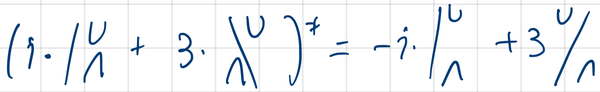
The positive definite Temperley-Lieb algebra (Planar Alg's, 2)

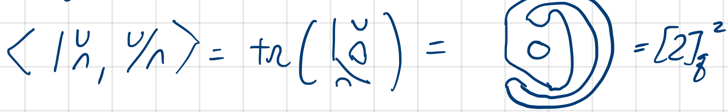
Def'n The involution $*$ on TL_n is (the conjugate-linear extension of) reflecting diagrams in a horizontal line:



Note: $\left(f^{(n)}\right)^* = f^{(n)}$ by uniqueness

Def'n A bilinear form on TL_n: $\langle x, y\rangle := tr(y^* x)$

eg:



exercise: D_1, D_2 are TL_n diagrams:

$\langle D_1, D_1\rangle = [2]_q^{\,n}$

$\langle D_1, D_2\rangle = [2]_q^{\,\ell}$ for $\ell < n$.

For subfactors, need to know: for what values of q is $\langle\cdot,\cdot\rangle$ pos. def on all TL_n?

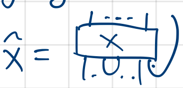
Def'n In a ~~vector~~ space V w/ bilinear form $\langle\cdot,\cdot\rangle$ $v \in V$ is negligible if $\forall w \in V$, $\langle v, w\rangle = 0$.

exercise: In TL_n, the set of negligible elements is an ideal.

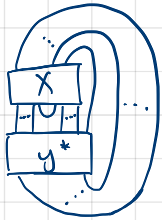
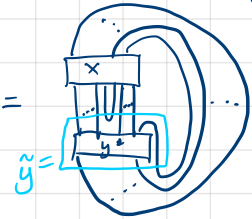
Lemma: For a fixed value of q, the "first" negligible element is $f^{(n)}$ $\exists n$

pf: Suppose TL_{n-1} has no non-zero negligibles, $x \in TL_n$ is non-zero & negligible.

x is uncappable: Let



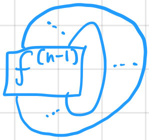
Take any $y \in TL_{n-1}$:

$$\langle \hat{x}, y \rangle = tr(y^* \hat{x}) =$$  = 

$$= \langle x, \tilde{y} \rangle = 0 \quad \exists \tilde{y} \in TL_n$$

Since x is uncappable, $x = c \cdot f^{(n)}$ by yesterday's lemma. $\square$

Exercise: If $f^{(n)}$ exists, $tr(f^{(n)}) = [n+1]_q$

Pf:  $= \frac{[n+1]}{[n]}$  $= \cdots = \frac{[n+1]}{[n]} \cdot \frac{[n]}{[n-1]} \cdots \frac{[3]}{[2]} [2] = [n+1]$

$\square$

Recall: the signature of a bilinear form $\langle \cdot, \cdot \rangle$ on V:
for an orthogonal basis v_i on V,

$$n_+ = \# \{ i \mid \langle v_i, v_i \rangle > 0 \}$$
$$n_0 = \# \{ i \mid \langle v_i, v_i \rangle = 0 \}$$
$$n_- = \# \{ i \mid \langle v_i, v_i \rangle < 0 \}$$

& (n_+, n_0, n_-), the signature of $\langle \cdot, \cdot \rangle$ on V, is invariant of our choice of basis

- $n_0 = 0, n_- = 0$: pos. def.
- $n_0 = 0, n_+ = 0$: neg. def.
- $n_0 = 0, n_+ \neq 0, n_- \neq 0$: indefinite
- $n_0 \neq 0$: degenerate

Thm: (a) If $[2]_q \geq 2$, TL_n is positive definite $\forall n$

(b) If $[2]_q = 2\cos\left(\frac{\pi}{n}\right)$ (so $q = e^{\pi i/n}$), $n \geq 2$

TL_k is pos. def. if $k < n$,

TL_k is degenerate if $k \geq n$

(c) If $0 < [2]_q < 2$ and $[2]_q \neq 2\cos\left(\frac{\pi}{n}\right)$ for any n,

$\exists\, m$ s.t. TL_m is neg. definite or indefinite.

Pf: matrix of inner products on TL_n: negligible elt. would be in kernel; non-trivial kernel $\Rightarrow$ det $= 0$.

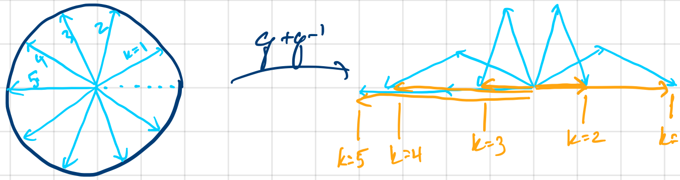
$\det(\{\langle D_i, D_j\rangle\})$ is a polynomial in $[2]_q$;
highest order term is $([2]^n)^{\cdot \# TL_n}$,
(exercise: $\# TL_n = n^{th}$ Catalan number)
from diagonal. (all other terms are lower from today's first exercise.) "Diagonal Dominance"

So for sufficiently large $[2]_q$, $\langle \cdot, \cdot \rangle$ is pos. def.

