# A Crash Course in Representation Theory

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All vector spaces are over the complex numbers. Representation theory in finite characteristic is harder!

## 1 Definitions and fundamental notions

**Definition 1.** Let $G$ be a group. A representation of $G$ is a linear action of $G$ on a vector space $V$. Such an action can be described in two equivalent ways:

- A map of groups $\rho_V : G \to \mathrm{GL}(V)$.
- An operation $G \times V \to V$ denoted by multiplication, such that

$$\begin{aligned} g(v + w) &= gv + gw \\ g(cv) &= cgv \\ e(v) &= v \\ g(hv) &= (gh)v \end{aligned}$$

  for any $g, h \in G$, any $v, w \in V$, and any $c \in \mathbb{C}$.

I will freely go between these two ways of thinking about a representation, but usually I'll use the second notation and drop the $\rho_V$.

**Definition 2.** A map of representations $\varphi : V \to W$ is a linear map such that for any $g \in G$, we have:

$$\varphi(gv) = g\varphi(v).$$

As usual an isomorphism $\varphi : V \to W$ is a map of representations that has an inverse map of representations $\psi : W \to V$. We denote the collection of all maps of representations $\mathrm{Hom}_G(V, W)$.

Note that in the above definition the action on the left-hand side is the action on $V$, while on the right-hand side it's the action on $W$. In other words, $\varphi(\rho_V(g)v) = \rho_W(g)\varphi(v)$.

**Definition 3.** A subrepresentation $W \subset V$ is a linear subspace $W$ of $V$ which is preserved by $G$ in the sense that for every $g \in G$ and $w \in W$ we have $gw \in W$.

If $W \subset V$ is a subrepresentation, then the quotient representation $V/W$ is the quotient vector space with the action $g(vW) = g(v)W$.

**Definition 4.** If $V$ and $W$ are representations of $G$, then we can define the following new representations:

- The trivial representation $\mathbf{1}$ where the vector space is $\mathbb{C}$ and every element of $G$ acts by the identity.
- The direct sum $V \oplus W$ where $g \in G$ acts by $g(v, w) = (gv, gw)$.
- The tensor product $V \otimes W$ where $g \in G$ acts on pure tensors by $g(v \otimes w) = gv \otimes gw$ and extending by linearity to other vectors.

- The dual representation $V^* = \mathrm{Hom}(V, \mathbb{C})$ where $g \in G$ sends $\varphi$ to the function $v \mapsto \varphi(g^{-1}v)$.
- The linear Hom representation $\mathrm{Hom}(V, W)$ where $g \in G$ sends $\varphi$ to the function $v \mapsto g\varphi(g^{-1}v)$.

**Exercise 5.** Check that $V^*$ as defined above is a representation. Would the formula $v \mapsto \varphi(gv)$ also define a representation?

**Definition 6.** If $V$ is a representation of $G$ then let the invariant subspace $V^G$ be

$$\{v \in V : \text{for all } g \in G \text{ we have } gv = v\}.$$

**Exercise 7.** Show that if $V$ is a finite dimensional representation of $G$, then $V^G$ is isomorphic to a direct sum of trivial representations.

**Exercise 8.** Show that if $V$ is a finite dimensional representation of $G$ then there's an isomorphism of representations $V \cong V^{**}$.

**Exercise 9.** If $U$, $V$, and $W$ are representations of $G$, then there's an isomorphism of representations

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

**Exercise 10.** If $V$ and $W$ are finite dimensional representations of $G$, then there's an isomorphism of representations

$$W \otimes V^* \cong \mathrm{Hom}(V, W).$$

**Exercise 11.** Show that the vector space $\mathrm{Hom}_G(V, W)$ is isomorphic to the underlying vector space of $\mathrm{Hom}(V, W)^G$.

**Definition 12.** We define:

- a non-zero representation $V$ is called *irreducible* if it has no proper nontrivial subrepresentations.
- a representation $V$ is called *completely reducible* if it is a direct sum of irreducible representations.
- a non-zero representation $V$ is called *indecomposable* if it cannot be written as $V = X \oplus Y$ for two non-zero proper subrepresentations $X, Y$.

**Warning 13.** *Note that irreducible representations are completely reducible. Even the representation* $0$ *is completely reducible because it is a direct sum of zero irreducible representations.*

**Exercise 14.** Any 1-dimensional representation is irreducible.

**Exercise 15.** If $W$ is completely reducible, then

$$W \cong \bigoplus_V V \otimes \mathrm{Hom}_G(V, W) \cong \bigoplus_V V^{\oplus \dim \mathrm{Hom}_G(V, W)}$$

where $V$ ranges over all irreducible representations. (You might also observe that the first isomorphism is natural in $W$.)

