Planar Algebras (Planar Algebras 3)

Temperley-Lieb wrap-up

Note: I've been sloppy about semisimplicity: can't talk about min. projections, min. central projections without knowing that we're in a semisimple setting.
(Semisimple has lots of equiv def'ns:
Jacobsen radical $= \{x \mid x \text{ acts as } 0 \text{ on all simple modules}\} = 0$;
semisimple $= \oplus$ simple algebras — take Artin-Wedderburn as definition of semisimple

Thm: TL_n is semisimple if $[2]_q \neq 2\cos(\frac{\pi}{n})$;
TL_n is not semisimple, but has s.s quotient, if $[2]_q = 2\cos(\frac{\pi}{n})$

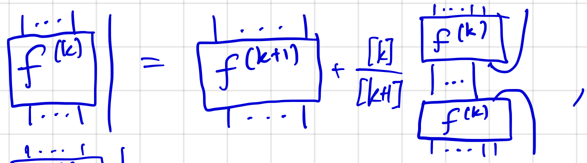
We were working out matrix algebra structure of TL_n:

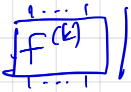
(a) $[2]_q > 2 : TL_k = \begin{cases} [f^{(0)}] \oplus [f^{(2)}] \oplus [f^{(4)}] \oplus \cdots \oplus [f^{(k)}], & k \text{ even} \\ [f^{(1)}] \oplus [f^{(3)}] \oplus \cdots \oplus [f^{(k)}], & k \text{ odd} \end{cases}$

(b) $[2]_q = 2\cos(\frac{\pi}{n}) : TL_k = \begin{cases} [f^{(0)}] \oplus [f^{(2)}] \oplus \cdots \oplus [f^{\min\{k, 2\cdot\lfloor\frac{n}{2}\rfloor\}}], & k \text{ even} \\ [f^{(1)}] \oplus [f^{(3)}] \oplus \cdots \oplus [f^{\min\{k, 2\lfloor\frac{n-1}{2}\rfloor+1\}}] & k \text{ odd} \end{cases}$

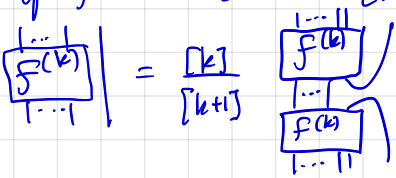
where $[f^{(k)}]$ = simple algebra whose min. projections are $\cong f^{(k)}$

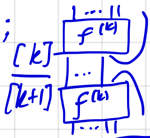
<u>Lemma</u>: (a) if $f^{(k)}$ exists and $[k+2] \neq 0$,

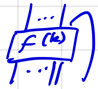


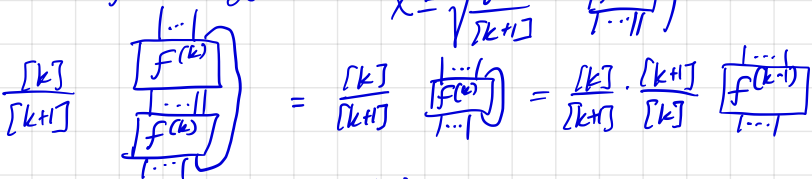
so  $\cong f^{(k+1)} \oplus f^{(k-1)}$

(b) if $f^{(k)}$ exists and $[k+2] = 0$, in $\widetilde{TL}$



Pf: (a)
This is Wenzl's relation;  is a projection

and $\cong f^{(k-1)}$ because $x = \sqrt{\frac{[k]}{[k+1]}}$  is isom:



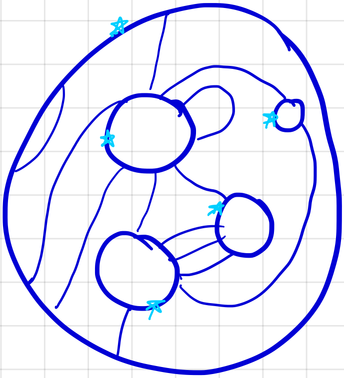
(b) If $[k+2] = 0$, $f^{(k+1)} = 0$ so Wenzl's relation says this, same calc. as before.

Thus we inductively know that the only min. projections are $\cong f^{(\text{even})}$ or $f^{(\text{odd})}$

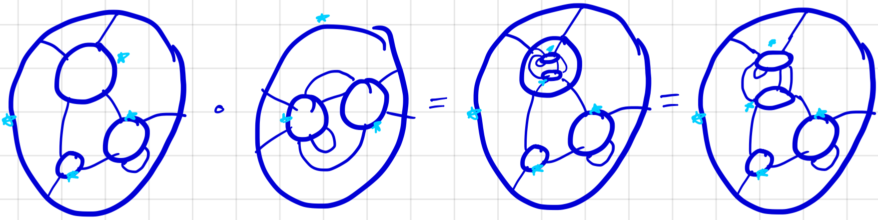
Planar Algebras!

