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Abstract	We give a diagrammatic presentation in terms of generators and relations of the representation category of $U_q(\mathfrak{sl}_n)$. More precisely, we produce all the relations among SL_n-webs, thus describing the full subcategory $\otimes$-generated by fundamental representations $\bigwedge^k \mathbb{C}^n$ (this subcategory can be idempotent completed to recover the entire representation category). Our result answers a question posed by Kuperberg (Commun Math Phys 180(1):109–151, 1996) and affirms conjectures of Kim (Graphical calculus on representations of quantum lie algebras, PhD thesis, University of California, Davis. arXiv: math.QA/0310143, 2003) and Morrison (A diagrammatic category for the representation theory of $U_q(\mathfrak{sl}_n)$. PhD thesis, University of California, Berkeley arXiv: 0704.1503, 2007). Our main tool is an application of quantum skew Howe duality.

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Webs and quantum skew Howe duality

Sabin Cautis · Joel Kamnitzer · Scott Morrison

1   **Abstract** We give a diagrammatic presentation in terms of generators and relations
2   of the representation category of $U_q(\mathfrak{sl}_n)$. More precisely, we produce all the relations
3   among SL_n-webs, thus describing the full subcategory $\otimes$-generated by fundamental
4   representations $\bigwedge^k \mathbb{C}^n$ (this subcategory can be idempotent completed to recover the
5   entire representation category). Our result answers a question posed by Kuperberg
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7   ical calculus on representations of quantum lie algebras, PhD thesis, University of
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10   nia, Berkeley arXiv: 0704.1503, 2007). Our main tool is an application of quantum
11   skew Howe duality.

12   **Contents**

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1 Introduction

40 The representation theory of SL_n is a pivotal tensor category, and it is natural to ask

41 for a presentation by generators and relations, as a pivotal tensor category.

42   There are two choices one needs to make. First, one can pass to a full subcategory

43 whose idempotent completion recovers the entire representation category. In particular,

44 in this paper we look at the full subcategory, denoted $\mathcal{R}ep(\mathrm{SL}_n)$, whose objects are

45 isomorphic to tensor products of the fundamental representations $\bigwedge^k \mathbb{C}^n$ of $\mathfrak{sl}_n$.

46   Second, we need to decide which generators to use. We take the natural maps

$$\textstyle\bigwedge^k \mathbb{C}^n \otimes \bigwedge^l \mathbb{C}^n \to \bigwedge^{k+l} \mathbb{C}^n \quad \text{and} \quad \bigwedge^{k+l} \mathbb{C}^n \to \bigwedge^k \mathbb{C}^n \otimes \bigwedge^l \mathbb{C}^n$$

48 which we depict diagrammatically as follows (where we read from the bottom up)

$$\tag{1.1}$$

50 It is relatively easy to show that these are indeed generators, i.e. that every SL_n-

51 linear map between tensor products of fundamental representations can be written as

52 tensor products and compositions of these maps, along with the duality, pairing, and

53 copairing maps [15, Prop. 3.5.8]. The question then, is to identify all the relations

54 between compositions of these generators.

55   Said another way, we have a pivotal category, the "free spider category" $\mathcal{FS}p(\mathrm{SL}_n)$,

56 of trivalent webs made up by glueing together the pieces in (1.1). The edges in these

57 webs are oriented and labelled by $\{1, \ldots, n-1\}$. Moreover, we have a full and dominant

58 functor $\mathcal{FS}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$. The question is to identify the pivotal ideal which
59 is the kernel of this functor.

60   We completely answer this question (Theorem 3.5) by giving generators of this
61 pivotal ideal in Sect. 2.2, Eqs. (2.3)–(2.8).

62 1.1 Some history

63 This problem has been studied previously. For $n = 2$, there are no trivalent vertices
64 and we do not need to label strands since the only label they could carry is 1. So,
65 in this case, the free spider category is essentially just the category of embedded 1-
66 manifolds up to isotopy. The kernel of the functor to representation theory is the ideal
67 generated by the relation $\bigcirc = 2$. If we were to ignore orientations, this gives us the
68 Temperley–Lieb category where the objects are indexed by $\mathbb{N}$ and the morphisms are
69 crossingless matchings.

70   For $n = 3$, relations generating the kernel were determined by Kuperberg [12]:

71

72   They allow one to remove circles, bigons, and squares. He introduced the term "SL$_3$
73 spider" for the resulting diagrammatic category.

74   For $n \geq 4$, relations generating the kernel were proposed by Kim in [10] (for $n = 4$)
75 and by the third author in [15] (for any n). However, they did not prove that their lists
76 of relations are complete.

77 1.2 Main result

78 We define the SL$_n$-spider, denoted $\mathcal{S}p(\mathrm{SL}_n)$, to be the quotient of $\mathcal{FS}p(\mathrm{SL}_n)$ by the
79 relations (2.3)–(2.8), which all involve webs with ≤ 4 boundary edges. Our main result
80 (Theorem 3.5) states that $\mathcal{S}p(\mathrm{SL}_n)$ is equivalent to $\mathcal{R}ep(\mathrm{SL}_n)$. In particular this shows
81 that the relations in [15] are complete. Quite surprisingly, we do not need the most
82 complicated relations from [15].

83 1.3 Skew Howe duality and webs

84 The core idea of our proof is to use skew Howe duality. We give a recipe for the
85 relations in $\mathcal{R}ep(\mathrm{SL}_n)$ as certain truncations of relations holding in $U(\mathfrak{gl}_m)$, for m
86 sufficiently large. We now give a quick overview of the argument.

87   Consider the commuting actions of $U(\mathfrak{sl}_n)$ and $U(\mathfrak{gl}_m)$ on $\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^m)$. Skew
88 Howe duality tells us that the resulting map

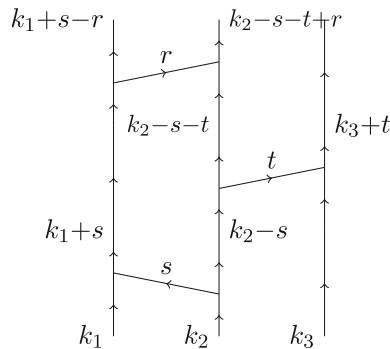🌊 Springer

Fig. 1 Ladder corresponding to $F_1^{(r)} F_2^{(t)} E_1^{(s)} 1_{k_1 k_2 k_3} \in \dot{U}(\mathfrak{gl}_3)$ (reading bottom to top in the diagram, right to left in the word)

$$\dot{U}(\mathfrak{gl}_m) \to \bigoplus_K \mathrm{End}_{U(\mathfrak{sl}_n)}\left(\bigwedge\nolimits^K (\mathbb{C}^n \otimes \mathbb{C}^m)\right) \tag{1.2}$$

is surjective, where $\dot{U}(\mathfrak{gl}_m)$ denotes Lusztig's idempotent form. Moreover, we prove that its kernel is the ideal generated by those weight space idempotents falling outside the weight support of $\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^m)$. This result is proved in Sect. 4.4. The quotient of $\dot{U}(\mathfrak{gl}_m)$ by this ideal is denoted $\dot{U}^n(\mathfrak{gl}_m)$.

Now, as $U(\mathfrak{sl}_n)$-representations, we have

$$\bigwedge\nolimits^K (\mathbb{C}^n \otimes \mathbb{C}^m) = \bigwedge\nolimits^K (\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) = \bigoplus_{\underline{k}: \sum \underline{k} = K} \bigwedge\nolimits^{k_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge\nolimits^{k_m} \mathbb{C}^n. \tag{1.3}$$

Thus combining (1.2) and (1.3) we obtain an isomorphism

$$\dot{U}^n(\mathfrak{gl}_m) \xrightarrow{\sim} \bigoplus_{\underline{k},\underline{l}: \sum \underline{k} = \sum \underline{l}} \mathrm{Hom}_{U(\mathfrak{sl}_n)}\left(\bigwedge\nolimits^{k_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge\nolimits^{k_m} \mathbb{C}^n, \bigwedge\nolimits^{l_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge\nolimits^{l_m} \mathbb{C}^n\right)$$

$$\tag{1.4}$$

Under this map, elements of $U(\mathfrak{gl}_m)$ are sent to particular webs, which we call ladders. Figure 1 illustrates the image of $F_1^{(r)} F_2^{(t)} E_1^{(s)} 1_{k_1 k_2 k_3} \in \dot{U}(\mathfrak{gl}_3)$ under this map.

