}$ with respect to the form $\wedge$:

$$\forall x\in E_i \qquad \sum_j \varepsilon_{ij}(\varepsilon'_{ij}\wedge x)=\sum_j(x\wedge\varepsilon_{ij})\varepsilon'_{ij}=x.$$

For each i, the trace of the product is an invariant form on L_i, and, corresponding to this form, we have a Casimir type element $\omega_i=\sum_j e_{ij}\otimes e'_{ij}$. This element belongs to $L\otimes L\otimes\mathbf{Z}[1/2]$, but $2\omega_i$ lies in $L\otimes L$. We have also a Casimir element $\pi\in X\otimes X$ defined by:

$$\pi=\sum(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}).$$

6.11 Lemma: *For each $i\in\{1,2,3\}$ and $x\in E_i$, we have the following:*

$$\sum_j \varepsilon_{ij}\otimes x.\varepsilon'_{ij}=2\sum_j e_{ij}(x)\otimes e'_{ij},$$

$$\sum_j x.\varepsilon_{ij}\otimes\varepsilon'_{ij}=-2\sum_j e_{ij}\otimes e'_{ij}(x).$$

Proof: Denote by τ the trace map. For every $\alpha\in L_i$ we have:

$$\sum_j \varepsilon_{ij}\tau((x.\varepsilon'_{ij})\alpha)=\sum_j\varepsilon_{ij}(\varepsilon'_{ij}\wedge\alpha(x))+\sum_j\varepsilon_{ij}(x\wedge\alpha(\varepsilon'_{ij}))$$

$$=\alpha(x)-\sum_j\varepsilon_{ij}(\alpha(x)\wedge\varepsilon'_{ij})=2\alpha(x)=2\sum_j e_{ij}(x)\tau(e'_{ij}\alpha)$$

and that gives the first formula. The second one is obtained in the same way. □

6.12 Lemma: *Let K be the fraction field of A. Then $\widetilde{\mathrm{D}}(2{,}1)\otimes K$ has an invariant bilinear form and the corresponding Casimir element is:*

$$\Omega=-a\omega_1-b\omega_2-c\omega_3+\pi.$$

Moreover Ω belongs to $\widetilde{\mathrm{D}}(2{,}1)\otimes\widetilde{\mathrm{D}}(2{,}1)$.

Proof: Let $x\otimes y\otimes z$ be an element of X. We have:

$$x\otimes y\otimes z(\pi)=\Sigma[x\otimes y\otimes z,\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k}]\otimes(\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k})$$

$$-\Sigma(\varepsilon_{1i}\otimes\varepsilon_{2j}\otimes\varepsilon_{3k})\otimes[x\otimes y\otimes z,\varepsilon'_{1i}\otimes\varepsilon'_{2j}\otimes\varepsilon'_{3k}]$$

$$=\frac{1}{2}(aZ_1+bZ_2+cZ_3)$$

where:

$$Z_1 = \Sigma x.\varepsilon_{1i}\ y{\wedge}\varepsilon_{2j}\ z{\wedge}\varepsilon_{3k} \otimes (\varepsilon'_{1i} \otimes \varepsilon'_{2j} \otimes \varepsilon'_{3k}) - \Sigma(\varepsilon_{1i} \otimes \varepsilon_{2j} \otimes \varepsilon_{3k}) \otimes x.\varepsilon'_{1i}\ y{\wedge}\varepsilon'_{2j}\ z{\wedge}\varepsilon'_{3k}$$

and Z_2 and Z_3 are obtained from Z_1 by cyclic permutation.

Using Lemma 6.11, Z_1 is easy to compute:

$$Z_1 = \Sigma x.\varepsilon_{1i} \otimes (\varepsilon'_{1i} \otimes y \otimes z) - \Sigma(\varepsilon_{1i} \otimes y \otimes z) \otimes x.\varepsilon'_{1i}$$

$$= -2\Sigma e_{1i} \otimes (e'_{1i}(x) \otimes y \otimes z) - 2\Sigma(e_{1i}(x) \otimes y \otimes z) \otimes e'_{1i} = 2x \otimes y \otimes z(\omega_1)$$

Therefore we have:

$$x \otimes y \otimes z(\Omega) = -ax \otimes y \otimes z(\omega_1) - bx \otimes y \otimes z(\omega_2) - cx \otimes y \otimes z(\omega_3) + aZ_1 + bZ_2 + cZ_3 = 0$$

and Ω, which is clearly L-invariant, is $\widetilde{\mathrm{D}}$(2,1)-invariant.

Let α, β, γ, be elements in L_1, L_2, L_3, congruent to the identity modulo 2. Then $2\omega_1$ is congruent to $\alpha \otimes \alpha$ modulo 2 and there exists an element $u \in L_1 \otimes L_1$ such that:

$$2\omega_1 = \alpha \otimes \alpha + 2u.$$

There is also elements $v \in L_2$ and $w \in L_3$ such that:

$$2\omega_2 = \beta \otimes \beta + 2v \qquad 2\omega_3 = \gamma \otimes \gamma + 2w.$$

So we have:

$$a\omega_1 + b\omega_2 + c\omega_3 = au + bv + cw + \frac{1}{2}(a\alpha + b\beta + c\gamma) \otimes \alpha + b\beta \otimes \frac{1}{2}(\beta - \alpha) + c\gamma \otimes \frac{1}{2}(\gamma - \alpha)$$

and Ω belongs to $\widetilde{\mathrm{D}}$(2,1)$\otimes\widetilde{\mathrm{D}}$(2,1).

