Subfactor exercises

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Let M be a finite von Neuman algebra with normal faithful trace τ. Let $L^2(M,\tau)$ denote the Hilbert space completion of M with respect to the inner product $\langle x, y\rangle = \tau(y^*x)$. Then M acts on $L^2(M,\tau)$, and this representation is called the *standard form of M*. This representation plays a special role in the theory of subfactors, and has a number of very special properties. Define the (conjugate linear) map $J : M \to M$ by $J(x) = x^*$. Since τ is a trace, J extends to an isometry on the Hilbert space completion. It is straightforward to check that $JMJ \subseteq M' \cap B(L^2(M,\tau))$. However, we also have the reverse inclusion, which gives the following theorem:

Theorem 0.1. *Let M be a II_1 factor. $M' \cap B(L^2(M,\tau)) = JMJ$.*

For a series of exercises which walk through a proof of this fact (and much more!), see David Penney's notes "On the 2-category of tracial von Neumann algebras", Section 1.2, which is available by google search. We will use this fact and the operator J extensively in the following series of exercises.

We also have the following theorem for finite von Neumann algebras, which was mentioned in Dima's lectures:

Theorem 0.2. *(Rank-Nullity Theorem). Suppose $M \subseteq B(H)$ is a finite von Neumann algebra with faithful normal tracial state τ. Then for $x \in M$, The projection onto the kernel $P_{Ker(x)}$ and the projection onto the image $P_{Im(x)}$ are both in M, and satisfy $\tau(P_{Ker(x)}) + \tau(P_{Im(x)}) = 1$.*

This can be proven using the *polar decomposition* of operators in a von Neumann algebra, but would take us too far afield. Instead, we content ourselves with the aesthetic satisfaction this theorem provides as a generalization of our familiar friend from linear algebra.

1. For a subset $M \subseteq B(H)$, show that $M' = M'''$

2. Let M be a II_1 factor. Assume $\xi \in M$. Prove that $\tau_{M'}(P_{\overline{M\xi}}) = \tau_M(P_{\overline{M'\xi}})$. (Hint: use J, the fact that $\tau(x) = \langle x \cdot 1, 1\rangle$ for $x \in M$ or in M', and the rank-nullity Theorem).

3. Work this out explicitly in the case where $M = M_{100\times 100}$, and $\tau(\cdot) = \frac{1}{100}Tr(\cdot)$.

4. Let $M = M_{n\times n}$ be the $n \times n$ matrices, and consider the module $M_{n\times m}$, thought of as a Hilbert space where matrix units form an orthonormal basis, and the action of M is the obvious one from matrix multiplication. Identify M', compute the dimension $\frac{\tau_{M'}(P_{\overline{My}})}{\tau_M(P_{\overline{M'y}})}$ for $y \neq 0$ and observe this does not depend on y.

5. Suppose $N \subseteq M$ is a finite index II_1 subfactor. Recall that e_N is the orthogonal projection onto $L^2(N,\tau) \subseteq L^2(M,\tau)$. It turns out that $e_N(M) \subseteq N$ inside $L^2(M,\tau)$, and induces a map called a *conditional expectation* $E : M \to N$, defined by $e_N(m \cdot 1) = E(m) \cdot 1$ that satisfies the following properties:

   - $E(x^*x) \geq 0$, and $E(x^*x) = 0$ implies $x = 0$
   - $E(1) = 1$
   - $E(n_1mn_2) = n_1E(m)n_2$ for all $n_1, n_2 \in N$, $m \in M$

   We remark that a-priori, $e_N(M) \subseteq L^2(N,\tau)$, and there is no obvious reason why it should land in the special subspace $N \subseteq L^2(N,\tau)$, however it does. For a proof of this, see the book "Introduction to Subfactors" by Jones and Sunder, Section 3.1.

   Show that $e_Nxe_N = E(x)e_N$ for all $x \in M$. Show that $N = \{e_N\}' \cap M$

6. Define $P := W^*(M, e_N)$, and $M_1 := JN'J$. Show that $P = M_1$ (Hint: use the fact that J commutes with e_N and the bicommutant theorem. Also note all commutants are taken in $B(L^2(M,\tau))$). This factor is called the *basic construction*

7. Compute $\tau_{M_1}(e_N)$.

8. Let G be a finite group acting outerly on a II_1 factor. Show that $M \rtimes G$ is isomorphic to the basic construction of the subfactor $M^G \subseteq M$. (Hint: Identify the Jones projection e_{M^G} in the crossed product.)

9. Show that if $[M : N] = 2$, then there exists an action of $\mathbb{Z}/2\mathbb{Z}$ on N such that $M \cong N \rtimes \mathbb{Z}/2\mathbb{Z}$ (Hint: Use the fact that if two projections p, q in a II_1 factor have the same trace, then there exists a unitary $u \in M$ such that $upu^* = q$)