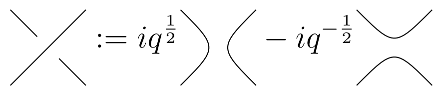
MSRI summer school: Planar Algebras
Quantum Symmetries Homework 3 June 22, 2017

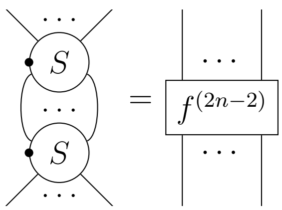
THE D_{2n} PLANAR ALGEBRA

Let $q = e^{i\pi/(4n-2)}$ and $\delta = 2\cos\frac{\pi}{4n-2}$. For this value of q, $[m]_q = [4n-2-m]_q$. We define a crossing in Temperley-Lieb as in the Kauffman bracket:

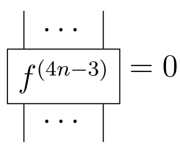


and notice that, with this definition, we are allowed to do Reidemeister type II and III moves on strings in our planar diagrams.

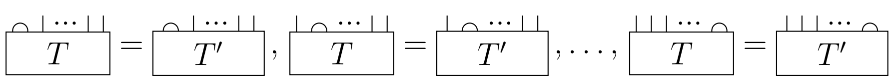
1. Show the following relations hold in $\mathcal{PA}(S)$:

   (a) 

   (Here $2n-2$ strands connect the two S boxes on the left hand side.)

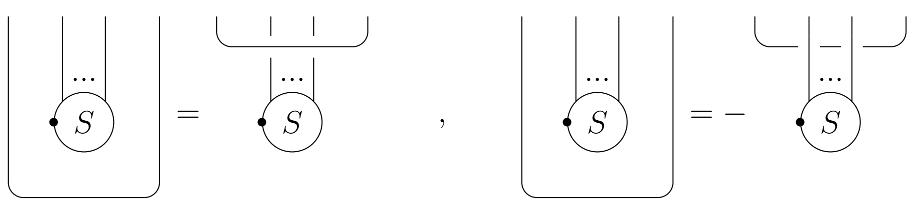
   (b) 

   (c) For $T, T' \in \mathcal{PA}(S)_{4n-4}$, if

   then $T = T'$. More generally, if $T, T' \in \mathcal{PA}(S)_m$ for $m \geq 4n-4$, and $4n-5$ consecutive cappings of T and T' are equal, then $T = T'$. (Hint: what do we know about $f^{(4n-4)}$?)

   (d) Show that $\mathcal{PA}(S)$ has a *partial braiding*: You can isotope a strand above an S box, but isotoping a strand below an S box introduces a factor of -1.

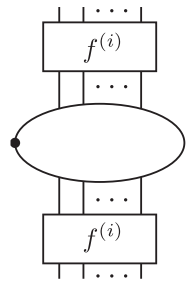
   (e) Any diagram in $\mathcal{PA}(S)$ is equal to a sum of diagrams involving at most one S.

2. Show that the principal graph of $\mathcal{PA}(S)$ is the Coxeter-Dynkin diagram D_{2n}:

   Recall that given two projections $\pi_1 \in P_{2n}$ and $\pi_2 \in P_{2m}$ we define $\operatorname{Hom}(\pi_1, \pi_2)$ to be the space $\pi_2 P_{n\to m} \pi_1$ ($P_{n\to m}$ is a convenient way of denoting P_{n+m}, drawn with n strands going down and m going up.) Now a projection π is called minimal if $\operatorname{Hom}(\pi, \pi)$ is

1-dimensional. (Note this is not quite the definition Emily gave in lecture, but it should have been.)

(a) The Jones-Wenzl idempotents $f^{(k)}$ for $k = 0, \ldots, f^{(2n-3)}$ are minimal. (Hint: Your proof should begin like this. "The space $\mathrm{Hom}\left(f^{(i)}, f^{(i)}\right)$ consists of all diagrams obtained by filling in the empty ellipse in the following diagram.



We want to show that any such diagram which is non-zero is equal to a multiple of the diagram gotten by inserting the identity into the empty ellipse...."

(b) The projections $P = \frac{1}{2}\left(f^{(2n-2)} + S\right)$ and $Q = \frac{1}{2}\left(f^{(2n-2)} - S\right)$ are minimal.

(c) If A and B are two distinct projections from the set

$$\{f^{(0)}, f^{(1)}, \ldots, f^{(2n-3)}, P, Q\}$$

then

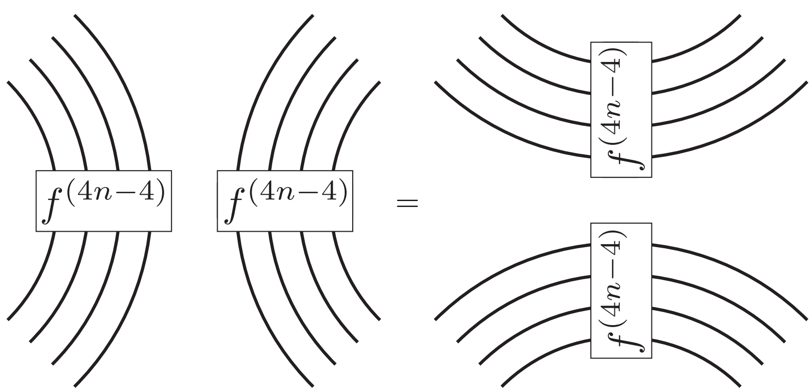
(d) The projection $f^{(k)} \otimes f^{(1)}$ is isomorphic to $f^{(k-1)} \oplus f^{(k+1)}$ for $k = 1, \ldots, 2n-4$. (Recall, from Noah's formulation of $\mathbf{Mat}(\mathbf{Kar}(\mathcal{C}))$, that these isomorphisms should be 2-by-1 matrices).

