Planar Algebras 7

Clarification: GPA embedding thm

If P exists, w/ p.graph Γ, $P \hookrightarrow GPA(\Gamma)$

Does $\exists$ SPA w/ p.graph Γ?

- read (some) generators S_i & rel'ns R_i from Γ
- Search for $\tilde{S}_i \in GPA(\Gamma)$ satisfying rel'ns.

Do $\tilde{S}_i$ exist?

need more rel'ns!

yes, not unique

yes, & unique

no

What PA do $\tilde{S}_i$ generate?

SPA w/ Γ DNE

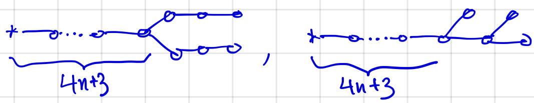
"too big"

SPA w/ Γ

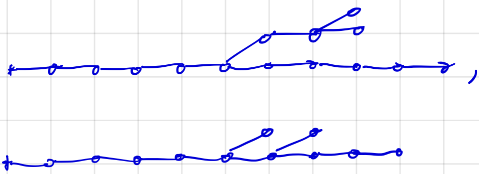
SPA w/ different p.graph

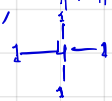
Thm: (Popa) at $\delta = 2$, only graphs are extended Dynkin diagrams; subfactors all exist, & for some p. graph is not quite a complete invariant
eg: Subfactors w/ p-graph $D_n^{(1)}$ are completely classified by $3 \in \mathbb{Z}/n\mathbb{Z}$

Thm (Haagerup): For subfactor $A \subseteq B$:
If $4 < [B:A] < 3+\sqrt{3}$
then p-graph is either

- (A_∞, A_∞) 
- H_n : 

- B_n : 

- AH : 

How do you prove something like this?
combinatorics of principal graphs
- If $\Gamma \subset \Gamma'$, $\|\Gamma\| < \|\Gamma'\|$ (connected)
  So, eg: 

- If (Γ_1, Γ_2) are a p. graph pair they have to 'fit together' nicely

- longest leg must be $*$-ed
- dimensions of vertices are $2\cos(\frac{\pi}{n})$ or ≥ 2 - in particular, 1 or greater
- dimensions of objects in a $\otimes$ cat must be cyclotomic integers ($= a_0 + a_1\zeta + \dots + a_{n-1}\zeta^{n-1}$ for ζ a root of 1)
- each graph describes fusion rules - are these associative?
- initial triple points are significant

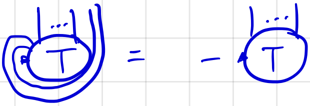
⋮

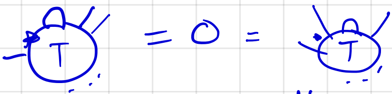
(see Jones-Morrison-Snyder Bulletin article)

Progress: A_∞ are TL: exist at all indices.
Bisch proved non-existence of B_n $\forall n$
Haagerup + Asaeda proved H_0, AH exist
Asaeda + Yasuda proved H_n DNE, $n \geq 2$
Bigelow Morrison P Snyder proved H_1 exists.

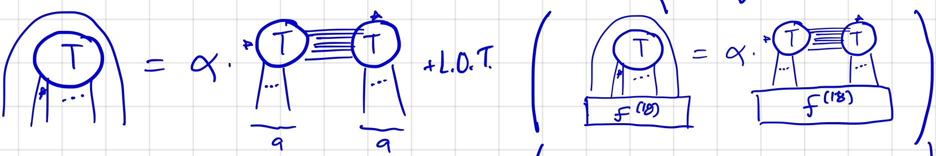
Def'n Consider the shaded non-degenerate spherical planar algebra $PA(T)$ generated by a single self-adjoint 16-strand generator T, w/ relations:

(1) $\delta \simeq \sqrt{4.377} \approx 2.09$ (largest real root of $\delta^6 - 8\delta^4 + 17\delta^2 - 5$)

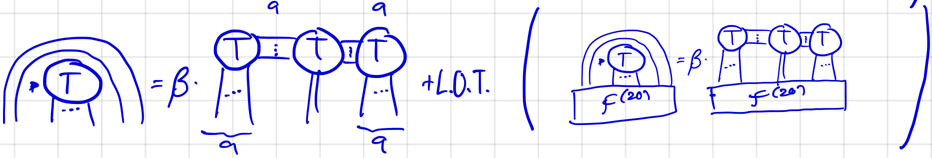
(2)  rotation

(3)  uncappable

(4) $T^2 = f^{(8)}$ "multiplication"

(5) 

"partial braiding"

(6) 

Thm $PA(T)$ is a subfactor planar algebra w/ principal graph pair H_1.