## 2 Representation theory of some nice groups

**Exercise 16.** Classify all finite dimensional irreducible representations of $\mathbb{Z}$.

*Hint:* Look at the eigenvectors of $\rho_V(1)$. □

**Exercise 17.** Classify all finite dimensional indecomposable representations of $\mathbb{Z}$.

*Hint:* Look at the Jordan decomposition of $\rho_V(1)$. □

**Exercise 18.** Show that not every finite dimensional representation of $\mathbb{Z}$ is completely reducible.

**Exercise 19.** Classify all finite dimensional representations of $\mathbb{Z}/n\mathbb{Z}$, and show that they're completely reducible.

*Hint:* $\rho_V(1)^n = \rho_V(n) = \rho_V(0) = 1$. □

**Definition 20.** The sign representation of the symmetric group $S_n$ is the 1-dimensional vector space with the action $\sigma v = \mathrm{sgn}(\sigma)v$ where $\mathrm{sgn}(\sigma)$ is the sign of the permutation $\sigma$.

The permutation representation of the symmetric group $S_n$ is the $n$-dimensional vector space with basis $e_i$, where $\sigma(e_i) = e_{\sigma i}$.

**Exercise 21.** Show that the permutation representation of $S_n$ has a trivial subrepresentation.

**Definition 22.** The standard representation of $S_n$ is the permutation representation quotiented by its trivial subrepresentation.

**Exercise 23.** Show that the trivial representation, the sign representation, and the standard representation of $S_3$ are irreducible.

**Exercise 24.** Show that any finite dimensional representation of the symmetric group $S_3$ is a direct sum of trivial representations, sign representations, and standard representations.

*Hint:* Break up $V$ into its eigenspaces for the action of $(123)$. Figure out how $(12)$ acts on these eigenspaces using that $(123)(12) = (12)(123)^2$. Specifically, show that the 1-eigenspace breaks up as a sum of trivial and sign representations, while the other two eigenspaces break up as a sum of standard representations. □

**Exercise 25.** Classify all finite dimensional representations of the dihedral group $D_{2n}$ and show that they're completely reducible.

*Hint:* $D_{2n}$ is generated by a rotation $r$ and a reflection $s$ satisfying $sr = r^{-1}s$. Follow the above outline, first decomposing into eigenspaces for $r$ and then figuring out how $s$ acts on these eigenspaces. □

## 3 Complete reducibility

**Exercise 26** (Schur's Lemma)**.** If $V$ and $W$ are irreducible representations and $\varphi : V \to W$ is a map of representations, then either $\varphi$ is the zero map or $\varphi$ is an isomorphism.

If $V$ is irreducible then $\mathrm{End}(V) = \mathbb{C}\mathrm{Id}$.

**Definition 27.** A unitary representation is a Hilbert space $V$ together with an action of $G$ on $V$ by unitary operators. A representation is called unitarizable if there exists some inner product on $V$ making the representation unitary.

**Exercise 28.** Any representation of $\mathbb{Z}/n\mathbb{Z}$ is unitarizable. Some representations of $\mathbb{Z}$ are not unitarizable.

**Exercise 29.** If $V$ is a unitary representation of $G$ and $W$ is a subrepresentation of $V$, show that the orthogonal complement $W^\perp$ is also a subrepresentation of $V$.

**Exercise 30.** Any finite dimensional unitary representation is completely reducible.

**Exercise 31.** Any finite dimensional representation of a finite group is unitarizable.

*Hint:* Take any inner product $\langle -,- \rangle$ and define a new inner product by

$$\frac{1}{\#G}\sum_{g\in G}\langle gv, gw\rangle.$$

Show that $G$ acts by unitary operators with respect to this new inner product. □

**Exercise 32.** Any finite dimensional representation of a finite group is completely reducible.

# 4 Character Theory

**Exercise 33.** Suppose that $G$ is a finite group and $V$ a representation of $G$, then $V^G$ is the image of the projection

$$\frac{1}{\#G}\sum_{g\in G}\rho_V(g).$$

**Exercise 34.** Suppose that $G$ is a finite group and $V$ a representation of $G$, then

$$\dim V^G = \frac{1}{\#G}\sum_{g\in G}\mathrm{Tr}\rho_V(g).$$

**Definition 35.** Suppose that $V$ is a $G$-representation, then we define

$$\chi_V(g) = \mathrm{Tr}(\rho_V(g)).$$

**Definition 36.** A function $f : G \to \mathbb{C}$ is called a class function if $f(gxg^{-1}) = f(x)$ for all $g, x \in G$.

**Exercise 37.** The dimension of the space of class functions is the number of conjugacy classes of $G$.

**Exercise 38.** $\chi_V$ is a class function.

**Exercise 39.** $\chi_V(1) = \dim V$.

**Exercise 40.** $\chi_1(g) = 1$.

**Exercise 41.** $\chi_{V\oplus W}(g) = \chi_V(g) + \chi_W(g)$

**Exercise 42.** $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$.

**Exercise 43.** $\chi_{V^*}(g) = \chi_V(g^{-1})$.

**Exercise 44.** If $G$ is a finite group, then $\chi_{V^*}(g) = \overline{\chi_V(g)}$.

**Exercise 45.** Suppose that $G$ is a finite group and $V$ a representation of $G$, then

$$\dim \mathrm{Hom}_G(V, W) = \frac{1}{\#G}\sum_{g\in G}\chi_V(g)\overline{\chi_W(g)}.$$

