A Crash Course in Representation Theory

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All vector spaces are over the complex numbers. Representation theory in finite characteristic is harder!

1 Definitions and fundamental notions

Definition 1. Let G be a group. A representation of G is a linear action of G on a vector space V. Such an action can be described in two equivalent ways:

- A map of groups $\rho_V : G \to \mathrm{GL}(V)$.
- An operation $G \times V \to V$ denoted by multiplication, such that

$$\begin{aligned} g(v+w) &= gv + gw \\ g(cv) &= cgv \\ e(v) &= v \\ g(hv) &= (gh)v \end{aligned}$$

  for any $g, h \in G$, any $v, w \in V$, and any $c \in \mathbb{C}$.

I will freely go between these two ways of thinking about a representation, but usually I'll use the second notation and drop the ρ_V.

Definition 2. A map of representations $\varphi : V \to W$ is a linear map such that for any $g \in G$, we have:

$$\varphi(gv) = g\varphi(v).$$

As usual an isomorphism $\varphi : V \to W$ is a map of representations that has an inverse map of representations $\psi : W \to V$. We denote the collection of all maps of representations $\mathrm{Hom}_G(V, W)$.

Note that in the above definition the action on the left-hand side is the action on V, while on the right-hand side it's the action on W. In other words, $\varphi(\rho_V(g)v) = \rho_W(g)\varphi(v)$.

Definition 3. A subrepresentation $W \subset V$ is a linear subspace W of V which is preserved by G in the sense that for every $g \in G$ and $w \in W$ we have $gw \in W$.

If $W \subset V$ is a subrepresentation, then the quotient representation V/W is the quotient vector space with the action $g(vW) = g(v)W$.

Definition 4. If V and W are representations of G, then we can define the following new representations:

- The trivial representation $\mathbf{1}$ where the vector space is $\mathbb{C}$ and every element of G acts by the identity.
- The direct sum $V \oplus W$ where $g \in G$ acts by $g(v, w) = (gv, gw)$.

- The tensor product $V \otimes W$ where $g \in G$ acts on pure tensors by $g(v \otimes w) = gv \otimes gw$ and extending by linearity to other vectors.
- The dual representation $V^* = \mathrm{Hom}(V, \mathbb{C})$ where $g \in G$ sends φ to the function $v \mapsto \varphi(g^{-1}v)$.
- The linear Hom representation $\mathrm{Hom}(V, W)$ where $g \in G$ sends φ to the function $v \mapsto g\varphi(g^{-1}v)$.

Exercise 5. Check that V^* as defined above is a representation. Would the formula $v \mapsto \varphi(gv)$ also define a representation?

Definition 6. If V is a representation of G then let the invariant subspace V^G be

$$\{v \in V : \text{for all } g \in G \text{ we have } gv = v\}.$$

Exercise 7. Show that if V is a finite dimensional representation of G, then V^G is isomorphic to a direct sum of trivial representations.

Exercise 8. Show that if V is a finite dimensional representation of G then there's an isomorphism of representations $V \cong V^{**}$.

Exercise 9. If U, V, and W are representations of G, then there's an isomorphism of representations

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

Exercise 10. If V and W are finite dimensional representations of G, then there's an isomorphism of representations

$$W \otimes V^* \cong \mathrm{Hom}(V, W).$$

Exercise 11. Show that the vector space $\mathrm{Hom}_G(V, W)$ is isomorphic to the underlying vector space of $\mathrm{Hom}(V, W)^G$.

Definition 12. We define:

- a non-zero representation V is called *irreducible* if it has no proper nontrivial subrepresentations.
- a representation V is called *completely reducible* if it is a direct sum of irreducible representations.
- a non-zero representation V is called *indecomposable* if it cannot be written as $V = X \oplus Y$ for two non-zero proper subrepresentations X, Y.

Warning 13. *Note that irreducible representations are completely reducible. Even the representation* 0 *is completely reducible because it is a direct sum of zero irreducible representations.*

Exercise 14. Any 1-dimensional representation is irreducible.

Exercise 15. If W is completely reducible, then

$$W \cong \bigoplus_V V \otimes \mathrm{Hom}_G(V, W) \cong \bigoplus_V V^{\oplus \dim \mathrm{Hom}_G(V,W)}$$

where V ranges over all irreducible representations. (You might also observe that the first isomorphism is natural in W.)

2 Representation theory of some nice groups

Exercise 16. Classify all finite dimensional irreducible representations of $\mathbb{Z}$.

Hint: Look at the eigenvectors of $\rho_V(1)$. $\square$

Exercise 17. Classify all finite dimensional indecomposable representations of $\mathbb{Z}$.