Because of the description of Ω, we have two K-basis of $\widetilde{\mathrm{D}}$(2,1) $\{u_i\}$ and $\{u'_i\}$ such that: $\Omega = \Sigma u_i \otimes u'_i$ and Ω is the Casimir element corresponding to a nonsingular bilinear form b. Since Ω is supersymmetric and invariant, the form b is also supersymmetric and invariant. $\square$

6.13 Theorem: *Let* $\mathbf{Z}[\sigma_2, \sigma_3]$ *be the graded subalgebra of* $\mathbf{Z}[a, b, c]/(a + b + c)$ *generated by* $\sigma_2 = ab + bc + ca$ *of degree 2 and* $\sigma_3 = abc$ *of degree 3. Then the character* χ_D *induced by* $\widetilde{\mathrm{D}}$*(2,1) equipped with the Casimir* Ω *is a graded character from* Λ *to* $\mathbf{Z}[\sigma_2, \sigma_3]$.

Moreover $\chi_D(x_n)$ *is given by the following formula (in* $\mathbf{Z}[[\sigma_2, \sigma_3]]$*):*

$$\Sigma_{n=1}^{\infty} \chi_D(x_n) = \frac{-12\sigma_3}{1 + 4\sigma_2 - 8\sigma_3}.$$

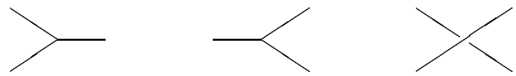
Proof: Consider a connected $(\emptyset, [p] \cup [q])$-diagram K where p and q are positive. It is not difficult to construct (by induction on the degree of K) a continuous function f from K to $[0, 1]$ such that:

— $[p] = f^{-1}(0)$ and $[q] = f^{-1}(1)$,

— f is affine on every edge of K,

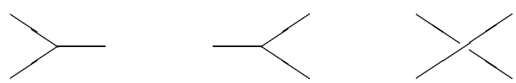
— for every 3-valent vertex $x \in K$, there are two edges a and b starting from x such that f restricted to $a \cup b$ is a local homeomorphism near x.

Therefore, using this map, we can cut K into elementary pieces and the morphism induced by K is a composite of morphisms of the form $\mathrm{Id} \otimes u \otimes \mathrm{Id}$, where u is one of the following morphisms:



Therefore if the bracket and the Casimir of a Lie superalgebra L are integral, the morphism $\Phi_L(K)$ from $L^{\otimes p}$ to $L^{\otimes q}$ is also integral. In particular if u lies in Λ, $\Phi_D(u)$ is a morphism from $\widetilde{\mathrm{D}}(2{,}1)^{\otimes 2}$ to $\widetilde{\mathrm{D}}(2{,}1)$, and $\chi_D(u)$ belongs to the algebra $A = \mathbf{Z}[a,b,c]/(a+b+c)$. Moreover the symmetric group $\mathcal{S}_3$ acts on $\widetilde{\mathrm{D}}(2{,}1)$. It permutes the E_i and the L_i and also the symbols a, b, c. For every $(\emptyset, [p] \cup [q])$-diagram K, the induced morphism $\Phi_D(K)$ commutes with this action and for every $u \in \Lambda$ the element $\chi_D(u)$ is invariant under this action and lies in $\mathbf{Z}[\sigma_2, \sigma_3]$.

On the other hand, $\widetilde{\mathrm{D}}(2{,}1)$ is a graded algebra. For each i, take a basis $\mathcal{B}_i$ of E_i in such a way that the exterior product of vectors in this basis is 1. The elements in $sl(E_i)$ with entries in $\mathbf{Z}[1/2]$ are of degree 0, the elements of X with entries in $\mathbf{Z}[1/2]$ are of degree 1 and a, b, c are of degree 2. Since the Casimir element Ω has degree 2, it is easy to see that morphims represented by the following diagrams:



have degrees 0, 2, 0. Therefore the degree of the morphism $\Phi_D(K)$ from $L^{\otimes p}$ to $L^{\otimes q}$ is $-p+q+r(K)$ where $r(K)$ is the number of 3-valent vertices of K. In particular for every $u \in \Lambda$, $\partial^\circ \Phi_D(u) = \partial^\circ \Phi_D(1) + r(u) - 1$ and: $\partial^\circ \chi_D(u) = r(u) - 1 = 2\partial^\circ(u)$.

Now, after dividing degrees in A by 2, χ_D becomes a graded character. In particular, $\chi_D(t)$ is trivial because $\mathbf{Z}[\sigma_2, \sigma_3]$ has no degree 1 element.