A planar diagram has
- outer boundary circle
- inner bndry circles
- non-xing strings between bndrys
- a marked region on each bndry

& we consider them up to (bndry-preserving) isotopy

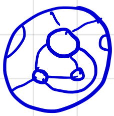


Composition of planar diagrams: if #pts on an inner/the outer bndry match up, we can put second inside first :

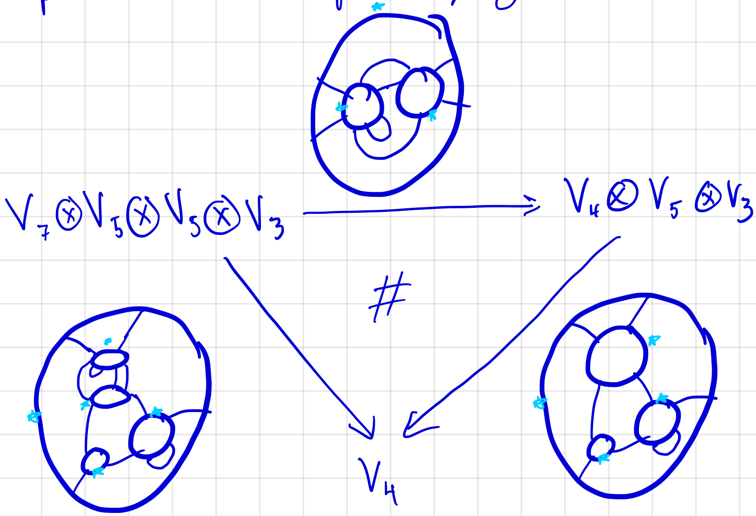


So, planar diagrams form an operad (or higher category). This is just a word for a gadget where composition is only allowed if certain indices match up.

A planar algebra is a family of vector spaces $\{V_k\}_{k\geq 0}$ which has an action of the planar operad:

 $: V_2 \otimes V_4 \otimes V_5 \longrightarrow V_7$

(compatible with composition, eg:



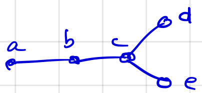
<u>EG 1:</u> $TL([2]_q) = \{TL_n([2])\}$ is a planar algebra!
draw diagrams in circles instead of rectangles; left side ; star
Planar diagrams act by insertion of diagrams
(+ take linear extension...)

<u>exercise:</u> negligible elements are a planar ideal in $TL([2]_q)$

<u>EG 2:</u> Planar Algebra of a graph

G = connected graph

$P_n = \{$ functionals $f: \{$ based length-$2n$ loops on $G\} \to \mathbb{C}\}$

<u>eg:</u> $G =$ 

$GPA(G)_0 = \{$ f'ns $f: \{a,b,c,d,e\} \to \mathbb{C}\}$
$GPA(G)_1 = \{$ f'ns $f:$ edges $\to \mathbb{C}\}$

action by planar diagrams?

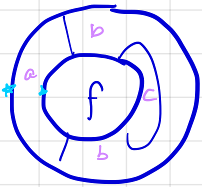
<u>action 1:</u> A <u>G-state</u> on a planar diagram assigns • vertices to regions
(• edges to strings)
<u>consistently</u> (so if region R touches string S, v(R) is on e(S).)

Of σ is a state, $\sigma|_{\text{brdry circle}}$ = a loop on G.

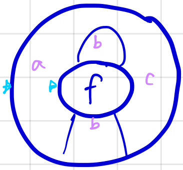
a planar diagram T acts on input f'nls $f_1, \dots, f_k$ by

$$T(f_1, F_2, \dots, f_k)(\ell) = \sum_{\substack{\text{states } \sigma \text{ on } T \\ \text{s.t. } \sigma|_0 = \ell}} \prod_{i=1}^{k} f_i(\sigma|_i)$$

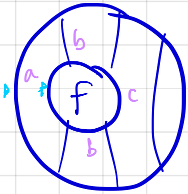
<u>eg:</u> Let $f = \delta_{abcb,-}$ $\left(f(\ell) = \begin{cases} 1 & \text{if } \ell = abcb \\ 0 & \text{else} \end{cases}\right)$



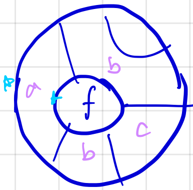
$(\ell) = \delta_{ab,\ell}$



$\equiv 0$ because $a \neq c$

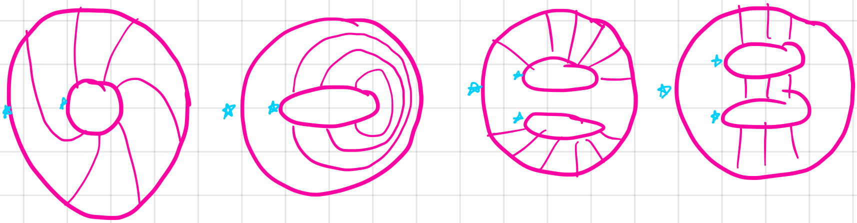


$(\ell) = \begin{cases} 1 & \text{if } \ell = abcbcba, \ abcdcba, \ abcecba \\ 0 & \text{else} \end{cases}$



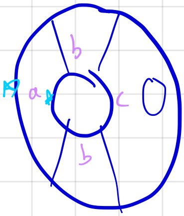
$$(\ell) = \begin{cases} 1 & \text{if } \ell = ababcba, \; abcbcba \\ 0 & \text{else} \end{cases}$$

exercise: what do the following diagrams do to $\delta_{\ell,-}$?

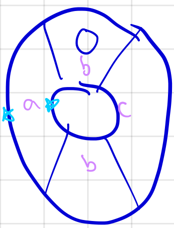


describe/ interpret them in words.

exercise: argue that this definition is a planar algebra, ie check that composition works as expected.



$(\ell) = 3 \cdot \delta_{abcd,-}$



$(\ell) = 2 \cdot \delta_{abcd,-}$