Hint: Look at the Jordan decomposition of $\rho_V(1)$. $\square$

Exercise 18. Show that not every finite dimensional representation of $\mathbb{Z}$ is completely reducible.

Exercise 19. Classify all finite dimensional representations of $\mathbb{Z}/n\mathbb{Z}$, and show that they're completely reducible.

Hint: $\rho_V(1)^n = \rho_V(n) = \rho_V(0) = 1$. $\square$

Definition 20. The sign representation of the symmetric group S_n is the 1-dimensional vector space with the action $\sigma v = \mathrm{sgn}(\sigma)v$ where $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

The permutation representation of the symmetric group S_n is the n-dimensional vector space with basis e_i, where $\sigma(e_i) = e_{\sigma i}$.

Exercise 21. Show that the permutation representation of S_n has a trivial subrepresentation.

Definition 22. The standard representation of S_n is the permutation representation quotiented by its trivial subrepresentation.

Exercise 23. Show that the trivial representation, the sign representation, and the standard representation of S_3 are irreducible.

Exercise 24. Show that any finite dimensional representation of the symmetric group S_3 is a direct sum of trivial representations, sign representations, and standard representations.

Hint: Break up V into its eigenspaces for the action of (123). Figure out how (12) acts on these eigenspaces using that $(123)(12) = (12)(123)^2$. Specifically, show that the 1-eigenspace breaks up as a sum of trivial and sign representations, while the other two eigenspaces break up as a sum of standard representations. $\square$

Exercise 25. Classify all finite dimensional representations of the dihedral group D_{2n} and show that they're completely reducible.

Hint: D_{2n} is generated by a rotation r and a reflection s satisfying $sr = r^{-1}s$. Follow the above outline, first decomposing into eigenspaces for r and then figuring out how s acts on these eigenspaces. $\square$

3 Complete reducibility

Exercise 26 (Schur's Lemma)**.** If V and W are irreducible representations and $\varphi : V \to W$ is a map of representations, then either φ is the zero map or φ is an isomorphism.

If V is irreducible then $\mathrm{End}(V) = \mathbb{C}\mathrm{Id}$.

Definition 27. A unitary representation is a Hilbert space V together with an action of G on V by unitary operators. A representation is called unitarizable if there exists some inner product on V making the representation unitary.

Exercise 28. Any representation of $\mathbb{Z}/n\mathbb{Z}$ is unitarizable. Some representations of $\mathbb{Z}$ are not unitarizable.

Exercise 29. If V is a unitary representation of G and W is a subrepresentation of V, show that the orthogonal complement $W^\perp$ is also a subrepresentation of V.

Exercise 30. Any finite dimensional unitary representation is completely reducible.

Exercise 31. Any finite dimensional representation of a finite group is unitarizable.

Hint: Take any inner product $\langle -, - \rangle$ and define a new inner product by

$$\frac{1}{\#G} \sum_{g \in G} \langle gv, gw \rangle.$$

Show that G acts by unitary operators with respect to this new inner product. □

Exercise 32. Any finite dimensional representation of a finite group is completely reducible.

4 Character Theory

Exercise 33. Suppose that G is a finite group and V a representation of G, then V^G is the image of the projection

$$\frac{1}{\#G} \sum_{g \in G} \rho_V(g).$$

Exercise 34. Suppose that G is a finite group and V a representation of G, then

$$\dim V^G = \frac{1}{\#G} \sum_{g \in G} \mathrm{Tr} \rho_V(g).$$

Definition 35. Suppose that V is a G-representation, then we define

$$\chi_V(g) = \mathrm{Tr}(\rho_V(g)).$$

Definition 36. A function $f : G \to \mathbb{C}$ is called a class function if $f(gxg^{-1}) = f(x)$ for all $g, x \in G$.

Exercise 37. The dimension of the space of class functions is the number of conjugacy classes of G.

Exercise 38. χ_V is a class function.

Exercise 39. $\chi_V(1) = \dim V$.

Exercise 40. $\chi_{\mathbf{1}}(g) = 1$.

Exercise 41. $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$

Exercise 42. $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$.

Exercise 43. $\chi_{V^*}(g) = \chi_V(g^{-1})$.

Exercise 44. If G is a finite group, then $\chi_{V^*}(g) = \overline{\chi_V(g)}$.