This allows us to write the generating relations of $\dot{U}^n(\mathfrak{gl}_m)$ in a diagrammatic form. By the above isomorphism, we see that these diagrammatic relations become the generating relations in $\mathcal{R}ep(\mathrm{SL}_n)$.

1.4 The quantum deformation and categorification

Our whole discussion above has a natural q-deformation. In other words, $\mathcal{R}ep(\mathrm{SL}_n)$ becomes the category of $U_q(\mathfrak{sl}_n)$-modules generated by tensor products of fundamental representations, while $\dot{U}(\mathfrak{gl}_m)$ is replaced by the quantum group $\dot{U}_q(\mathfrak{gl}_m)$.

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108   In the previous section we assumed $q = 1$ in order to simplify the notation. However,
109 in the rest of the paper we will always consider this quantum deformation, with q taken
110 as a formal variable. In Sect. 4.2 we will discuss in detail the quantum skew Howe
111 duality results we need, which may be of independent interest.

112   The q-deformation of the linear maps (1.2) can be rephrased as giving us a functor
113 $\Psi_m^n : \dot{\mathcal{U}}_q(\mathfrak{gl}_m) \to \mathcal{S}p(\mathrm{SL}_n)$ where $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ is the categorical version of $\dot{U}_q(\mathfrak{gl}_m)$
114 (replacing weight space idempotents with distinct objects). The categories $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$
115 and $\mathcal{S}p(\mathrm{SL}_n)$ can both be categorified. First, $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ can be lifted to the 2-category
116 $\dot{\mathbb{U}}_q(\mathfrak{gl}_m)$, defined by Khovanov–Lauda [11] and Rouquier [20]. On the other hand, the
117 spider category $\mathcal{S}p(\mathrm{SL}_n)$ can be lifted to a 2-category $\mathcal{F}\mathrm{oam}_n$ where the 2-morphisms
118 consist of "foams" between webs, introduced by Khovanov [9] (for n = 3) and partially
119 described for general n by Mackaay et al. [19].

120   Thus, it is natural to ask if the functor Ψ_m^n lifts to a 2-functor

$$121 \qquad \widetilde{\Psi}_m^n : \dot{\mathbb{U}}_q(\mathfrak{gl}_m) \to \mathcal{F}\mathrm{oam}_n.$$

122 In the cases $n = 2$ and 3 such a 2-functor has recently been studied by Mackaay et al.
123 [18] and by Lauda et al. [13].

124   There is something more specific one can say. It is easy to see that the functor Ψ_m^n
125 factors through the quotient $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ of $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ by the ideal generated by the identity
126 morphisms 1_λ where λ is not n-bounded (see Sect. 4.1 for the definition of n-bounded
127 weights). The point of taking this quotient is that, combining Theorems 4.4 and 3.5
128 together with diagram (3.9), we have that

$$129 \qquad \Psi_m^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \to \mathcal{S}p(\mathrm{SL}_n)$$

130 is fully faithful. Thus the spider category is essentially a quotient of the limit category
131 $\dot{\mathcal{U}}_q(\mathfrak{gl}_\infty)$

$$132 \qquad \mathcal{S}p(\mathrm{SL}_n) \cong \dot{\mathcal{U}}_q^n(\mathfrak{gl}_\infty).$$

133   One can likewise define $\dot{\mathbb{U}}_q^n(\mathfrak{gl}_m)$ as the quotient of $\dot{\mathbb{U}}_q(\mathfrak{gl}_m)$ by the ideal generated
134 by the identity 2-morphisms $\mathrm{id}_{1_\lambda} : 1_\lambda \to 1_\lambda$ where λ is not n-bounded. It should
135 be easy to check that $\widetilde{\Psi}_m^n$ factors through this quotient. We then speculate that the
136 2-functor

$$137 \qquad \widetilde{\Psi}_m^n : \dot{\mathbb{U}}_q^n(\mathfrak{gl}_m) \to \mathcal{F}\mathrm{oam}_n$$

138 is fully faithful on 2-morphisms. This would mean that the foam categories are essen-
139 tially alternative descriptions of certain quotients of the limit 2-category $\dot{\mathbb{U}}_q(\mathfrak{gl}_\infty)$.

140 ## 1.5 Braiding and knot invariants

141   The category $\mathcal{R}ep(\mathrm{SL}_n)$ is in fact a braided monoidal category where the braiding
142 comes from the R-matrix of the quantum group $U_q(\mathfrak{sl}_n)$. In Sect. 6.1, we express this
143 braiding using webs.

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More precisely, we define a braided monoidal category structure on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet) = \oplus_m \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ using Lusztig's quantum Weyl groups elements. We then prove (Theorem 6.5) that the functor $\Phi^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet) \to \mathcal{R}ep(\mathrm{SL}_n)$ carries the braiding in $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$ to the braiding in $\mathcal{R}ep(\mathrm{SL}_n)$. In particular, this shows that under quantum skew Howe duality, Lusztig's quantum group element $T \in \dot{\mathcal{U}}_q(\mathfrak{gl}_2)$ is taken to the R-matrix braidings $\beta : \bigwedge_q^k \mathbb{C}_q^n \otimes \bigwedge_q^l \mathbb{C}_q^n \to \bigwedge_q^l \mathbb{C}_q^n \otimes \bigwedge_q^k \mathbb{C}_q^n$. This last fact was previously proven by the first and second authors and Licata in [2], following the approach from [22] who established the analogous result for symmetric Howe duality. The proofs from [22] and [2] involve a somewhat lengthy computation, whereas the categorical approach in this paper provides a far more conceptual proof.

The braided monoidal category structure on $\mathcal{R}ep(\mathrm{SL}_n)$ leads to quantum knot invariants. The results of this paper show that these knot invariants can be computed using webs. Of course, this was known by the work of Murakami et al. [16]. Our work extends theirs in the sense that while they gave a partition function evaluating closed webs, we have shown that the category of open webs, with the relations described here, is itself equivalent to the representation category. We would like to have an evaluation algorithm, showing directly that any closed web can be evaluated to a scalar by repeated application of the relations here; one has been proposed by Jeong and Kim [8]. Grant [5] also gives a somewhat indirect evaluation algorithm using a subset of our relations. While the tensor category defined by those relations is thus evaluable, it seems possible that it is degenerate (i.e. there are negligible morphisms, which when paired with any other morphism to give a closed diagram, give zero) and hence not equivalent to the representation category. Sikora's work in [21] gives an alternative presentation of the representation category as a *braided* pivotal category, using only the standard representation and the determinant map $\otimes^n \mathbb{C}_q^n \to \bigwedge_q^n \mathbb{C}_q^n$ and its dual as generators. Again, it is not clear whether the diagram category with the given relations is degenerate or not.

2 The categories $\mathcal{FS}p(\mathrm{SL}_n)$ and $\mathcal{S}p(\mathrm{SL}_n)$
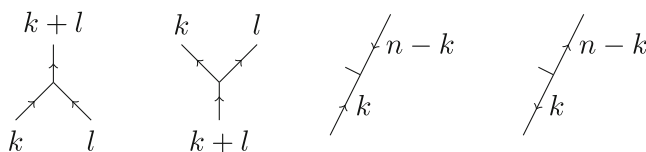
We will denote by $[n]_q$ the quantum integer $q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$. More generally, we have quantum binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q \ldots [1]_q}{([n-k]_q \ldots [1]_q)([k]_q \ldots [1]_q)}.$$

We also adopt the convention that $[-n]_q = -[n]_q$, which is clear if we write $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

2.1 The free spider category $\mathcal{FS}p(\mathrm{SL}_n)$

The free spider category $\mathcal{FS}p(\mathrm{SL}_n)$ has as objects sequences $\underline{k}$ in $\{1^\pm, \ldots, (n-1)^\pm\}$, and as morphisms ($\mathbb{C}(q)$-linear combinations of) oriented planar graphs locally modeled on the following four types of vertices:

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182 with all labels drawn from the set $\{1, \ldots, n-1\}$. The third and fourth graphs depict
183 bivalent vertices, called 'tags', which are not rotationally symmetric, meaning that
184 the tag provides a distinguished side. The bottom boundary of any planar graph in
185 $\mathrm{Hom}(\underline{k}, \underline{l})$ is $\underline{k}$ with the strand oriented up for each positive entry, and the strand
186 oriented down for each negative entry. Similarly, the top boundary is determined by $\underline{l}$
187 in the same way.

