

# Webs and skew Howe duality

(Joint with Sabri Cautis and  
Joel Kamnitzer  
arXiv: 1210.6437 )

In quantum topology we've come across many beautiful examples of skein theory

Skein theory 'is' the study of pivotal categories from the perspective of generators and relations

## Examples

- Compute the  $U_{q \leq 3}$  quantum knot invariants by replacing

$$\text{Diagram} \rightsquigarrow q \} f - q^3 \text{ Diagram}$$

$$\text{Diagram} \rightsquigarrow q^{-1} \} f - q^{-3} \text{ Diagram}$$

and evaluating using

$$0 = q^2 + 1 + q^{-2} \quad \text{Diagram} = (q + q^{-1}) f$$

$$\text{Diagram} = f + \text{Diagram}$$

- Construct exotic subfactors
  - (e.g. Extended Haagerup. arXiv:0909.4099)
    - by guessing generators and relations for an associated pivotal category
    - and then using skein theory (in particular, a nonlocal evaluation algorithm called the 'jellyfish algorithm') to show it is non-trivial and has the right properties.

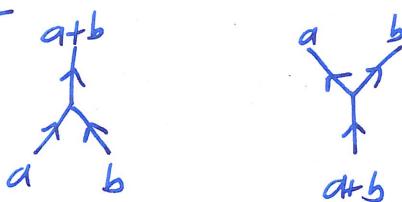
Today: we have a skein theory for Rep $\mathbb{Q}_{\leq 1}$

and the construction uses some nice representation theory, 'skew Howe duality'

Our skein theory consists of:

edges:— oriented, labelled  $0, \dots, n$  (corresponding to  $\Lambda^k(\mathbb{C}^n)$ )

vertices:—



(corresponding to the maps  $\Lambda^a \otimes \Lambda^b \rightarrow \Lambda^{a+b}$

and  $\Lambda^{a+b} \rightarrow \Lambda^a \otimes \Lambda^b$ )

relations:—

$$\text{Diagram showing a loop with two edges labeled } k \text{ and } l \text{ meeting at a vertex. The total label is } k+l. \text{ It is equated to } \left[ \begin{matrix} k+l \\ l \end{matrix} \right]_q / k+l.$$

$$\text{Diagram showing a loop with two edges labeled } l \text{ and } k \text{ meeting at a vertex. The total label is } k+l. \text{ It is equated to } \left[ \begin{matrix} n-k \\ l \end{matrix} \right]_q / k.$$

$$\text{Diagram showing a triangle with three edges labeled } a, b, c \text{ meeting at a vertex. The total label is } a+b+c. \text{ It is equated to the same triangle with edges } a, b, c \text{ and total label } a+b+c.$$

$$\text{Diagram showing a rectangle with four edges labeled } k, l, r, s \text{ meeting at vertices. The top edges are labeled } k+r \text{ and } l+s-r. \text{ The bottom edges are labeled } k \text{ and } l. \text{ It is equated to a sum of terms involving binomial coefficients: } \sum_t \left[ \begin{matrix} k-l+r-s \\ t \end{matrix} \right]_q.$$

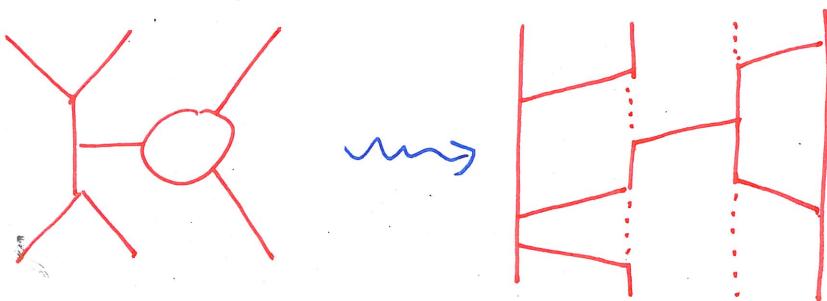
$$\text{Diagram showing a rectangle with four edges labeled } k, l, r, s \text{ meeting at vertices. The top edges are labeled } s-t \text{ and } r-t. \text{ The bottom edges are labeled } k \text{ and } l.$$

It's relatively easy to see that all maps in  $\text{Rep}_{\mathbb{U}_{q,\text{sl}_n}}$  can be built in this way, and further that the above relations hold.

But is it a presentation of the category?

(i.e. do two diagrams correspond to the same morphism only if they are related by a sequence of diagrams?)

The essential idea is that using isotopy and relations we can rewrite any diagram in ladder form.



The rungs of these ladders behave exactly like the  $E_i$ 's and  $F_i$ 's in  $U_{q,\text{sl}_n}$ .

Now some representation theory:

Skew Howe duality says that

$$GL(m) \subset \Lambda^{\bullet}(\mathbb{C}^m \otimes \mathbb{C}^n) \supset SL(n)$$

generate each others commutants.

(And with work, there's a quantum version.)