6.14 Lemma: *The endomorphism Ψ satisfies the following:*

$$\Psi(\omega_1) = -4a\omega_1 + \frac{3}{2}\pi, \qquad \Psi(\omega_2) = -4b\omega_2 + \frac{3}{2}\pi, \qquad \Psi(\omega_3) = -4c\omega_3 + \frac{3}{2}\pi,$$

$$\Psi(\pi) = -4(a^2\omega_1 + b^2\omega_2 + c^2\omega_3).$$

Proof: We have:

$$\Psi(\omega_1) = -a\sum [e_{1i}, e_{1j}] \otimes [e_{1j}, e_{1i}] + \sum [e_{1i}, \varepsilon_{1j} \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}] \otimes [\varepsilon'_{1j} \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}, e'_{1i}].$$

The coefficient of $-a$ in this formula is the image of the Casimir of sl_2 under the corresponding homomorphism Ψ. Then it is equal to $2\chi_{sl_2}(t)\omega_1 = 4\omega_1$, and:

$$\Psi(\omega_1) = -4a\omega_1 - \Sigma(e_{1i}(\varepsilon_{1j}) \otimes \varepsilon_{2k} \otimes \varepsilon_{3l}) \otimes (e'_{1i}(\varepsilon'_{1j}) \otimes \varepsilon'_{2k} \otimes \varepsilon'_{3l}).$$

Because of Lemma 6.11, we have:

$$\sum_{ij} e_{1i}(\varepsilon_{1j}) \otimes e'_{1i}(\varepsilon'_{1j}) = \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes \varepsilon_{1j}.\varepsilon'_{1i}(\varepsilon'_{1j})$$

$$= \frac{1}{2}\sum_{ij} \varepsilon_{1i} \otimes (\varepsilon_{1j}\ \varepsilon'_{1i}\wedge\varepsilon'_{1j} + \varepsilon'_{1i}\ \varepsilon_{1j}\wedge\varepsilon'_{1j})$$

$$= \frac{1}{2}\sum_{j} \varepsilon'_{1j} \otimes \varepsilon_{1j} + \frac{1}{2}\sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i} \sum_{j} \varepsilon_{1j}\wedge\varepsilon'_{1j}$$

$$= -\frac{1}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j} - \sum_{i} \varepsilon_{1i} \otimes \varepsilon'_{1i} = -\frac{3}{2}\sum_{j} \varepsilon_{1j} \otimes \varepsilon'_{1j}$$

and that implies the first formula. For computing $\Psi(\omega_2)$ and $\Psi(\omega_3)$, just apply a cyclic permutation.

Since Ω is the Casimir and t is zero in this case, we have:

$$0 = \Psi(\Omega) = 4a^2\omega_1 + 4b^2\omega_2 + 4c^2\omega_3 + \Psi(\pi)$$

and that proves the lemma. □

6.15 Lemma: *The module $S^2\widetilde{\mathrm{D}}(2,1)\otimes K$ decomposes into a direct sum $U_0 \oplus U_1 \oplus U_2 \oplus U_3$. The module U_0 is isomorphic to K and generated by the Casimir. The homomorphism Ψ respects this decomposition. It acts trivially on U_0. On U_1, U_2, U_3 it acts by multiplication by $2a, 2b, 2c$ respectively.*

Proof: Let V_0 be the sub K-module of $S^2\widetilde{\mathrm{D}}(2,1)\otimes K$ generated by $\omega_1, \omega_2, \omega_3, \pi$. The morphism Ψ induces an endomorphism of V_0. The matrix of this endomorphism in the basis $(2\omega_1, 2\omega_2, 2\omega_3, \pi)$ is:

$$\begin{pmatrix} -4a & 0 & 0 & -2a^2 \\ 0 & -4b & 0 & -2b^2 \\ 0 & 0 & -4c & -2c^2 \\ 3 & 3 & 3 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are $0, 2a, 2b, 2c$ and corresponding eigenvectors are:

$$\Omega = -a\omega_1 - b\omega_2 - c\omega_3 + \pi,$$

$$2a(b-c)\omega_1 + 6b^2\omega_2 - 6c^2\omega_3 - 3(b-c)\pi,$$

$$2b(c-a)\omega_2 + 6c^2\omega_3 - 6a^2\omega_1 - 3(c-a)\pi,$$

$$2c(a-b)\omega_3 + 6a^2\omega_1 - 6b^2\omega_2 - 3(a-b)\pi.$$

Let F_p be the simple sl_2-module of dimension $p+1$. This module is the symmetric power S^pF_1 and $F_2 = sl_2$. Denote by $[p, q, r]$ the isomorphism class of the L-module $F_p \otimes F_q \otimes F_r$. These elements form a basis of the Grothendieck algebra $\mathrm{Rep}(L)$ of representations of L. In this algebra we have:

$$[L] = [2, 0, 0] + [0, 2, 0] + [0, 0, 2], \qquad [X] = [1, 1, 1],$$

$$[S^2L] = 3[0, 0, 0] + [4, 0, 0] + [0, 4, 0] + [0, 0, 4] + [2, 2, 0] + [2, 0, 2] + [0, 2, 2],$$
$$[\Lambda^2X] = [0, 0, 0] + [2, 2, 0] + [2, 0, 2] + [0, 2, 2],$$
$$[L \otimes X] = 3[1, 1, 1] + [3, 1, 1] + [1, 3, 1] + [1, 1, 3].$$

