Basic examples of II_1 factors

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For an in-depth account of von Neumann algebras and II_1 factors in particular, we refer you to the excellent book "An introduction to II_1 factors", by Claire Anantharaman and Sorin Popa, as well as the comprehensive online notes "Notes on von Neumann algebras" by Jesse Peterson.

We assume all Hilbert spaces are seperable, that is there exists a countable orthonormal basis.

Recall that for a subset $A \subseteq B(H)$, the *commutant* $A' := \{b \in B(H) : ba = ab \ \forall a \in A\}$. A von Neumann algebra is a $*$-closed subalgebra $A \subseteq B(H)$ such that $A = A''$. Since clearly $A \subseteq A''$, we can view A'' as a kind of algebraic closure of A, and so a von Neumann algebra is a sub-algebra of $B(H)$ closed in this sense.

There are also topologies on $B(H)$ which give a more analytic flavor for this closure. In particular, we define a linear functional $\omega : B(H) \to \mathbb{C}$ to be *normal* if there exists sequences of vectors $(\xi_i)_{i\in\mathbb{N}}$, $(\eta_j)_{j\in\mathbb{N}}$, $\xi_i, \eta_j \in H$ satisfying $\sum ||\xi_i||^2 < \infty$ and $\sum ||\eta_j||^2 < \infty$, such that $\omega(a) = \sum_{i\in\mathbb{N}} \langle a\xi_i, \eta_i\rangle$.

We say a net of operators $\{A_\lambda\}_{\lambda\in\Lambda}$ converges in the weak-$*$ topology to an operator A if for all normal linear functionals ω, the numbers $|\omega(A_\lambda - A)| \to 0$. This notion of convergence defines a topology on $B(H)$ called the *weak $*$-topology.*

Theorem 0.1. *(Von Neumann bi-commutant theorem). Let $A \subseteq B(H)$ be a unital, $*$-closed subalgbera. Then A'' is the closure of A in the weak-$*$ topology.*

In particular, this says that the "algebraic closure" of a $*$-subalgebra A is the "topological closure" with respect to the weak-$*$ topology.

Let A be an associative $*$-algebra, and $\phi : A \to \mathbb{C}$ be a linear functional. ϕ is:

- *positive* if $\phi(a^*a) \geq 0$ for all $a \in A$.

- *faithful* if $\phi(a^*a) = 0$ implies $a = 0$.
- *tracial* if $\phi(ab) = \phi(ba)$ for all $a, b \in A$.
- called a *state* if $\phi(1) = 1$.

Philisophicaly, self-adjoint elements (i.e. $a = a^*$) of a von Neumann algebra are meant to represent "observable functions" of a quantum mechanical system, and states are meant to represent posible configurations of your system. Evaluating a state ϕ on an operator $a \in A$ produces a real number $\phi(a)$, which corresponds to the expected value of measuring the observable a in the state ϕ.

In this course, we will primarily be interesting in II_1 factors. A *factor* is a von Neumann algebra with trivial center. A von Neumann algebra A is called a II_1 *factor* if it is infinite dimensional, and there exists a unique normal faithful tracial state. It turns out that this condition automatically implies A is a factor (has trivial center).

Note that the $n \times n$ matrices $M_n(\mathbb{C})$ have a unique faithful normal tracial state, the normalized ordinary matrix trace, but these algebras are finite dimensional so are not type II_1 factors.

1 Constructing von Neumann algebras from associative $*$-algebras

How do we construct von Neumann algebras, and in particular II_1 factors? One way is to take an associative *-algebra, represent it as bounded operators on a Hilbert space, and take is double commutant.

To this end, if A is a unital, associtative $*$-algebra, a linear functional $\phi : A \to \mathbb{C}$ is called *bounded* if for every $x \in A$, there exists a constant $c_x > 0$ so that $\phi(y^*x^*xy) \leq c_x\phi(y^*y)$.

If ϕ is a faithful, bounded state, we can define the Hilbert space $L^2(A, \phi)$ to be the completion of A with respect to the positive definite sesquilinear form $\langle a, b\rangle = \phi(b^*a)$. Note that since ϕ is faithful, the inclusion $A \subseteq L^2(A, \phi)$ is injective.

