The Temperley-Lieb Algebra (Planar Algebras, I)

Quantum Integers:

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

so $[2]_q = q + q^{-1}$

$[3]_q = q^2 + 1 + q^{-2}$

$[4]_q = q^3 + q + q^{-1} + q^{-3}$

$[1]_q = ?$

When $q = 1$, $[n]_q = n$

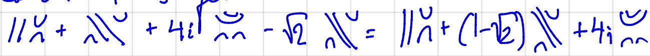
Exercise: $[2]_q [n]_q = [n-1]_q + [n+1]_q$

The Temperley-Lieb algebra:

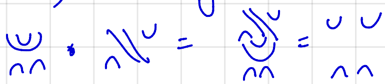
TL_n diagram: non-xing pairings on $2n$ points

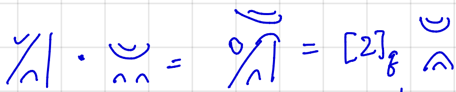
TL_n is a $\mathbb{C}$-vector space w/ basis TL_n diagrams

TL_n is an algebra: + is formal:



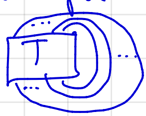
- $\cdot$ is by stacking:



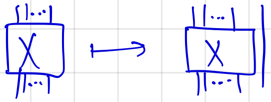
& 

(every closed circle ⇝ multiply by $[2]_q$)

$|||\cdots|$, denoted $\mathbb{1}_n$, is the multiplicative identity.

- it has a trace: $tr(T) =$ 

We have inclusions $TL_n \hookrightarrow TL_{n+1}$ (for fixed values of q)



So we tend to think of $\{TL_n\}$ as a family of algebras

Recall: The Artin-Wedderburn thm says that (if hypotheses hold)

$$TL_n \cong \bigoplus_{i \in I} M_{n_i}(\mathbb{C}) \qquad \exists\, n_i \in \mathbb{N}$$

$$TL_3 = \mathrm{Span}_{\mathbb{C}}\{\, |||,\ \overset{\cup}{\cap}|,\ |\overset{\cup}{\cap},\ \overset{\cup}{\cap}\!\!\diagdown,\ \diagup\!\!\overset{\cup}{\cap} \,\}$$

So.. as a matrix algebra, $TL_3 = ?$

5 dim: so either $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ or $\mathbb{C} \oplus M_2\mathbb{C}$

How do we tell? $p \in A$ is a central projection if $p^2 = p$ and $\forall a \in A$, $pa = ap$.

What is a minimal central projection? p s.t. $\forall$ central projections q, $qp = 0$ or $qp = p$.

Min. central projections of $\mathbb{C}^{\oplus 5}$ are $(1,0,0,0,0), (0,1,0,0,0) \ldots$

Min. central projections of $\mathbb{C} \oplus M_2\mathbb{C}$ are $\left(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$, $\left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$

Q: What are the minimal central projections in TL_n?

Q: Let's do it for TL_2:

Two projections: $\frac{1}{[2]}\overset{\cup}{\cap}$, $|| = \mathbb{1}_2$

but! $||$ is not minimal

$$|| \cdot \frac{1}{[2]}\overset{\cup}{\cap} = \frac{1}{[2]}\overset{\cup}{\cap}$$

so, $|| - \frac{1}{[2]}\overset{\cup}{\cap}$ is the other minimal projection.

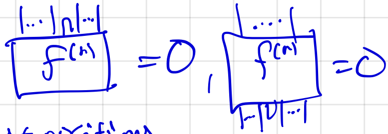
Exercise: What are min. central projections for TL_3?

Def'n: The Jones-Wenzl projection is $f^{(n)} \in TL(n)$ s.t.:

(0) $f^{(n)} \neq 0$

(1) $f^{(n)} f^{(n)} = f^{(n)}$

(2) $f^{(n)}$ is uncappable:



$\forall$ positions of cap.

(N.B.: this says something about an element of $TL_{n \to (n-2)}$ or $TL_{(n-2) \to n}$

Thm: $f^{(n)}$ exists, is unique.

Lemma 1: D_1, D_2 are diagrams in TL_n. $D_1 D_2 = \mathbb{1}_n \Rightarrow D_1 = D_2 = \mathbb{1}_n$

Pf: # through strands of $D_1 \cdot D_2 \leq$

min({# through strands of D_i})

and $\mathbb{1}_n$ is only diagram w/ n through strands.

Lemma 2: If P is a projection, coeff. of $\mathbb{1}_n$ in P is 0 or 1.

proof:

$$P = k \cdot \mathbb{1}_n + \underbrace{(P - k \cdot \mathbb{1}_n)}_{\text{lin. combo of diagrams w/ } < n \text{ thru strands}}$$

$$P = P^2 = k^2 \mathbb{1}_n + \underbrace{2k(P - k\mathbb{1}_n)}_{\text{coeff of } \mathbb{1}_n \text{ here} = 0} + \underbrace{(P - k \cdot \mathbb{1}_n)^2}_{\text{coeff of } \mathbb{1}_n \text{ here} = 0}$$

so, $k^2 = k \Rightarrow k = 0$ or 1. $\square$

Note: f uncappable, D_k is non-$\mathbb{1}_n$ TL_n diagram: $f \cdot D_k = D_k \cdot f = 0$

Ex: Coeff of $\mathbb{1}_n$ in an uncappable projection is 1.