188 *Example* We can build a trivalent vertex with one incoming edge labelled by $n-2$
189 and two outgoing edges labelled by $n-1$, for example as

190

$$(2.1)$$

191 In this example there are various choices about which direction each tag points. Once
192 we impose the relations in the spider category, these choices will all become equal,
193 up to a sign, via Eq. (2.3).

194 *Example* There are several ways to build a trivalent vertex with all edges oriented
195 inwards. For instance,

196

$$(2.2)$$
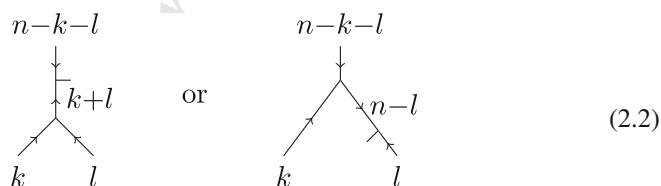
197 There are choices both in where around the trivalent vertex to place the tag, and on
198 which side of the edge the tag lies. Again, these will all become equal (possibly up to
199 a sign), via Eqs. (2.3) and (2.10).

200 We will often draw diagrams with edges also labelled by 0 or n. This is a notational
201 convenience, to be interpreted as follows. Edges labelled by 0 and n are to be deleted.
202 Trivalent vertices involving a 0 edge become simple strands and trivalent vertices
203 involving an edge labelled by n are replaced with tags:

Any trivalent vertices with all edges labelled either 0 or n can be deleted. We will occasionally utilize diagrams with an edge labelled less than 0 or greater than n; by convention these diagrams are 0.

2.2 Definition of the spider category $\mathcal{S}p(\mathrm{SL}_n)$

The spider category $\mathcal{S}p(\mathrm{SL}_n)$ is the quotient of $\mathcal{FS}p(\mathrm{SL}_n)$ by the following relations:

$$ \qquad (2.3) $$

$$ = \begin{bmatrix} k+l \\ l \end{bmatrix}_q \qquad (2.4) $$

$$ = \begin{bmatrix} n-k \\ l \end{bmatrix}_q \qquad (2.5) $$

$$ \qquad (2.6) $$

$$
\begin{aligned}
&\vcenter{\hbox{[square web diagram: outer labels $k-s-r$ (top left), $l+s+r$ (top right), middle labels r, $k-s$, $l+s$, s, bottom labels k, l]}}
= \begin{bmatrix} r+s \\ r \end{bmatrix}_q
\vcenter{\hbox{[web diagram: top labels $k-s-r$, $l+s+r$, middle label $r+s$, bottom labels k, l]}}
\end{aligned}
\tag{2.7}
$$

$$
\vcenter{\hbox{[square web diagram: top labels $k-s+r$, $l+s-r$, middle labels r, $k-s$, $l+s$, s, bottom labels k, l]}}
= \sum_t \begin{bmatrix} k-l+r-s \\ t \end{bmatrix}_q
\vcenter{\hbox{[square web diagram: top labels $k-s+r$, $l+s-r$, middle labels $s-t$, $k+r-t$, $l-r+t$, $r-t$, bottom labels k, l]}}
\tag{2.8}
$$

together with the mirror reflections and the arrow reversals of these. These relations are often referred to as the 'switching a tag' (2.3), 'removing a bigon' (2.4) and (2.5), '$I = H$' (2.6), 'square removal' (2.7) and 'square switch' (2.8).

Remark In the relations above we allow strands to be labelled by 0 and n. As before, this means that 0-strands should be deleted and n-strands replaced by tags.

Remark The relations above are redundant. For example, relation (2.7) is not necessary, following readily from relations (2.4) and (2.6) (alternatively, (2.4) is a special case of (2.7) with $l = k - s - r = 0$). Relation (2.5) is a special case of relation (2.8), with some edges labelled by 0 or n. Moreover, relation (2.8) for $r, s > 1$ follows from the square switch relation with $r = s = 1$ (and the rest of the relations). There is an easy diagrammatic proof for these facts or they can be proven as consequences of our main theorem. We give the above over-complete list of relations because they would be needed if we worked over $\mathbb{Z}[q, q^{-1}]$ rather than $\mathbb{C}(q)$.

Lemma 2.1 *The following are consequences of the relations above:*

$$
\vcenter{\hbox{[circle diagram with label k]}} = \begin{bmatrix} n \\ k \end{bmatrix}_q
\tag{2.9}
$$

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$$(2.10)$$

$$(2.11)$$

$$- [2]_q \qquad + \qquad = 0$$

$$(2.12)$$

where we use the convention that any non-vertical unlabelled strand carries a 1, *while*
the vertical strands have arbitrary compatible labels.

Proof The first identity follows from relation (2.5) with $k = 0$ after deleting the 0-strings. The second follows from (2.6) after replacing n-strands with tags. The third also follows from (2.5) with $l = n - k$ after replacing the n-strand with a matching pair of tags.

Finally, to prove (2.12), we apply the $I = H$ relation along the leftmost upright to obtain

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252 where, to get the second equality, we apply Eq. (2.8) to the central square with
253 $(k, l, r, s) = (2, k, k, 1)$. Both coefficients here are equal to $+1$. Finally an appli-
254 cation of Eq. (2.7) on each 2-strand gives the desired identity. □

255 *Remark* Later, we will use Eq. (2.12) in the proof of Proposition 5.3, where it will
256 correspond to the quantum Serre relation 4.3 in $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$.

257 *Remark* Many more local relations hold, as consequences of these. In particular [15]
258 described another classes of relations, the 'Kekulé' relations:

259

260 for each $\Sigma b \leq j \leq \Sigma a + n - 1$ (each edge label is the signed sum of the blue
261 arrows on either side, $\underline{\sum} a = \sum a - \min a$, and $\overline{\sum} b = \sum b - \min b$). We do not
262 know a diagrammatic argument deriving the Kekulé relations from the relations pre-
263 sented here; nevertheless, such a derivation must exist, by our main theorem. (For the
264 simplest Kekulé relation, originally found by Kim [10], we do have such a deriva-
265 tion.)

266 Further, the main theorem of this paper in particular implies that any closed spi-
267 der diagram can be reduced to a scalar multiple of the empty diagram, by successive
268 application of the given relations, but we do not have such an evaluation algorithm at
269 this point.

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3 Statement and proof of the main theorem

Recall that $U_q(\mathfrak{sl}_n)$ is a $\mathbb{C}(q)$-algebra with generators E_i, F_i, K_i for $i = 1, \ldots, n-1$ and the following relations

$$K_i K_j = K_j K_i, \quad K_j E_i K_j^{-1} = q^{\langle i,j \rangle} E_i, \quad K_j F_j K_j^{-1} = q^{-\langle i,j \rangle} F_i$$

$$\left[E_i, F_j\right] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

$$[2]_q E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \text{ if } |i - j| = 1, \quad [E_i, E_j] = 0 \text{ if } |i - j| > 1$$

where

$$\langle i, j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It is a Hopf algebra with the coproduct given by

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i \tag{3.1}$$

the antipode by

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i \tag{3.2}$$

and the conunit by

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0, \quad \epsilon(F_i) = 0. \tag{3.3}$$

We will study the category $\mathcal{R}ep(\mathrm{SL}_n)$ whose objects are representation of $U_q(\mathfrak{sl}_n)$ isomorphic to tensor products of the fundamental representations $\bigwedge_q^k \mathbb{C}_q^n$ of $U_q(\mathfrak{sl}_n)$.