1. Show that for an associative $*$-algebra A, if ϕ is a faithful, bounded state, the left multiplication operator $L(a) : A \to A$ given by $L(a)(b) := ab$ extends to a bounded linear operator on $L^2(A, \phi)$ for each $a \in A$.
2. Show that $L : A \to B(L^2(A, \phi))$ is an injective $*$-algebra homomorphism. We define the von Neumann algebra $A_\phi := L(A)''$.

3. Define $R(a) : A \to A$ by $R(a)(b) = ba$, for $a, b \in A$. Show that if ϕ is tracial, then $R : A \to B(L^2(A,\phi))$ is an anti-*- homomorphism (i.e. $R(ab) = R(b)R(a)$ and $R(b^*) = R(b)^*$), and $R(A) \subseteq L(A)'$.

4. Show that if ϕ is tracial, then it extends to a faithful, normal, tracial positive linear functional on A_ϕ (Hints: To extend ϕ to a normal linear functional on A_ϕ find a vector $\xi \in L^2(A,\phi)$ such that $\omega_\xi(\cdot) := \langle \cdot\xi, \xi\rangle$ is equal to ϕ. Use continuity to show it is tracial. To show that ϕ is faithful, show that if $x\xi = 0$ then $xR(a)\xi = 0$, and use the fact that $R(A)\xi = A$ is dense.).

We remark that in the very last exercise, we did not need the boundedness condition when considering ϕ as a faithful, normal, positive linear functional on a von Neumann algebra. It turns out such functionals are automatically bounded.

2 Group von Neumann algebras

The point of this exercise is to provide a large supply of II_1 factors, constructed from groups whose non-trivial conjugacy classes are all infinite. Let G be a countable group, and let $\mathbb{C}[G]$ denote the free complex vector space with basis given by the elements of G. Then elements of $\mathbb{C}[G]$ are of the form $\sum x_g g$, where the x_g are complex numbers, with only finitely many $x_g \neq 0$. The operations

$$\left(\sum x_g g\right) \cdot \left(\sum y_h h\right) := \sum \left(\sum_g x_g y_{g^{-1}k}\right) u_k$$

$$\left(\sum x_g g\right)^* = \sum \overline{x_g} g^{-1}$$

make $\mathbb{C}[G]$ into an associative $*$-algebra. Define the linear functional $\tau(\sum x_g g) = x_e$, where $e \in G$ is the identity.

1. Show that τ is faithful, tracial state on $\mathbb{C}[G]$. We define the *group von Neumann algebra* $L(G) := \mathbb{C}[G]_\tau$.

2. Let $\ell^2(G)$ be the Hilbert space of square summable functions on G. Show that there exists a unitary $V : L^2(\mathbb{C}[G], \tau) \to \ell^2(G)$ (recall a unitary between Hilbert spaces H and K is a bounded, invertible operator $V : H \to K$ such that $V^* = V^{-1}$).

3. Let $\delta_e \in \ell^2(G) \cong L^2(\mathbb{C}[G], \tau)$ denote the characteristic function of the identity element. Show that if x is in the center of $L(G)$, then the function $VxV^*\delta_e \in \ell^2(G)$ is constant on conjugacy classes of G.

4. A group G has *infinite conjugacy classes* (abbreviated ICC) if every non-trivial conjugacy class is infinite. Show that $L(G)$ is a factor if and only if G is ICC.

3 The hyperfinite II_1 factor and finite group actions

There exists a unique II_1 factor R which contains a sequence of (unital) inclusions of finite dimensional von Neumann algebras $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq R$ such that $(\bigcup A_n)'' = R$. This factor is called the *hyperfinite* II_1 *factor*. It is fundamental for many reasons, and in some sense is the "smallest" II_1 factor since it is contained in any other seperable II_1 factors. Uniqueness of such a factor is a non-trivial theorem, but here we outline a particular model which easily allows for the construction of finite group actions on R.

Recall that the matrix algebras $M_n(\mathbb{C})$ are finite dimensional type I factors, and we have isomorphisms $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \cong M_{nm}(\mathbb{C})$. Matrix algebras have a unique normalized tracial state $\tau(\cdot) := \frac{Tr(\cdot)}{n}$, which is just the normalization of the ordinary trace on matrices familiar from linear algebra.