Exercise: $f^{(n)} \cdot f^{(m)} = f^{(\max\{n,m\})}$

Lemma 3: If P, Q are both uncappable, coeff of $\mathbb{1}_n$ in P is 1, then Q is some multiple of P.

proof:

$$P = \mathbb{1}_n + (P - \mathbb{1}_n)$$
$$Q = c \cdot \mathbb{1}_n + (Q - \mathbb{1}_n)$$

$$PQ = \mathbb{1}_n \cdot Q + (P - \mathbb{1}_n) \cdot Q = Q + 0 = Q$$

and

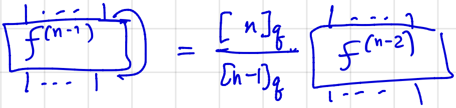
$$PQ = c \cdot P \cdot \mathbb{1}_n + P \cdot (Q - \mathbb{1}_n) = c \cdot P + 0 = c \cdot P$$

Hence $c \cdot P = Q$. $\square$

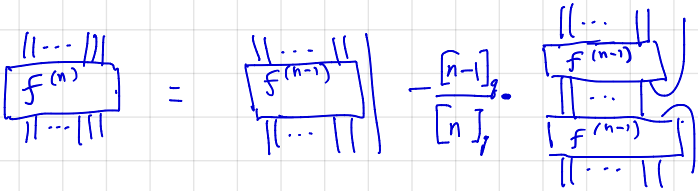
Cor: If $f^{(n)}$ exists, it is unique.

Lemma 4:

(a_n) If $f^{(n-1)}$ exists,



(b_n) If $[n]_q \neq 0$ and $f^{(n-1)}$ exists, then $f^{(n)}$ exists + satisfies Wenzl's relation:



Exercise: Generate $f^{(3)}$, compare it to the min. central projections of TL_3.

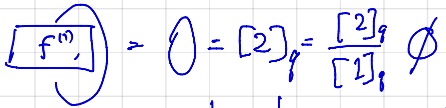
proof: by induction:

base case: $n=0$: $f^{(0)} = \emptyset$

$n=1$: $f^{(1)} = |$,

$n=2$: $f^{(2)} = || - \frac{1}{[2]_q}$

(a_2)



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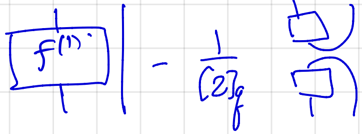
(b_2) $f^{(2)} =$



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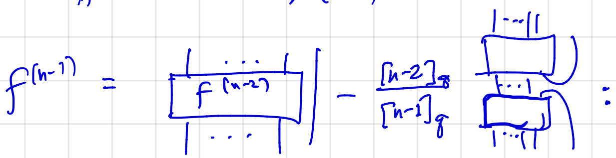
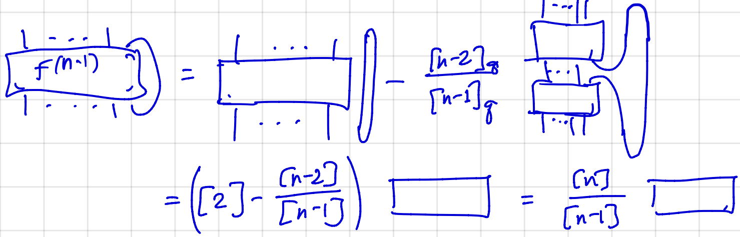
ind. step: $(a_{n-1}), (b_{n-1}) \Rightarrow (a_n) \Rightarrow (b_n)$

Given (b_{n-1}): $f^{(n-1)} =$



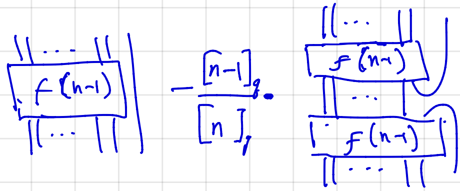
:

(a_n)

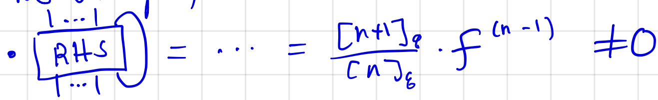


(b_n)

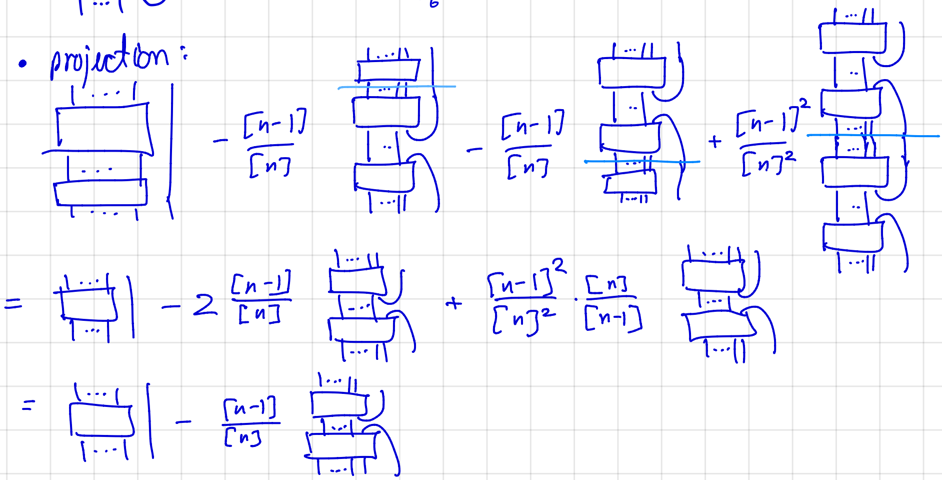
RHS =



has JW properties:



• projection:



• Exercise: Show that RHS is uncappable.

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