Webs and skew Howe duality

What is a pivotal category?

$\Longrightarrow$ a tensor category with 'nice' duals

Duality means a functor $*: \mathcal{C} \to \mathcal{C}^{op \cdot op}$
with $** = id$

along with ~~structure~~ structure maps

'pairing' : $V^* \otimes V \to \mathbb{1}$

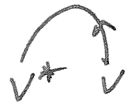
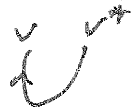
and 'copairing': $\mathbb{1} \to V \otimes V^*$

satisfying the axioms you'd expect (e.g. from linear algebra)

Examples

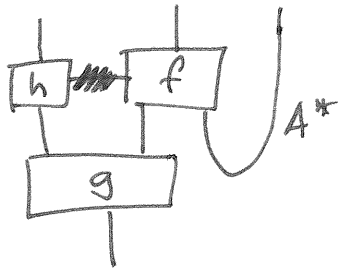
- $\operatorname{Rep} G$, for G a finite group
- $\operatorname{Rep} U_q \mathfrak{g}$ representations of a quantum group
- A-A bimodules generated by ${}_A B_A$, where $A \subset B$ is a finite index subfactor
- Deligne's interpolated symmetric groups $\operatorname{Rep} S_t$.

Pivotal categories have a diagrammatic calculus

algebra	pictures
composition	vertical stacking
$\otimes$-product	horizontal juxtaposition
$*$	π-rotation
$p_V: V^* \otimes V \to \mathbb{1}$	
$c_V: \mathbb{1} \longrightarrow V \otimes V^*$	

<u>Example</u>

$$(h \otimes f \otimes id_{A^*}) \circ (g \otimes c_A) \longleftrightarrow$$



Fundamental theorem of pivotal categories:
if two diagrams are planar isotopic, the morphisms they represent are equal.

Which diagrams do we need to describe a particular pivotal category?

<u>Kuperberg's program</u>:

Give generators and relations (ie a presentation as a pivotal category) for every pivotal category you know.

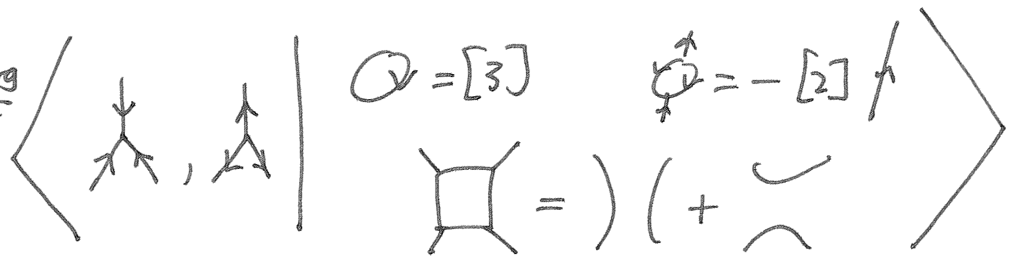
<u>Examples</u>

- $\mathrm{Rep}^{uni}\, U_q\mathfrak{sl}_2 = TL(q+q^{-1})$

  a pivotal category with no generators, just unlabelled strings, and one relation:

  $\bigcirc = q+q^{-1}$.

  (actually, you need to idempotent complete TL)

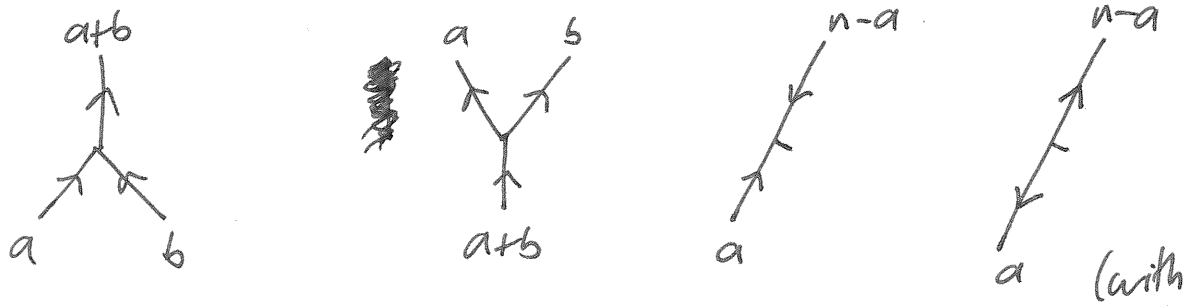
- $\mathrm{Rep}\, U_q\mathfrak{sl}_3 \overset{\text{Kuperberg}}{=\!=}$ 

- this is more or less how we constructed the extended Haagerup subfactor.