3.1 Some generating morphisms

Denote by $x_1, \ldots, x_n$ the usual basis of the standard $U_q(\mathfrak{sl}_n)$-module $\mathbb{C}_q^n$. Note that $S_q^2 \mathbb{C}_q^n$ is spanned by

$$x_i \otimes x_j + q x_j \otimes x_i, \quad \text{for } i < j, \text{ and } x_i^2, \text{ for all } i.$$

We define the **quantum exterior algebra** of $\mathbb{C}_q^n$

$$\bigwedge_q^\bullet(\mathbb{C}_q^n) := T\mathbb{C}_q^n / \langle S_q^2(\mathbb{C}_q^n) \rangle$$

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to be the tensor algebra, over $\mathbb{C}(q)$, of $\mathbb{C}_q^n$ modulo the quantum symmetric square (see [1]). The space $\bigwedge_q^\bullet(\mathbb{C}_q^n)$ is a graded $U_q(\mathfrak{sl}_n)$-module algebra and we denote the product by $\wedge_q$. Thus in $\bigwedge_q^\bullet(\mathbb{C}_q^n)$ we have that

$$x_i \wedge_q x_j + q x_j \wedge_q x_i = 0 \quad \text{for } i < j, \text{ and } x_i \wedge_q x_i = 0, \text{ for all } i.$$

If $S = \{k_1, \ldots, k_a\} \subset \{1, \ldots, n\}$, with $k_1 > \cdots > k_a$, we write $x_S = x_{k_1} \wedge_q \cdots \wedge_q x_{k_a} \in \bigwedge_q^a(\mathbb{C}_q^n)$. The set x_S where S ranges over k element subsets of $\{1, \ldots, n\}$ forms a basis for $\bigwedge_q^k(\mathbb{C}_q^n)$. Note that E_i, F_i and K_i act as follows:

$$E_i(x_k) = \begin{cases} x_{k-1} & \text{if } i = k-1 \\ 0 & \text{otherwise} \end{cases} \quad F_i(x_k) = \begin{cases} x_{k+1} & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$$

$$K_i(x_k) = \begin{cases} q x_k & \text{if } k = i \\ q^{-1} x_k & \text{if } k = i+1 \\ x_k & \text{otherwise.} \end{cases}$$

Together with the comultiplication this determines how E_i, F_i, K_i act on x_S.

We now define a generating set of morphisms in $\mathcal{R}ep(\mathrm{SL}_n)$. If S, T are two disjoint subsets of $\{1, \ldots, n\}$ we define

$$\ell(S, T) = |\{(i, j) : i \in S, j \in T \text{ and } i < j\}|.$$

Note that $\ell(S, T) + \ell(T, S) = |S||T|$. We define $M_{k,l} : \bigwedge_q^k(\mathbb{C}_q^n) \otimes \bigwedge_q^l(\mathbb{C}_q^n) \to \bigwedge_q^{k+l}(\mathbb{C}_q^n)$ to be the multiplication map $\wedge_q$, so that we have

$$M_{k,l}(x_S \otimes x_T) = x_S \wedge_q x_T = \begin{cases} (-q)^{\ell(S,T)} x_{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that $M_{k,l}$ is a $U_q(\mathfrak{sl}_n)$-module map by the definition of the quantum exterior algebra.

On the other hand, we define a $\mathbb{C}(q)$-linear map $M'_{k,l} : \bigwedge_q^{k+l}(\mathbb{C}_q^n) \to \bigwedge_q^k(\mathbb{C}_q^n) \otimes \bigwedge_q^l(\mathbb{C}_q^n)$ as follows

$$M'_{k,l}(x_S) = (-1)^{kl} \sum_{T \subset S} (-q)^{-\ell(S \smallsetminus T, T)} x_T \otimes x_{S \smallsetminus T}$$

where T ranges over k-element subsets of S. Finally, we define $D_k : \bigwedge_q^k(\mathbb{C}_q^n) \to (\bigwedge_q^{n-k}(\mathbb{C}_q^n))^*$ by

$$D_k(x_S)(x_T) = \begin{cases} (-q)^{\ell(S,T)} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

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Lemma 3.1 *The maps $M'_{k,l}$ and D_k defined above are morphisms of $U_q(\mathfrak{sl}_n)$-modules.*

Proof We prove that D_k is a map of $U_q(\mathfrak{sl}_n)$-modules (the proof for $M'_{k,l}$ is similar). First we need to show that

$$E_i(D_k(x_S))(x_T) = D_k(E_i(x_S))(x_T) \tag{3.4}$$

for any $S, T \subset \{1, \ldots, n\}$. On the one hand, if $S = \{k_1 > \cdots > k_a\}$ then

$$E_i(x_S) = \begin{cases} x_{k_1} \wedge_q \cdots \wedge_q x_{k_j-1} \wedge_q x_{k_{j+1}} \wedge_q \cdots \wedge_q x_{k_a} & \text{if } k_j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the right side of (3.4) equals

$$\begin{cases} (-q)^{\ell(S',T)} & \text{if } S' = T^c \\ 0 & \text{otherwise} \end{cases}$$

where $S' := S \smallsetminus \{i+1\} \cup \{i\}$ and $T^c = \{1, \ldots, n\} \setminus T$ is the complement. On the other hand, using the antipode S, the left hand side of (3.4) equals

$$D_k(x_S)(-E_i K_i^{-1}(x_T)) = D_k(x_S)(-E_i(x_T)) \cdot \begin{cases} q^{-1} & \text{if } i \in T, i+1 \notin T \\ q & \text{if } i+1 \in T, i \notin T \\ 1 & \text{otherwise} \end{cases}$$

$$= \begin{cases} (-q)^{\ell(S,T')+1} & \text{if } S^c = T' \\ 0 & \text{otherwise} \end{cases}$$

where $T' := T \smallsetminus \{i+1\} \cup \{i\}$. Here we used that

$$K_i(x_T) = \begin{cases} q x_T & \text{if } i \in T, i+1 \notin T \\ q^{-1} x_T & \text{if } i+1 \in T, i \notin T \\ x_T & \text{otherwise.} \end{cases}$$

Relation (3.4) now follows since one can check that $S' = T^c$ if and only if $S' \cap T^c$ and if this is the case then $\ell(S', T) = \ell(S, T') + 1$. The analogues of (3.4) for F_i and K_i follow similarly. $\qquad\square$

The following result is perhaps of independent interest.

Lemma 3.2 *The space $\bigwedge_q^\bullet(\mathbb{C}_q^n)$ carries the structure of a coassociative coalgebra, where comultiplication is given by the map*

$$\Delta = \bigoplus_{k+l=N} M'_{k,l} : \bigwedge_q^N(\mathbb{C}_q^n) \to \bigoplus_{k+l=N} \bigwedge_q^k(\mathbb{C}_q^n) \otimes \bigwedge_q^l(\mathbb{C}_q^n)$$

and the counit $\epsilon : \bigwedge_q^\bullet(\mathbb{C}_q^n) \to \mathbb{C}$ by $x_S \mapsto \delta_{S,\emptyset}$.

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Proof To check coassociativity, let A, B, C be disjoint and consider the coefficient of $x_A \otimes x_B \otimes x_C$ in $(\Delta \otimes 1) \circ \Delta(x_S)$. Checking the definition we see that it equals

$$(-1)^{|A \cup B||C| + |A||B|}(-q)^{-\ell(C, A \cup B) - \ell(B,A)} = (-1)^{|A||C| + |B||C| + |A||B|}$$
$$\times (-q)^{-\ell(C,A) - \ell(C,B) - \ell(B,A)}$$

which equals its coefficient in $(1 \otimes \Delta) \circ \Delta(x_S)$.

The counit identity follows from

$$x_S \mapsto \sum_{k,l}(-1)^{kl} \sum_{T \subset S, |T| = k}(-q)^{-\ell(S \smallsetminus T, T)} x_T \otimes x_{S \smallsetminus T} \mapsto (-q)^{-\ell(S, \emptyset)} x_\emptyset \otimes x_S = x_S.$$

$\square$

3.2 Definition of the functor $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$

We now define the functor from the spider to the representation category $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$. At the level of objects we take

$$(k_1^{\epsilon_1}, \ldots, k_m^{\epsilon_m}) \mapsto \left(\bigwedge_q^{k_1} \mathbb{C}_q^n\right)^{\epsilon_1} \otimes \cdots \otimes \left(\bigwedge_q^{k_m} \mathbb{C}_q^n\right)^{\epsilon_m}.$$

where $\epsilon_i \in \{1, -1\}$ and we interpret W^{-1} as the dual representation W^*. For generating morphisms we take

As a special case, this forces us to define Γ_n on tags by

Theorem 3.3 *This defines a pivotal functor* $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$.

