Homework on monoidal categories

June 14, 2017

Exercise 1. Show that fdVec and Mat are equivalent as categories.

Hint: Pick a basis for every finite dimensional vector space. □

Exercise 2. Show that every category is equivalent to a skeletal category.

Exercise 3. Show that $TL(d = 1)$ is monoidally equivalent to $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$.

Exercise 4. If $\mathcal{C}$ is a monoidal category, $\mathcal{D}$ is a category, and we have an equivalence $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$, show you can make D into a tensor monoidal in such a way that F and G are a monoidal equivalence.

(This is an instance of the idea of ‘transport of structure‘.)

Exercise 5. Conclude that every monoidal category is equivalent to a skeletal monoidal category.

Exercise 6. Recall that a monoidal functor is a pair: a functor and a natural isomorphism $F(X \otimes Y) \to F(X) \otimes F(Y)$. Classify all monoidal functors from $\mathrm{Vec}(G)$ to $\mathrm{Vec}(G)$ where the underlying functor is the identity functor.

Hint: Your answer should involve group cohomology of G. □

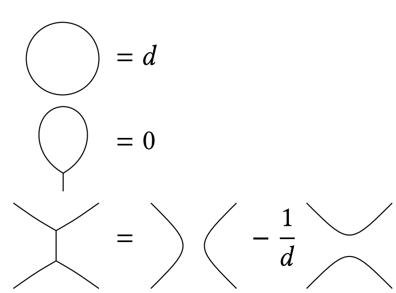
Exercise 7. Find the fusion rules for $TL(q + q^{-1})$ at q a root of unity.

Conclude that the last surviving Jones-Wenzl idempotent is invertible. This gives a 2-object subcategory where all the objects are invertible — how does it fit into our classification?

Hint: Suppose we are a root of unity such that $f^{(n+1)}$ is the first negligible Jones-Wenzl idempotent. Prove that $f^{(k)} \otimes f^{(1)} \cong f^{(k-1)} \oplus f^{(k+1)}$ unless $k = 0$ or $k = n$. What happens in those cases? From these observations, give a formula for how $f^{(a)} \otimes f^{(b)}$ breaks up as a direct sum of Jones-Wenzl idempotents. □

Exercise 8. Show that the golden category, as defined in lecture, is semisimple with two simple objects 1 and X and in particular that $X \otimes X \oplus 1 \oplus X$.

Recall that the golden category has as objects finite subsets of an interval, and the morphisms are planar trivalent graphs modulo the local relations



Exercise 9. (Checking every detail here is tedious; use your judgement!)

Show that every monoidal category C is monoidally equivalent to the strict monoidal category $\mathsf{List}C$.

Here $\mathsf{List}C$ has as objects words in the objects of C, and

$$\mathsf{List}C([x_1, x_2, \ldots x_n] \to [y_1, y_2, \ldots y_m]) = C(x_1 \otimes (x_2 \otimes \cdots \otimes (x_n \otimes 1)) \to y_1 \otimes (y_2 \otimes \cdots \otimes (y_m \otimes 1))).$$

Part of the exercise is to define the tensor product of morphisms in $\mathsf{List}C$. The functors between C and $\mathsf{List}C$ should send x to $[x]$ and $[x_1, x_2, \ldots x_n]$ to $x_1 \otimes (x_2 \otimes \cdots \otimes (x_n \otimes 1))$. You'll need to specify what the functors do on morphisms, and make them into monoidal functors by specifying tensorators. Finally you'll need to show that these functors form an equivalence; you can do this directly, or show one of the functors is fully faithful and essentially surjective.