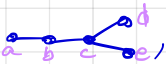
Planar Algebras 4

More Graph Planar algebras!

$G =$ 

, $GPA(G)_n = \{ \text{f'ns } \{\text{length-}n \text{ loops on } G\} \to \mathbb{C} \}$

$$T(f_1, \dots, f_k)(\ell) := \sum_{\substack{\text{states } \sigma \\ \text{on } T, \\ \sigma|_0 = \ell}} \prod_{i=1}^{k} f_i(\sigma|_i);$$

eg: $\bigcirc\,(v) \longrightarrow \deg(v)$

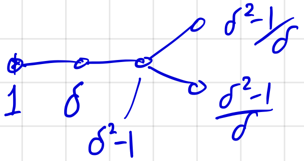
exercise: convince yourself this is a planar algebra (isotopy invariant? composition?)

Goal: define another action of planar diagrams on $GPA(G)$, which has closed loops count for a fixed quantity ('loop parameter' δ)

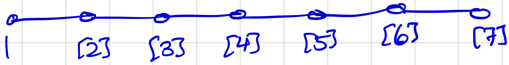
Def: The Perron-Frobenius eigenvector of a graph is a weight $d: V(G) \to \mathbb{R}$ s.t.
(1) $d(v) > 0 \quad \forall v$
(2) $\exists\, \delta \in \mathbb{R}_{>0}$ s.t. $\delta \cdot d(v) = \sum_{v' \text{ adjacent to } v} d(v')$

Thm (Perron-Frobenius): such a δ & d exist:
δ is norm-largest eigenvalue of adjacency matrix A
w is the associated eigenvector

(observe that multiplication by A is a contraction on lines in the first quadrant ("quadrant") in $\mathbb{R}^n$; therefore it has a fixed point)

E_6: 

$$\delta(\delta^2-1) = \delta + 2\cdot\frac{\delta^2-1}{\delta} \Rightarrow$$
$$\delta^2(\delta^2-1) = \delta^2 + 2(\delta^2-1) \Rightarrow$$
$$\delta^4 - 4\delta^2 - 2 = 0 \Rightarrow \delta^2 = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

EG: 

if $[8]=0$

action 2: Suppose G is a bipartite graph, so all loops have even length. Then $GPA(G)_{\text{odd } n} = 0$.
a planar diagram T acts on input f'nls $f_1, \dots, f_k$ by

$$T(f_1, f_2, \dots, f_k)(\ell) = \sum_{\substack{\text{states } \sigma \text{ on } T \\ \text{s.t. } \sigma|_0 = \ell}} \prod_{i=1}^{k} c(T,\sigma)\, f_i(\sigma|_i)$$

with $c(T,\sigma) := \prod_{t \in E(T)} \sqrt{\dfrac{d(\sigma(t_{\text{convex}}))}{d(\sigma(t_{\text{concave}}))}}$

... ?
Draw T in 'standard form': all disks are rectangles, parallel to axes,
same # strings meeting top & bottom (none on sides),
*-marked regions on left side, smooth strands.
now $E(T)$ = set of local min & max of strands of T
$t_{\text{convex}} = \sigma$(region on convex side of t)
$t_{\text{concave}} = \sigma$(—— " —— concave —— " ——)

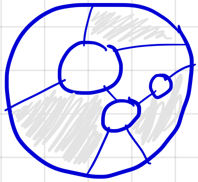
Q: Is this still a planar algebra?
A: yes, but ugh, we don't want to prove it.

exercise: the examples we did for action 1?
redo them for action 2 (including exercises).

exercise: Let T be a diagram with a closed loop, $\tilde{T}$ same diagram with loop removed.
Show $T(f_1, \dots, f_k) = \delta \cdot \tilde{T}(f_1, \dots, f_k)$
(hint: this is not hard!)

Def'n A *subfactor planar algebra* has

① an action by *checkerboard-shaded* planar diagrams



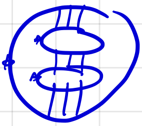
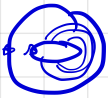
on $\{V_{2n,\pm}\}$

(+ or − depending on position of $*$: unshaded or shaded)

② an *adjoint* $*$ on each $V_{2n,\pm}$, compatible w/ reflection of diagrams

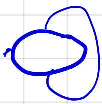
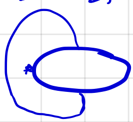
③ $V_{0,\pm}$ are 1-dimensional; $V_{2n,\pm}$ are fin dim.
↳ identify w/ $\mathbb{C}$ by $\emptyset \mapsto 1$

④ $\langle x,y \rangle := tr(y^*x)$ is pos. def.

(mult:  $: V_{2n} \otimes V_{2n} \to V_{2n}$, tr:  $: V_{2n} \to V_0$

⑤ *Sphericality*:

$$tr_R : V_{2,+} \to V_{0,+} \cong \mathbb{C} \; = \; tr_L : V_{2,-} \to V_{0,-} \cong \mathbb{C}$$

 = 

Since $V_{0,\pm} \cong \mathbb{C}$, closed loops must count for $\delta_\pm$.
$\delta_+ = \delta_-$ by sphericality.
Therefore, we have a map $TL_n \to V_{2n}$ for any subfactor planar algebra:

exercise: If G is a bipartite graph, then $GPA(G)$, with 2^{nd} action, is almost a subfactor planar algebra. Verify all conditions except one.

Subfactors $\rightsquigarrow$ SPAs!

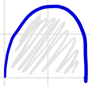
$A \subseteq B$

In $Bim(A)$, let $X = {}_A L^2 B_B$, $\bar{X} = {}_B L^2 B^{op}_A$

$V_{2n} = \mathrm{End}(\underbrace{X \otimes \bar{X} \otimes \cdots \otimes (X \text{ or } \bar{X})}_{n})$ $\quad$ $\left(X^{\otimes 0} \ ? := {}_A L^2 A_A \right.$
$\quad \bar{X}^{\otimes 0} \ ? := {}_B L^2 B_B$

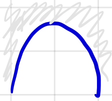
action by shaded planar tangles?

X labels a strand; unshaded regions $\rightsquigarrow A$, shaded $\rightsquigarrow B$



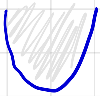
: $X \otimes \bar{X} = {}_A L^2 B_A \longrightarrow {}_A L^2 A_A$

conditional expectation



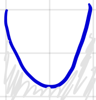
: $\bar{X} \otimes X = {}_B L^2 B^{op}_A \otimes L^2 B_B \longrightarrow {}_B L^2 B_B$

multiplication on B



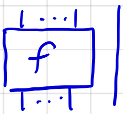
: ${}_A L^2 A_A \longrightarrow {}_A L^2 B_A$

inclusion



: ${}_B L^2 B_B \longrightarrow {}_B L^2 B^{op} \underset{A}{\otimes} L^2 B_B$

... Pimsner-Popa basis...



$\rightsquigarrow f \otimes \mathrm{id}_X$

Thm (Jones) these ingredients fit together consistently to give a SPA. (5)

Thm (Popa) (Hard!) From a SPA, can construct a subfactor w/ same principal graph.