The Jones tower and Temperley-Lieb algebras

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We recall from the last round of exercises and lectures that if $N \subseteq M$ is a finite index inclusion of II_1 factors, and $e_N \in B(L^2(M,\tau))$ is the orthogonal projection onto $L^2(N,\tau) \subseteq L^2(M,\tau)$ we have the following

1. $e_N \in N'$
2. $N = M \cap \{e_N\}'$
3. $W^*(e_N, M) = JN'J$ is the basic construction.
4. $W^*(M, e_n)$ is a II_1 factor, and $[W^*(M,e_n) : M] = [M : N]$
5. $\tau_{W^*(M,e_n)}(e_N) = [M : N]^{-1}$

We return briefly to the notion of *conditional expectation.* Recall that if $N \subseteq M$ is an inclusion of II_1 factors, we claimed (without proof) that $e_N(M) \subseteq N$, which is not obvious. Indeed, we know that by definition $e_N(M) \subseteq L^2(N)$, but apriori we do not know that $e_N(M)$ lands inside the special, non-closed subspace $N \subseteq L^2(N)$. For details, we recommend consulting "Introduction to Subfactors" by Jones and Sunder, Section 3.1. Assuming this, then we can define a map $E_N : M \to N$, by $E(m) := e_N(m) \in N$. This is positive, faithful, unital, and N-bimodular (see previous exercises for precise statements).

1. Show $\tau_M(nm) = \tau_N(nE_N(m))$ for all $n \in N$ and $m \in M$. In particular, setting $n = 1$, we see that $\tau_N(E_N(x)) = \tau_M(x)$
2. Show $E_N(x)e_N = e_N x e_N$ as operators on $L^2(M)$ (This was an exercise on the previous set).
3. Let $E_M : W^*(M, e_N) \to M$ denote the conditional expectation defined as above. Show $E_M(e_N) = [M : N]^{-1}1$.

Now, we switch notation slightly, and begin with a finite index inclusion $M_0 \subseteq M_1$ of II_1 factors. Then we define $M_2 := W^*(M_1, e_1)$, where $e_1 : L^2(M_1) \to L^2(M_0)$ is the orthogonal projection as discussed above. Now, since $M_1 \subseteq M_2$ has finite index ($[M_2 : M_1] = [M_1 : M_2]$), we can do this construction again, and define $M_3 := W^*(M_2, e_2)$, where now $e_2 : L^2(M_2) \to L^2(M_1)$ is again an orthogonal projection.

Iterating this process, we obtain a tower of finite index inclusions $M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots$, where $M_{i+1} := W^*(M_i, e_i) \subseteq B(L^2(M_i))$. Notice that we are changing Hilbert spaces when we construct the next factor, but the algebras that were already constructed come along for the ride.

Since there is a unique trace at each stage in the tower, we can define the inductive limit von Neumann algebra $M_\infty = (\cup_i M_i)''$, where the double commutant is taken in the GNS representation of $\cup_i M_i$ with respect to the inductive limit of the unique traces on the M_i. It turns out that this trace is unique, and M_∞ is a II_1 factor.

Then we can view $\{e_i\}_{i=1}^\infty$ as a sequence of projections in M_∞, called the *Jones projections*, not to be confused with the *Jones-Wenzl* projections encountered previously. These projections have the following properties, the first 2 which follow from previous exercises.

1. $\tau(e_n) = [M_1 : M_0]^{-1}$ for all n.
2. $\tau(xe_n) = [M_1 : M_0]^{-1}\tau(x)$ for all $x \in M_n$
3. Show $e_n \in (M_{n-1})' \cap M_{n+1}$ for all $n \geq 1$, hence $e_n e_m = e_m e_n$ if $|m - n| > 1$.
4. Show $e_{n+1}e_n e_{n+1} = [M_1 : N_0]^{-1} e_{n+1}$ for all $n \geq 1$.
5. Show $e_n e_{n+1} e_n = [M_1 : N_0]^{-1}$ for all $n \geq 1$. (Hint: use the fact that $u = [M_1 : N_0]^{-\frac{1}{2}} e_n e_{n+1}$ is a partial isometry, satisfying $u^*u = e_{n+1}$. Then identify the equivalent projection uu^* as a sub-projection of another one, and use faitfulness of the trace.

The algebras generated by these projections are not actually new to us. Indeed, they are familiar friends in disguise, which hopefully will demonstrate the concrete ties between subfactor theory and planar algebras. Define the abstract $*$-algebra $A_n(\delta) := Alg(\{e_i\}_{i=1,n-1})$ to be the linear span of projections satisfying the relations $(3), (4), (5)$, with $\delta = [M_1 : N_0]^{-\frac{1}{2}}$.

1. Define a word in the set $\{e_i\}_{i=1,n-1}$ to be *reduced* if it is of minimal length under the grammatical replacement rules $e_i^2 \leftrightarrow e_i,\ e_i e_j \leftrightarrow$

$e_j e_i$, and $e_i e_{i\pm 1} e_i \leftrightarrow e_i$. Show that if $e_{i_1} e_{i_2} \dots e_{i_k}$ is a reduced word, then $m = max\{i_1, \dots, i_k\}$ occurs only once on the list $i_1, \dots i_k$.

2. Show that any reduced word can be uniquely written in the form $(e_{j_1} e_{j_1 - 1} \dots e_{k_1})(e_{j_2} e_{j_2 - 1} \dots e_{k_2}) \dots (e_{j_p} e_{j_p - 1} \dots e_{k_p})$, where j_p is the maximum index, $j_i \geq k_i$, $j_{i+1} > j_i$, and $k_{i+1} > k_i$, and conclude that the set of reduced words form a basis for $A_n(\delta)$.

3. Define a map from the set of words on the $\{e_i\}_{i-1,n}$ to $TL_n(\delta)$ by extending the assignment

$$e_i \mapsto \frac{1}{\delta} \quad \Big| \, i-1 \, \Big| \; \substack{\smile \\ \frown} \; \Big| \, n-i-1 \, \Big| \quad \in TL_n(\delta)$$

. Show this map extends to a $*$-algebra isomorphism. (Hint: To obtain a homomorphism is straightforward. To get an isomorphism, show that every TL_n can be written as a "reduced" word as in the previous exercise to show that the obvious inverse map is well defined.)

4. A reduced word is *cyclically reduced* if it is of minimal among all cyclic permutations of the word, subject to the grammatical replacement rules of the previous exercise. Show that every cyclically reduced word is of the form $w = e_{i_1} \dots e_{i_k}$ where $|i_j - i_l| \geq 2$ for all $j \neq l$. Conclude there is a unique trace on $A_n(\delta)$ such that $\tau(w) = \frac{1}{\delta^{2k}}$, for any cyclically reduced word of length k.

5. Suppose that δ^2 is the index of a finite index subfactor $M_0 \subseteq M_1$. Show that on the finite dimensional C^* generated by the image of $A_n(\delta)$ in M_∞, the trace τ from M_∞ is equal to $\frac{1}{\delta^n} Tr(\cdot)$, where $Tr(\cdot)$ is the picture trace on TL_n. Since τ is positive definite, conclude that $\delta^2 \in \{4\cos^2(\frac{\pi}{n}) \;:\; n \geq 3\} \cup [4, \infty)$.