The module V_0 is the submodule $3[0, 0, 0] + [0, 0, 0]$ of $S^2\tilde{\mathrm{D}}(2{,}1)$. Set $V_0' = V_0$ and define by induction submodules V_p' to be the image of $X \otimes V_{p-1}'$ under the action map $L \otimes S^2\tilde{\mathrm{D}}(2{,}1) \to S^2\tilde{\mathrm{D}}(2{,}1)$. Then set: $V_p = V_0' + \ldots + V_p'$. For every $p \geq 0$, V_p is a L-module. It is not difficult to prove the following:

$$[V_0] = 4[0, 0, 0], \qquad [V_1] = 4[0, 0, 0] + 3[1, 1, 1],$$

$$[V_2] = 4[0, 0, 0] + 3[1, 1, 1] + [2, 2, 0] + [2, 0, 2] + [0, 2, 2] \ \Rightarrow \Lambda^2X \in V_2,$$
$$[V_3] = 4[0, 0, 0] + 3[1, 1, 1] + [2, 2, 0] + [2, 0, 2] + [0, 2, 2] + [3, 1, 1] + [1, 3, 1] + [1, 1, 3],$$
$$\Rightarrow \Lambda^2X \oplus L \otimes X \in V_3.$$

Then there is a unique L-submodule W of $S^2L \subset S^2\tilde{\mathrm{D}}(2{,}1)$ such that $V_3 \oplus W = S^2\tilde{\mathrm{D}}(2{,}1)$. If V is the $\tilde{\mathrm{D}}(2{,}1)$-submodule of $S^2\tilde{\mathrm{D}}(2{,}1)$ generated by V_0, the module $S^2\tilde{\mathrm{D}}(2{,}1)/V$ is a quotient of W and then has no odd degree component. Therefore this module is trivial and $S^2\tilde{\mathrm{D}}(2{,}1)$ is generated by V_0 as a $\tilde{\mathrm{D}}(2{,}1)$-module, and that implies that $S^2\tilde{\mathrm{D}}(2{,}1)$ is the direct sum of $\tilde{\mathrm{D}}(2{,}1)$-modules generated by the eigenvectors above and the lemma is proven. $\square$

Denote by G_i, $i = 0, 1, 2, 3$, the projectors onto U_i. If σ is the map $x \otimes y \mapsto (-1)^{\partial^\circ x \partial^\circ y} y \otimes x$, G_1 is the following:

$$G_1 = \frac{1}{2}(1 + \sigma)\frac{\Psi(\Psi - 2b)(\Psi - 2c)}{2a(2a - 2b)(2a - 2c)}$$

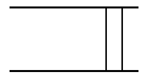
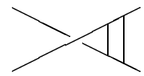
and G_1 is homogenous of degree 0. The same holds for G_2 and G_3.

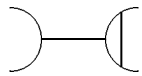
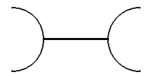
We have the following:

$$\sum_{p=0}^{\infty} \Psi^p = (\sum_{p=0}^{\infty} \Psi^p)\frac{1 - \sigma}{2} + (\sum_{p=0}^{\infty} \Psi^p)(G_0 + G_1 + G_2 + G_3)$$

$$= (\sum_{p=0}^{\infty} \Psi^p)\frac{1 - \sigma}{2} + G_0 + \frac{1}{1 - 2a}G_1 + \frac{1}{1 - 2b}G_2 + \frac{1}{1 - 2c}G_3$$

If we identify any diagram to its corresponding morphism between $\widetilde{\mathrm{D}}(2{,}1)$-modules, we get:

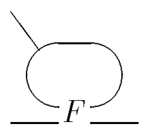
$\Psi^2(1-\sigma) =$  $-$ 

$=$  $= t$  $= 0$

Hence:

$$\sum_{p=0}^{\infty}\Psi^p = \frac{1-\sigma}{2} + \Psi\frac{1-\sigma}{2} + G_0 + \frac{1}{1-2a}G_1 + \frac{1}{1-2b}G_2 + \frac{1}{1-2c}G_3.$$

For every endomorphism F of $\widetilde{\mathrm{D}}(2{,}1)^{\otimes 2}$, denote by $T(F)$ the morphism from $\widetilde{\mathrm{D}}(2{,}1)^{\otimes 2}$ to $\widetilde{\mathrm{D}}(2{,}1)$ represented by:

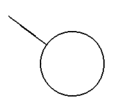


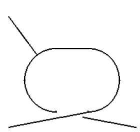
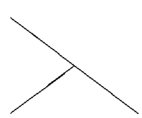
This morphism is the Lie bracket multiplied by a scalar $\Theta(F)$. For every $p \geq 0$, $\chi_D(x_p) = \Theta(\Psi^p)$. Hence:

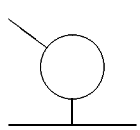
$$\sum_{p=0}^{\infty}\chi_D(x_p) = \frac{1}{2}\Theta(1) - \frac{1}{2}\Theta(\sigma) + \frac{1}{2}\Theta(\Psi) - \frac{1}{2}\Theta(\Psi\sigma)$$

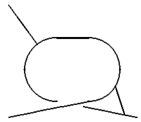
$$+\Theta(G_0) + \frac{1}{1-2a}\Theta(G_1) + \frac{1}{1-2b}\Theta(G_2) + \frac{1}{1-2c}\Theta(G_3)$$