Let us define $A_n := \otimes^n M_2(\mathbb{C})$, which is abstractly isomorphic to $M_{2^n}(\mathbb{C})$. Then we have a unital inclusion of $*$-algebras $i_n : A_n \to A_{n+1}$ defined by $i_n(x) = x \otimes 1$. Let $A_\infty = \bigcup_n A_n$ be the inductive limit $*$-algebra.

1. Show that A_∞ admits a unique faithful, tracial, bounded state $\tau : A_\infty \to \mathbb{C}$.

2. Prove that the von Neumann algebra $R := L(A_\infty)'' \subseteq B(L^2(A_\infty, \tau))$ has a unique normal faithful tracial state, and thus by infinite dimensionality, is a II_1 factor.

3. A *-automorphism $\alpha \in Aut(M)$ is called *inner* if there is a unitary element $u \in M$ such that $\alpha(x) = u^*xu$. An automorphism is called *outer* if it is not inner. An automorphism α is called *free* if for any element $x \in M$ satisfying $yx = x\alpha(y)$ for all $y \in M$, then $x = 0$. Show that if M is a factor, then α is outer if and only if α is free.

4. Using the model of R constructed above, show that any finite group G admits a homomorphism $G \to Aut(R)$ such that the image of each group element is outer. Such a homomorphism is called an *outer action.*

4 Crossed products.

The purpose of this exercise is to show, given an outer action of a group on a II_1 factor M, we can construct a new II_1 factor $M \rtimes G$ which contains M with finite index. This is one of the motivating examples for the theory of finite index subfactors. Let M be a II_1 factor, and let $Aut(M)$ denote the set of $*$-automorphisms of M. Let G be a finite group, and suppose we have a homomorphism $G \to Aut(M)$, $g \to \alpha_g$. We define the smashed product algebra $M\#_\alpha G := \bigoplus_{g\in G} M$ as a vector space. Let u_g denote the copy of 1 in the g-component, where we identify the component of the identity element e with M itself so that with $u_e = 1_M$. Then an arbitrary vector can be writted as $x = \sum_{g\in G} x_g u_g$, where $x_g \in M$ are called the *Fourier coefficients*.

We define a product and $*$-operation as follows:

$$\left(\sum_{g\in G} x_g u_g\right) \cdot \left(\sum_{h\in G} y_h u_h\right) := \sum_{k\in G}\left(\sum_{h\in G} x_h \alpha_h(y_{h^{-1}k})\right) u_k$$

$$\left(\sum_{g\in G} x_g u_g\right)^* = \sum_{g\in G} \alpha_g(x^*_{g^{-1}}) u_g$$

It is easy to see that these operations imply $u_g^* = u_{g^{-1}}$, $u_g \cdot u_h = u_{gh}$, and $u_g \cdot x \cdot u_g^* = \alpha_g(x)$.

1. Verify $M\#_\alpha G$ is a unital, associative $*$-algebra.

2. Show that every automorphism of a II_1 factor M preserves the trace (here you can assume that $\tau\circ\alpha(\cdot)$ is a normal linear functional. Indeed this assumption is extraneous since this is automatically true, but the proof is beyond the scope of these exercises).

3. Define $\tau(\sum_{g\in G} x_g u_g) = \tau(x_e)$. Show that τ is a faithful, tracial, bounded state on $M\#_\alpha G$.

4. We define the *crossed product von Neumann algebra* $M \rtimes_\alpha G := L(M\#_\alpha G)''$. Show that $M' \cap M \rtimes_\alpha G = \mathbb{C}1$ if and only if each α_g is outer (Hint: use the characterization of outer automorphism from the previous exercise). Conclude that if each α_g is outer, $M \rtimes_\alpha G$ is a II_1 factor.

5. The *fixed point algebra* of an action $g \to \alpha_g$ is defined to be the set $M^G = \{m \in M \ : \ \alpha_g(m) = m \ \forall g \in G\}$. Show that M^G is a von Neumann algebra.

6. Show that if M is a II_1 factor and each α_g is outer, then $(M^G)' \cap M = \mathbb{C}1$, which in particular implies M^G is also a II_1 factor.