Advances in Mathematics **171**, 59–85 (2002)
doi:10.1006/aima.2002.2071

Triality, Exceptional Lie Algebras and Deligne Dimension Formulas

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Communicated by Andrei Zelevinski

Received November 14, 2001; accepted December 20, 2001

We give a computer-free proof of the Deligne, Cohen and de Man formulas for the dimensions of the irreducible $\mathfrak{g}$-modules appearing in $\mathfrak{g}^{\otimes k}$, $k \leqslant 4$, where $\mathfrak{g}$ ranges over the exceptional complex simple Lie algebras. We give additional dimension formulas for the exceptional series, as well as uniform dimension formulas for other representations distinguished by Freudenthal along the rows of his magic chart. Our proofs use the triality model of the magic square, which we review and present a simplified proof of its validity. We conclude with some general remarks about obtaining "series" of Lie algebras in the spirit of Deligne and Vogel. © 2002 Elsevier Science (USA)

1. INTRODUCTION

The goal of this paper is to give a partial explanation to some astonishing observations made by Deligne about the exceptional complex simple Lie algebras [7]. Deligne, following a remark of Vogel, noticed that the tensor powers $\mathfrak{g}^{\otimes k}$, for $\mathfrak{g}$ an exceptional complex simple Lie algebra, decomposed uniformly into irreducible $\mathfrak{g}$-modules when $k \leqslant 4$. Parametrizing the exceptional series $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{g}_2, \mathfrak{d}_4, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ by the inverse Coxeter number

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0001-8708/02 $35.00

λ, he, together with Cohen and de Man, gave the dimensions of the corresponding irreducible modules in terms of rational functions of λ. These rational functions, computed by Cohen *et al.* [5], had the "miraculous" property that both the numerators and denominators were products of *linear* functions of λ.

Inspired by the work of Freudenthal and Tits, we thought it might be interesting to parametrize the exceptional series by $a = \dim_{\mathbb{C}}\mathbb{A}$, where $\mathbb{A}$ is, respectively, the complexification of $0, \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ for the last five algebras in the exceptional series (so $a = 0, 1, 2, 4, 8$). A first indication that this might be fruitful was the simple relation $\lambda = -\frac{2}{a+2}$. The parameter a simplified the Deligne dimension formula because every time a power of λ appears in the denominator (which is always), its contribution to the degree of the denominator is erased upon the change of variable, so that using a, the denominators have lower degree and the numerators the same degree.

The presence of only linear forms in the Deligne dimension formulas also suggests that one should attempt to apply the Weyl dimension formula in a uniform way, which is what we have done.

To do this, we had to find a suitable variant of the Vinberg construction of the exceptional Lie algebras in terms of normed division algebras. The construction we use highlights the *triality* principle, since we put a natural Lie algebra structure on the direct sum:

$$\begin{tikzcd}[row sep=small] & & \mathbb{A}_3\otimes\mathbb{B}_3 \\ \mathfrak{g}(\mathbb{A},\mathbb{B}) = \mathbb{A}_1\otimes\mathbb{B}_1 \arrow[r, dash] & \mathfrak{t}(\mathbb{A})\times\mathfrak{t}(\mathbb{B}) \arrow[ur, dash] \arrow[dr, dash] & \\ & & \mathbb{A}_2\otimes\mathbb{B}_2 \end{tikzcd}$$

where $\mathfrak{t}(\mathbb{A})$ is a certain triality algebra associated to $\mathbb{A}$. This structure was actually discovered by Allison [1], and rediscovered by Barton and Sudbery [2], Dadok and Harvey [6], and ourselves. We give a much more direct and simple proof, which was also obtained independently by Dadok and Harvey [6].

All this leads to a simple description of the exceptional root systems, the key point for the dimension formulas being that the roots of $\mathfrak{g}(\mathbb{A},\mathbb{O})$ are naturally partitioned into intervals whose endpoints are linear functions of a. This allows one to explicitly write down infinite series of formulas generalizing those of Deligne, Cohen and de Man, see Theorem 3.3. For example, specializing to just Cartan powers of the adjoint representation we obtain:

PROPOSITION 1.1. *Let* $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, *with* $a = -\frac{4}{3}, -1, -\frac{2}{3}, 0, 1, 2, 4, 8$, *respectively. Then for all* $k \geqslant 0$,

$$\dim \mathfrak{g}^{(k)} = \frac{3a+2k+5}{3a+5} \frac{\binom{k+2a+3}{k}\binom{k+\frac{5a}{2}+3}{k}\binom{k+3a+4}{k}}{\binom{k+\frac{a}{2}+1}{k}\binom{k+a+1}{k}}.$$

Caution. By definition, our binomial coefficients $\binom{k+x}{k} = (1+x)\cdots(k+x)/k!$ *are rational polynomials of degree k in x. They are not equal to zero when x is not a non-negative integer.*

The perspective also naturally uncovers the representations distinguished by Freudenthal and dimension formulas for their Cartan powers, see Theorems 4.3 and 5.3. In particular, it leads to new models for the standard representations in the second and third rows of Freudenthal's magic square.

In a companion paper to this one [14], we discuss the decomposition formulas of Deligne and Vogel from a geometric perspective. We are able to account for nearly all the factors that appear in their decompositions using elementary algebraic geometry. This paper is the fourth in a series exploring connections between representation theory and the projective geometry of rational homogeneous varieties (see also [11–13]).

2. TRIALITY AND THE TITS–VINBERG CONSTRUCTION

For $\mathbb{A}$ a normed algebra over a field k, let

$$T(\mathbb{A}) = \{\theta = (\theta_1, \theta_2, \theta_3) \in SO(\mathbb{A})^3, \quad \theta_3(xy) = \theta_1(x)\theta_2(y)\ \forall x, y \in \mathbb{A}\}.$$

There are three natural actions of $T(\mathbb{A})$ on $\mathbb{A}$ corresponding to its three projections on $SO(\mathbb{A})$, and we denote these representations by $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$.

If $\mathbb{A}$ is a real Cayley algebra, it is a classical fact that $T(\mathbb{A})$ is an algebraic group of type D_4. In this case the representations $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ are non-equivalent and they are exchanged by the outer automorphism t of $T(\mathbb{A})$ of order 3 defined by $t.\theta = (\theta_2, \theta_3, \theta_1)$. This is the famous *triality principle*, encoded in the triple symmetry of the Dynkin diagram for D_4. For the other real normed division algebras $\mathbb{A}$, we get the following types for the

Lie algebra $\mathfrak{t}(\mathbb{A})$ of $T(\mathbb{A})$, see [2]:

$$\begin{array}{cccc} \mathfrak{t}(\mathbb{R}) & \mathfrak{t}(\mathbb{C}) & \mathfrak{t}(\mathbb{H}) & \mathfrak{t}(\mathbb{O}). \\ 0 & \mathbb{R}^2 & \mathfrak{so}_3 \times \mathfrak{so}_3 \times \mathfrak{so}_3 & \mathfrak{so}_8 \end{array}$$

Now let $\mathbb{A}$ and $\mathbb{B}$ be two normed algebras. We define on

$$\mathfrak{g} = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus (\mathbb{A}_1 \otimes \mathbb{B}_1) \oplus (\mathbb{A}_2 \otimes \mathbb{B}_2) \oplus (\mathbb{A}_3 \otimes \mathbb{B}_3)$$

a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebra structure by the following conditions:

- $\mathfrak{g}_0 = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B})$;
- the bracket of an element of $\mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B})$ with one of $\mathbb{A}_i \otimes \mathbb{B}_i$ is given by the actions of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_i$ and $\mathfrak{t}(\mathbb{B})$ on $\mathbb{B}_i$, that is

$$[\theta^{\mathbb{A}}, u_i \otimes v_i] = \theta_i^{\mathbb{A}}(u_i) \otimes v_i, \qquad [\theta^{\mathbb{B}}, u_i \otimes v_i] = u_i \otimes \theta_i^{\mathbb{B}}(v_i);$$

- the bracket of two elements in $\mathbb{A}_i \otimes \mathbb{B}_i$ is given by the natural map $\Lambda^2(\mathbb{A}_i \otimes \mathbb{B}_i) = \Lambda^2\mathbb{A}_i \otimes S^2\mathbb{B}_i \oplus S^2\mathbb{A}_i \otimes \Lambda^2\mathbb{B}_i \to \Lambda^2\mathbb{A}_i \oplus \Lambda^2\mathbb{B}_i \to \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B})$, where the first arrow follows from the quadratic forms given on $\mathbb{A}_i$ and $\mathbb{B}_i$, and the second arrow is dual to the map $\mathfrak{t}(\mathbb{A}) \to \Lambda^2\mathbb{A}_i \subset \mathrm{End}(\mathbb{A}_i)$ (and similarly for $\mathbb{B}$) prescribing the action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_i$ (which, by definition, preserves the quadratic form on $\mathbb{A}_i$). Here duality is taken with respect to a $\mathfrak{t}(\mathbb{A})$-invariant quadratic form on the reductive algebra $\mathfrak{t}(\mathbb{A})$, and the quadratic form on $\Lambda^2\mathbb{A}_i$ induced by that on $\mathbb{A}_i$;
- finally, the bracket of an element of $\mathbb{A}_i \otimes \mathbb{B}_i$ with one of $\mathbb{A}_j \otimes \mathbb{B}_j$, for $i \neq j$, is given by the following rules, with obvious notations:

$$[u_1 \otimes v_1, u_2 \otimes v_2] = u_1 u_2 \otimes v_1 v_2 \in \mathbb{A}_3 \otimes \mathbb{B}_3,$$

$$[u_2 \otimes v_2, u_3 \otimes v_3] = u_3 \bar{u_2} \otimes v_3 \bar{v_2} \in \mathbb{A}_1 \otimes \mathbb{B}_1,$$

$$[u_3 \otimes v_3, u_1 \otimes v_1] = \bar{u_1} u_3 \otimes \bar{v_1} v_3 \in \mathbb{A}_2 \otimes \mathbb{B}_2.$$

THEOREM 2.1. *This bracket defines a structure of semi-simple Lie algebra on* $\mathfrak{g}$, *whose type is given by Freudenthal's magic square. Moreover, each* $\mathfrak{h}_i = \mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B}) \oplus \mathbb{A}_i \otimes \mathbb{B}_i$ *is a subalgebra of maximal rank of* $\mathfrak{g}$.

Our definition above of the Lie bracket on $\mathfrak{g}$ is much simpler than that in [2] since it does not involve Jordan algebras and their derivations as the Tits construction does. As a result, below we present a simpler proof of the fact that $\mathfrak{g}$ is indeed a Lie algebra.