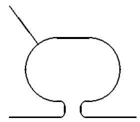
We have:

$T(1) =$  $= 0 \qquad \Rightarrow \qquad \Theta(1) = 0,$

$T(\sigma) =$  $= -$  $\qquad \Rightarrow \qquad \Theta(\sigma) = -1,$

$T(\Psi) =$  $= 0 \qquad \Rightarrow \qquad \Theta(\Psi) = 0,$

$T(\Psi\sigma) =$  $= 0 \qquad \Rightarrow \qquad \Theta(\Psi\sigma) = 0,$

$T(G_0) =$  $\qquad \Rightarrow \qquad \Theta(G_0) = 1,$

and there exist three elements u, v, w in K of degree 0 such that:

$$\sum_{p=0}^{\infty} \chi_D(x_p) = \frac{3}{2} + \frac{u}{1-2a} + \frac{v}{1-2b} + \frac{w}{1-2c}.$$

Therefore there exist three elements $z_0, z_1, z_2 \in K$ of degree 0, 1, 2 (in the new graduation with $\partial^\circ a = \partial^\circ b = \partial^\circ c = 1$) such that:

$$\sum_{p=0}^{\infty} \chi_D(x_p) = \frac{3}{2} + \frac{z_0 + z_1 + z_2}{1 + 4\sigma_2 - 8\sigma_3}.$$

But $\chi_D(x_0) = \chi_D(x_1) = \chi_D(x_2) = 0$. Then:

$$0 = \frac{3}{2} + z_0, \qquad 0 = z_1, \qquad 0 = z_2 - 4\sigma_2 z_0,$$

and:

$$\sum_{p=0}^{\infty} \chi_D(x_p) = \frac{3}{2} + \frac{-3/2 - 6\sigma_2}{1 + 4\sigma_2 - 8\sigma_3} = \frac{-12\sigma_3}{1 + 4\sigma_2 - 8\sigma_3}.$$

$\square$

Remark: There exist two exceptional Lie superalgebras with a Casimir element, G(3) and F(4) [Kc], but the morphism Ψ has the same eigenvalues and the same superdimensions of eigenspaces than sl_2 and sl_3. Therefore on the subalgebra of Λ generated by the x_n's these two Lie superalgebras induce the same character as sl_2 and sl_3. Presumably, they don't give any new character.

There exist also two families of simple Lie superalgebras with a Casimir element: $psl(n,n)$ and the Hamiltonian algebra $H(n)$ for $n > 4$ [Kc]. And that is a complete list of simple Lie superalgebras having a nontrivial Casimir element.

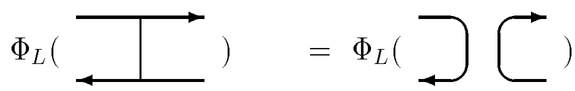
The Hamiltonian algebra $L = H(n)$ has for $n > 4$ the following property: it has a $\mathbf{Z}$-graduation compatible with the Lie bracket, and the Casimir has a nonzero degree. Therefore for any element $u \in \Lambda$ of positive degree, the induced element $\chi_L(u)$ has a nonzero degree. But it is an element of the coefficient field. Hence the character χ_L is the augmentation character.

In the case of the Lie superalgebra $psl(n,n)$, we can proceed as follows:

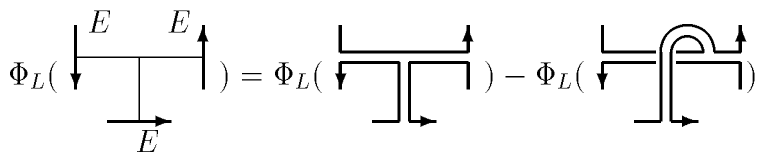
Consider a k-supermodule E of superdimension m with a homogenous basis $\{e_i\}$, and the Lie superalgebra $L = gl(E)$ with the standard associated basis $\{e_{ij}\}$. Set:

$$\Omega = \sum (-1)^{\partial^\circ e_j} e_{ij} \otimes e_{ji}$$

This element Ω is the Casimir element associated to the standard form on L, and it is easy to show the following formula (see the proof of theorem 6.4):



where oriented parts of these diagrams are coloured by E, and that implies:



Therefore, it is possible to compute Φ_L using surfaces like in the sl-case. If K is the diagram corresponding to $1 \in \Lambda$ and if u is an element of Λ, we have:

$$\Phi_L(uK) = P(m,1)\Phi_L(K) \qquad \text{where } P(t,\beta) = \chi_{sl}(u)$$

If the superdimension of E is zero, the image of Ω in $L/\mathrm{Id} \otimes L/\mathrm{Id}$ lies in $L' \otimes L'$ where $L' = psl(E)$, and we get:

$$\Phi_{L'}(uK) = P(0,1)\Phi_{L'}(K).$$

Then the character associated to $psl(E)$ is the composite

$$\Lambda \stackrel{\chi_{sl}}{\to} \mathbf{Z}[t,\beta] \stackrel{\gamma}{\to} k$$

where γ is the ring homomorphism sending t to 0 and β to 1. So the character associated to $psl(n,n)$ is a consequence of the character χ_{sl}. Therefore it doesn't depend on the dimension n of the even part of E, and to compute it, it is enough to take $n = 1$. In this case, the bracket is trivial on $psl(E)$ and the character $\chi_{psl(n,n)}$ is the augmentation character, as in the Hamiltonian case.