Today, let's do $U_q\mathfrak{sl}_n$.

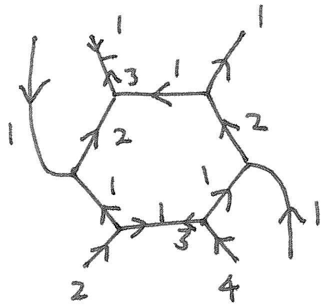
(Some history: Kim conjectured a presentation for $n=4$ in 2004, I conjectured a presentation for all n in 2007, arXiv:1210.6437 with Sabin Cautis and Joel Kamnitzer completes the story.)

The *free* SL_n spider just consists of planar diagram locally modelled on the following four vertices:



(with labels between 0 and n)

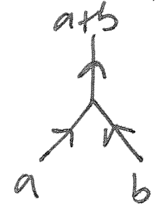
Example



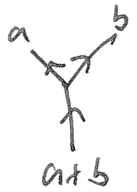
is a morphism

$$2^+4^+1^+ \longrightarrow 1^-1^-1^+$$

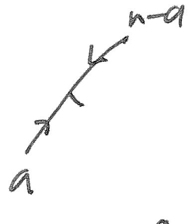
There's a functor (unique up to some normalizations) to $\mathrm{Rep}\,SL_n$



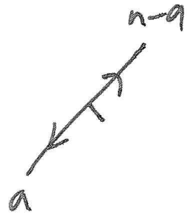
$\longmapsto \left(\Lambda^a \otimes \Lambda^b \longrightarrow \Lambda^{a+b}\right)$



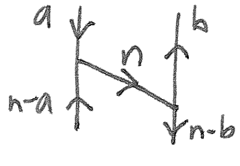
$\longmapsto \left(\Lambda^{a+b} \longrightarrow \Lambda^a \otimes \Lambda^b\right)$

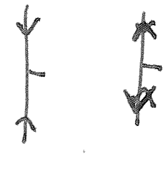


$\longmapsto \left(\Lambda^a \longrightarrow (\Lambda^{n-a})^*\right)$

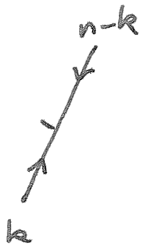


$\longmapsto \left((\Lambda^a)^* \longrightarrow \Lambda^{n-a}\right)$

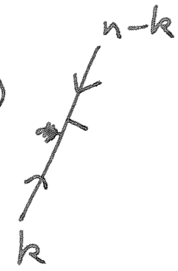
- Edges labelled 0 and n are a notational convenience: delete 0 edges, replace 

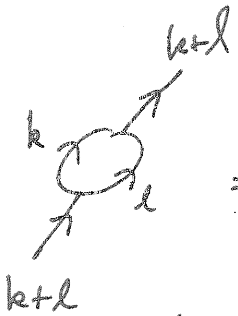
with 

Next we impose some relations:

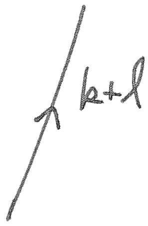


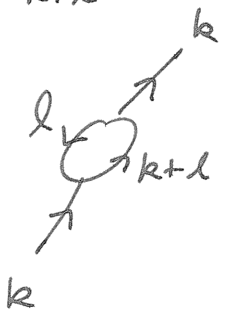
$$= (-1)^{k(n-k)}$$



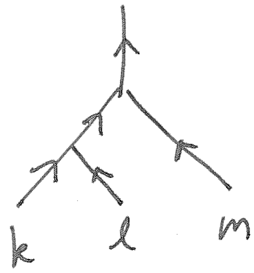


$$= \begin{bmatrix} k+l \\ l \end{bmatrix}$$

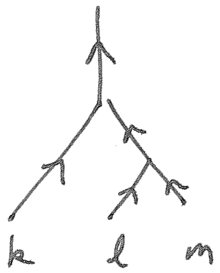


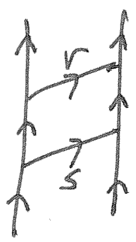


$$= \begin{bmatrix} n-k \\ l \end{bmatrix}$$

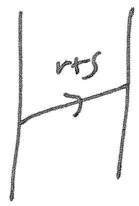


$$=$$

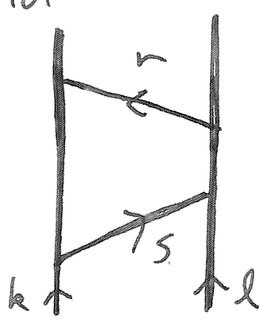




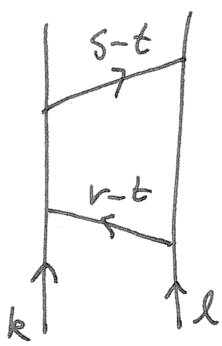
$$= \begin{bmatrix} r+s \\ s \end{bmatrix}$$



and



$$= \sum_t \begin{bmatrix} k-l+r-s \\ t \end{bmatrix}$$



Theorem $\Gamma_n : Sp(SL_n) \to Rep(SL_n)$ is an equivalence of pivotal categories.