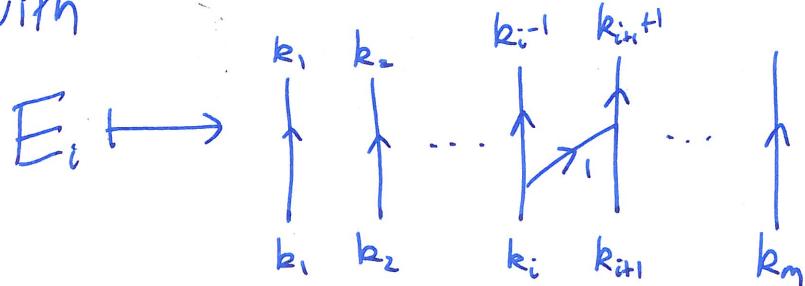
$$\text{Now } \Lambda^k(\mathbb{C}^m \otimes \mathbb{C}^n) = \Lambda^k(\underbrace{\mathbb{C}^n \oplus \dots \oplus \mathbb{C}^n}_{m \text{ times}})$$

$$= \bigoplus_{\sum k_i = k} \Lambda^{k_1} \mathbb{C}^n \otimes \dots \otimes \Lambda^{k_m} \mathbb{C}^n$$

So in particular we have

$$U_{q,m} \rightarrow \text{Hom}_{SL(n)} (\Lambda^{k_1} \mathbb{C}^n \otimes \dots \otimes \Lambda^{k_m} \mathbb{C}^n \rightarrow \Lambda^{l_1} \mathbb{C}^n \otimes \dots \otimes \Lambda^{l_m} \mathbb{C}^n)$$

with



Further, we can explicitly identify the kernel of

$$U_{\mathfrak{gl}_m} \longrightarrow \text{End}_{SL(n)}(\Lambda^k(C^m \otimes C^n))$$

as

$$I_{\lambda(K)} = \langle 1_v \mid v \text{ not dominated by } \lambda(K) \rangle$$

where

$$\lambda(K) = (n, n, \dots, n, r, 0, \dots, 0), \text{ summing to } K.$$

(This follows from something more general:

Skew Howe duality tells us

$$\Lambda^k(C^m \otimes C^n) \cong \bigoplus_{\mu \leq \lambda(K)} V(\mu) \otimes V(\mu^\pm)$$

(as  $GL(m)$ - $SL(n)$  bimodules)

$$\text{so } \text{End}_{SL(n)} \Lambda^k(C^m \otimes C^n) = \bigoplus_{\mu \leq \lambda(K)} \text{End}_{\text{---}}(V(\mu))$$

$$\text{and } U_{\mathfrak{gl}_m}/I_\lambda \cong \bigoplus_{\mu \leq \lambda} \text{End}(V(\mu)).$$

Here's the diagram which proves everything:

$$\begin{array}{ccccc} \text{Ladders}_m^n & \longrightarrow & \mathcal{U}_{\mathfrak{gl}^m}/I_{\lambda(K)} & & \\ \downarrow & \searrow & \downarrow & & \\ \text{FreeSpider}(SL(n)) & \longrightarrow & \text{Spider}(SL(n)) & \xrightarrow{\Gamma} & \text{Rep}(SL(n)) \end{array}$$

Note that we pick different  $m$  and  $K$  for different elements!

$\Gamma$  is surjective — for any morphism in  $\text{Rep}(SL(n))$ , it comes from something in some  $\mathcal{U}_{\mathfrak{gl}^m}/I_{\lambda(K)}$

$\Gamma$  is injective — for  $\alpha \in \text{Spider}(SL(n))$  with  $\Gamma(\alpha) = 0$ , lift it to a ladder for some  $m$ .

This goes to zero in  $\text{Rep}(SL(n))$ , so also zero in  $\mathcal{U}_{\mathfrak{gl}^m}/I_{\lambda(K)}$ , so also zero in  $\text{Spider}(SL(n))$ .

From the standard relations in  $U_{q,\mathbb{S}^n}$  and this ideal,  
we can immediately read off the desired relations.

Curiously, we don't need a relation corresponding to  
the Serre relation; once we allow planar isotopy  
it follows from the others:

$$\begin{array}{c} \text{Diagram 1: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop. The bottom-right corner has a small loop. There are dashed green lines connecting the corners.} \end{array} = \begin{array}{c} \text{Diagram 2: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop labeled '2'. The bottom-right corner has a small loop. A central diamond shape is formed by dashed green lines connecting the corners.} \end{array}$$

$$= \begin{array}{c} \text{Diagram 3: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop labeled '2'. The bottom-right corner has a small loop. A central diamond shape is formed by dashed green lines connecting the corners.} \end{array} + \begin{array}{c} \text{Diagram 4: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop labeled '2'. The bottom-right corner has a small loop. A central diamond shape is formed by dashed green lines connecting the corners.} \end{array}$$

$$= \begin{array}{c} \text{Diagram 5: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop. The bottom-right corner has a small loop. A central diamond shape is formed by dashed green lines connecting the corners.} \end{array} + \begin{array}{c} \text{Diagram 6: } \\ \text{A rectangle with two vertical red lines and two horizontal red lines. The top-left corner has a small loop. The bottom-right corner has a small loop. A central diamond shape is formed by dashed green lines connecting the corners.} \end{array}.$$