(e) The projection $f^{(2n-3)} \otimes f^{(1)}$ is isomorphic to $f^{(2n-4)} \oplus P \oplus Q$.

(f) $P \otimes f^{(1)} \cong f^{(2n-3)}$ and $Q \otimes f^{(1)} \cong f^{(2n-3)}$.

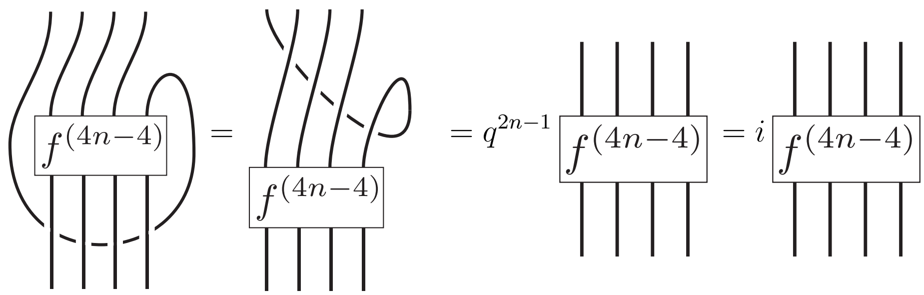
3. Feeling brave? Let's show consistency of $\mathcal{PA}(S)$ by hand. First, you'll want to know the following:

(a) Show that strands cabled by $f^{(4n-4)}$ can be reconnected.

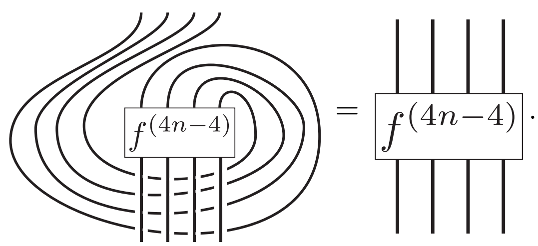


This relation also holds if superimposed on top of, or behind, another Temperley-Lieb diagram; (this is true for any relation in the Temperley-Lieb algebra.)

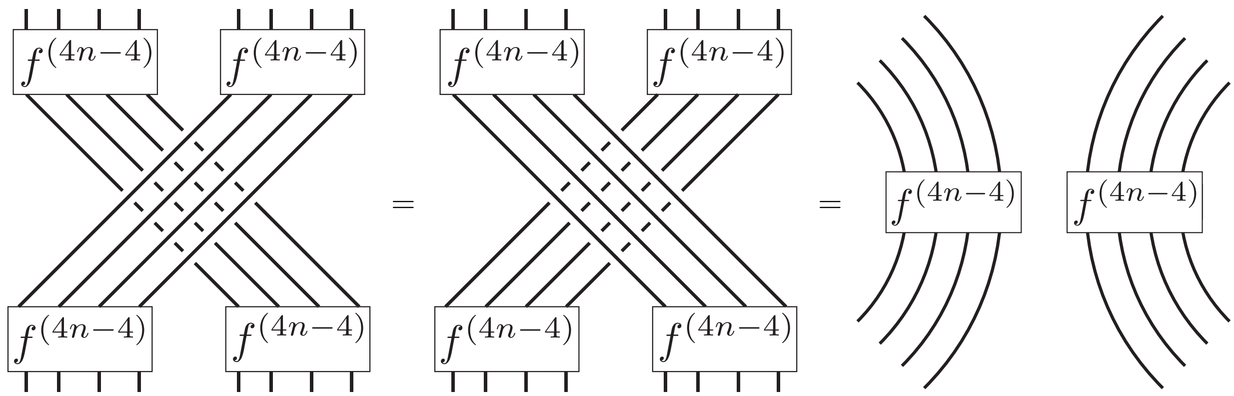
(b) Show that



(the twisted strand here indicates just a single strand, while the 3 parallel strands actually represent 4n-5 strands) and as an easy consequence



(c) Show that overcrossings, undercrossings and the 2-string identity cabled by $f^{(4n-4)}$ are all the same.



4. Here's an outline of how you could prove that the result of the evaluation algorithm is independent of the many choices made.

   1. If two applications of the algorithm use the same pairing of S boxes, and the same arcs [choice of a path from each generator S to the outside region], but replace the pairs in different orders, we get the same answer.
   2. If we apply the algorithm to a diagram with exactly two S boxes, then we can isotope the arc connecting them without affecting the answer.
   3. Isotoping any arc does not change the answer.

4. Changing the point at which an arc attaches to an S box does not change the answer.
5. Two applications of the algorithm which use different pairings of the S boxes give the same answers.