Proof We will build a commutative diagram

$$\begin{tikzcd}
Lad^n_m \arrow[r] \arrow[dd] & \dot{U}_q(\mathfrak{gl}_m) \arrow[d] \arrow[ddr, "\Phi_m"] & \\
& \dot{U}^n_q(\mathfrak{gl}_m) \arrow[d, "\Psi^n_m"'] \arrow[dr, "\Phi^n_m"] & \\
FSp(SL_n) \arrow[r] & Sp(SL_n) \arrow[r, "\Gamma_n"'] & Rep\, SL_n
\end{tikzcd}$$

(Here Lad^n_m, $\dot{U}^n_q(\mathfrak{gl}_m)$, and many arrows, are still to be defined.)

① Γ_n is surjective on Hom spaces, because Φ^n_m is and the triangle commutes.

② Γ_n is injective:

Suppose $\Gamma_n(\omega)=0$ and, without loss of generality, the source and target of ω are oriented upwards.

We can find some m, and a lift of ω to $\tilde{\omega} \in Lad^n_m$.

Now $\Phi^n_m(\tilde{\omega})=0$, by commutativity, but Φ^n_m is injective so $\tilde{\omega}=0$ and $\omega=0$ $\square$

We define the map Γ_n on generators, and check the relations hold via some q-combinatorics.

Why is there a map $U_q(\mathfrak{gl}_m) \to \mathrm{Rep}\, SL_n$?

$\Rightarrow$ skew Howe duality.

$\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$ carries commuting actions of SL_n and GL_m; indeed, these are each others commutants, so we have

$$\mathrm{End}_{U\mathfrak{sl}_n}(\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)) \twoheadleftarrow U\mathfrak{gl}_m.$$

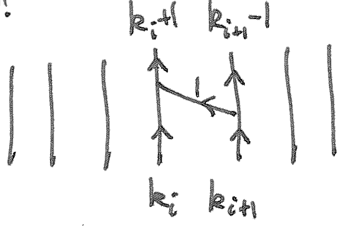
Now $\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m) = \Lambda^\bullet(\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) = \bigoplus_{\underline{k}} \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$,

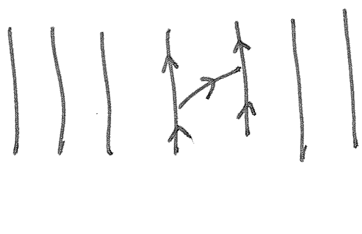
so $\mathrm{End}_{U\mathfrak{sl}_n}(\Lambda^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)) = \bigoplus_{\underline{k},\underline{l}} \mathrm{Hom}(\Lambda^{\underline{k}}\mathbb{C}^n \to \Lambda^{\underline{l}}\mathbb{C}^n)$.

Let's use ~~the~~ Lusztig's 'idempotent form' $\dot{U}_q\mathfrak{gl}_m$. Then

$$\mathrm{Hom}(\Lambda^{\underline{k}}\mathbb{C}^n \to \Lambda^{\underline{l}}\mathbb{C}^n) \twoheadleftarrow 1\!\!1_{\underline{l}}\, \dot{U}_q\mathfrak{gl}_m\, 1\!\!1_{\underline{k}}$$

Where do generators go?

$E_i \longmapsto$ 

$F_i \longmapsto$ 

What is the kernel of $U\mathfrak{gl}_m \to \bigoplus_K \mathrm{End}_{U\mathfrak{sl}_n}(\Lambda^K(\mathbb{C}^n \otimes \mathbb{C}^m))$?