The following tables list the possible types for $\mathfrak{g}$ and $\mathfrak{h}$, respectively, over $\mathbb{C}$. The first table is Freudenthal's magic square.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{sl}_2$	$\mathfrak{sl}_3$	$\mathfrak{sp}_6$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{sl}_3$	$\mathfrak{sl}_3\times\mathfrak{sl}_3$	$\mathfrak{sl}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_6$	$\mathfrak{sl}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{C}\times\mathfrak{sl}_2$	$\mathfrak{sl}_2\times\mathfrak{sp}_4$	$\mathfrak{so}_9$
$\mathbb{C}$	$\mathbb{C}\times\mathfrak{sl}_2$	$\mathfrak{sl}_2\times\mathbb{C}^2\times\mathfrak{sl}_2$	$\mathbb{C}\times\mathfrak{sl}_2\times\mathfrak{sl}_4$	$\mathbb{C}\times\mathfrak{so}_{10}$
$\mathbb{H}$	$\mathfrak{sl}_2\times\mathfrak{sp}_4$	$\mathfrak{sl}_2\times\mathbb{C}\times\mathfrak{sl}_4$	$\mathfrak{sl}_2\times\mathfrak{sl}_2\times\mathfrak{so}_8$	$\mathfrak{sl}_2\times\mathfrak{so}_{12}$
$\mathbb{O}$	$\mathfrak{so}_9$	$\mathbb{C}\times\mathfrak{so}_{10}$	$\mathfrak{sl}_2\times\mathfrak{so}_{12}$	$\mathfrak{so}_{16}$

Proof. We must check that the Jacobi identity holds in $\mathfrak{g}$. We begin with a few remarks. Denote by $\Psi_i : \Lambda^2\mathbb{A}_i \to \mathfrak{t}(\mathbb{A})$ the map dual to the action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_i$ with respect to an invariant non-degenerate quadratic form $K_{\mathfrak{t}(\mathbb{A})}$ on $\mathfrak{t}(\mathbb{A})$, and the quadratic form on $\Lambda^2\mathbb{A}_i$ induced by the quadratic form $Q = Q_{\mathbb{A}_i}$ on $\mathbb{A}_i$. We have

$$K_{\mathfrak{t}(\mathbb{A})}(\Psi_i(u\wedge v),\theta) = Q(u,\theta_i(v)) \qquad \forall u,v\in\mathbb{A}_i, \quad \forall\theta\in\mathfrak{t}(\mathbb{A}).$$

The action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_1$ factors through the natural representation of $SO(\mathbb{A})$, while the actions on $\mathbb{A}_2$ and $\mathbb{A}_3$ are induced by the left and right multiplications of $\mathbb{A}$ on itself. More precisely, we have the following formulas:

$$\Psi_1(u\wedge v)_1 x = 4Q(u,x)v - 4Q(v,x)u,$$

$$\Psi_1(u\wedge v)_2 x = \bar{v}(ux) - \bar{u}(vx),$$

$$\Psi_1(u\wedge v)_3 x = (xu)\bar{v} - (xv)\bar{u}.$$

(For the case of octonions, these formulas can be deduced from [15, Lecture 15]. The other cases are easy.) Using the compatibility of our construction with the automorphism of $\mathfrak{t}(\mathbb{A})$ which exchanges the three representations $\mathbb{A}_i$, we are reduced to verifying this identity between homogeneous elements in the following cases:

(1) $(\mathfrak{t}(\mathbb{A}),\mathfrak{t}(\mathbb{A}),\mathfrak{t}(\mathbb{A}))$—this is just the Jacobi identity inside $\mathfrak{t}(\mathbb{A})$;

(2) $(\mathfrak{t}(\mathbb{A}),\mathfrak{t}(\mathbb{A}),\mathbb{A}_1\otimes\mathbb{B}_1))$—this case follows from the equivariance of the action of $\mathfrak{t}(\mathbb{A})$ on $\mathbb{A}_1$;

(3) $(\mathfrak{t}(\mathbb{A}),\mathbb{A}_1\otimes\mathbb{B}_1,\mathbb{A}_1\otimes\mathbb{B}_1))$—this case follows from the equivariance of Ψ_1;

(4) $(\mathbb{A}_1 \otimes \mathbb{B}_1, \mathbb{A}_1 \otimes \mathbb{B}_1, \mathbb{A}_1 \otimes \mathbb{B}_1))$—here we must check that for $a, b, c, d, e, f \in \mathbb{A}_1$,

$$[[a \otimes b, c \otimes d], e \otimes f] + [[c \otimes d, e \otimes f], a \otimes b] + [[e \otimes f, a \otimes b], c \otimes d] = 0.$$

But the first of these brackets, for example, can be computed as follows:

$$\begin{aligned}[[a \otimes b, c \otimes d], e \otimes f] &= Q(b,d)\Psi_1(a \wedge c)_1 e \otimes f + Q(a,c)e \otimes \Psi_1(b \wedge d)_1 f \\ &= 4Q(b,d)Q(a,e)c \otimes f - 4Q(b,d)Q(c,e)a \otimes f \\ &\quad + 4Q(a,c)Q(b,f)e \otimes d - 4Q(a,c)Q(d,f)e \otimes b,\end{aligned}$$

and the result easily follows;

(5) $(\mathfrak{t}(\mathbb{A}), \mathbb{A}_1 \otimes \mathbb{B}_1, \mathbb{A}_2 \otimes \mathbb{B}_2))$—here we need to check that

$$\begin{aligned}&[[\theta, a \otimes b], c \otimes d] - [\theta, [a \otimes b, c \otimes d]] + [a \otimes b, [\theta, c \otimes d]] \\ &= [\theta_1(a) \otimes b], c \otimes d] - [\theta, ac \otimes bd] + [a \otimes b, \theta_2(c) \otimes d]] \\ &= \{\theta_1(a)c - \theta_3(ac) + a\theta_2(c)\} \otimes bd = 0,\end{aligned}$$

and this follows from the infinitesimal triality principle for θ.

(6) $(\mathbb{A}_1 \otimes \mathbb{B}_1, \mathbb{A}_1 \otimes \mathbb{B}_1, \mathbb{A}_2 \otimes \mathbb{B}_2))$—here we compute

$$\begin{aligned}&[[a \otimes b, c \otimes d], e \otimes f] + [[c \otimes d, e \otimes f], a \otimes b] + [[e \otimes f, a \otimes b], c \otimes d] \\ &= [Q(b,d)\Psi_1(a \wedge c) + Q(a,c)\Psi_1(b \wedge d), e \otimes f] + [ce \otimes df, a \otimes b] \\ &\quad - [ae \otimes bf, c \otimes d] \\ &= Q(b,d)\Psi_1(a \wedge c)_2 e \otimes f + Q(a,c)e \otimes \Psi_1(b \wedge d)_2 f + \bar{a}(ce) \otimes \bar{b}(df) \\ &\quad - \bar{c}(ae) \otimes \bar{d}(bf).\end{aligned}$$

To check that this is zero, we split this expression into its symmetric and antisymmetric parts with respect to a and c. To control the symmetric part, we simply let $c = a$, and since $\bar{a}(ae) = Q(a,a)e$, we are left with

$$Q(a,a)e \otimes \left\{\Psi_1(b \wedge d)_2 f + \bar{b}(df) - \bar{d}(bf)\right\} = 0.$$

Now the antisymmetric part is

$$\begin{aligned}&2Q(b,d)\Psi_1(a \wedge c)_2 e \otimes f + \bar{a}(ce) \otimes \bar{b}(df) - \bar{c}(ae) \otimes \bar{d}(bf), \\ &\quad - \bar{c}(ae) \otimes \bar{b}(df) + \bar{a}(ce) \otimes \bar{d}(bf),\end{aligned}$$

which is symmetric in b and d. So to check that it vanishes, we can let $b = d$ and we are left with

$$2Q(b,b)\big\{\Psi_1(a\wedge c)_2 e + \bar{a}(ce) - \bar{c}(ae)\big\}\otimes f = 0.$$

(7) $(\mathbb{A}_1\otimes\mathbb{B}_1, \mathbb{A}_2\otimes\mathbb{B}_2, \mathbb{A}_3\otimes\mathbb{B}_3))$—here we compute

$$\begin{aligned}
&[[a\otimes b, c\otimes d], e\otimes f] + [[c\otimes d, e\otimes f], a\otimes b] + [[e\otimes f, a\otimes b], c\otimes d]\\
&\quad = [ac\otimes bd, e\otimes f] + [e\bar{c}\otimes f\bar{d}, a\otimes b] + [\bar{a}e\otimes \bar{b}f, c\otimes d]\\
&\quad = Q(bd,f)\Psi_3(ac\wedge e) + Q(f\bar{d},b)\Psi_1(e\bar{c}\wedge a) + Q(\bar{b}f,d)\Psi_2(\bar{a}e\wedge c)
\end{aligned}$$

plus a symmetric expression with values in $\mathfrak{t}(\mathbb{B})$. But we have, $Q(bd,f) = Q(f\bar{d},b) = Q(\bar{b}f,d)$, so we just need to check that

$$\Psi_3(ac\wedge e) + \Psi_1(e\bar{c}\wedge a) + \Psi_2(\bar{a}e\wedge c) = 0.$$

This follows from the triality principle by duality: indeed, for every $\theta\in\mathfrak{t}(\mathbb{A})$, we have

$$\begin{aligned}
K_{\mathfrak{t}(\mathbb{A})}(\Psi_3(ac\wedge e),\theta) &= -\,Q(\theta_3(ac),e)\\
&= -\,Q(\theta_1(a)c + a\theta_2(c), e)\\
&= -\,Q(\theta_1(a), e\bar{c}) - Q(\theta_2(c),\bar{a}e)\\
&= -\,K_{\mathfrak{t}(\mathbb{A})}(\Psi_1(e\bar{c}\wedge a),\theta) - K_{\mathfrak{t}(\mathbb{A})}(\Psi_2(\bar{a}e\wedge c),\theta),
\end{aligned}$$

and the result follows.

This proves that we have endowed $\mathfrak{g}$ with a Lie algebra structure. This algebra is reductive. There is a natural quadratic form $\mathcal{Q}$ on $\mathfrak{g}$ defined by the fact that the factors of $\mathfrak{g}$ are mutually orthogonal, each one being endowed with its natural quadratic form.