Proof By Lemma 3.1 we have a well defined map $\mathcal{F}\mathcal{S}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$. It remains to show that this maps factors through $\mathcal{S}p(\mathrm{SL}_n)$, which means checking relations (2.3)–(2.8).

Relation (2.4). We need to compute $M_{k,l} \circ M'_{k,l} = \begin{bmatrix} k+l \\ k \end{bmatrix}_q \mathrm{id}$. For S with $|S| = k+l$ we have

$$M_{k,l} \circ M'_{k,l}(x_S) = M_{k,l}\left((-1)^{kl} \sum_{T \subset S} (-q)^{-\ell(S \smallsetminus T, T)} x_T \otimes x_{S \smallsetminus T}\right)$$

$$= (-1)^{kl} \sum_{T \subset S} (-q)^{\ell(T, S \smallsetminus T)}(-q)^{-\ell(S \smallsetminus T, T)} x_S$$

$$= (-1)^{kl}(-q)^{-kl} \sum_{T \subset S} (-q)^{2\ell(T, S \smallsetminus T)} x_S$$

$$= \begin{bmatrix} k+l \\ k \end{bmatrix}_q x_S$$

using that $\ell(T, S \smallsetminus T) + \ell(S \smallsetminus T, T) = |S \smallsetminus T||T| = kl$ and using Lemma 3.4 to obtain the last equality.

Relation (2.5) The bigon on the left hand side of (2.5) is the composition of a cup, two trivalent vertices and a cap. Because Γ_n is defined as a pivotal functor it takes the cup and cap to the "cup" and "cap" in $\mathcal{R}ep(\mathrm{SL}_n)$. We will consider $(\bigwedge_q^k \mathbb{C}_q^n)^*$ to be the left dual of $\bigwedge_q^k \mathbb{C}_q^n$ with dual basis x_T^* (here T ranges over k-element subsets of $\{1, \ldots, n\}$). Thus the "cap" map

$$(\textstyle\bigwedge_q^k \mathbb{C}_q^n)^* \otimes \bigwedge_q^k \mathbb{C}_q^n \to \mathbb{C}(q)$$

is just given by $x_T^* \otimes x_U \mapsto \delta_{T,U}$.

On the other hand the "cup" map is given by the canonical copairing followed by the inverse of the pivotal isomorphism. The pivotal isomorphism in the category $\mathcal{R}ep(\mathrm{SL}_n)$ is given by the element $K_{2\rho} = K_1^{n-1} \ldots K_{n-1}$. Thus, we see that the "cup" map is given by

$$\mathbb{C}(q) \to (\textstyle\bigwedge_q^k \mathbb{C}_q^n)^* \otimes \bigwedge_q^k \mathbb{C}_q^n$$

$$1 \mapsto \sum_{|T|=k} q^{k(n-k)-2\ell(T,T^c)} x_T^* \otimes x_T$$

where we use that $K_{2\rho}^{-1} x_T = q^{\ell(T^c,T)-\ell(T,T^c)} x_T = q^{k(n-k)-2\ell(T,T^c)} x_T$.

Thus, for $S \subset \{1, \ldots, n\}$ with $|S| = k$, the left hand side of (2.5) acts on x_S as follows:

$$x_S \mapsto q^{l(n-l)} \sum_{|T|=l} (-q)^{-2\ell(T,T^c)} x_T^* \otimes x_T \otimes x_S$$

$$\mapsto q^{l(n-l)} \sum_{|T|=l, S \cap T=\emptyset} (-q)^{\ell(T,S)-2\ell(T,T^c)} x_T^* \otimes x_{T \cup S}$$

$$\mapsto (-1)^{kl} q^{l(n-l)} \sum_{|T|=l, S \cap T=\emptyset} (-q)^{\ell(T,S)-2\ell(T,T^c)}$$

$$\times \sum_{U \subset T \cup S} (-q)^{-\ell((T \cup S) \smallsetminus U, U)} x_T^* \otimes x_U \otimes x_{(T \cup S) \smallsetminus U}$$

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389
$$\mapsto (-1)^{kl} q^{l(n-l)} \sum_{|T|=l, S \cap T = \emptyset} (-q)^{\ell(T,S) - \ell(S,T) - 2\ell(T,T^c)} x_S$$

390
$$= (-1)^{kl} q^{l(n-l)} \sum_{|T|=l, S \cap T = \emptyset} (-q)^{-\ell(T,S) - \ell(S,T) - 2\ell(T,T^c \setminus S)} x_S$$

391
$$= q^{l(n-l-k)} \sum_{T \subset S^c} q^{-2\ell(T,T^c \setminus S)} x_S$$

392
$$= \begin{bmatrix} n-k \\ l \end{bmatrix}_q x_S$$

393 where we write S^c and T^c for the complements of S and T in $\{1, \ldots, n\}$. The result
394 follows.

395 **Relation** (2.6) This follows immediately from the fact that $\bigwedge_q^\bullet(\mathbb{C}_q^n)$ forms an asso-
396 ciative algebra (it is a quotient of a tensor algebra) and the arrow reversal follows from
397 the fact that it forms a coassociative coalgebra (Lemma 3.2).

398 **Relation** (2.8) We will prove (2.8) in the case when $r = s = 1$ (this suffices by the
399 second remark in §2.2), which amounts to the following diagram.

400

401
$$\tag{3.5}$$

402 It suffices to check this on $x_S \otimes x_T$ for some arbitrary $S, T \subset \{1, \ldots, n\}$ with $|S| = k$
403 and $|T| = l$. The left hand side of (3.5) acts as follows:

404
$$x_S \otimes x_T \mapsto (-1)^{k-1} \sum_{r \in S} (-q)^{-\ell(r, S \setminus r)} x_{S \setminus r} \otimes x_r \otimes x_T$$

405
$$\mapsto (-1)^{k-1} \sum_{r \in S \setminus T} (-q)^{-\ell(r, S \setminus r) + \ell(r, T)} x_{S \setminus r} \otimes x_{T \cup r}$$

406
$$\mapsto (-1)^{k+l-1} \sum_{r \in S \setminus T} \sum_{r' \in T \cup r} (-q)^{-\ell(r, S \setminus r) + \ell(r, T) - \ell((T \cup r) \setminus r', r')}$$

407
$$\times x_{S \setminus r} \otimes x_{r'} \otimes x_{(T \cup r) \setminus r'}$$

408
$$\mapsto (-1)^{k+l-1} \sum_{r \in S \setminus T} \sum_{r' \in (T \setminus S) \cup r} (-q)^{-\ell(r, S \setminus r) + \ell(r, T) - \ell((T \cup r) \setminus r', r') + \ell(S \setminus r, r')}$$

409
$$\times x_{(S \setminus r) \cup r'} \otimes x_{(T \cup r) \setminus r'}.$$

410 We can rewrite this sum depending on whether $r = r'$ or $r \neq r'$ in the latter case we
411 get

$$(-1)^{k+l-1} \sum_{r \in S \smallsetminus T} \sum_{r' \in T \smallsetminus S} (-q)^{-\ell(r, S \smallsetminus r)+\ell(r,T)-\ell(T \smallsetminus r', r')+\ell(S,r')} x_{(S \smallsetminus r) \cup r'} \otimes x_{(T \cup r) \smallsetminus r'}.$$
(3.6)

414 where, for convenience, we assumed $r > r'$. In the former case we get

$$(-1)^{k+l-1} \sum_{r \in S \smallsetminus T} (-q)^{-\ell(r, S \smallsetminus r)+\ell(r,T)-\ell(T,r)+\ell(S \smallsetminus r, r)} x_S \otimes x_T$$

$$= (-1)^{k+l-1} \sum_{r \in S \smallsetminus T} (-q)^{-2\ell(r, S \smallsetminus r)+2\ell(r,T)-l+k-1} x_S \otimes x_T$$

$$= q^{-l+k-1} \sum_{r \in S \smallsetminus T} q^{2(\ell(r, T \smallsetminus S)-\ell(r, S \smallsetminus (T \sqcup \{r\})))} x_S \otimes x_T$$