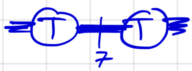
Q: How do we choose generators + rel'ns?

Count dimensions of $PA(T)_{2i}$ $\begin{cases} = \dim(TL_i) & i<8 \\ = \dim(TL_8)+1 & i=8 \\ = \dim(TL_9)+18 & i=9 \\ = \dim(TL_{10})+181 & i=10 \end{cases}$

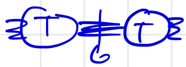
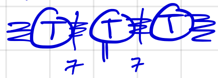
$\Rightarrow$ 1 generator T which is a 16-box

- & eigenvector for rotation
- & $T^2 \in \mathrm{span}\{TL, T\}$

all 18-boxes are in span TL, $ATL(T)$

in particular  $\in \mathrm{span}(TL, ATL(T))$

among 20-boxes, $TL \oplus ATL(T)$ has codim 1.

so  & 

Bigelow's insight is to turn these last 2 rel'ns around: instead of reducing # of Ts, increase # of Ts & pull past strand

Pf of thm:

(A) embed in GPA: non-zero, spherical, pos. def.

I'm sweeping lots of stuff under the rug here - this calculation was computer-assisted.

To prove rel'n $A=B$ holds in GPA, show $\langle A-B, A-B\rangle = 0$

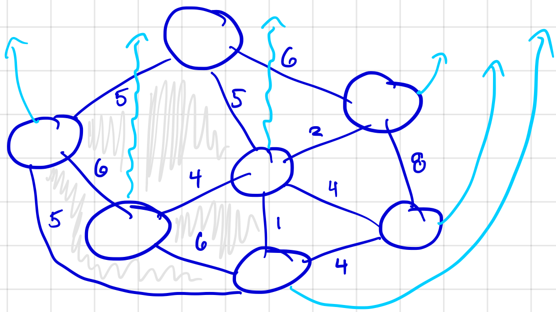
... boils down to evaluating networks w/ at most 4 Ts

Now $\tilde{T}$ is represented as a $10{,}000 \times 10{,}000$ matrix - knowing $tr(\tilde{T})$, $tr(\tilde{T}^2)$, $tr(\tilde{T}^3)$, $tr(\tilde{T}^4)$ suffices.

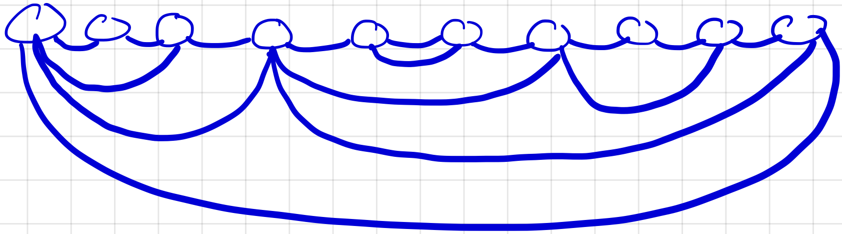
(Why matrices? aren't we looking @ functionals on loops?)

(B) evaluation algorithm: $\dim\left(PA(T)_{0,\underline{1}}\right) = 1$

Bigelow's jellyfish algorithm:



1. choose route out, apply partial braidings as needed...



<u>Claim:</u> in 'triangulated' 'n-gon' graph,
$\exists$ a pair of Ts connected by
8 (or more!) edges.

pf: every triangulated n-gon has a corner
b/c triangulation of n-gon requires n-2 triangles

2. Cancel a pair, simplify, recancel ...

3. End up w/ 0 or 1 T.

(c) At this index p-graph can only be (A_∞, A_∞) or H_1. Since it's too big to be TL, it's H_1. $\square$