If TL has a non-zero negligible ideal, $f^{(n)}$ is the first negligible element;
so $0 = \langle f^{(n)}, \mathbb{1}_n \rangle = tr(f^{(n)}) = [n+1]_q$:

$$0 = \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}} = \frac{q^{-n-1}(q^{2n+2} - 1)}{q - q^{-1}} \Rightarrow q^{2n+2} = 1, \ q \neq 1$$

hence $q = e^{l\pi i / 2n+2} = e^{l\pi i / n+1}$, $[2]_q = 2\cos\left(\frac{l\pi}{n}\right) \leq 2\cos\left(\frac{\pi}{n}\right) < 2$.



(a) To cease being pos. def, pass thru a degenerate point. So $\forall n$, $\forall [2]_q \geq 2$, TL_n is pos. def.

(b) if $k < n$, $2\cos\left(\frac{l\pi}{k}\right) < 2\cos\left(\frac{\pi}{n}\right)$: So TL_k is pos. def.
if $k \geq n$, negligibles $\neq \{0\}$: $\langle f^{(n)}, D_i \rangle = 0$ if D_i has a cup or cap, or if $D_i = \mathbb{1}_n$; so $\forall D_i$. So TL_n is degenerate.

(c) <u>exercise:</u> Suppose $0 < [2]_q < 2$ and $[2]_q \neq 2\cos(\frac{\pi}{n})$ for any n. Show that there is some m s.t. (1) TL_m has no negligible elements, and
(2) $\langle f^{(m)}, f^{(m)} \rangle = [m+1]_q < 0$.
Conclude that TL_m is not positive definite.

<u>Thm:</u> If $[2]_q = 2\cos(\frac{\pi}{n})$, $\widetilde{TL} = TL / \text{ideal of negligibles}$ is positive definite.

To do this, we need to look @ what's left after we remove negligibles, so first we need to know what's there!
What's in TL_n <u>other</u> than Jones-Wenzl s?
(other Jones-Wenzls, hah.)

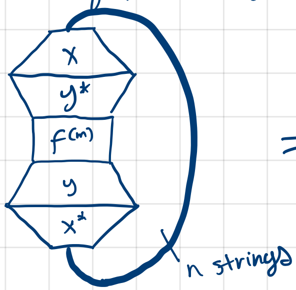
<u>Def'n</u> Fix $[2]_q$. Projections $p \in TL_n$, $q \in TL_m$ are <u>isomorphic</u>, $p \simeq q$, if $\exists$ an <u>isomorphism</u> $x \in TL_{n \to m}$ s.t.
$p = xx^*$, $q = x^*x$.
(equiv: $p = xqx^*$, $q = x^*px$)

<u>eg:</u> $\frac{1}{[2]_q} \,|\, {}^{\cup}_{\cap} \simeq f^{(1)}$ because let $x = \sqrt{\frac{1}{[2]}} \cdot /\!\!\cap$.
Then
$$xx^* = \frac{1}{[2]_q} \rangle\!{}^{\cup}_{/\!\cap} = \frac{1}{[2]_q} \,|\, {}^{\cup}_{\cap},$$
$$x^*x = \frac{1}{[2]_q} \langle\!\bigcirc = | \qquad \checkmark$$

<u>Lemma:</u> If $p \cong f^{(n)}$, $q \cong f^{(m)}$ and $n < m$, then $\langle p, q \rangle = 0$.

pf: x is isom. p to $f^{(n)}$
y is isom. q to $f^{(m)}$

$$\langle p,q\rangle = tr(q^* p) = tr(y f^{(m)} y^* x x^*) = tr(x^* y f^{(m)} y^* x)$$

= 

$= 0$: since $m > n$, every diagram caps $f^{(m)}$ somehow.

□

Thm: every minimal projection in $\widetilde{TL}_n$ is $\cong f^{(m)}$ $\exists m \le n$

pf: induction!
base case: TL_1 has only one projection, $f^{(1)} = |$
inductive step: assume every min. proj. in $\widetilde{TL}_{n-1} \cong f^{(m)}$ $\exists m \le n-1$.
we can get all projections by including prior projections up, then decomposing into minimal projections: $\boxed{X}\,|$

any minimal projection is a subprojection of I_n.

$$||||||| = \boxed{I_{n-1}}\,| = \boxed{Z_1 \oplus Z_2 \oplus \cdots \oplus Z_k}\,|$$

Z_i: min central projections of TL_{n-1}

& a min. projection is $\le \boxed{Z_i}\,|$ for some i.

• If $f^{(k)}$ exists,

$$\left[\boxed{f^{(k)}}\,|\right] = \begin{cases} [f^{(k+1)}] \oplus [f^{(k-1)}] & \text{if } [k+2]_q \neq 0 \\ [f^{(k-1)}] & \text{if } [k+2]_q = 0 \end{cases}$$

TO BE CONTINUED!