LEMMA 2.2. *The following non-degenerate quadratic form on* $\mathfrak{g}$ *is* $\mathfrak{g}$-*invariant*:

$$K = K_{\mathfrak{t}(\mathbb{A})} + K_{\mathfrak{t}(\mathbb{B})} + \sum_{i=1}^{3} Q_{\mathbb{A}_i}\otimes Q_{\mathbb{B}_i}.$$

Since the center of $\mathfrak{g}$ is trivial, we conclude that $\mathfrak{g}$ is semi-simple. Moreover, any Cartan subalgebra of $\mathfrak{t}(\mathbb{A})\times\mathfrak{t}(\mathbb{B})$ will be a Cartan subalgebra of $\mathfrak{g}$: in particular, $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\mathfrak{t}(\mathbb{A})) + \mathrm{rank}(\mathfrak{t}(\mathbb{B}))$. Finally, knowing the ranks and dimensions of the semi-simple Lie algebra $\mathfrak{g}$ and its reductive subalgebra $\mathfrak{h}$, we easily check that their types are given by Freudenthal's square and the table below. ∎

The triality Lie algebras can be generalized to r-ality for all r to recover the generalized Freudenthal chart (see [11]). For $r > 3$ we have

$$\mathfrak{t}_r(\mathbb{R}) = 0, \qquad \mathfrak{t}_r(\mathbb{C}) = \mathbb{C}^{\oplus(r-1)}, \qquad \mathfrak{t}_r(\mathbb{H}) = \mathfrak{sl}_2^{\times r}$$

and

$$\mathfrak{g}_r(\mathbb{A}, \mathbb{B}) = \mathfrak{t}_r(\mathbb{A}) \times \mathfrak{t}_r(\mathbb{B}) \oplus (\oplus_{1 \leqslant i < j \leqslant r} \mathbb{A}_{ij} \otimes \mathbb{B}_{ij}).$$

This model is useful for more generalized dimension formulas, see Section 7.

3. THE EXCEPTIONAL SERIES

From now on, we work over the complex numbers.

For $\mathbb{B} = \mathbb{O}$, our construction gives the last line of Freudenthal square. Let us describe the root system of $\mathfrak{g}$. For this we choose Cartan subalgebras of $\mathfrak{so}_8$ and $\mathfrak{t}(\mathbb{A})$. Their product is a Cartan subalgebra of $\mathfrak{g}$, and the corresponding root spaces in $\mathfrak{g}$ are the root spaces in $\mathfrak{so}_8$ and $\mathfrak{t}(\mathbb{A})$ and the weight spaces of the tensor products $\mathbb{A}_i \otimes \mathbb{O}_i$. Thus the roots of $\mathfrak{g}$ are

- the roots of $\mathfrak{so}_8$,
- the roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\mu + \nu$ with μ a weight of $\mathbb{A}_i$ and ν a weight of $\mathbb{O}_i$.

To get a set of positive roots we choose linear forms l and $l_{\mathbb{A}}$ on the root lattices that are strictly positive on positive roots. More precisely, we choose $l = l_1\varepsilon_1^* + l_2\varepsilon_2^* + l_3\varepsilon_3^* + l_4\varepsilon_4^*$ with $l_1 \gg l_2 \gg l_3 \gg l_4$. (Here and in what follows, we use the notations and conventions of [3].) Then the linear form $ml + l_{\mathbb{A}}$, where $m \gg 1$, will be positive on the following set of positive roots of $\mathfrak{g}$:

- the positive roots of $\mathfrak{so}_8$,
- the positive roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\mu + \nu$ with μ being a weight of $\mathbb{A}_i$ and ν a weight of $\mathbb{O}_i$ such that $l(\nu) > 0$.

These weights ν of $\mathbb{O}_i$ such that $l(\nu) > 0$ are given by the following tables:

$\mathbb{O}_1$	$\mathbb{O}_2$	$\mathbb{O}_3$
$1-1\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$	$\frac{1}{2}-1\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,1\end{smallmatrix}$	$\frac{1}{2}-1\begin{smallmatrix}\diagup\,1\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$
$0-1\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$	$\frac{1}{2}-1\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,0\end{smallmatrix}$	$\frac{1}{2}-1\begin{smallmatrix}\diagup\,0\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$
$0-0\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$	$\frac{1}{2}-0\begin{smallmatrix}\diagup\,\frac{1}{2}\\ \diagdown\,0\end{smallmatrix}$	$\frac{1}{2}-0\begin{smallmatrix}\diagup\,0\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$
$0-0\begin{smallmatrix}\diagup\,-\frac{1}{2}\\ \diagdown\,\frac{1}{2}\end{smallmatrix}$	$\frac{1}{2}-0\begin{smallmatrix}\diagup\,-\frac{1}{2}\\ \diagdown\,0\end{smallmatrix}$	$\frac{1}{2}-0\begin{smallmatrix}\diagup\,0\\ \diagdown\,-\frac{1}{2}\end{smallmatrix}$

For example, the first weight in the first column is $\alpha_1 + \alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{2}\alpha_4$.

Remark. From this explicit description of the root system of $\mathfrak{g}$, it is quite easy to extract the set of simple roots, from which one can readily obtain the Dynkin diagram of $\mathfrak{g}$. Observe in particular that if we normalize the invariant scalar product of the (dual of the) Cartan algebra of $\mathfrak{so}_8$ in such a way that the root lengths equal two, then the length of a root of the form $\mu + \nu$ equals $(\mu, \mu) + (\nu, \nu) = 1 + (\nu, \nu)$. For $\mathbb{A} \neq \mathbb{R}$, this is larger than one, so it must in fact equal two. This fixes the relative normalization of the invariant scalar product on the Cartan subalgebra of $\mathfrak{t}(\mathbb{A})$, and shows that $\mathfrak{g}$ must be simply laced. On the contrary, for $\mathbb{A} = \mathbb{R}$ we get roots of length one ($\mathfrak{g} = \mathfrak{f}_4$!), and there is no problem of normalization.

PROPOSITION 3.1.

1. *With the ordering above, the following are, in order, the three highest roots of* $\mathfrak{g}$:

$$\tilde{\alpha}_{\mathbb{O}} = 1-2\begin{smallmatrix}\diagup\,1\\ \diagdown\,1\end{smallmatrix} \qquad \beta_1 = 1-1\begin{smallmatrix}\diagup\,1\\ \diagdown\,1\end{smallmatrix}, \qquad \beta_2 = 1-1\begin{smallmatrix}\diagup\,0\\ \diagdown\,1\end{smallmatrix}$$

They are all the simple roots of $\mathfrak{g}$ *annihilated by the torus of* $\mathfrak{t}(\mathbb{A})$, *in fact the next highest root is* $\beta_3 = \omega_1 + \mu^+$ *where* μ^+ *is the highest weight of* $\mathbb{A}_1$.

2. *Any positive weight of* $\mathfrak{g}$ *annhilated by the torus of* $\mathfrak{t}(\mathbb{A})$ *is a linear combination of the following four weights*:

$$\omega(\mathfrak{g}) = 1-2\begin{smallmatrix}\diagup\,1\\ \diagdown\,1\end{smallmatrix} \quad \omega(X_2) = 2-3\begin{smallmatrix}\diagup\,2\\ \diagdown\,2\end{smallmatrix} \quad \omega(X_3) = 3-4\begin{smallmatrix}\diagup\,2\\ \diagdown\,3\end{smallmatrix} \quad \omega(Y_2^*) = 2-2\begin{smallmatrix}\diagup\,1\\ \diagdown\,1\end{smallmatrix}$$

They occur, respectively, $\omega(\mathfrak{g}) = \omega_2$ *as the highest weight of* $\mathfrak{g}$, $\omega(X_2) = \omega_1 + \omega_3 + \omega_4$ *as the highest weight of* $\Lambda^2\mathfrak{g}$, $\omega(X_3) = 2\omega_1 + 2\omega_3$ *as the highest weight of* $\Lambda^3\mathfrak{g}$, *and* $\omega(Y_2^*) = 2\omega_1$ *as the highest weight of* $S^2\mathfrak{g} - \mathfrak{g}^{(2)}$.

3. *The half-sum of the positive roots of* $\mathfrak{g}$ *is* $\rho = \rho_{\mathfrak{t}(\mathbb{A})} + \rho_{\mathfrak{t}(\mathbb{O})} + a\gamma_{\mathfrak{t}(\mathbb{O})}$, *where* $\gamma_{\mathfrak{t}(\mathbb{O})} = 2\omega_1 + \omega_4$.

The names of the weights are borrowed from [4]. We show below that the representations are indeed those of [4] for $\mathfrak{e}_8$ below, the other cases are safely left to the reader.

Proof. Everything is clear except for the assertion about $\omega(Y_2^*)$, which is a consequence of the following observations.

Let μ be a weight of $S^2\mathfrak{g}$ such that $2\tilde{\alpha} \geqslant \mu > 2\omega_1$. Such a weight μ must be the sum of two positive roots γ and δ. Suppose that $\gamma, \delta \neq \tilde{\alpha}$. Then γ, δ have coefficients at most one, hence μ has coefficients at most two, when expressed in terms of simple roots. Since $\mu > 2\omega_1 = 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$, we have $\mu = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$ (up to exchanging α_3 and α_4) hence $\gamma = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ and $\delta = \alpha_1 + \alpha_2 + \alpha_3$. Since in that case $\mu - \tilde{\alpha}$ is not a root, this implies that each possible μ has multiplicity one inside $S^2\mathfrak{g}$. But it also has multiplicity one inside the irreducible component of highest weight $2\tilde{\alpha}$.

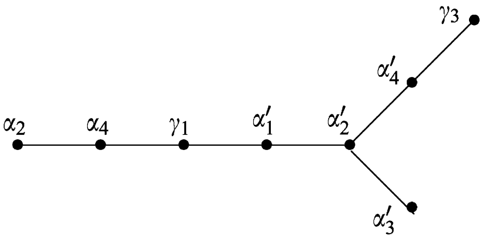
The situation is different for $2\omega_1$, whose multiplicity is at least 3 since there are already 3 different ways to write it as the sum of two roots of $\mathfrak{so}_8$. To conclude, we just need to check that the multiplicity of $2\omega_1$ is strictly larger than its multiplicity inside the irreducible $\mathfrak{g}$-module of highest weight $2\tilde{\alpha}$. But since $2\omega_1$ and $2\tilde{\alpha}$ both have support on the weight lattice of $\mathfrak{so}_8$, it follows from Kostant's multiplicity formula that this multiplicity can be computed directly in $\mathfrak{so}_8$, where we check that it is two. We are done. ∎

Remark. Consider the weights ω of $\mathfrak{g}$ that have support on the Cartan subalgebra of $\mathfrak{so}_8$. Obviously, they must belong to the weight lattice of $\mathfrak{so}_8$, but there are more conditions imposed by the roots of $\mathfrak{g}$ of the form $\mu + \nu$, ν a weight of $\mathbb{O}_i$, μ a weight of $\mathbb{A}_i$: namely, $2(\omega, \nu)/(\mu + \nu, \mu + \nu)$ must be an integer. We have $(\mu, \mu) = 1$, and $(\nu, \nu) = 1$ as well (except in the case where $\mathbb{A} = \mathbb{R}$, for which $\nu = 0$). Thus our conditions reduce to $(\omega, \nu) \in \mathbb{Z}$ for each ν. If we write $\omega = o_1\omega_1 + o_2\omega_2 + o_3\omega_3 + o_4\omega_4$, this means that the integers o_1, o_2, o_3, o_4 must be such that $o_1 + o_3, o_1 + o_4$ and $o_3 + o_4$ are even. This defines a sub-lattice of index four of the weight lattice of $\mathfrak{so}_8$, and it is straightforward to check that the cone of positive weights in this lattice is precisely the cone of non-negative linear combinations of the four weights $\omega(\mathfrak{g})$, $\omega(X_2)$, $\omega(X_3)$ and $\omega(Y_2^*)$.