7. Independence of characters

In last section, we constructed eight characters χ_i, $i = 1 \ldots 8$ corresponding to families sl, osp, E_6, E_7, E_8, F_4, G_2 and $\widetilde{\mathrm{D}}(2,1)$. These characters are graded algebra homomorphisms from Λ to A_i, where $A_1 = \mathbf{Z}[t,\beta]$, $A_2 = \mathbf{Z}[t,\alpha]$, $A_3 = A_4 = A_5 = A_6 = A_7 = \mathbf{Q}[t]$, $A_8 = \mathbf{Z}[\sigma_2,\sigma_3]$.

7.1 Theorem: *For every $i = 1 \ldots 8$, there exists an element $u \in \Lambda$ such that $\chi_i(u) \neq 0$ and $\chi_j(u)$ vanishes for every $j \neq i$.*

7.2 Corollary: *For every $i = 1 \ldots 8$, there exists a primitive element u in the module of chord diagrams $\mathcal{A}$ such that u is nontrivial in $\mathcal{A} \otimes \mathbf{Q}$ and for every simple*

Lie superalgebra L of a type different from i and every L-module M, the invariant of u induced by (L, M) is trivial.

7.3 Remark: It was conjectured in [BN] that every element in $\mathcal{A}$ is detected by invariants coming from Lie algebras in series A, B, C, D. This conjecture is false. There is a weaker conjecture saying that invariants coming from simple Lie algebras detect every element in $\mathcal{A}$. That's also false because of the Lie superalgebra $\widetilde{\mathrm{D}}(2,1)$. Actually we have the following result:

7.4 Theorem: *There exists a primitive element of degree 23 in $\mathcal{A}$ which is rationally nontrivial and killed by every weight function obtained by a semisimple Lie (super)algebra and a finite dimensional representation.*

Proof of Theorem 7.1: For every $i = 1 \dots 8$, there is an element of smallest degree in Λ killed by χ_i.

The *sl* **case:** Consider the character $\chi_1 = \chi_{sl}$ from Λ to $\mathbf{Z}[t, \beta]$. We have:

$$\chi_1(x_3) = t^3 + 12t\beta, \qquad \chi_1(x_5) = t^5 + 32t^3\beta + 48t\beta^2,$$

and the element

$$X_1 = 3tx_5 - 6t^3x_3 - x_3^2 + 4t^6$$

is killed by χ_1.

The *osp* **case:** For the character $\chi_2 = \chi_{osp}$ from Λ to $\mathbf{Z}[t, \alpha]$, we have:

$\chi_2(x_3) = t^3 - 3t^2\alpha + 30t\alpha^2 - 24\alpha^3,$

$\chi_2(x_5) = t^5 - 5t^4\alpha + 80t^3\alpha^2 - 184t^2\alpha^3 + 408t\alpha^4 - 288\alpha^5,$

$\chi_2(x_7) = t^7 - 7t^6\alpha + 294t^5\alpha^2 - 844t^4\alpha^3 + 1608t^3\alpha^4 - 2128t^2\alpha^5 + 4576t\alpha^6 - 3456\alpha^7,$

$\chi_2(x_9) = t^9 - 9t^8\alpha + 1092t^7\alpha^2 - 3328t^6\alpha^3 + 7440t^5\alpha^4 - 13216t^4\alpha^5 + 18048t^3\alpha^6$
$\quad -17920t^2\alpha^7 + 55680t\alpha^8 - 47616\alpha^9,$

and:

$$\begin{aligned} X_2 = \quad & -108x_9 + 3267x_7t^2 - 1920x_5x_3t - 20913x_5t^4 \\ & +372x_3^3 + 8906x_3^2t^3 + 13748x_3t^6 - 3352t^9 \end{aligned}$$

is killed by χ_2.

Exceptional cases: For $i = 3 \dots 7$, $\chi_i(x_3)$ has following values:

$$\mathrm{E}_6 : \frac{7}{8}t^3, \qquad \mathrm{E}_7 : \frac{64}{81}t^3, \qquad \mathrm{E}_8 : \frac{11}{15}t^3, \qquad \mathrm{F}_4 : \frac{79}{81}t^3, \qquad \mathrm{G}_2 : \frac{41}{24}t^3.$$

Therefore we may take:

$$X_3 = 8x_3 - 7t^3, \qquad X_4 = 81x_3 - 64t^3, \qquad X_5 = 15x_3 - 11t^3$$

$$X_6 = 81x_3 - 79t^3, \qquad X_7 = 24x_3 - 41t^3,$$

and X_i is killed by χ_i for each $i = 3 \dots 7$.

It is also possible to take the following element:

$$X_e = 45tx_5 - 18x_3^2 - 71t^3x_3 + 32t^6.$$

We can express it in term of the dimension d of the Lie algebra, and every exceptional character kill this element.