Exercise 45. Suppose that G is a finite group and V a representation of G, then

$$\dim \mathrm{Hom}_G(V, W) = \frac{1}{\#G} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)}.$$

Hint: Recall

$$\mathrm{Hom}_G(V,W) \cong \mathrm{Hom}(V,W)^G \cong (W \otimes V^*)^G.$$

Now calculate this using Exercises 34, 42, and 44. $\square$

Definition 46. We define an inner product on class functions by

$$\langle \varphi, \psi \rangle = \frac{1}{\#G} \sum_{g \in G} \varphi(g)\overline{\psi(g)}.$$

Exercise 47. The characters χ_V for V irreducible are orthonormal.

Definition 48. The group ring $\mathbb{C}[G]$ is formal linear combinations $\sum_{g\in G} a_g e_g$ for $a_g \in \mathbb{C}$ with multiplication defined $e_g e_h = e_{gh}$ extended linearly. The regular representation is $\mathbb{C}[G]$ as a vector space with the action given by $ge_h = e_{gh}$.

Exercise 49. The regular representation breaks up as a direct sum over all irreducible representations

$$\mathbb{C}[G] \cong \bigoplus_V V^{\oplus \dim V}.$$

Hint: Calculate the character of the regular representation, and compute its inner product with each irreducible representation. $\square$

Exercise 50. As a ring, the group ring is isomorphic to a sum of matrix rings:

$$\mathbb{C}[G] \cong \bigoplus_V M_{\dim V}(\mathbb{C}).$$

Hint: Compute the ring $\mathrm{End}_G(\mathbb{C}[G], \mathbb{C}[G])$ in two ways. $\square$

Exercise 51. The number of conjugacy classes is equal to the number of irreducible representations.

Hint: Compute the dimension of the center of $\mathbb{C}[G]$ in two ways. $\square$

Exercise 52. The characters χ_V for V irreducible form an orthonormal basis for the space of class functions.

Definition 53. The character table of G is a square matrix whose columns are index by the conjugacy classes $[g]$, whose rows are indexed by the irreducible representations, and whose entries are the numbers $\chi_V(g)$. The rows of the character table are orthonormal if you weight each entry by $\#[g]/\#G$.

Exercise 54 (Column Orthogonality)**.** The columns of the character table are orthogonal with respect to the usual dot product. Each column dotted with itself is $\#G/\#[g]$.

Hint: Row orthogonality says that $MDM^\dagger = \mathrm{Id}$ where M is the character table and D is a certain diagonal matrix. Column orthogonality says $M^\dagger M = D^{-1}$. $\square$

Exercise 55. The 1-dimensional representations are indexed by the dual group of the abelianization of G, that is by

$$\mathrm{Hom}(G/[G,G], \mathbb{C}).$$

Exercise 56. Compute the character table of S_3.

Hint: Find the 1-dimensional representations and then use column orthogonality. $\square$

Exercise 57. Compute the character table of each of the non-abelian groups of order 8.

5 Fusion graphs

Exercise 58 (Frobenius Reciprocity)**.** If X, Y, Z are representations, then

$$\mathrm{Hom}_G(X, Y \otimes Z) \cong \mathrm{Hom}_G(X \otimes Z^*, Y).$$

Hint: This can either be proved directly by constructing an explicit map, or by calculating characters. □

Definition 59. If G is a group and V a self-dual representation, then we define the fusion graph of V to be the graph whose vertices are the irreducible representations of G and where vertices Y and Z are connected by $\dim \mathrm{Hom}(Y, Z \otimes V)$ edges. (This gives a well-defined undirected graph because of Frobenius reciprocity.)

Exercise 60. Compute the fusion graph for the standard 2-dimensional representation of S_3.

Exercise 61. Compute the fusion graph for the defining 2-dimensional representation of D_{2n}.

Hint: One approach is to use your earlier classification of representations of D_{2n}. Another approach is to inductively compute the character table and fusion graph by repeatedly tensoring with the defining 2-dimensional representation (you will need to also know about the 1-dimensional representations). □

Exercise 62. Compute the character table of the 12-element group T of rotational symmetries of the tetrahedron.

Exercise 63. There is a double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Let $\tilde{T}$ be the 'binary tetrahedral group', which is the 24 element group consisting of all the elements of $\mathrm{SU}(2)$ which lie above elements of the group T from the previous exercise. The group $\tilde{T}$ has a defining 2-dimensional representation of $\tilde{T}$ coming from the defining action of $SU(2)$ on $\mathbb{C}^2$. Compute the fusion graph for $\tilde{T}$ with respect to its defining representation.

Hint: Since T is a quotient of $\tilde{T}$ every representation of T gives one of $\tilde{T}$. You also have the defining 2-dimensional representation. This should be enough to calculate the fusion graph. □