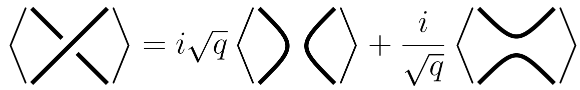
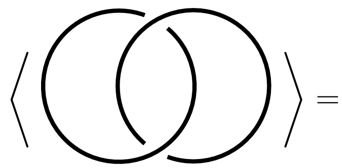
MSRI summer school: Quantum Symmetries | **Planar Algebras Homework 1** | **June 13, 2017**

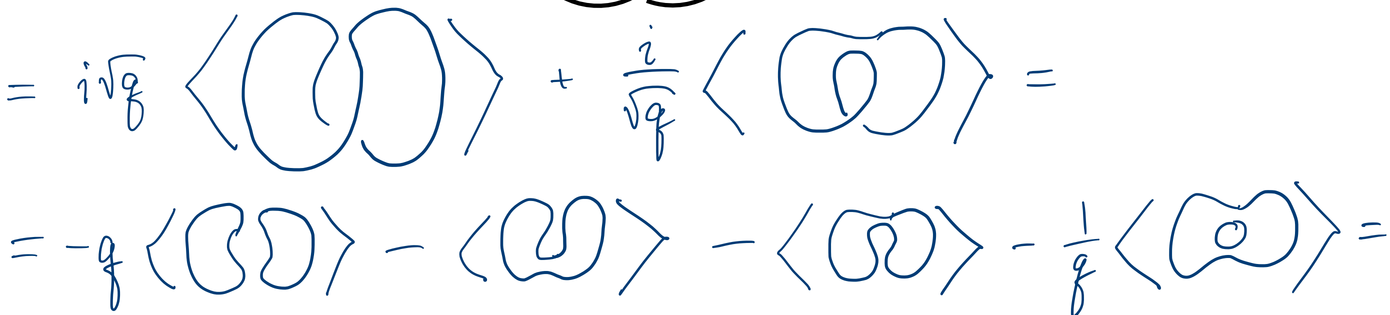
Some Knot Theory

1. To construct the Kauffman bracket of a knot, we start with a planar knot diagram, and produce an elements of TL_0, by ‘resolving’ each crossing:



For example,

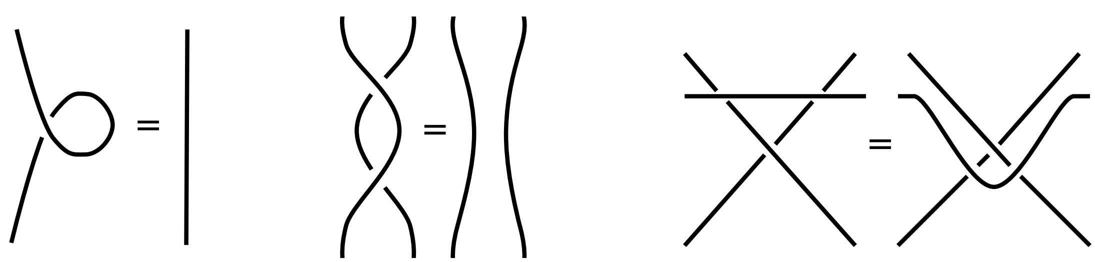




$$= -q[2]^2 - [2] - [2] - \frac{1}{q}[2]^2 = -q^3 - 5q - 5q^{-1} - q^{-3}$$

Compute

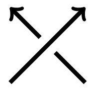
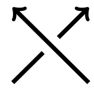
2. It’s a theorem that two knot diagrams represent the same knot if you can get from one to the other in a series of Reidemeister moves (called I, II and III):



Show that Reidemeister moves II and III don’t change the value of the Kauffman bracket, but Reidemeister move I does.

3. We can either live with the Kauffman bracket’s failure to be preserved by Reidemeister move I, and view it as an invariant of a framed link; or we can fix it.

To calculate the *writhe* of a knot or a link, we first orient all components, and then assign a value of $+1$ or -1 to each crossing based on whether it's right-handed or left-handed:

 gets $+1$ and  gets -1

(a) Show that the writhe of a knot or link is invariant under Reidemeister moves II and III.

(b) We saw that the bracket polynomial was also invariant under Reidemeister moves II and III. Can you combine the bracket polynomial and the writhe to make a polynomial for knot projections which is invariant under all Reidemeister moves?

Lecture 2 Exercises

4. If D_1 and D_2 are TL_n diagrams,

$$\langle D_1, D_2 \rangle = \begin{cases} [2]_q^n \text{ if } D_1 = D_2 \\ [2]_q^m \text{ with } m < n \text{ if } D_1 \neq D_2 \end{cases}$$

5. In TL_n, show that the set of negligible elements is an ideal.

6. If $f^{(n)}$ exists, show that $\mathrm{tr}(f^{(n)}) = [n+1]_q$

7. Show that the number of TL_n diagrams is equal to the n^{th} Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, either directly, or by finding a bijection with something else the Catalan numbers count (for instance, allowed configurations of n open and n closed parenthesis: if $n = 3$, ()(()) is allowed and ())(() is not.)

8. Suppose $0 < [2]_q < 2$ and $[2]_q \neq 2\cos(\frac{\pi}{n})$ for any n. Show that there is some m such that

   1. TL_m has no negligible elements, and
   2. $\langle f^{(m)}, f^{(m)} \rangle = [m+1]_q < 0$.

   Conclude that TL_m is not positive definite.

Lecture 1 Exercises

9. Verify that $[2]_q[n]_q = [n+1]_q + [n-1]_q$.

10. Find the minimal central projections in TL_3

11. Show that the coefficient of $\mathbf{1}_n$ in an uncappable projection is 1.

12. Show that $f^{(n)}f^{(m)} = f^{(\max\{n,m\})}$.

13. Generate $f^{(3)}$, either from Wenzl's relation or the definition of the Jones-Wenzl projection, and show that it is one of the minimal central projections in TL_3 that you found above.

14. Finish our proof that the Jones-Wenzl projection $f^{(n)}$ exists if $[n]_q \neq 0$, by showing that the right hand side of Wenzl's relation is uncappable.