Super case: The character χ_8 kills the element t. Then we set: $X_8 = t$. $\square$

For $i \neq j$, the element $\chi_i(X_j)$ is nontrivial. That's easy to check for $j = 3 \ldots 8$. For $j = 1$ or 2, it's better to use the following general fact:

7.5 Lemma: *Let χ_i be one of the previous characters. Then there exist two elements u and v of degree 2 and 3 in A_i such that the following holds for every $n \geq 0$:*

$$\chi_i(x_{n+3}) = t\chi_i(x_{n+2}) + u(2\chi_i(x_{n+1}) - t^{n+1}) + v(2\chi_i(x_n) - t^n - 2(2t)^n).$$

The values of u and v are the following:

$$\begin{array}{rcc}
sl: & u = 2\beta & v = -2t\beta \\
osp: & u = -t\alpha + 6\alpha^2 & v = -4t\alpha^2 + 8\alpha^3 \\
\mathrm{E}_6: & u = -5t^2/72 & v = -t^3/36 \\
\mathrm{E}_7: & u = -7t^2/81 & v = -4t^3/243 \\
\mathrm{E}_8: & u = -22t^2/225 & v = -2t^3/225 \\
\mathrm{F}_4: & u = -4t^2/81 & v = -10t^3/243 \\
\mathrm{G}_2: & u = 7t^2/72 & v = -5t^3/36 \\
\widetilde{\mathrm{D}}(2,1): & u = -2\sigma_2 & v = 4\sigma_3
\end{array}$$

If L is any exceptional Lie algebra of dimension d, one has:

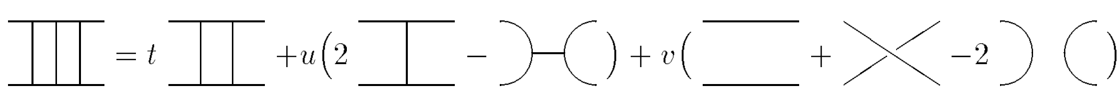
$$u = w - \frac{t^2}{9} \qquad v = -\frac{2wt}{3} \qquad \textit{where} \quad w = \frac{10t^2}{3(d+2)}.$$

Proof: Let L be a simple Lie superalgebra equipped with a Casimir element. The corresponding operator Ψ from $L^{\otimes 2}$ to itself has three eigenvalues λ, μ, ν on S^2L/Ω if L is of type 1, 2 or 8 (sl, osp or $\widetilde{\mathrm{D}}$(2,1)). And in each of these cases the sum $\lambda + \mu + \nu$ is equal to t. In cases $3, \ldots, 7$, L is an exceptional Lie algebra and Ψ has only two eigenvalues λ, μ on S^2L/Ω. Moreover: $\lambda + \mu = t/3$. Then we may add a 0-dimensional eigenspace corresponding to an eigenvalue $\nu = 2t/3$. Hence, in each cases, Ψ has three eigenvalues λ, μ, ν on S^2L/Ω and $\lambda + \mu + \nu = t$.

Set:

$$\lambda\mu + \mu\nu + \nu\lambda = -2u \qquad \lambda\mu\nu = 2v$$

Then $\Psi^3 - t\Psi^2 - 2u\Psi - 2v$ acts trivially on S^2L/Ω. Since we know the action of Ψ on $\Lambda^2 L$ and on the Casimir, we get the following equality after applying the functor Φ_L:

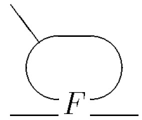


The equality is easy to check on S^2L/Ω and $\Lambda^2 L$. On the submodule generated by Ω, it's a consequence of the formula:

$$2t^3 = 2ut + v(1 - \mathrm{superdim} L),$$

which is always true. Then it's enough to multiply this formula by Ψ^n and apply the transformation:

$$F \qquad \mapsto$$



and we get the desired formula. $\square$

With this formula, it is possible to express $\chi(x_i)$ in $\mathbf{Z}[t,u,v]$ for every character χ corresponding to a simple Lie superalgebra equipped with a Casimir:

$\chi(x_0) = 0$,
$\chi(x_1) = 2t$,
$\chi(x_2) = t^2$,
$\chi(x_3) = t^3 + 3tu - 3v$,
$\chi(x_4) = t^4 + 4t^2u - 4tv$,
$\chi(x_5) = t^5 + 5t^3u - 11t^2v + 6tu^2 - 6uv$,
$\chi(x_6) = t^6 + 6t^4u - 26t^3v + 14t^2u^2 - 8tuv - 6v^2$,
$\chi(x_7) = t^7 + 7t^5u - 57t^4v + 24t^3u^2 - 22t^2uv + 12tu^3 - 14tv^2 - 12u^2v$,
$\chi(x_8) = t^8 + 8t^6u - 120t^5v + 36t^4u^2 - 64t^3uv + 40t^2u^3 - 36t^2v^2 - 16tu^2v - 24uv^2$,
$\chi(x_9) = t^9 + 9t^7u - 247t^6v + 50t^5u^2 - 166t^4uv + 88t^3u^3 - 88t^3v^2 - 32t^2u^2v + 24tu^4 - 68tuv^2 - 24u^3v - 12v^3$.