Example. Consider the case of $\mathfrak{e}_8$, i.e., $\mathbb{A} = \mathbb{O}$. We denote the roots and weights of $\mathfrak{t}(\mathbb{A}) = \mathfrak{so}_8$ with primes. We first determine the set of simple roots of $\mathfrak{e}_8$. They must be among the simple roots α_i of $\mathfrak{t}(\mathbb{O})$, the simple roots α'_j of $\mathfrak{t}(\mathbb{A})$, and the weights

$$\gamma_1 = \omega_3 - \omega_4 - \omega'_1, \qquad \gamma_2 = \omega_1 - \omega_4 - \omega'_3, \qquad \gamma_3 = \omega_1 - \omega_3 - \omega'_4,$$

which are the smallest positive roots inside $\mathbb{A}_1 \otimes \mathbb{O}_1, \mathbb{A}_2 \otimes \mathbb{O}_2, \mathbb{A}_3 \otimes \mathbb{O}_3$, respectively. But $\alpha_3 = \alpha_4 + 2(\omega_3 - \omega_4) = \alpha_4 + 2\gamma_1 + 2\omega'_1$, where $2\omega'_1$ belongs to the root lattice of $\mathfrak{so}_8$, showing that α_3 cannot be a simple root of $\mathfrak{g}$. Neither can α_1 for the same reason. The same conclusion holds for γ_2 because of the relation $\gamma_2 - \gamma_1 - \gamma_3 = \omega'_1 + \omega'_4 - \omega'_3 = \alpha'_1 + \alpha'_2 + \alpha'_4$. Since we know we must have 8 simple roots, they must be $\alpha_2, \alpha_4, \alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \gamma_1, \gamma_3$. Using them, we easily deduce the Dynkin diagram of e_8: we have a subdiagram of type $\mathfrak{so}_8$ corresponding to $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4$, and we attach to it 4 other nodes according to the non-zero scalar products (α_2, α_4), (α_4, γ_1), (γ_1, α'_1) and (γ_3, α'_4):



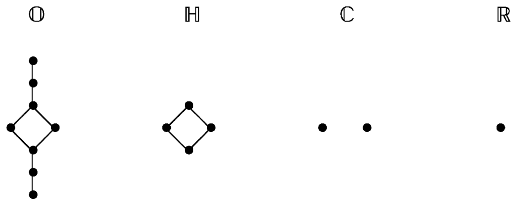
It is now a simple computation to express the weights $\omega(\mathfrak{g})$, $\omega(X_2)$, $\omega(X_3)$ and $\omega(Y_2^*)$ in terms of our simple roots. We obtain

$$\omega(\mathfrak{g}) = 2-3-4-5-6\begin{array}{l} \;/4/2 \\ \;\backslash 3 \end{array} = \omega_8(\mathfrak{e}_8), \qquad \omega(X_2) = 3-6-8-10-12\begin{array}{l} \;/8/4 \\ \;\backslash 6 \end{array} = \omega_7(\mathfrak{e}_8)$$

$$\omega(X_3) = 4-8-10-13-16\begin{array}{l} \;/11/6 \\ \;\backslash 8 \end{array} = \omega_6(\mathfrak{e}_8), \qquad \omega(Y_2^*) = 2-4-6-8-10\begin{array}{l} \;/7/4 \\ \;\backslash 5 \end{array} = \omega_1(\mathfrak{e}_8).$$

This shows our terminology agrees with that of [4] in the case of $\mathfrak{e}_8$.

Now we make a few observations on the weights of $\mathbb{A}_i$. First note that since $\mathbb{A}_i$ has an invariant quadratic form, the set of its weights is symmetric with respect to the origin. In particular, their sum is zero. The weight structure is as follows:



In particular, when μ describes the weights of $\mathbb{A}_i$, the integer (ρ, μ) takes each value in the interval $[1 - \frac{a}{2}, \frac{a}{2} - 1]$ once (this is the empty interval for $a = 1$), plus the value zero once more. We call this set of values $v(\mathbb{A})$.

Remark. Recall that the short roots of $\mathfrak{f}_4$ define a root system of type $\mathfrak{so}_8$. In our description of the root systems of $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$, we see that each long root of $\mathfrak{f}_4$ has been "unfolded" into a set of a roots.

Now look at the inner products of the weights $\omega(\mathfrak{g}), \omega(X_2), \omega(X_3)$ and $\omega(Y_2^*)$ with the positive roots of $\mathfrak{g}$. Since these four weights come from $\mathfrak{so}_8$ only, the pairing is zero on the roots coming from $\mathfrak{t}(\mathbb{A})$. Moreover, on the roots of the form $\mu + v$, the pairing depends only on μ. We get the following possibilities:

0122	(12)	1232	$(3\frac{5}{2})$
1000	(10)	1110	$(2\frac{1}{2})$
0100	(10)	0110	$(1\frac{1}{2})$
0120	(11)	0010	$(0\frac{1}{2})$
1122	(22)	1221	$(3\frac{3}{2})$
1100	(20)	1121	$(2\frac{3}{2})$
1120	(21)	0121	$(1\frac{3}{2})$
1220	(31)	0001	$(0\frac{1}{2})$
1242	(33)	1231	(32)
1222	(32)	1111	(21)
1342	(43)	0111	(11)
2342	(53)	0011	(01)

The first column comes from the positive roots α of $\mathfrak{so}_8$: the first four integers are the scalar products $(\omega(\mathfrak{g}), \alpha)$, $(\omega(X_2), \alpha)$, $(\omega(X_3), \alpha)$ and $(\omega(Y_2^*), \alpha)$, and we added in parentheses the values of $u = (\rho_{\mathfrak{t}(\mathbb{O})}, \alpha)$ and $v = (\gamma_{\mathfrak{t}(\mathbb{O})}, \alpha)$. Each possibility occurs exactly once, and $(\rho, \alpha) = u + av$.

The second column comes from the weights μ of the three modules $\mathbb{O}_i$: we denote by Σ the set of these weights. Again the first four integers are $(\omega(\mathfrak{g}), \mu)$, $(\omega(X_2), \mu)$, $(\omega(X_3), \mu)$ and $(\omega(Y_2^*), \mu)$, and in parentheses we gave $u = (\rho_{\mathfrak{t}(\mathbb{O})}, \mu)$ and $v = (\gamma_{\mathfrak{t}(\mathbb{O})}, \mu)$. The main difference with the first column is that here, each possibility occurs for the a positive roots of $\mathfrak{g}$ of the form $\mu + \nu$. And the values taken by ρ on these a positive roots will be the set $v(\mathbb{A})$ translated by $u + av$.

This is precisely the information we need to apply the Weyl dimension formula for an irreducible $\mathfrak{g}$-module whose highest weight ω has support on the Cartan subalgebra of $\mathfrak{so}_8$. Indeed, the positive roots of $\mathfrak{g}$ coming from $\mathfrak{t}(\mathbb{A})$ will not contribute. Each positive root α of $\mathfrak{so}_8$ will contribute by $(\rho + \omega, \alpha)/(\rho, \alpha)$, where (ρ, α) is linear in a. Finally, for each $\mu \in \Sigma$, the contribution of the positive roots of $\mathfrak{g}$ of the form $\mu + \nu$ will be

$$\prod_{\nu} \frac{(\rho + \omega, \mu + \nu)}{(\rho, \mu + \nu)} = \frac{(\rho + \omega, \mu)}{(\rho, \mu)} \prod_{j=1-\frac{a}{2}}^{j=\frac{a}{2}-1} \frac{(\rho + \omega, \mu) + j}{(\rho, \mu) + j}.$$

It is convenient to write this last product as a quotient of two binomial coefficients, and we finally get the following formula.

THEOREM 3.2. *The dimension of the irreducible* $\mathfrak{g}$-*module with highest weight* $\omega = p\omega(\mathfrak{g}) + q\omega(X_2) + r\omega(X_3) + s\omega(Y_2^*)$ *is given by the following formula*:

$$\dim V_\omega$$

$$= \prod_{\alpha \in \Delta_+(\mathfrak{so}_8) \cup \Sigma} \frac{(a\gamma_{\mathfrak{t}(\mathbb{O})} + \rho_{\mathfrak{t}(\mathbb{O})} + \omega, \alpha)}{(a\gamma_{\mathfrak{t}(\mathbb{O})} + \rho_{\mathfrak{t}(\mathbb{O})}, \alpha)} \prod_{\beta \in \Sigma} \frac{\binom{(a\gamma_{\mathfrak{t}(\mathbb{O})} + \rho_{\mathfrak{t}(\mathbb{O})} + \omega, \beta) + \frac{a}{2} - 1}{(\omega, \beta)}}{\binom{(a\gamma_{\mathfrak{t}(\mathbb{O})} + \rho_{\mathfrak{t}(\mathbb{O})} + \omega, \beta) - \frac{a}{2}}{(\omega, \beta)}}.$$

For each choice of p, q, r, s, *this formula gives a rational function of* a, *whose numerator and denominator are products of* $6p + 12q + 16r + 10s + 24$ *linear forms.*

This formula includes and provides a wide generalization of 15 of the 25 dimension formulas of [4]. Since it applies to actual non-trivial irreducible representations of $\mathfrak{d}_4, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, one could not hope to apply it to their representations that are zero, negative or reducible (i.e., two copies of the same representation) for one of these algebras. When one removes such representations from the list of 25, only the 15 we are able to account for remain, so in that sense this is the best possible formula.

The formula can be made more explicit as follows. Each term *abcd* (uv) in the table above contributes to the product a term $(x+u+av)/(u+av)$, where $x = ap + bq + cr + ds$. If the term is from the second column, it also contributes

$$\frac{(u+av+\frac{a}{2})\cdots(u+av+\frac{a}{2}+x-1)}{(u+av+1-\frac{a}{2})\cdots(u+av-\frac{a}{2}+x)}$$

$$=\frac{(u+av+x-\frac{a}{2}+1)\cdots(u+av+x+\frac{a}{2}-1)}{(u+av+1-\frac{a}{2})\cdots(u+av+\frac{a}{2}-1)},$$

where the numerator and denominator of the rational function on the left are products of x linear forms, and those of the rational function on the right are products of $a-1$ linear forms.

Specializing to multiples of the highest root, we obtain the formula of Proposition 1.1 of the Introduction. We have so far proved this proposition only for $a \geqslant 0$, but we give a second proof in Section 6 that is valid for the entire series.

