Planar Algebras 5

Def'n The principal graph of a SPA records "fusion rules" for projections:

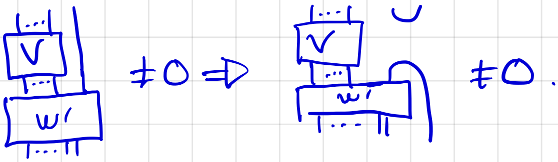
V: {isom. classes of min. projections}

(isom again means: $p \in V_{2n,+}$ & $q \in V_{2m,+}$ projections, $\exists\, x \in V_{n \to m,+}$ s.t. $xx^* = p$, $x^*x = q$)

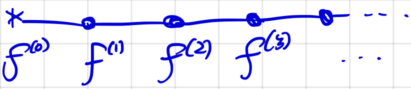
convention: 1st vertex from left is $\emptyset \in V_{0,\pm}$ marked w/ $*$

E: v connects to w if $v \otimes \mathrm{id}_x$ (= ) contains a subprojection $\cong w$.

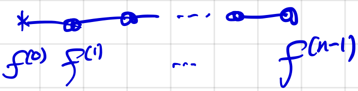
this is not directed because:



Eg: $TL([2] > 2)$:



$\widetilde{TL}\left([2] = 2\cos\left(\frac{\pi}{n+1}\right)\right)$:



Bratelli diagrams!

$TL_0 : \cong \mathbb{C}$

$\cap$

$TL_1 : \cong \mathbb{C}$

$\cap$

$TL_2 : \cong \mathbb{C} \oplus \mathbb{C}$

$\cap$

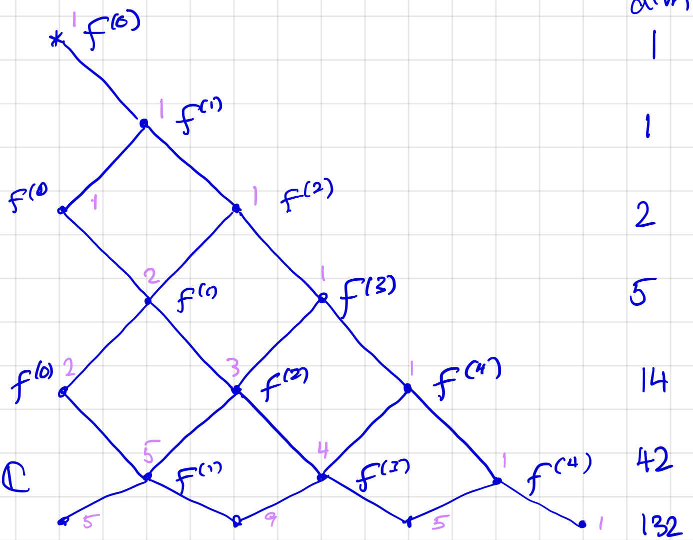
$TL_3 : \cong M_2\mathbb{C} \oplus \mathbb{C}$

$\cap$

$TL_4 : \cong M_2\mathbb{C} \oplus M_3\mathbb{C} \oplus \mathbb{C}$

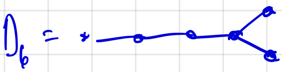
$\cap$

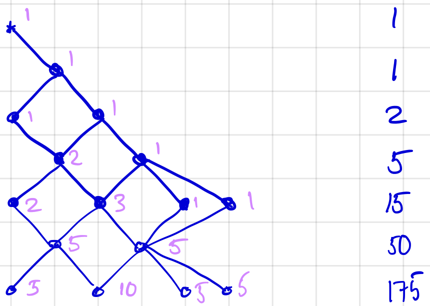
$TL_5 : \cong M_5\mathbb{C} \oplus M_4\mathbb{C} \oplus \mathbb{C}$



In general, given a principal graph, build a Bratelli diagram as above to know simple summands.

EG: D_6 =





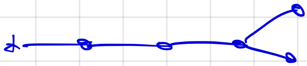
75
−32
43

Note: $\widetilde{TL}_n \longrightarrow V_{2n}$ is injective!

