Exercise 1. Let $\mathscr{A} = \mathbb{C}[t]$ be the algebra of polynomials in one variable t, and identify $\mathscr{A} \otimes \mathscr{A}$ with the algebra $\mathbb{C}[x, y]$ of (commutative) polynomials in two variables x, y. Define a derivation $\partial : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ by

$$\begin{aligned} \partial(t) &= 1 \otimes 1 \\ \partial(pq) &= p\partial(q) + \partial(p)q, \qquad \forall p, q \in \mathscr{A}. \end{aligned}$$

Show that this defines ∂ uniquely; show moreover that with the identifications above,

$$\partial(p) = \frac{p(x) - p(y)}{x - y}, \qquad \forall p \in \mathscr{A}.$$

(In fact, this is an if and only if).

Exercise 2. Let μ be a probability measure on $[0, 1]$ (e.g., the uniform measure). With the notation of the previous exercise, regard ∂ as an unbounded operator $L^2([0, 1], \mu) \to L^2([0, 1], \mu)$. Show that if ∂ is closable, then μ is non-atomic. (Hint: if $\mu(\{a\}) \neq 0$ then $\mu \times \mu(\{(a, a)\}) \neq 0$; now manufacture a family of functions f_n so that $\|f_n\|_{L^2(\mu)} \to 0$ by $\|\partial f_n\|_{L^2(\mu\times\mu)} \not\to 0$ as $n \to \infty$).

Exercise 3. Let ν be a probability measure on $\mathbb{R}$ and denote by $G_\nu(z)$ the Cauchy transform

$$G_\nu(z) = \int \frac{d\nu(t)}{z - t}.$$

The goal is to show that that for any interval $I = [a, b] \subset \mathbb{R}$,

$$\lim_{y\downarrow 0} \int_I -\frac{1}{\pi}\Im[G_\nu(x + iy)]dx = \nu((a, b)) + \frac{1}{2}(\nu(a) + \nu(b)).$$

In particular, if ν has a continuous density ρ at a point $x \in \mathbb{R}$, then $\rho(x) = \lim_{y\downarrow 0} -\pi^{-1}\Im[G_\nu(x + iy)]$.

(a) Show that

$$\begin{aligned} -\pi^{-1}G_\nu(x + iy) &= -\pi^{-1} \int \Im[(iy + x - t)^{-1}]d\nu(t) \\ &= \frac{1}{\pi} \int \frac{y}{(x - t)^2 + y^2} d\nu(t) = \nu * \alpha_y \end{aligned}$$

where

$$\alpha_y = \frac{y}{x^2 + y^2}$$

is the Cauchy law with scale parameter y.

(b) If X and Y are random variables so that $X \sim \nu$ and $Y \sim \alpha_1$ then

$$\nu * \alpha_y \sim X + yY$$

so that

$$\int_I -\frac{1}{\pi}\Im(G_\nu(x + iy))dx = \mathbb{E}[1_{X+yY\in[a,b]}].$$

Now as $y \downarrow 0$, show that

$$1_{X+yY\in[a,b]} \to 1_{X\in(a,b)} + 1_{X=a\ \&\ Y\geq 0} + 1_{X=b\ \&\ Y\leq 0},$$

which gives the advertised formula for the limit (Hint: X and Y are independent and the probability that $Y \geq 0$ is 1/2).

Exercise 4. Let $P = TL(\delta)$ be the Temperley-Lieb-Jones planar algebra with loop parameter δ. Recall that $Gr_0(P)$ is defined on drawing diagrams in P with all strings up and multiplying them by putting the diagrams next to each other.

Let $x = \cup \in P$; thus $x^2 = \cup\cup$, $x^3 = \cup\cup\cup$ and so on.

Denote

$$\tau(q) = \sum_{D\in TL} \begin{array}{c}\boxed{\;D\;}\\ |\cdots||\\ \boxed{\qquad q \qquad}\end{array} .$$

Let $\alpha_m = \tau(q^m)$.

(a) Derive a recursive formula for α_m (hint: follow the leftmost string of the first $\cup$ in x^m). Show that

$$\|x\| = \limsup_{n\to\infty} [\tau(x^{2n})]^{1/2n}$$

is finite.

(b) Let $G(z) = \tau((z-x)^{-1})$. Show that $G(z) = \sum \alpha_m z^{-m+1}$ and is in particular analytic for $|z| > \|x\|$. (Hint: expand the resolvent $(z-x)^{-1} = z(1-(x/z))^{-1}$ as a power series in x/z). Note that $G(z) \sim z^{-1}$ as $|z| \to \infty$.

(c) Let $\Phi(z) = \sum \alpha_m z^m = zG(z^{-1})$. Use your result from (a) to show that $\Phi(z)$ satisfies the quadratic equation

$$\Phi(z) - 1 = z(\delta - 1)\Phi(z) + z\Phi(z)^2$$

and find $G(z)$ (the choice of which root of the quadratic equation to take is dictated by the asymptotics of $G(z)$ as $z \to \infty$).

(d) Let $dP(\lambda)$ denote the projection-valued spectral measure of x so that

$$x = \int \lambda dP(\lambda).$$

Conclude that if $d\mu(\lambda) = \tau(dP(\lambda))$ then

$$G(z) = \tau[(x-z)^{-1}] = \int \frac{1}{\lambda - z} d\mu(\lambda)$$

is the Cauchy transform of the measure μ.

(e) Use the previous exercise to show that μ is the measure

$$d\nu(t) = \frac{1}{2\pi\delta} t^{-1} \sqrt{4\delta - (t-(1+\delta))^2}\, \chi_{[(1-\sqrt{\delta})^2, (1+\sqrt{\delta})^2]} dt;$$

in particular the law is non-atomic.