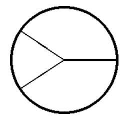
It is easy now to compute elements $\chi(X_i)$ in term of polynomials in $\mathbf{Z}[t,u,v]$:

$\chi(X_1) = -9t^4u - 9t^3v + 9t^2u^2 - 9v^2 = -9(tu+v)(t^3 - tu + v)$,
$\chi(X_2) = -648t^6v - 2292t^5u^2 - 5184t^4uv + 5184t^3u^3 - 9396t^3v^2 + 3240t^2u^2v - 2592tu^4 + 2916tuv + 2592u^3v - 8748v^3$
$\qquad = -324(t^3 - tu + v)(2t^3v + 8t^2u^2 + 18tuv - 8u^3 + 27v^2)$,
$\chi(X_3) = t^3 + 24(tu - v)$,
$\chi(X_4) = 17t^3 + 243(tu - v)$,
$\chi(X_5) = 4t^3 + 45(tu - v)$,
$\chi(X_6) = 2t^3 + 243(tu - v)$,
$\chi(X_7) = -17t^3 + 72(tu - v)$,
$\chi(X_e) = -12t^6 - 96t^4u - 174t^3v + 108t^2u^2 - 54tuv - 162v^2$
$\qquad = -6(t^3 - tu + v)(2t^3 + 18tu + 27v)$
$\chi(X_8) = t$.

The element $t^3 - tu + v$ is never zero and the three elements $tu+v$, $2t^3v + 8t^2u^2 + 18tuv - 8u^3 + 27v^2$, $2t^3 + 18tu + 27v$ vanish only in cases sl, osp and exceptional respectively. Therefore $\chi_i(X_j)$ vanishes if and only if $i = j$ and $\chi_i(X_e)$ vanishes only for $i = 3\ldots 7$.

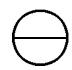
Denote by U_i the product in Λ of all X_j, $j \neq i$. Since each algebra A_i is an integral domain, U_i is killed by every χ_j, $j \neq i$, but $\chi_i(U_i)$ is nontrivial. That proves theorem 7.1. $\square$

Proof of Corollary 7.2: Let u be the following primitive element of $\mathcal{A}_2$:



Then the map $\lambda \mapsto \lambda u$ is a rational injection from Λ to the module $\mathcal{P}$ of primitives of $\mathcal{A}$ (see Corollary 4.7). Hence for every $i = 1 \ldots 8$, the element $U_i u$ is a primitive element in $\mathcal{A}$, rationally nontrivial and for every $j \neq i$, $\chi_j(U_i) = 0$. Therefore for every simple Lie superalgebra L equipped with a Casimir element, which is not of type i, and every L-module M, the corresponding invariant of $U_i u$ is trivial too. If $i = 8$, we may as well replace U_8 by $U'_8 = X_1 X_2 X_e$. We get the same result but $\partial^\circ U'_8 = 21 < \partial^\circ U_8 = 30$. $\square$

Proof of Theorem 7.4: Consider the element $U'_8 u$. It's a primitive element in $\mathcal{A}$ of degree 23 and it's rationally nontrivial. Let L be a simple Lie superalgebra equipped with a Casimir element. If L is not of type 8, the corresponding invariant of $U'_8 u$ is trivial. If L is of type 8 (i.e. $L = \widetilde{\mathrm{D}}(2,1)$), we have:

$\Phi_L(U'_8 u) = \chi_8(U'_8)\Phi_L(u) = \chi_8(U'_8)\chi_8(t)\Phi_L($  $)$

But $\chi_8(t) = 0$. Therefore $U'_8 u$ is killed by Φ_L.

If $L = \oplus L_i$ is semisimple, $\Phi_L(U'_8 u) = \sum \Phi_{L_i}(U'_8 u) = 0$ because $U'_8 u$ is primitive. $\square$

Remark: It is possible to prove that the morphism from $\mathbf{Q}[t, x_3, x_5, \ldots]$ to $\Lambda \otimes \mathbf{Q}$ is an isomorphism in degree < 8 (essentially because the dimension of $\mathcal{P}$ is known in degree < 10). But the sum of all characters χ_i from $\mathbf{Q}[t, x_3, x_5, \ldots]$ to the direct sum of algebras $A_i \otimes \mathbf{Q}$ is injective in degree < 10 and its kernel is generated (over $\mathbf{Q}$) by the following polynomial:

$$P = -9x_3x_7 + 36t^3x_7 + 9x_5^2 + 9t^2x_3x_5 - 252t^5x_5 - 14tx_3^3 + 101t^4x_3^2 + 152t^7x_3 - 32t^{10}$$

Therefore the morphism from $\mathbf{Q}[t, x_3, x_5, \ldots]$ to $\Lambda \otimes \mathbf{Q}$ is injective in degree < 10, and the polynomial P is the first possible candidat for a polynomial in the odd x_i''s which is trivial in $\Lambda \otimes \mathbf{Q}$.

Conjecture: *The morphism from $\mathbf{Q}[t, x_3, x_5, \ldots]$ to $\Lambda \otimes \mathbf{Q}$ is an isomorphism.*

The first possible counterexample of this conjecture is the polynomial P above. Then the first question is:

Question: Is P trivial or not in $\Lambda\Gamma$

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