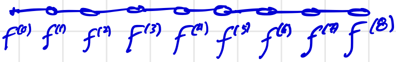
$\langle \cdot , \cdot \rangle_{SPA} = \langle \cdot , \cdot \rangle_{TL}$

What is $D_6 - \widetilde{TL}$?

(If $\mathcal{P}$ were a D_6 planar algebra, what would the non-$\widetilde{TL}$ part look like?



$\delta = 2\cos(\pi/10)$: $[10]=0$

$\widetilde{TL} =$ 

$\mathcal{P}_{2k,+} \ominus \widetilde{TL}_k :=$ $\perp$ complement of $\widetilde{TL}_k$ in $\mathcal{P}_{2k,+}$

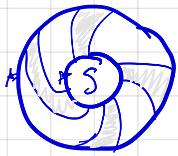
$\dim(\mathcal{P}_{2k,+} \ominus \widetilde{TL}_k) = 0, \quad k = 0,1,2,3$

$\dim(\mathcal{P}_{8,+} \ominus \widetilde{TL}_4) = 1$

$\dim(\mathcal{P}_{10,+} \ominus \widetilde{TL}_5) = 8$

$\dim(\mathcal{P}_{12,+} \ominus \widetilde{TL}_6) = 43$

So, there's a "new" element in $\mathcal{P}_{8,+}$; take the self-adjoint generator of $\mathcal{P}_{8,+} \ominus \widetilde{TL}_4$; call it S.

S must be an eigenvector for rotation:  $= \lambda \cdot S$

and, $S \perp \widetilde{TL}_4 \Rightarrow S$ is uncappable:

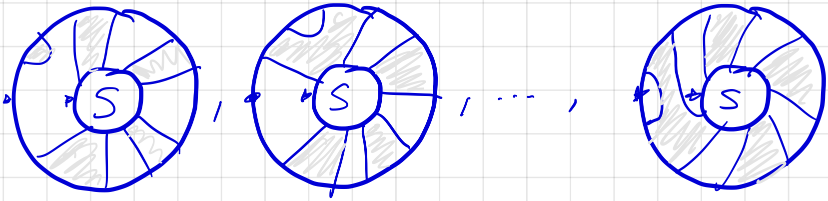
If $\hat{S} =$  $\neq 0$ then $\exists\, y \in \mathcal{P}_{6,+}$ s.t.

$\langle \hat{S}, y \rangle \neq 0$

But, $\mathcal{P}_{6,+} = \widetilde{TL}_3$ so $\langle \hat{S}, y \rangle = \langle S, \tilde{y} \rangle = 0$

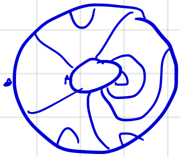
$\exists\, \tilde{y} \in \widetilde{TL}_4$.

What about $P_{10,+} \ominus TL_5$? are there 8 new elements here?



actually 10! but not all linearly independent. so really 8. (show by taking $\langle \cdot, \cdot \rangle$)

In general, we can ask what happens when we put S inside an "annular Temperley-Lieb" diagram:



But since S is uncappable, only non-zero $ATL(S)$ elements have 8 thru strings.

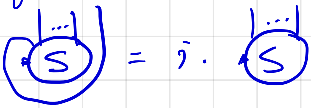
ex: How many ATL diagrams from k to n strings, are there?

And ATL is only a tiny fraction of planar tangles...

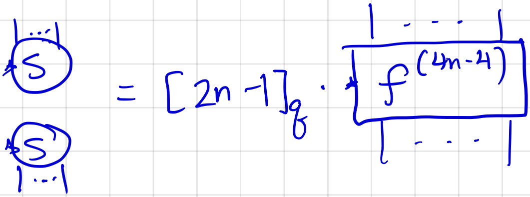
So does S planar generate all of $P_{2n,\pm}$?

Def'n. Consider the unshaded planar algebra $PA(S)$: generated by a single $(4n-4)$-strand generator S, w/ relations:

(1) $[2]_q = 2\cos(\pi/4n-2) = \delta$

(2)  rotation

(3)  uncappable

(4) 

Thm $PA(S)$ is a subfactor planar algebra
- closed diagrams are 1-dimensional
- spherical
- $S^* = S$, and $\langle \cdot , \cdot \rangle$ is positive definite
- principal graph is Dynkin diagram D_{2n}: