Homework on monoidal categories

June 14, 2017

Exercise 1. Show that fdVec and Mat are equivalent as categories.

Hint: Pick a basis for every finite dimensional vector space. □

Exercise 2. Show that every category is equivalent to a skeletal category.

Exercise 3. Show that $TL(d = 1)$ is monoidally equivalent to $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$.

Exercise 4. If $\mathcal{C}$ is a monoidal category, $\mathcal{D}$ is a category, and we have an equivalence $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$, show you can make D into a tensor monoidal in such a way that F and G are a monoidal equivalence.

(This is an instance of the idea of 'transport of structure'.)

Exercise 5. Conclude that every monoidal category is equivalent to a skeletal monoidal category.

Exercise 6. Recall that a monoidal functor is a pair: a functor and a natural isomorphism $F(X \otimes Y) \to F(X) \otimes F(Y)$. Classify all monoidal functors from $\mathrm{Vec}(G)$ to $\mathrm{Vec}(G)$ where the underlying functor is the identity functor.

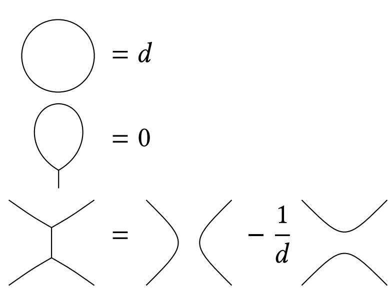
Hint: Your answer should involve group cohomology of G. □

Exercise 7. Find the fusion rules for $TL(q + q^{-1})$ at q a root of unity.

Conclude that the last surviving Jones-Wenzl idempotent is invertible. This gives a 2-object subcategory where all the objects are invertible — how does it fit into our classification?

Hint: Suppose we are a root of unity such that $f^{(n+1)}$ is the first negligible Jones-Wenzl idempotent. Prove that $f^{(k)} \otimes f^{(1)} \cong f^{(k-1)} \oplus f^{(k+1)}$ unless $k = 0$ or $k = n$. What happens in those cases? From these observations, give a formula for how $f^{(a)} \otimes f^{(b)}$ breaks up as a direct sum of Jones-Wenzl idempotents. □

Recall that the golden category has as objects finite subsets of an interval, and the morphisms are planar trivalent graphs modulo the local relations



where $d = \frac{1+\sqrt{5}}{2}$.

Exercise 8. Show that the golden category, as defined in lecture and above, is semisimple with two simple objects 1 and X and in particular that $X \otimes X \oplus 1 \oplus X$.

Exercise 9. (Checking every detail here is tedious; use your judgement!)

Show that every monoidal category C is monoidally equivalent to the strict monoidal category $\mathsf{List}C$.

Here $\mathsf{List}C$ has as objects words in the objects of C, and

$$\mathsf{List}C([x_1, x_2, \dots x_n] \to [y_1, y_2, \dots y_m]) = C(x_1 \otimes (x_2 \otimes \cdots \otimes (x_n \otimes 1)) \to y_1 \otimes (y_2 \otimes \cdots \otimes (y_m \otimes 1))).$$

Part of the exercise is to define the tensor product of morphisms in $\mathsf{List}C$. The functors between C and $\mathsf{List}C$ should send x to $[x]$ and $[x_1, x_2, \dots x_n]$ to $x_1 \otimes (x_2 \otimes \cdots \otimes (x_n \otimes 1))$. You'll need to specify what the functors do on morphisms, and make them into monoidal functors by specifying tensorators. Finally you'll need to show that these functors form an equivalence; you can do this directly, or show one of the functors is fully faithful and essentially surjective.