418 On the other hand, the first term on the right hand side of (3.5) acts as

$$x_S \otimes x_T \mapsto (-1)^{l-1} \sum_{r' \in T} (-q)^{-\ell(T \smallsetminus r', r')} x_S \otimes x_{r'} \otimes x_{T \smallsetminus r'}$$

$$\mapsto (-1)^{l-1} \sum_{r' \in T \smallsetminus S} (-q)^{-\ell(T \smallsetminus r'), r')+\ell(S,r')} x_{S \cup r'} \otimes x_{T \smallsetminus r'}$$

$$\mapsto (-1)^{k+l-1} \sum_{r' \in T \smallsetminus S} \sum_{r \in (S \smallsetminus T) \cup r'} (-q)^{-\ell(T \smallsetminus r', r')+\ell(S,r')-\ell(r,(S \cup r') \smallsetminus r)}$$

$$\times x_{(S \cup r') \smallsetminus r} \otimes x_r \otimes x_{T \smallsetminus r'}$$

$$\mapsto (-1)^{k+l-1} \sum_{r' \in T \smallsetminus S} \sum_{r \in (S \smallsetminus T) \cup r'} (-q)^{-\ell(T \smallsetminus r', r')+\ell(S,r')-\ell(r,(S \cup r') \smallsetminus r)+\ell(r, T \smallsetminus r')}$$

$$\times x_{(S \cup r') \smallsetminus r} \otimes x_{(T \smallsetminus r') \cup r}$$

425 Again, we have two cases, depending on whether $r = r'$ or $r \neq r'$. In the latter case
426 we get the same expression as in (3.6). In the former case we end up with

$$(-1)^{k+l-1} \sum_{r \in T \smallsetminus S} (-q)^{-\ell(T \smallsetminus r, r)+\ell(S,r)-\ell(r,S)+\ell(r, T \smallsetminus r)} x_S \otimes x_T$$

$$= q^{k-l-1} \sum_{r \in T \smallsetminus S} q^{2(-\ell(r,S)+\ell(r, T \smallsetminus r)+1)} x_S \otimes x_T$$

429 Thus it suffices to prove the following: for any S, T of size k, l respectively,

$$\sum_{r \in S} q^{2(\ell(r,T)-\ell(r, S \smallsetminus r))} - \sum_{r \in T} q^{2(-\ell(r,S)+\ell(r, T \smallsetminus r)+1)} = q^{l-k+1}[k-l]_q. \quad (3.7)$$

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Since $\ell(r, T \smallsetminus r) = \ell(r, (T \smallsetminus S) \smallsetminus r) + \ell(r, S \cap T)$ and $\ell(r, S) = \ell(r, S \smallsetminus T) + \ell(r, S \cap T)$ we find that

$$\ell(r, T \smallsetminus r) - \ell(r, S) = \ell(r, (T \smallsetminus S) \smallsetminus r) - \ell(r, S \smallsetminus T).$$

Thus, in proving (3.7) we can assume that S and T are disjoint.

We proceed by induction on $\min(k, l)$. The base case of our induction will be when $k = 0$ or $l = 0$. In this case, it is easy to see that (3.7) holds. Now assume that $k, l > 0$. Consider the elements of $S \cup T$ arranged in order. We can find a pair of consecutive entries one from S and one from T. More precisely, there exists $s \in S$ and $t \in T$ such that no element of S or T lies in between s and t. Suppose that $s < t$ (if $s > t$ then the argument is similar). Then

$$\ell(s, T) = \ell(t, T \smallsetminus t) + 1 \quad \text{and} \quad \ell(s, S \smallsetminus s) = \ell(t, S)$$

and thus the left hand side of (3.7) is unchanged by the removal of s from S and t from T and so by induction (3.7) holds.

Finally, relation (2.3) is straightforward while the proof of relation (2.7) is similar to that of (2.4) above and we omit it. $\qquad\square$

Lemma 3.4 *For any $n \geq k \in \mathbb{N}$ we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{-k(n-k)} \sum_{S \subset T, |S|=k} q^{2\ell(S, T \smallsetminus S)} = q^{k(n-k)} \sum_{S \subset T, |S|=k} q^{-2\ell(S, T \smallsetminus S)} \quad (3.8)$$

where $T = \{1, \ldots, n\}$.

Proof We prove the first equality as the second follows in the same way. The proof is by induction on n. It is an elementary exercise to show that the left hand side of (3.8) satisfies the recursion relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{-n+k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

It remains to show that the right hand side of (3.8) also satisfies this recursion. To do this we break up the sum into two depending on whether $n \in S$. We have

$$q^{-k(n-k)} \sum_{\substack{S \subset T \\ |S|=k, n \notin S}} q^{2\ell(S, T \smallsetminus S)} = q^{-k(n-k)} q^{2k} \sum_{\substack{S \subset T \smallsetminus n \\ |S|=k}} q^{2\ell(S, (T \smallsetminus n) \smallsetminus S)} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$$

where the last equality follows by induction. Similarly one finds that

$$q^{-k(n-k)} \sum_{\substack{S \subset T \\ |S|=k, n \in S}} q^{2\ell(S, T \smallsetminus S)} = q^{-k(n-k)} \sum_{\substack{S \smallsetminus n \subset T \smallsetminus n \\ |S \smallsetminus n|=k-1}} q^{2\ell(S \smallsetminus n, T \smallsetminus S)} = q^{-n+k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

The result follows by induction. □

3.3 The main result

Theorem 3.5 *The functor* $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$ *is an equivalence of pivotal categories.*

Proof We will use the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{L}ad_m^n & \longrightarrow & \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \\
\downarrow & \Psi_m^n \downarrow & \searrow{\Phi_m^n} \\
\mathcal{F}\mathcal{S}p(\mathrm{SL}_n) & \longrightarrow \mathcal{S}p(\mathrm{SL}_n) \xrightarrow{\Gamma_n} & \mathcal{R}ep(\mathrm{SL}_n)
\end{array}
\tag{3.9}
$$

where the three categories in the bottom row were defined in Sect. 2, Φ and $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ are defined in Sect. 4 while $\mathcal{L}ad_m^n$ and Ψ are defined in Sect. 5.

We now explain why Γ_n is an equivalence of categories. Since it is clearly an isomorphism on objects we must show that it is fully faithful.

Surjectivity (fullness) of Γ_n on Hom spaces follows from the fullness of the functor Φ_m^n, which is proven in Theorem 4.4.[1] More precisely, given any two objects V, W in $\mathcal{R}ep(\mathrm{SL}_n)$ we can find some m such that there exist n-bounded weights $\underline{k}, \underline{l}$ of $U_q(\mathfrak{gl}_m)$ such that $\Phi_m^n(\underline{k}) = V$ and $\Phi_m^n(\underline{l}) = W$. The fullness of Φ_m^n tells us that the map

$$
\Phi_m^n : 1_{\underline{l}}\dot{\mathcal{U}}_q(\mathfrak{gl}_m)1_{\underline{k}} \to \mathrm{Hom}_{U_q(\mathfrak{sl}_n)}(V, W)
$$

is surjective (i.e. all the morphisms come from ladders with m uprights). The commutativity of the right triangle (established in Proposition 5.4) shows us that these morphisms all come from webs in $\mathcal{S}p(\mathrm{SL}_n)$.

Next we show that Γ_n is injective (faithful) on Hom spaces. It suffices to do this on the Hom spaces between objects of the form $(k_1^+, \ldots, k_m^+)$ (that is, objects which are all oriented upwards), because every object is isomorphic (via a morphism built solely out of tags) to such an object. Let w be a morphism in $\mathcal{S}p(\mathrm{SL}_n)$ between upwards oriented objects such that $\Gamma_n(w) = 0$. By Theorem 5.5 and the commutativity of the left square (immediate from the definition of Ψ in Proposition 5.3), we can find some m and some $\tilde{w} \in \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ such that $\Psi_m^n(\tilde{w}) = w$ by finding a ladder $\tilde{w}$ equivalent to the web w. Then by the commutativity of the right triangle, we see that $\Phi_m^n(\tilde{w}) = 0$. However, by Theorem 4.4, Φ_m^n is faithful which means $\tilde{w} = 0$ and hence $w = 0$ as desired. □

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[1] The fullness of Γ_n was proven in Proposition 3.5.8 of [15] using Schur–Weyl duality instead of skew Howe duality, but the argument is essentially the same.