$\Rightarrow$ it's generated by weight space idempotents

$$\{ \mathbb{1}_{\underline{k}} \mid \underline{k} \text{ is not a weight of } \Lambda^\bullet \mathbb{C}^n \otimes \mathbb{C}^m \}$$

Proof: Define I_λ to be the ideal in $\dot{U}(\mathfrak{gl}_m)$ generated by $\{ \mathbb{1}_\nu \mid \lambda \text{ does not dominate } \nu \}$

Lemma $\quad \dot{U}(\mathfrak{gl}_m)/I_\lambda = \bigoplus_{\mu \leq \lambda} \mathrm{End}(V(\mu))$

Proof The LHS is finite dimensional, so by Artin-Wedderburn it suffices to ~~identify the~~ show that ~~the~~ its category of modules is semisimple with simples $V(\mu)$. for $\mu \leq \lambda$. $\square$

Now, skew Howe duality tells us that as $U\mathfrak{sl}_n \otimes U\mathfrak{gl}_m$ representations,

$$\Lambda^K(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_\mu V(\mu^t) \otimes V(\mu)$$

where μ ranges over n-bounded $(0 \leq \mu_i \leq n)$ weights of $U\mathfrak{gl}_m$ and $\sum \mu = K$.

Thus $\quad \mathrm{End}_{U\mathfrak{sl}_n} \Lambda^K(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_\mu \mathrm{End}(V(\mu))$

$$= \dot{U}(\mathfrak{gl}_m)/I_{\lambda(K)}$$

where $\lambda(K) = (n, n, \ldots, n, r, 0, \ldots, 0)$, $\quad \sum \lambda(K) = K$.

$\square$

(The same argument works for $U_q\, \mathfrak{gl}_m$.)

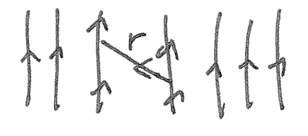
That gives us

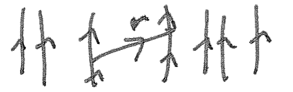
$$\begin{tikzcd} \dot{U}_q(\mathfrak{gl}_m) \arrow[d] \arrow[dr] & \\ \dot{U}^n_q(\mathfrak{gl}_m) \arrow[r, bend right, "\Phi^n_m"] & \mathrm{Rep}(SL_n) \end{tikzcd}$$

Next we need a map

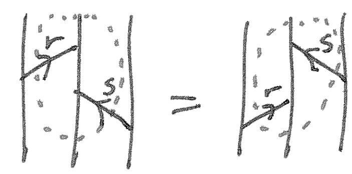
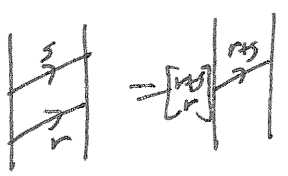
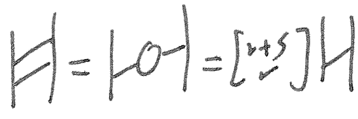
$$\dot{U}^n_q(\mathfrak{gl}_m) \xrightarrow{\Psi^n_m} Sp(SL_n)$$

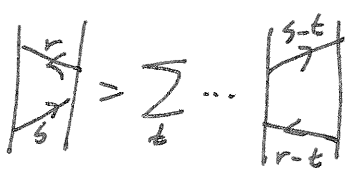
which we construct on generators:

$E_i^{(r)} \longmapsto$ 

$F_i^{(r)} \longmapsto$ 

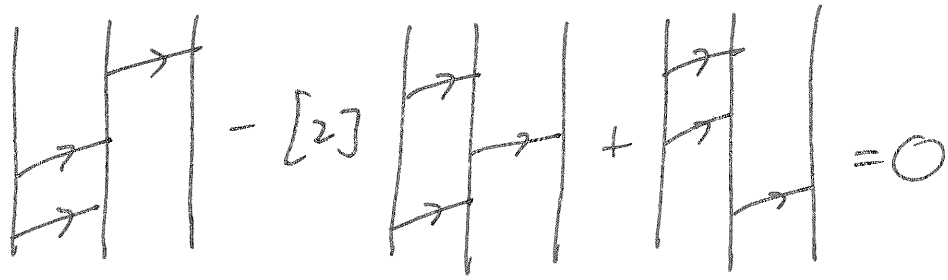
We then need to verify that the relations of $\dot{U}_q(\mathfrak{gl}_m)$ hold.

- $E_i^{(r)} F_j^{(s)} = F_j^{(s)} E_i^{(r)}$ $\quad i \neq j$ 
- $E_i^{(r)} E_j^{(s)} = E_j^{(s)} E_i^{(r)}$ $\quad |i-j|>1$ 
- $E_i^{(s)} E_i^{(r)} = \begin{bmatrix} r+s \\ r \end{bmatrix} E_i^{(r+s)}$  proof: 
- $E_i^{(r)} F_i^{(s)} 1\!\!1_{\underline{k}} = \sum_t \begin{bmatrix} \langle \underline{k}, \alpha_i \rangle + r - s \\ t \end{bmatrix} F_i^{(s-t)} E_i^{(r-t)} 1\!\!1_{\underline{k}}$