In Deligne's notations, $\mathfrak{g}^{(k)}$ is Y_k. Using his parameter λ we get

$$\dim Y_k = \frac{(2k-1)\lambda-6}{k!\lambda^k(\lambda+6)} \prod_{j=1}^{k} \frac{((j-1)\lambda-4)((j-2)\lambda-5)((j-2)\lambda-6)}{(j\lambda-1)((j-1)\lambda-2)}.$$

Note that the q-analogs of our formulas (see e.g. [10, Proposition 10.10, p. 183]) are immediate consequences of our methods. For example,

$$\dim_q \mathfrak{g}^{(k)} = \frac{1-q^{3a+2k+5}}{1-q^{3a+5}} \frac{\begin{bmatrix} k+2a+3 \\ k \end{bmatrix}_q \begin{bmatrix} k+\frac{5a}{2}+3 \\ k \end{bmatrix}_q \begin{bmatrix} k+3a+4 \\ k \end{bmatrix}_q}{\begin{bmatrix} k+\frac{a}{2}+1 \\ k \end{bmatrix}_q \begin{bmatrix} k+a+1 \\ k \end{bmatrix}_q},$$

where $\begin{bmatrix} k+l \\ k \end{bmatrix}_q = \frac{(1-q^{l+1})\cdots(1-q^{l+k})}{(1-q)\cdots(1-q^k)}$ is the usual Gauss polynomial.

Note that $\dim \mathfrak{g}^{(k)}$ can be interpreted as the Hilbert function of the unique closed G-orbit inside $\mathbb{P}\mathfrak{g}$, which we call the adjoint variety and denote by X_{ad}. In particular, $\dim \mathfrak{g}^{(k)}$ grows like $(\deg X_{ad})k^d/d!$, where d is the dimension of X_{ad}. From this and the above proposition we recover that

$d = 6a + 9$, and we deduce a funny formula for the degree:

$$\deg X_{ad} = 2\frac{(\frac{a}{2}+1)!(a+1)!(6a+9)!}{(\frac{5a}{2}+3)!(2a+3)!(3a+5)!}.$$

After Freudenthal [9], we, respectively, call $X_{F-\text{planes}}$, $X_{F-\text{lines}}$ and $X_{F-\text{points}}$ the closed orbits in $\mathbb{P}X_2$, $\mathbb{P}X_3$ and $\mathbb{P}Y_2^*$. Specializing to the Cartan powers, Theorem 3.2 gives the Hilbert functions of these varieties, respectively,

$$\dim X_2^{(k)} = \frac{k+a+1}{a+1}\frac{k+a+2}{a+2}\frac{2k+2a+3}{2a+3}\frac{3k+3a+4}{3a+4}\frac{3k+3a+5}{3a+5}$$

$$\times\frac{\binom{k+\frac{3a}{2}+1}{k}\binom{k+\frac{3a}{2}+2}{k}\binom{k+2a+1}{k}\binom{k+2a+2}{k}\binom{2k+\frac{5a}{2}+3}{2k}\binom{2k+3a+3}{2k}}{\binom{k+1}{k}\binom{k+\frac{a}{2}}{k}\binom{k+\frac{a}{2}+1}{k}\binom{2k+a+2}{2k}\binom{2k+\frac{3a}{2}+2}{2k}},$$

$$\dim X_3^{(k)} = \frac{2k+\frac{3a}{2}+1}{\frac{3a}{2}+1}\frac{2k+\frac{3a}{2}+2}{\frac{3a}{2}+2}\frac{2k+\frac{3a}{2}+3}{\frac{3a}{2}+3}\frac{4k+3a+3}{3a+3}\frac{4k+3a+4}{3a+4}\frac{4k+3a+5}{3a+5}$$

$$\times\frac{\begin{gathered}\binom{k+a}{k}\binom{k+a+1}{k}\binom{k+a+2}{k}\binom{k+\frac{3a}{2}-1}{k}\binom{k+\frac{3a}{2}}{k}\binom{k+\frac{3a}{2}+1}{k}\\ \times\binom{2k+2a+1}{2k}\binom{2k+2a+2}{2k}\binom{2k+2a+3}{2k}\binom{3k+\frac{5a}{2}+3}{3k}\binom{3k+3a+2}{3k}\end{gathered}}{\begin{gathered}\binom{k+1}{k}\binom{k+2}{k}\binom{k+\frac{a}{2}-1}{k}\binom{k+\frac{a}{2}}{k}\binom{k+\frac{a}{2}+1}{k}\binom{2k+a}{2k}\\ \times\binom{2k+a+1}{2k}\binom{2k+a+2}{2k}\binom{3k+\frac{3a}{2}+3}{3k}\binom{3k+2a+2}{3k}\end{gathered}},$$

$$\dim Y_2^{*(k)} = \frac{2k+\frac{5a}{2}+3}{\frac{5a}{2}+3}$$

$$\times\frac{\binom{k+2a}{k}\binom{k+2a+1}{k}\binom{k+2a+3}{k}\binom{k+\frac{5a}{2}+2}{k}\binom{2k+3a+5}{2k}}{\binom{k+\frac{a}{2}-1}{k}\binom{k+\frac{a}{2}+1}{k}\binom{k+\frac{a}{2}+2}{k}\binom{k+a+1}{k}\binom{k+a+3}{k}\binom{2k+2a}{2k}},$$

one recovers that $\dim X_{F-\text{planes}} = 9a+11$, $\dim X_{F-\text{lines}} = 11a+9$, $\dim X_{F-\text{points}} = 9a+6$, and that their degrees are

$$\deg X_{F-\text{planes}} = 2^{3a+3}3^2 \frac{(9a+11)!a!\frac{a}{2}!(\frac{a}{2}+1)!}{(\frac{3a}{2}+1)!(2a+1)!(2a+3)!(\frac{5a}{2}+3)!(3a+5)!},$$

$$\deg X_{F-\text{lines}} = 2^{3a+6}3^{2a} \frac{(11a+9)!(\frac{a}{2}-1)!\frac{a}{2}!(\frac{a}{2}+1)!}{(\frac{3a}{2}-1)!(\frac{3a}{2}+1)!(2a+3)!(\frac{5a}{2}+3)!},$$

$$\deg X_{F-\text{points}} = 2^{a+6} \frac{(6a+9)!(\frac{a}{2}+1)!(\frac{a}{2}+2)!(a+1)!(a+3)!}{(2a+1)!(2a+3)!(\frac{5a}{2}+2)!(3a+5)!}.$$

4. THE SUBEXCEPTIONAL SERIES

In this section we let $\mathbb{B} = \mathbb{H}$. Then $\mathfrak{t}(\mathbb{B}) \simeq \mathfrak{so}_3 \times \mathfrak{so}_3 \times \mathfrak{so}_3 \simeq \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$. Note that $\mathfrak{t}(\mathbb{B})$ can be naturally identified with $\mathrm{Im}(\mathbb{H})^{\oplus 3}$, acting on $\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbb{H}_3$ by

$$(a,b,c) \mapsto (L_b - R_c, L_c - R_a, L_a - R_b),$$

where L_a, R_a denote the operators of left and right multiplication by a, respectively (see [2]). This means that if we denote by U_1, U_2, U_3 the natural 2-dimensional representations of our three copies of $\mathfrak{sl}_2$, then $\mathbb{H}_1 = U_2 \otimes U_3$, $\mathbb{H}_2 = U_3 \otimes U_1$ and $\mathbb{H}_3 = U_1 \otimes U_2$. Therefore, the roots of $\mathfrak{g}$ are

- the roots $\pm\alpha^1$, $\pm\alpha^2$, $\pm\alpha^3$ of $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$,
- the roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\pm\frac{1}{2}\alpha^i \pm \frac{1}{2}\alpha^j + \mu$, where μ is a weight of $\mathbb{A}_k$ and $\{i,j,k\} = \{1,2,3\}$.

To get a set of positive roots we choose linear forms l and $l_{\mathbb{A}}$ on the root lattices, that are strictly positive on positive roots. More precisely, we choose $l = l_1\alpha^{1*} + l_2\alpha^{2*} + l_3\alpha^{3*}$ with $l_1 \gg l_2 \gg l_3$. Then the linear form $ml + l_{\mathbb{A}}$, where $m \gg 1$, will be positive on the following set of positive roots of $\mathfrak{g}$:

- α^1, α^2, α^3,
- the positive roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\frac{1}{2}\alpha^i \pm \frac{1}{2}\alpha^j + \mu$, where μ is a weight of $\mathbb{A}_k$, with $\{i,j,k\} = \{1,2,3\}$ and $i<j$.

An important difference with the exceptional series is that we have a nice geometric model for one of the distinguished $\mathfrak{g}$-modules $V = \mathbb{A}_1 \otimes U_1 \oplus \mathbb{A}_2 \otimes U_2 \oplus \mathbb{A}_3 \otimes U_3 \oplus U_1 \otimes U_2 \otimes U_3$.

THEOREM 4.1. *There is a natural structure of* $\mathfrak{g}$-*module on*

$$V \;=\; \mathbb{A}_1 \otimes U_1 \;\text{——}\; U_1 \otimes \begin{array}{l} \qquad\qquad \mathbb{A}_3 \otimes U_3 \\ \qquad \diagup \\ U_3 \\ U_2 \\ \qquad \diagdown \\ \qquad\qquad \mathbb{A}_2 \otimes U_2 \end{array}$$

This $\mathfrak{g}$-*module* V *is simple of dimension* $6a + 8$.

Proof. We define the action of $\mathfrak{g}$ on V as follows. There is already a natural action of the subalgebra $\mathfrak{t}(\mathbb{A}) \times \mathfrak{t}(\mathbb{B})$, and up to the ternary symmetry we just need to define an action of $\mathbb{H}_1 \otimes \mathbb{A}_1 = U_2 \otimes U_3 \otimes \mathbb{A}_1$. This action is provided by the natural maps

$$\begin{aligned}
&(U_2 \otimes U_3 \otimes \mathbb{A}_1) \otimes (U_1 \otimes U_2 \otimes U_3) \to U_1 \otimes \mathbb{A}_1,\\
&(U_2 \otimes U_3 \otimes \mathbb{A}_1) \otimes (U_1 \otimes \mathbb{A}_1) \to U_1 \otimes U_2 \otimes U_3,\\
&(U_2 \otimes U_3 \otimes \mathbb{A}_1) \otimes (U_2 \otimes \mathbb{A}_2) \to U_3 \otimes \mathbb{A}_3,\\
&(U_2 \otimes U_3 \otimes \mathbb{A}_1) \otimes (U_3 \otimes \mathbb{A}_3) \to U_2 \otimes \mathbb{A}_2,
\end{aligned}$$

which are easily defined using the invariant quadratic forms on U_1, U_2, U_3 and $\mathbb{A}_1$, and the natural multiplication map $\mathbb{A}_1 \otimes \mathbb{A}_2 \to \mathbb{A}_3$. The verification that this defines a module structure over the algebra $\mathfrak{g}$ is just a computation. ∎

Remark. There is another more natural but less direct proof of this result, based on the observation that $\mathfrak{g} = \mathfrak{g}(\mathbb{A}, \mathbb{H})$ is the semi-simple part of a parabolic subalgebra $\mathfrak{p}(\mathbb{A})$ of $\mathfrak{g}(\mathbb{A}, \mathbb{O})$. Hence a natural action of $\mathfrak{g}$ on the quotient $\mathfrak{g}(\mathbb{A}, \mathbb{O})/\mathfrak{p}(\mathbb{A})$, which is just $\mathbb{C} \oplus V$. The same idea also gives another proof of Theorem 5.1.