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4 The functor $\Phi_m^n : \dot{\mathcal{U}}_q(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathbf{SL}_n)$

4.1 $U_q(\mathfrak{gl}_m)$ and its idempotent form $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$

We begin with the definition of $U_q(\mathfrak{gl}_m)$. It is defined much the same way as $U_q(\mathfrak{sl}_m)$, except that we enlarge the "torus" by having invertible group-like generators $L_1, \ldots, L_n$ with $K_i = L_i L_{i+1}^{-1}$. In this way the weight spaces of $U_q(\mathfrak{gl}_m)$ are labelled by $\mathbb{Z}^m$.

We will also use Lusztig's idempotent form $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$. We regard $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ as a $\mathbb{C}(q)$-linear category with objects $\underline{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m$. The identity morphism of the object $\underline{k}$ is denoted $1_{\underline{k}}$ and we write $1_{\underline{l}} \dot{\mathcal{U}}_q(\mathfrak{gl}_m) 1_{\underline{k}}$ for the space of morphisms.

The morphisms are generated by $E_i^{(r)} 1_{\underline{k}} \in 1_{\underline{k}+r\alpha_i} \dot{\mathcal{U}}_q(\mathfrak{gl}_m) 1_{\underline{k}}$ and $F_i^{(r)} 1_{\underline{k}} \in 1_{\underline{k}-r\alpha_i} \dot{\mathcal{U}}_q(\mathfrak{gl}_m) 1_{\underline{k}}$, for $i = 1, \ldots, m-1$ and $r \in \mathbb{N}$ (here $\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$ where the 1 appears in position i). Notice that $\operatorname{Hom}(\underline{k}, \underline{l}) = 0$ unless $\sum k_i = \sum l_i$. When the specific weight space is not important (or is obvious from the context) we will write E_i instead of $E_i 1_{\underline{k}}$, F_i instead of $F_i 1_{\underline{k}}$ etc.

These morphisms satisfy the following set of relations:

$$E_i^{(r)} F_i^{(s)} 1_{\underline{k}} = \sum_t \begin{bmatrix} \langle \underline{k}, \alpha_i \rangle + r - s \\ t \end{bmatrix}_q F_i^{(s-t)} E_i^{(r-t)} 1_{\underline{k}} \tag{4.1}$$

$$E_i^{(r)} F_j^{(s)} 1_{\underline{k}} = F_j^{(s)} E_i^{(r)} 1_{\underline{k}}, \quad \text{if } i \neq j \tag{4.2}$$

$$E_i E_j E_i 1_{\underline{k}} = (E_i^{(2)} E_j + E_j E_i^{(2)}) 1_{\underline{k}}, \quad \text{if } |i - j| = 1, \text{ and likewise with } F'\text{s}, \tag{4.3}$$

$$E_i^{(r)} E_j^{(s)} 1_{\underline{k}} = E_j^{(s)} E_i^{(r)} 1_{\underline{k}} \quad \text{if } |i - j| > 1, \text{ and likewise with } F'\text{s}, \tag{4.4}$$

$$E_i^{(s)} E_i^{(r)} = \begin{bmatrix} r + s \\ r \end{bmatrix}_q E_i^{(r+s)} \quad \text{and likewise with } F'\text{s}. \tag{4.5}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{Z}^m$.

Remark Since we work over $\mathbb{C}(q)$, we do not need the generators $E_i^{(r)}$, $F_i^{(r)}$ for $r > 1$ as

$$E_i^{(r)} = \frac{E_i^r}{[r]_q \ldots [1]_q} \quad \text{and} \quad F_i^{(r)} = \frac{F_i^r}{[r]_q \ldots [1]_q}.$$

We decided to list these extra generators since they appear naturally from the webs perspective. Moreover, these extra generators are needed for Lusztig's $\mathbb{Z}[q, q^{-1}]$ form of the quantum group, though additional "Serre-like" relations (similar to 4.3) are needed in that setting.

A representation V of $U_q(\mathfrak{gl}_m)$ where the L_i act semisimply with all eigenvalues powers of q is equivalent to a functor from $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$ to the category of vector spaces which takes the object $\underline{k}$ to the weight space $V_{\underline{k}} := \{v \in V : L_i v = q^{k_i} v \text{ for all } i\}$.

519   We will be interested in a certain truncation of $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$. We say that a weight $\underline{k}$ is

520 an n-**bounded** if $0 \leq k_i \leq n$ for all i. We denote by $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ the quotient of $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$

521 where we set to zero all objects which are not n-bounded. In other words, we quotient

522 by the 2-sided ideal of morphisms generated by all $1_{\underline{k}}$ such that $\underline{k}$ is not n-bounded.

4.2 Quantum skew Howe duality

524 The vector space $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$ carries commuting actions of $U(\mathfrak{sl}_n)$ and $U(\mathfrak{gl}_m)$.

525 **Theorem 4.1** *The usual skew Howe duality [6,7] can be summarized as follows.*

526 1. *There is an isomorphism of $U(\mathfrak{sl}_n)$ representations*

$$\bigwedge{}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m) \cong \left(\bigwedge{}^\bullet\mathbb{C}^n\right)^{\otimes m} \tag{4.6}$$

528   *under which the $\underline{k}$ weight space for the action of $U(\mathfrak{gl}_m)$ on the left hand side is*
529   *identified with $\bigwedge^{k_1}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_m}\mathbb{C}^n$.*
530 2. *For each K, the actions of $U(\mathfrak{gl}_m)$ and $U(\mathfrak{sl}_n)$ on $\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)$ generate each*
531   *other's commutant.*
532 3. *As a representation of $U(\mathfrak{gl}_m) \otimes U(\mathfrak{sl}_n)$, we have a decomposition*

$$\bigwedge{}^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_\mu V(\mu^t) \otimes V(\mu)$$

534   *where μ varies over all n-bounded weights of $U(\mathfrak{gl}_m)$. Here μ^t is the transpose of*
535   *μ, regarded as a weight of $U(\mathfrak{sl}_n)$.*

536   We will need to generalize this result to the quantum setting. Unfortunately, there
537 is not much literature concerning quantum skew Howe duality, so we will develop the
538 theory here, following the ideas of Berenstein–Zwicknagl [1].

539   We consider $\mathbb{C}_q^n \otimes \mathbb{C}_q^m$ as a representation of $U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_m) = U_q(\mathfrak{sl}_n \oplus \mathfrak{gl}_m)$.
540 Let us write $x_1, \ldots, x_n$ for the standard basis of $\mathbb{C}_q^n$ and $y_1, \ldots, y_m$ for the standard
541 basis of $\mathbb{C}_q^m$. Then $\mathbb{C}_q^n \otimes \mathbb{C}_q^m$ has a basis given by $z_{ij} := x_i \otimes y_j$.

542   We define the quantum exterior algebra of this representation to be the quotient of
543 its tensor algebra by the ideal generated by its quantum symmetric square,

$$\bigwedge{}_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m) := T(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)/\langle S_q^2(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)\rangle$$

545 Following the proof of [1, Prop. 2.33], we have that

$$S_q^2(\mathbb{C}_q^n \otimes \mathbb{C}_q^m) = (S_q^2\mathbb{C}_q^n \otimes S_q^2\mathbb{C}_q^m) \oplus (\bigwedge{}_q^2\mathbb{C}_q^n \otimes \bigwedge{}_q^2\mathbb{C}_q^m).$$

547 Continuing to follow the proof of [1, Prop. 2.33], we see that $S_q^2(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is spanned
548 by

$$(x_i \otimes x_i) \otimes (y_l \otimes y_l)$$
$$(x_i \otimes x_i) \otimes (y_l \otimes y_p + qy_p \otimes y_l), \quad \text{for } l < p$$
549 $$(x_i \otimes x_j + qx_j \otimes x_i) \otimes (y_l \otimes y_l), \quad \text{for } i < j$$
$$(x_i \otimes x_j + qx_j \otimes x_i) \otimes (y_l \otimes y_p + qy_p \otimes y_l), \quad \text{for } i < j, l < p$$
$$(qx_i \otimes x_j - x_j \otimes x_i) \otimes (qy_l \otimes y_p - y_p \otimes y_l), \quad \text{for } i < j, l < p.$$