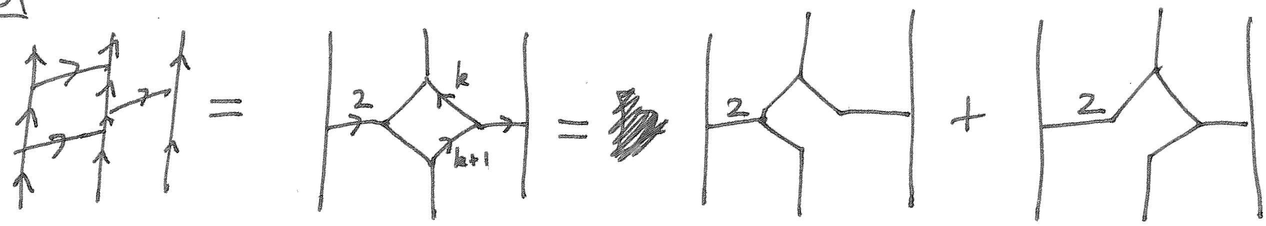
 (this is a relation in the spider already)

and finally the Serre relation

$$E_2 E_1 E_1 - [2] E_1 E_2 E_1 + E_1 E_1 E_2 = 0$$



proof



(applying the square switch relation)

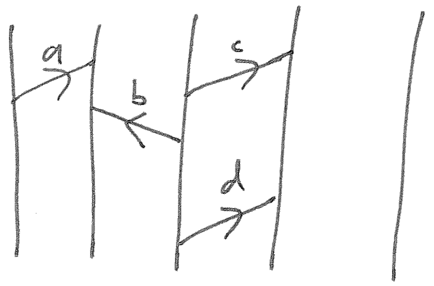
$\square$

Our final obligation is the rectangle

$$\begin{tikzcd}
\mathrm{Lad}^n_m \arrow[r] \arrow[d] & \dot{U}^n_q(\mathfrak{gl}_m) \arrow[d, "\Psi^n_m"'] \\
FSp(SL_n) \arrow[r] & Sp(SL_n) \arrow[ul, dotted]
\end{tikzcd}$$

and constructing the lift $w \mapsto \tilde{w} \in \mathrm{Lad}^n_m$.

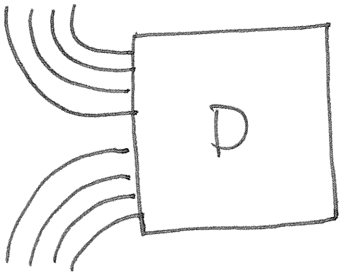
A ladder diagram is simply one of the form

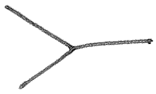


and it is clear what all the maps in the rectangle are.

How do we convert an arbitrary spider diagram into a ladder diagram?

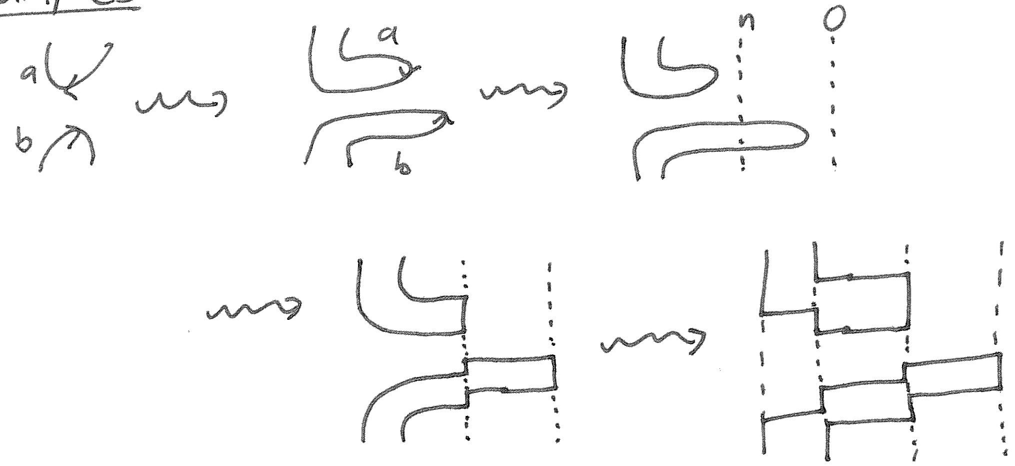
First, isotope all the vertices out to the right:



and then isotope D so everything is in generic position w.r.t. the x coordinate, and further every trivalent vertex looks like  .

Now superimpose a vertical 0 or n strand to the right of each elementary piece, and make the local modification so a neighbourhood looks like a ladder.

Examples



and.