The natural $\mathfrak{g}$-invariant symplectic form Ω on V may be written as

$$\Omega = \Omega_{U_1 \otimes U_2 \otimes U_3} + \sum_{i=1}^{3} \Omega_i,$$

where $\Omega_{U_1\otimes U_2\otimes U_3}$ is just the tensor product of the determinants on U_1, U_2, U_3, and Ω_i is the symplectic form on $U_i \otimes \mathbb{A}_i$ induced by the determinant on U_i and the quadratic form Q_i on $\mathbb{A}_i$.

PROPOSITION 4.2. 1. *With the ordering above, the three highest roots of* $\mathfrak{g}$ *are* $\omega(\mathfrak{g}) = \tilde{\alpha} = \alpha^1 = 2\omega^1$, $\omega(V) = \omega^1 + \omega^2 + \omega^3$ *and* $\omega^1 + \omega^2 - \omega^3$.

They are all the simple roots of $\mathfrak{g}$ *annhilated by the torus of* $\mathfrak{t}(\mathbb{A})$, *in fact the next highest root is* $\omega^1 + \mu^+$ *where* μ^+ *is the highest weight of* $\mathbb{A}_1$.

2. *Any positive weight of* $\mathfrak{g}$ *annhilated by the torus of* $\mathfrak{t}(\mathbb{A})$ *is a linear combination of the following three weights*: $\omega(\mathfrak{g}) = \tilde{\alpha} = \alpha^1 = 2\omega^1$, $\omega(V) = \omega^1 + \omega^2 + \omega^3$ *and* $\omega(V_2) = 2\omega^1 + 2\omega^2$.

They occur, respectively, as the highest weight of $\mathfrak{g}$, V, *and* $\Lambda^2 V$.

3. *The half-sum of the positive roots of* $\mathfrak{g}$ *is* $\rho = \rho_{\mathfrak{t}(\mathbb{A})} + \rho_{\mathfrak{t}(\mathbb{H})} + a\gamma_{\mathfrak{t}(\mathbb{H})}$, *where* $\gamma_{\mathfrak{t}(\mathbb{H})} = 2\omega^1 + \omega^2$.

The values of the pairings of the weights $\omega(\mathfrak{g}), \omega(V)$ and $\omega(V_2)$ with the positive roots of $\mathfrak{g}$ are obtained as follows. Since these three weights come from $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, their value is zero on the roots coming from $\mathfrak{t}(\mathbb{A})$. Moreover, on the roots of the form $\frac{1}{2}\alpha^i \pm \frac{1}{2}\alpha^j + \mu$, their values do not depend on μ. We get the following possibilities:

212	(12)	112	$(1\frac{3}{2})$
012	(11)	100	$(0\frac{1}{2})$
010	(10)	111	(11)
		101	(01)
		011	$(1\frac{1}{2})$
		001	$(0\frac{1}{2})$

The first column comes from the positive roots of $\mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, each possibility occurs exactly once. The second column comes from the weights of the three modules $\mathbb{H}_i$: we denote this set by Γ. Each possibility occurs for exactly a positive roots of $\mathfrak{g}$. In parentheses, are the values of the pairings with $\rho_{\mathfrak{t}(\mathbb{H})}$ and $\gamma_{\mathfrak{t}(\mathbb{H})}$. Applying the Weyl dimension formula as above we get the following result.

THEOREM 4.3. *The dimension of the irreducible* $\mathfrak{g}$-*module with highest weight* $\omega = p\omega(\mathfrak{g}) + q\omega(V) + r\omega(V_2)$ *is given by the following function*:

$$\dim V_\omega = \prod_{\alpha\in\Delta_+(\mathfrak{sl}_2^3)\cup\Gamma} \frac{(a\gamma_{\mathfrak{t}(\mathbb{H})} + \rho_{\mathfrak{t}(\mathbb{H})} + \omega, \alpha)}{(a\gamma_{\mathfrak{t}(\mathbb{H})} + \rho_{\mathfrak{t}(\mathbb{H})}, \alpha)} \prod_{\beta\in\Gamma} \frac{\binom{(a\gamma_{\mathfrak{t}(\mathbb{H})} + \rho_{\mathfrak{t}(\mathbb{H})} + \omega, \beta) + \frac{a}{2} - 1}{(\omega, \beta)}}{\binom{(a\gamma_{\mathfrak{t}(\mathbb{H})} + \rho_{\mathfrak{t}(\mathbb{H})} + \omega, \beta) - \frac{a}{2}}{(\omega, \beta)}}.$$

For each choice of p, q, r, *this formula gives a rational function of* a, *whose numerator and denominator are products of* $4p + 3q + 6r + 9$ *linear forms.*

COROLLARY 4.4. *Let* V *be the distinguished module, of dimension* $6a + 8$, *of a semi-simple Lie algebra* $\mathfrak{g}$ *in the subexceptional series, with* $a = -\frac{2}{3}, 0, 1, 2, 4, 8$. *Then*

$$\dim \mathfrak{g}^{(k)} = \frac{2k+2a+1}{2a+1} \frac{\binom{k+\frac{3a}{2}-1}{k}\binom{k+\frac{3a}{2}+1}{k}\binom{k+2a}{k}}{\binom{k+\frac{a}{2}-1}{k}\binom{k+\frac{a}{2}+1}{k}},$$

$$\dim V^{(k)} = \frac{a+k+1}{a+1} \frac{\binom{k+2a+1}{k}\binom{k+\frac{3a}{2}+1}{k}}{\binom{k+\frac{a}{2}}{k}},$$

$$\dim V_2^{(k)} = \frac{(4k+3a+2)}{(k+1)(3a+2)} \frac{\binom{k+a}{k}\binom{k+a+1}{k}\binom{k+\frac{3a}{2}-1}{k}\binom{k+\frac{3a}{2}}{k}\binom{2k+2a+1}{2k}}{\binom{k+\frac{a}{2}-1}{k}\binom{k+\frac{a}{2}}{k}\binom{2k+a}{2k}}.$$

Let $X \subset \mathbb{P}V$, $X_{ad} \subset \mathbb{P}\mathfrak{g}$, $X_{F-\text{planes}} \subset \mathbb{P}V_2$ denote the closed orbits. We recover from the Hilbert functions above that $\dim X_{ad} = 4a + 1$, $\dim X = 3a + 3$, $\dim X_{F-\text{lines}} = 5a + 2$ and

$$\deg X_{ad} = \frac{(4a+1)!2(\frac{a}{2}-1)!(\frac{a}{2}+1)!}{(2a+1)(\frac{3a}{2}-1)!(\frac{3a}{2}+1)!(2a)!},$$

$$\deg X = \frac{(3a+3)!(\frac{a}{2}+1)!}{(2a+1)!(\frac{3a}{2}+1)!},$$

$$\deg X_{F-\text{lines}} = \frac{(5a+2)!2^{a+3}(\frac{a}{2})!(\frac{a}{2}-1)!}{(3a+2)!(a+1)!(2a+1)!(\frac{3a}{2})!(\frac{3a}{2}-1)!}.$$

5. THE SEVERI SERIES

Now we let $\mathbb{B} = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$, which is naturally the plane $\Pi \subset \mathbb{C}^3$ of equation $z_1 + z_2 + z_3 = 0$, acting diagonally on $\mathbb{C}_1 \times \mathbb{C}_2 \times \mathbb{C}_3$. There is a natural identification of $\mathbb{B}_i$ with $\mathbb{C}_j \otimes \mathbb{C}_k^{-1} \oplus \mathbb{C}_j^{-1} \otimes \mathbb{C}_k$ for $\{i, j, k\} = \{1, 2, 3\}$, as $\mathfrak{t}(\mathbb{B})$-modules.

Let us denote by $\omega_1, \omega_2, \omega_3$ the highest weights of the action of $\mathfrak{t}(\mathbb{B})$ on $\mathbb{C}_1$, $\mathbb{C}_2$, $\mathbb{C}_3$, which are subject to the relation $\omega_1 + \omega_2 + \omega_3 = 0$. Then the roots of $\mathfrak{g}$ are:

- the roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\pm(\omega_j - \omega_k) + \mu$, where μ is a weight of $\mathbb{A}_i$ and $\{i, j, k\} = \{1, 2, 3\}$.

To get a set of positive roots we choose linear forms l and $l_{\mathbb{A}}$ on the root lattices that are strictly positive on positive roots. More precisely, we choose $l = l_1\omega_{1^*} + l_2\omega^{2^*}$ with $l_1 \gg l_2 \gg 0$. Then the linear form $l + l_{\mathbb{A}}$ will be positive on the following set of positive roots of $\mathfrak{g}$:

- the positive roots of $\mathfrak{t}(\mathbb{A})$,
- the weights $\omega_j - \omega_k + \mu$, where μ is a weight of $\mathbb{A}_i$, $j<k$ and $\{i, j, k\} = \{1, 2, 3\}$.

As for the subexceptional series, we have a nice geometric model for the distinguished $\mathfrak{g}$-modules:

Theorem 5.1. *There is a natural structure of* $\mathfrak{g}$*-module on*

$$\begin{tikzcd}
 & & \mathbb{A}_3 \otimes \mathbb{C}_3^{-1} \\
W \;=\; \mathbb{A}_1 \otimes \mathbb{C}_1^{-1} \arrow[r, dash] & \mathbb{C}_1^2 \oplus \begin{matrix} \mathbb{C}_3^2 \\ \mathbb{C}_2^2 \end{matrix} \arrow[ur, dash] \arrow[dr, dash] & \\
 & & \mathbb{A}_2 \otimes \mathbb{C}_2^{-1}
\end{tikzcd}$$

This $\mathfrak{g}$*-module* W *is simple of dimension* $3a + 3$.