*Hint:* Recall

$$\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}(V, W)^G \cong (W \otimes V^*)^G.$$

Now calculate this using Exercises 34, 42, and 44. □

**Definition 46.** We define an inner product on class functions by

$$\langle\varphi, \psi\rangle = \frac{1}{\#G}\sum_{g\in G}\varphi(g)\overline{\psi(g)}.$$

**Exercise 47.** The characters $\chi_V$ for $V$ irreducible are orthonormal.

**Definition 48.** The group ring $\mathbb{C}[G]$ is formal linear combinations $\sum_{g\in G} a_g e_g$ for $a_g \in \mathbb{C}$ with multiplication defined $e_g e_h = e_{gh}$ extended linearly. The regular representation is $\mathbb{C}[G]$ as a vector space with the action given by $g e_h = e_{gh}$.

**Exercise 49.** The regular representation breaks up as a direct sum over all irreducible representations

$$\mathbb{C}[G] \cong \bigoplus_V V^{\oplus \dim V}.$$

*Hint:* Calculate the character of the regular representation, and compute its inner product with each irreducible representation. ◻

**Exercise 50.** As a ring, the group ring is isomorphic to a sum of matrix rings:

$$\mathbb{C}[G] \cong \bigoplus_V M_{\dim V}(\mathbb{C}).$$

*Hint:* Compute the ring $\mathrm{End}_G(\mathbb{C}[G], \mathbb{C}[G])$ in two ways. ◻

**Exercise 51.** The number of conjugacy classes is equal to the number of irreducible representations.

*Hint:* Compute the dimension of the center of $\mathbb{C}[G]$ in two ways. ◻

**Exercise 52.** The characters $\chi_V$ for $V$ irreducible form an orthonormal basis for the space of class functions.

**Definition 53.** The character table of $G$ is a square matrix whose columns are index by the conjugacy classes $[g]$, whose rows are indexed by the irreducible representations, and whose entries are the numbers $\chi_V(g)$. The rows of the character table are orthonormal if you weight each entry by $\#[g]/\#G$.

**Exercise 54** (Column Orthogonality)**.** The columns of the character table are orthogonal with respect to the usual dot product. Each column dotted with itself is $\#G/\#[g]$.

*Hint:* Row orthogonality says that $MDM^\dagger = \mathrm{Id}$ where $M$ is the character table and $D$ is a certain diagonal matrix. Column orthogonality says $M^\dagger M = D^{-1}$. ◻

**Exercise 55.** The 1-dimensional representations are indexed by the dual group of the abelianization of $G$, that is by

$$\mathrm{Hom}(G/[G, G], \mathbb{C}).$$

**Exercise 56.** Compute the character table of $S_3$.

*Hint:* Find the 1-dimensional representations and then use column orthogonality. ◻

**Exercise 57.** Compute the character table of each of the non-abelian groups of order 8.

## 5 Fusion graphs

**Exercise 58** (Frobenius Reciprocity)**.** If $X$, $Y$, $Z$ are representations, then

$$\mathrm{Hom}_G(X, Y \otimes Z) \cong \mathrm{Hom}_G(X \otimes Z^*, Y).$$

*Hint:* This can either be proved directly by constructing an explicit map, or by calculating characters. ◻

**Definition 59.** If $G$ is a group and $V$ a self-dual representation, then we define the fusion graph of $V$ to be the graph whose vertices are the irreducible representations of $G$ and where vertices $Y$ and $Z$ are connected by $\dim \mathrm{Hom}(Y, Z \otimes V)$ edges. (This gives a well-defined undirected graph because of Frobenius reciprocity.)

**Exercise 60.** Compute the fusion graph for the standard 2-dimensional representation of $S_3$.

**Exercise 61.** Compute the fusion graph for the defining 2-dimensional representation of $D_{2n}$.

*Hint:* One approach is to use your earlier classification of representations of $D_{2n}$. Another approach is to inductively compute the character table and fusion graph by repeatedly tensoring with the defining 2-dimensional representation (you will need to also know about the 1-dimensional representations). ◻

**Exercise 62.** Compute the character table of the 12-element group $T$ of rotational symmetries of the tetrahedron.

**Exercise 63.** There is a double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Let $\tilde{T}$ be the 'binary tetrahedral group', which is the 24 element group consisting of all the elements of $\mathrm{SU}(2)$ which lie above elements of the group $T$ from the previous exercise. The group $\tilde{T}$ has a defining 2-dimensional representation of $\tilde{T}$ coming from the defining action of $SU(2)$ on $\mathbb{C}^2$. Compute the fusion graph for $\tilde{T}$ with respect to its defining representation.

*Hint:* Since $T$ is a quotient of $\tilde{T}$ every representation of $T$ gives one of $\tilde{T}$. You also have the defining 2-dimensional representation. This should be enough to calculate the fusion graph. $\square$