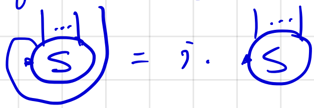
# Planar Algebras 6

**Def'n** Consider the unshaded planar algebra $PA(S)$: generated by a single self-adjoint $(4n-4)$-strand generator $S$, w/ relations:

(1) $[2]_q = 2\cos\left(\pi/4n-2\right) = \delta$

(2)

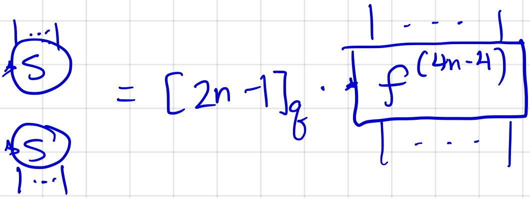



rotation

(3)


uncappable

(4)




two-S rel'n

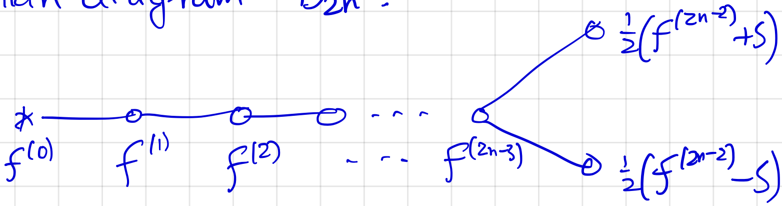
**Thm** $PA(S)$ is a subfactor planar algebra

(A) closed diagrams are 1-dimensional

(B) spherical

(C) $\langle \cdot , \cdot \rangle$ is positive definite

(D) principal graph is Dynkin diagram $D_{2n}$:



Pf: w/ $\times := i\sqrt{q}\ )( \ - \frac{i}{\sqrt{q}} \ \underset{\frown}{\smile}$,

we can show

(exercise)

(A) This makes evaluation algorithm easy:

To show $P_{0,\pm}$ (given by generators and relations) is $\leq$ 1-dim, give an algorithm that takes any closed network of generators + reduces it to a multiple of the empty diagram

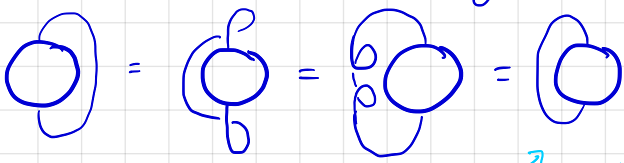
For a diagram in $PA(S)$,

1. Pull all copies of S to outside region (under strings), rotate so $*$ is in outside region.
2. Apply 2-S relation in pairs until 1 or 0 S's remain
3. 1 S: $= 0$ b/c S is uncappable
   0 S: evaluate in TL

Thus $\dim(PA(S)_0) \leq 1$.
Is it zero?
No - manually prove consistency (that if we made any different choices in algorithm, we'd get the same result.)

(exercise?)

(B) sphericality is easy because of $X$:



twists in opposite directions cancel

(C) As in exercises, construct an orthogonal tree basis

(D) At this value of $\delta$, we have Jones-Wenzls

$$f^{(0)}, f^{(1)}, \dots, f^{(4n-4)}$$

+ new projections from $S$: $P = \frac{1}{2}(f^{(2n-2)} + S)$

$Q = \frac{1}{2}(f^{(2n-2)} - S)$

<u>exercise</u>: $f^{(i)} \cong f^{(4n-4-i)}$ in $PA(S)$

(eg: two-$S$ relation says that $S$ is an isomorphism between $f^{(0)}$ + $f^{(4n-4)}$

<u>exercise</u>: $f^{(i)}$ are still minimal, except for $f^{(2n-2)}$

morally: took 

+ added on a morphism $S$ folding graph in half + doubling the vertex at the fold

"de-equivariantization"

$\square$

Does this scale? i.e. is this a good way to construct subfactors?

Almost - one weird trick lets us skip consistency, sphericality, and pos. def:

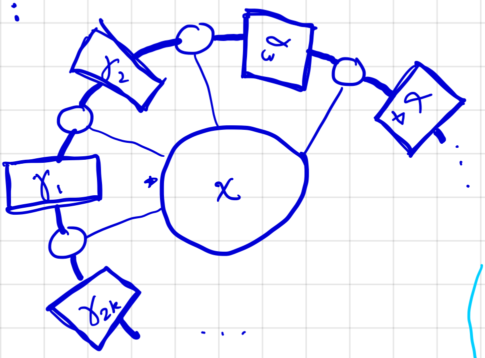
Planar Algebra Embedding Thm: (folklore, Jones-Penneys, Morrison-Walker?)

Suppose $\mathcal{P}$ is a subfactor planar algebra w/ principal graph $\Gamma$. There is a copy of $\mathcal{P}$ inside the planar algebra $GPA(\Gamma)$.

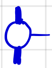
Idea: to get $\mathcal{P} \longrightarrow GPA(\Gamma)$

need $x \in \mathcal{P}_{2k,\pm} \longmapsto \{ f'\mathrm{ns}\ \{\text{length } 2k \text{ based loops on } \Gamma\} \to \mathbb{C} \}$

ie, given $x$ & $\{\gamma_1, \gamma_2, \dots, \gamma_{2k}\}$ produce a number:





what's in  ? morphism $\gamma_i \otimes id \to \gamma_{i+1}$

or $\gamma_{i+1} \longrightarrow \gamma_i \otimes id$

depending on their depth in the principal graph.

(5)

Check that this is a planar algebra morphism - eek! $\square$

## Classification of subfactor PAs by principal graph:

subfactors have 3 related invariants: index
principal graph
standard invariant (PA)

& if subfactor is amenable (or p.graph is finite), recover p-graph from PA, index from p.graph

recall that the $\sqrt{\text{index}}$ = op. norm of p-graph
= PF eigenval of adj. matrix

Thm: only graphs w/ $\delta < 2$ are ADE Dynkin diagrams

Thm (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)
$A_n$, $D_{2n}$, $E_6$, $E_8$ subfactors exist & are unique (up to complex conj.)
$D_{2n+1}$, $E_7$ DNE