Proof. We just need to define the action of a typical factor $\mathbb{A}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3^{-1}$ on W. This action is given by the natural maps

$$\begin{aligned}
(\mathbb{A}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3^{-1}) \otimes \mathbb{C}_3^2 \quad &\rightarrow \quad \mathbb{A}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3 = \mathbb{A}_1 \otimes \mathbb{C}_1^{-1}, \\
(\mathbb{A}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3^{-1}) \otimes (\mathbb{A}_1 \otimes \mathbb{C}_1^{-1}) \quad &\rightarrow \quad \mathbb{C}_1^{-1} \otimes \mathbb{C}_2 \otimes \mathbb{C}_3^{-1} = \mathbb{C}_2^2, \\
(\mathbb{A}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3^{-1}) \otimes (\mathbb{A}_2 \otimes \mathbb{C}_2^{-1}) \quad &\rightarrow \quad \mathbb{A}_3 \otimes \mathbb{C}_3^{-1},
\end{aligned}$$

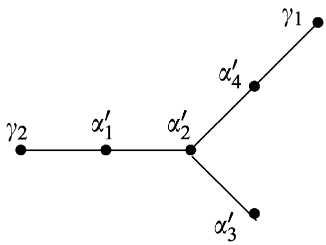
where we use for the first two arrows the fact that $\mathbb{C}_1 \otimes \mathbb{C}_2 \otimes \mathbb{C}_3$ is a trivial $\mathfrak{t}(\mathbb{B})$-module, for the second arrow the quadratic form on $\mathbb{A}_1$, and for the last arrow the multiplication map $\mathbb{A}_1 \otimes \mathbb{A}_2 \rightarrow \mathbb{A}_3$. The action on the other factors is equal to zero. We leave to the reader the computations that are necessary to check that this is indeed a Lie algebra action of $\mathfrak{g}$. The fact that we get a simple module is obvious. ∎

PROPOSITION 5.2. *The highest root of* $\mathfrak{g}$ *is* $\omega(\mathfrak{g}) = \tilde{\alpha} = \omega_1 - \omega_3 + \mu_2$, *where* μ_2 *is the highest weight of* $\mathbb{A}_2$.

The highest weight of W *is* $\omega(W) = 2\omega_1$, *its lowest weight is* $-\omega(W^*) = 2\omega_3$.

The half-sum of the positive roots of $\mathfrak{g}$ *is* $\rho = \rho_{\mathfrak{t}(\mathbb{A})} + a\gamma_{\mathfrak{t}(\mathbb{C})}$, *where* $\gamma_{\mathfrak{t}(\mathbb{C})} = \omega_1 - \omega_3$.

EXAMPLE. Let us treat in detail the case where $\mathbb{A} = \mathbb{O}$, leading to e_6 and its minimal representation. The simple roots of $\mathfrak{g}$ are those of $\mathfrak{t}(\mathbb{O}) = \mathfrak{so}_8$, say α_1', α_2', α_3', α_4', and $\gamma_1 = \omega_1 - \omega_2 - \mu_3$, $\gamma_2 = \omega_2 - \omega_3 - \mu_1$, where $\mu_1 = \omega_1'$, $\mu_3 = \omega_4'$ denote the highest weights of $\mathbb{O}_1$, $\mathbb{O}_3$, respectively. We get the following Dynkin diagram:



It is then straightforward to compute $\omega(W)$ and $\omega(W^*)$ in terms of the simple roots. We obtain

$$\omega(W) = \begin{array}{ccccc} & & & & \tfrac{4}{3} \\ & & & \tfrac{5}{3} & \\ \tfrac{2}{3} - \tfrac{4}{3} - 2 & & & & \\ & & & 1 & \end{array} = \omega_1(\mathfrak{e}_6), \qquad \omega(W^*) = \begin{array}{ccccc} & & & & \tfrac{2}{3} \\ & & & \tfrac{4}{3} & \\ \tfrac{4}{3} - \tfrac{5}{3} - 2 & & & & \\ & & & 1 & \end{array} = \omega_6(\mathfrak{e}_6).$$

Since the highest root of $\mathfrak{g}$ does depend on $\mathbb{A}$, we will not obtain any rational expression in a for the dimension of $\mathfrak{g}$ and its Cartan powers using this model. However, we will obtain such a formula for the irreducible $\mathfrak{g}$-modules whose highest weights are linear combinations of $\omega(W)$ and $\omega(W^*)$.

For this we need to compute the values of $\omega(W)$ and $\omega(W^*)$ on the positive roots of $\mathfrak{g}$. These values are zero on the roots coming of $\mathfrak{t}(\mathbb{A})$. To compute the other ones, we consider on Π the restriction of the canonical metric on $\mathbb{C}^3$. Computing the dual metric we get $(\omega_i, \omega_i) = \frac{1}{3}$ and $(\omega_i, \omega_j) = -\frac{1}{6}$ for $1 \leqslant i \neq j \leqslant 3$. It is then straightforward to apply Weyl's dimension formula and obtain:

THEOREM 5.3. *The dimension of the irreducible* $\mathfrak{g}$*-module with highest weight* $\omega = p\omega(W) + p^*\omega(W^*)$ *is given by the following function*:

$$\begin{aligned}
\dim V_\omega &= \prod_{i \in v(\mathbb{A})} \frac{p+\frac{a}{2}+i}{\frac{a}{2}+i} \frac{p+p^*+a+i}{a+i} \frac{p^*+\frac{a}{2}+i}{\frac{a}{2}+i} \\
&= \frac{(2p+a)(p+p^*+a)(2p^*+a)}{a^3} \frac{\binom{p+a-1}{p}\binom{p+p^*+\frac{3a}{2}-1}{p+p^*}\binom{p^*+a-1}{p^*}}{\binom{p+p^*+\frac{a}{2}}{p+p^*}}
\end{aligned}$$

6. OTHER MODELS FOR THE EXCEPTIONAL SERIES

There exist two other series of models for the exceptional Lie algebras similar to the constructions we used in Section 2. In Section 2, we exploited on the triality phenomenon, which is reflected in the threefold symmetry of the Dynkin diagram of $\mathfrak{so}_8$. For our two other series, we use the simplest Dynkin diagram with twofold symmetry, which is that of $\mathfrak{sl}_3$, and the simplest one with “onefold symmetry,” which is that of $\mathfrak{sl}_2$. This leads to the three series

$$\mathfrak{g}(\mathbb{A}) = \mathfrak{so}_8 \oplus \mathfrak{t}(\mathbb{A}) \oplus \mathbb{O}_1 \otimes \mathbb{A}_1 \;\oplus\; \mathbb{O}_2 \otimes \mathbb{A}_2 \;\oplus\; \mathbb{O}_3 \otimes \mathbb{A}_3,$$

$$\mathfrak{g}(\mathbb{A}) = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3(\mathbb{A}) \oplus \mathbb{C}^3 \otimes \mathfrak{h}_3(\mathbb{A}) \;\oplus\; \mathbb{C}^{3*} \otimes \mathfrak{h}_3(\mathbb{A})^*,$$

$$\mathfrak{g}(\mathbb{A}) = \mathfrak{sl}_2 \oplus \mathfrak{sp}_6(\mathbb{A}) \oplus \mathbb{C}^2 \otimes \Lambda^{\langle 3 \rangle}\mathbb{A}^6.$$

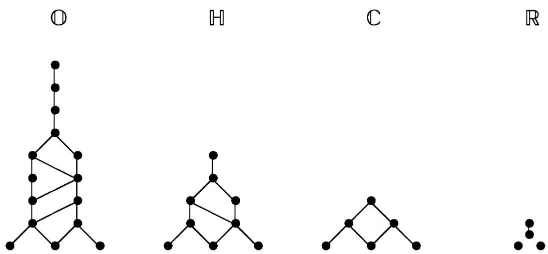
Here $SL_3(\mathbb{A})$, respectively, denotes the Lie groups $1, 1, \mathfrak{S}_3$ and the four groups on the second row of Freudenthal's magic chart. $\mathfrak{h}_3(\mathbb{A})$ denotes the Jordan algebra of Hermitian matrices of order 3 over $\mathbb{A}$ in the last four cases and $\emptyset$, homotheties, and diagonal 3×3 matrices in the first three cases (see [11]). Similarly, $\mathfrak{sp}_6(\mathbb{A})$, respectively, denotes $0, \mathfrak{sl}_2, \mathfrak{sl}_2^{\oplus 3}$ and the Lie algebras appearing in the third row of Freudenthal's chart. $\Lambda^{\langle 3 \rangle}\mathbb{A}^6$, respectively, denotes 0, $S^3\mathbb{C}^2$ and the subexceptional representations V.

These series show the same remarkable uniformity properties in the distributions of the root heights necessary for nice dimension formulas. But the formulas one obtains only concern representations whose highest weights are supported on the weight lattice of the fixed subalgebra of each series, namely $\mathfrak{so}_8$, $\mathfrak{sl}_3$ and $\mathfrak{sl}_2$ respectively. The rank of this subalgebra is maximal for the first series so we will not be able to extract more information from the other two series.

Let us consider, nevertheless, our models in the second series, involving the action of $\mathfrak{sl}_3(\mathbb{A})$ on the Jordan algebra $\mathfrak{h}_3(\mathbb{A})$. A natural Cartan subalgebra of $\mathfrak{g}(\mathbb{A})$ is obtained as the direct sum of Cartan subalgebras of $\mathfrak{sl}_3$ and $\mathfrak{sl}_3(\mathbb{A})$. We choose a linear form on its dual which takes positive values on the positive roots of $\mathfrak{sl}_3(\mathbb{A})$, and very large positive values on those of $\mathfrak{sl}_3$. Then the positive roots of $\mathfrak{g}(\mathbb{A})$ are those of $\mathfrak{sl}_3(\mathbb{A})$, those of $\mathfrak{sl}_3$, along with the weights $\omega_1 + \mu$, $\omega_2 - \mu$ and $\omega_1 - \omega_2 - \mu$, where μ is a weight of $\mathfrak{h}_3(\mathbb{A})$. In particular, the highest root and the half-sum of the positive roots are

$$\begin{aligned}\tilde{\alpha}_{\mathfrak{g}(\mathbb{A})} &= \tilde{\alpha}_{\mathfrak{sl}_3} = \omega_1 + \omega_2,\\ \rho_{\mathfrak{g}(\mathbb{A})} &= \rho_{\mathfrak{sl}_3(\mathbb{A})} + \rho_{\mathfrak{sl}_3} + \dim \mathfrak{h}_3(\mathbb{A})\omega_1 = \rho_{\mathfrak{sl}_3(\mathbb{A})} + (3a+4)\omega_1 + \omega_2.\end{aligned}$$

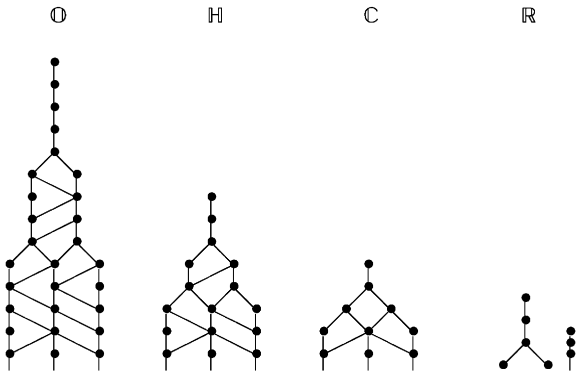
We need to understand the distribution of the weights of the $\mathfrak{sl}_3(\mathbb{A})$-modules $\mathfrak{h}_3(\mathbb{A})$. They are as follows:



The vertices of these diagrams indicate the weights with non-negative height (where the number (ρ, ω) is the height of a weight ω), while an edge indicates the action of a simple reflection (the $\mathfrak{h}_3(\mathbb{A})$ are all minuscule modules, so that their sets of weights are just the orbits of the highest ones). The complete diagram is obtained by a symmetry along the line of height zero.

The first three diagrams look very similar: there are three weights of height zero, two weights on each height between 1 and $\frac{a}{2}$, then one weight on each height up to a. This means that these three diagrams as being given by the superposition of intervals $[-a, a]$, $[-\frac{a}{2}, \frac{a}{2}]$, plus a 0. For $a = 1$, this gives weights in height $-1, -\frac{1}{2}, 0, \frac{1}{2}, 1$, with multiplicity two for the zero height: this is precisely our fourth diagram (where we have to use the normalization of [3] divided by two).

We can analyze in a similar way our third series of models, for which the weight distributions in the $\mathfrak{sp}_6(\mathbb{A})$-module $\Lambda^{\langle 3 \rangle}\mathbb{A}^6$ are again remarkably uniform. They are as follows:



Again the fourth diagram is somewhat special: it splits into two orbits of the Weyl group, the corresponding $\mathfrak{sp}_6$-module being non-minuscule. Nevertheless, the first three diagrams are strikingly similar: there are three strands of height from $\frac{1}{2}$ to $\frac{a+1}{2}$, then two strands of length $\frac{a}{2}$, and a last strand of length $\frac{a}{2}+1$. Said otherwise, the heights describe three intervals, namely $[-\frac{3a+3}{2}, \frac{3a+3}{2}]$, $[-\frac{2a+1}{2}, \frac{2a+1}{2}]$ and $[-\frac{a+1}{2}, \frac{a+1}{2}]$, and this, even for $a = 1$. It is then very simple to apply the Weyl dimension formula to compute the dimension of $\mathfrak{g}(\mathbb{A})$ and its Cartan powers. A proof of Proposition 1.1 stated in the Introduction, valid for the entire exceptional series, follows.

7. THE GENERAL SET UP

We say a collection of reductive Lie algebras $\mathfrak{g}(t)$ parametrized by t and equipped with representations $(V_{\lambda_1}(t), \ldots, V_{\lambda_p}(t))$ is a *series in the dimension sense of Deligne* if there exists a formula for $\dim V_{m_1\lambda_1+\cdots+m_p\lambda_p}$ that is a rational function whose numerator and denominator are products of linear functions of t. In this case, once one fixes $m_1, \ldots, m_p$, the dimension formula looks like the Weyl dimension formula (see below). We discuss other notions of series in [14].

How to construct such series?

One way would be to start with a fixed Lie algebra $\mathfrak{f}$, and consider A-graded Lie algebras $\mathfrak{g}$ (where A is an abelian group), containing $\mathfrak{f}$ as a component of $\mathfrak{g}_0$. If the grading comes from marking some nodes on the extended Dynkin diagram of $\mathfrak{g}$, then $\mathfrak{f}$ will be given by a union of connected components of the diagram obtained by removing the marked nodes. If one only marks one node, so one has an $\mathbb{Z}_2$-grading, then $\mathfrak{g}_0 = \mathfrak{f} + \mathfrak{h}$ where $\mathfrak{h}$ is whatever else is left over after the nodes and components of $\mathfrak{f}$ are removed. In this case, $\mathfrak{g}_1 = V \otimes W(t)$ where V (resp. W) is the representation of $\mathfrak{f}$ (resp. $\mathfrak{h}(t)$) with highest weight the sum of fundamental weights corresponding to nodes adjacent to the marked node. For example, if one takes the node(s) next to the longest root, $\mathfrak{f} = \mathfrak{sl}_2$, and one can in particular recover the last series of models of the exceptional Lie algebras in the preceding section. If one takes the next node(s) over, then $\mathfrak{f} = \mathfrak{sl}_3$ and one can recover the preceeding series.

In order to have a series in the dimension sense of Deligne, the Lie algebras $\mathfrak{h}$ and representations U that remain must satisfy additional conditions explained below.

Write $\mathfrak{g}(t) = \mathfrak{f} + \mathfrak{h}(t) + W(t)$ where $\mathfrak{g}_0(t) = \mathfrak{f} + \mathfrak{h}(t)$ so $W(t)$ is an $\mathfrak{f} + \mathfrak{h}(t)$-module. We will need that $W(t) = \Sigma_j V_j \otimes U_j(t)$ where the V_j are irreducible $\mathfrak{f}$-modules all of the same dimension and the $U_j(t)$ are irreducible $\mathfrak{h}(t)$-modules also all of the same dimension $u(t)$. We will also need that rank $\mathfrak{g}(t) = \text{rank}\, \mathfrak{f} + \text{rank}\, \mathfrak{h}(t)$ so we may chose Cartan subalgebras such that $\mathfrak{t}_{\mathfrak{g}} = \mathfrak{t}_f \oplus \mathfrak{t}_{\mathfrak{h}}$. When there is no confusion, we supress the t. The roots of $\mathfrak{g}(t)$ are

- the roots of $\mathfrak{f}$,
- the roots of $\mathfrak{h}$,
- the weights $\mu + v$ with μ a weight of some V_j and v a weight of $U_j(t)$.

To get a set of positive roots we choose linear forms l and l_t on the root lattices, that are strictly positive on positive roots and heavily favor the roots of $\mathfrak{f}$, so that the positive roots are:

- the positive roots of $\mathfrak{f}$,
- the positive roots of $\mathfrak{h}$,
- the weights $\mu + v$, with μ being a weight of V such that $l(\mu) > 0$ and v a weight of U.

We may write the half-sum of the positive roots as $\rho_{\mathfrak{g}(t)} = \rho_f + \rho_{\mathfrak{h}} + u(t)\gamma$, where γ is one-half the sum of the positive weights of the V_j's (positive in the sense that l takes positive values on them: we denote by $\Delta_+(V)$ the set of these weights). We must reparametrize, if necessary, so that u is a linear function of t.

Now let ω be a weight of $\mathfrak{g}$ supported on $\gamma \in \mathfrak{t}_f$. This means that ω is a weight of $\mathfrak{f}$ satisfying the integrality condition that $2(\omega, \mu)/(\mu, \mu) \in \mathbb{Z}$ for all $\mu \in \Delta_+(V)$. (So in particular, p above can at most be equal to the rank of $\mathfrak{f}$.)

We apply the Weyl dimension formula to ω. The contribution of the roots of $\mathfrak{h}$ to the product is trivial. The contribution of the roots of $\mathfrak{f}$ is

$$\prod_{\alpha \in \Delta_+(\mathfrak{f})} \frac{(\rho_{\mathfrak{f}} + u(t)\gamma + \omega, \alpha)}{(\rho_{\mathfrak{f}} + u(t)\gamma, \alpha)}.$$

The contribution of the other roots is more complicated, and that is to control this contribution that we need to add our most serious hypothesis: we require that when t varies, the integers $(\rho_{\mathfrak{g}(t)}, \mu + v)$, for each set of values of (λ_i, μ), not all zero, is the union of a fixed number of intervals $[n_i(t)+1, m_i(t)]$, where $n_i(t)$ and $m_i(t)$ are *linear* functions of t. We allow that for some values of t, $m_i(t)<n_i(t)$, which is to be interpreted as deleting the interval $[m_i(t)+1, n_i(t)]$. Then the contribution of such an interval to the Weyl dimension formula is

$$\frac{\binom{(\omega,\mu)+m_i}{(\omega,\mu)}}{\binom{(\omega,\mu)+n_i}{(\omega,\mu)}}.$$

Putting these contributions together, we see that we have a series in the dimension sense of Deligne.

EXAMPLE. Here is a classical example. Let $\mathfrak{g}(t)=\mathfrak{so}_{2t+4}$, $\mathfrak{f}=\mathfrak{sl}_2$, $\mathfrak{h}(t)=\mathfrak{sl}_2\times\mathfrak{so}_{2t}$, $V=\mathbb{C}^2$, $U=\mathbb{C}^2\otimes\mathbb{C}^{2t}$. Let $\mathfrak{f}$ have root α and the $\mathfrak{sl}_2$ in $\mathfrak{h}(t)$ have root β. We use α_j to describe the roots of $\mathfrak{so}_{2l}$ and sometimes the ε_j's instead. The positive roots of $\mathfrak{g}(t)$ are:

- α,
- β, $\Delta_+(\mathfrak{so}_{2t})$,
- the weights $\frac{1}{2}\alpha\pm\frac{1}{2}\beta\pm\varepsilon_j$, $j=1,\dots,l$.

We have $\rho_{\mathfrak{g}_{\mathfrak{f}}}=\frac{1}{2}\alpha$, $\gamma=\frac{1}{2}\alpha$ and $\rho_{\mathfrak{h}}=\frac{1}{2}\beta+(l-1)\varepsilon_1+(l-2)\varepsilon_2+\cdots+\varepsilon_{l-1}$. Thus, taking inner products such that $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$, $(\alpha,\alpha)=(\beta,\beta)=2$, we have $(\rho_{\mathfrak{g}(t)},\mu+v)$ filling the intervals $[1,2t-1]$, $[2,2t]$, plus the isolated values t and $t+1$. Thus applying our general formula we obtain

$$\dim\mathfrak{g}^{(k)}=\frac{(2k+2t+1)(k+t)(k+t+1)}{(2t+1)t(t+1)(k+1)}\binom{k+2t-1}{k}\binom{k+2t}{k},$$

which is easy to obtain by directly applying the Weyl dimension formula.

EXAMPLE. The generalized third row. With the notations of Section 4, we have

$$\mathfrak{g}_r(\mathbb{A},\mathbb{H})=\mathfrak{t}_r(\mathbb{A})\times\mathfrak{sl}_2^{\times r}\oplus(\oplus_{i<j}U_i\otimes U_j\otimes\mathbb{A}_{ij}).$$

With our conventions, the positive roots of $\mathfrak{g}(t)$ are:

- the positive roots α_i, $1\leqslant i\leqslant r$, of $\mathfrak{sl}_2^{\times r}$,
- the positive roots of $\mathfrak{t}_r(\mathbb{A})$,
- the weights $\omega_i-\omega_j+\mu_{ij}$, $i<j$.

Write the half-sum of positive roots as $\rho = \rho_{\mathfrak{t}_r(\mathbb{H})} + \rho_{\mathfrak{t}_r(\mathbb{A})} + a\gamma$ with $2\gamma = (r-1)\alpha_1 + (r-2)\alpha_2 + \cdots + \alpha_{r-1}$. Applying our method once again, we obtain the three-parameter formula

$$\dim \mathfrak{g}_r(\mathbb{H}, \mathbb{A})^{(k)} = \frac{2k + a(r-1) + 1}{a(r-1) + 1} \frac{\binom{k + \frac{ar}{2} - 1}{k} \binom{k + ar - a}{k} \binom{k + \frac{ar-a}{2}}{k} \binom{k + ar + 1 - \frac{3a}{2}}{k}}{\binom{k + \frac{a}{2} - 1}{k} \binom{k + \frac{ar}{2} + 1 - a}{k} \binom{k + \frac{ar-a}{2}}{k}}.$$

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