550 A little manipulation proves that $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is the quotient of the free algebra on
551 the set $\{z_{ij}\}$ modulo the relations

552 $$z_{ij} \wedge_q z_{ij} = 0$$
553 $$z_{ij} \wedge_q z_{lj} = -qz_{lj} \wedge_q z_{ij} \quad \text{if } i < l$$
554 $$z_{ij} \wedge_q z_{ip} = -qz_{ip} \wedge_q z_{ij} \quad \text{if } j < p$$
555 $$z_{ij} \wedge_q z_{lp} = -z_{lp} \wedge_q z_{ij} \quad \text{if } i < l, j < p$$
556 $$z_{ij} \wedge_q z_{lp} = -z_{lp} \wedge_q z_{ij} + (q - q^{-1})z_{ip} \wedge_q z_{lj} \quad \text{if } i < l, j > p$$

557 From the general theory from [1], we see that the algebra $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ carries
558 commuting actions of $U_q(\mathfrak{sl}_n)$ and $U_q(\mathfrak{gl}_m)$ (equivalently it carries an action of the
559 quantum group $U_q(\mathfrak{sl}_n \oplus \mathfrak{gl}_m)$). The generators of $E_p, F_p, L_p \in U_q(\mathfrak{gl}_m)$ act on the
560 generators z_{ij} of $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ in the obvious fashion

561 $$E_p z_{ij} = \begin{cases} z_{i,j-1} & \text{if } p = j - 1 \\ 0 & \text{otherwise} \end{cases}, \quad F_p z_{ij} = \begin{cases} z_{i,j+1} & \text{if } p = j \\ 0 & \text{otherwise} \end{cases},$$

562 $$L_p z_{ij} = \begin{cases} qz_{ij} & \text{if } p = j \\ z_{ij} & \text{otherwise} \end{cases}$$

563 and similarly for the generators of $U_q(\mathfrak{sl}_n)$.

564 Recall from [1] that if V is a representation of a quantum group $U_q(\mathfrak{g})$, then $\bigwedge_q^\bullet(V)$
565 always admits a $q = 1$ specialization, denoted $\overline{\bigwedge_q^\bullet(V)}$, which will be a quotient of
566 $\bigwedge^\bullet \overline{V}$ (as a $U(\mathfrak{g})$-module). For certain special V, we actually specialize to the entire
567 exterior algebra—this is true in our case.

568 **Theorem 4.2** 1. *The specialization $\overline{\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)}$ is isomorphic, as a $U(\mathfrak{gl}_m) \otimes$*
569 *$U(\mathfrak{sl}_n)$-module, to $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$.*
570 2. *For each K, the actions of $U_q(\mathfrak{gl}_m)$ and $U_q(\mathfrak{sl}_n)$ on $\bigwedge_q^K(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ generate each*
571 *other's commutant.*
572 3. *As a representation of $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{sl}_n)$, we have a decomposition*

573 $$\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m) = \bigoplus_\mu V(\mu^t) \otimes V(\mu)$$

where μ varies over all n-bounded weights of $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$.

4. We have an isomorphism as $U_q(\mathfrak{sl}_n)$ representations $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m) \cong \bigwedge_q^\bullet(\mathbb{C}_q^n)^{\otimes m}$.
   Moreover under this isomorphism, the $\underline{k}$ weight space for the action of $U(\mathfrak{gl}_m)$ on
   the left hand side is identified with $\bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$.

In fact the last part of this theorem can be strengthened to an algebra isomorphism, but we will not need this here.

Proof We begin with statement (1). It suffices to show that $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ has the correct graded dimension. To prove this, note that $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is the quadratic dual of the more familiar quantum matrix algebra $S_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. By [1, Prop. 2.33], this algebra is flat and thus Koszul by [1, Prop. 2.28]. By numerical Koszul duality,

$$h\left(\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m), t^{-1}\right) h(S_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m), t) = 1$$

where $h(V^\bullet, t) = \sum_k \dim V^k t^k$ denotes graded dimension.

Since $S_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is flat, $h(S_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m), t) = h(S^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m), t)$ and thus $h(\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m), t) = h(\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m), t)$ as desired.

By statement (1), we know that $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ decomposes into irreducible $U_q(\mathfrak{gl}_m \oplus \mathfrak{sl}_n)$ representations in the same manner as $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$ decomposes into $U(\mathfrak{gl}_m \oplus \mathfrak{sl}_n)$-modules. This immediately implies statement (3) which in turn implies (2).

Now we consider statement (4). For each $1 \le j \le m$, we define an algebra map $T_j : \bigwedge_q^\bullet(\mathbb{C}_q^n) \to \bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ by taking generators x_i to z_{ij}. This is well-defined as an algebra map because the relations in $\bigwedge_q^\bullet(\mathbb{C}_q^n)$ are taken to relations in $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. Moreover T_j is a map of $U_q(\mathfrak{sl}_n)$ representations. Let us write

$$z_{S,j} = T_j(x_S) = z_{k_1,j} \wedge_q z_{k_2,j} \wedge_q \cdots \wedge_q z_{k_a,j}$$

where $S = \{k_1 > \cdots > k_a\} \subset \{1, \ldots, n\}$. By multiplying together the T_j, we define

$$T : \bigwedge_q^\bullet(\mathbb{C}_q^n) \otimes \cdots \otimes \bigwedge_q^\bullet(\mathbb{C}_q^n) \to \bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$$
$$v_1 \otimes \cdots \otimes v_m \mapsto T_1(v_1) \wedge_q \cdots \wedge_q T_m(v_m)$$

Since the multiplication map on $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is $U_q(\mathfrak{sl}_n)$-equivariant, T is a map of $U_q(\mathfrak{sl}_n)$ representations. For $S_1, \ldots, S_m \subset \{1, \ldots, n\}$, we consider $T(x_{S_1} \otimes \cdots \otimes x_{S_m}) = z_{S_1,1} \wedge_q \cdots \wedge_q z_{S_m,m}$. If we let S_j range over all subsets, these clearly span $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. This means that they form a basis since the number of such elements equals the dimension of $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$. Thus we see that the map T is an isomorphism since it takes a basis to a basis. □

Lemma 4.3 *The action of E_p, $F_p \in U_q(\mathfrak{gl}_m)$ on $\bigwedge_q^\bullet(\mathbb{C}_q^n \otimes \mathbb{C}_q^m)$ is given by*

$$E_p(z_{S_1,1} \wedge_q \cdots \wedge_q z_{S_m,m}) = (-1)^{|S_{p+1}|+1} \sum_{r \in S_{p+1}} (-q)^{-\#\{s \in S_{p+1}: s < r\}} z_{S_1,1}$$

$$\wedge_q \cdots \wedge_q z_{S_p,p} \wedge_q z_{r,p} \wedge_q z_{S_{p+1} \smallsetminus r, p+1} \wedge_q \cdots \wedge_q z_{S_m,m} F_p(z_{S_1,1} \wedge_q \cdots \wedge_q z_{S_m,m})$$

$$= (-1)^{|S_p|+1} \sum_{r \in S_p} (-q)^{-\#\{s \in S_p: s > r\}} z_{S_1,1}$$

$$\wedge_q \cdots \wedge_q z_{S_p \smallsetminus r, p} \wedge_q z_{r,p+1} \wedge_q z_{S_{p+1},p+1} \wedge_q \cdots \wedge_q z_{S_m,m}.$$

Proof We prove the first assertion in the case $m = 2$ and $p = 1$ since it simplifies notation and the general case is the same (the second assertion follows similarly). Recall that $z_{S,i} = z_{k_1,i} \wedge_q \cdots \wedge_q z_{k_a,i}$ where $S = \{k_1 > \cdots > k_a\}$. Since we are taking products we need to use the Hopf algebra structure of $U_q(\mathfrak{gl}_m)$ which is given in (3.1). In particular, $\Delta(E_p) = E_p \otimes K_p + 1 \otimes E_p$. We also note that

$$K_p(z_{ij}) = \begin{cases} q z_{ij} & \text{if } j = p \\ q^{-1} z_{ij} & \text{if } j = p+1 \\ z_{ij} & \text{otherwise.} \end{cases}$$

Suppose $S_1 = \{k_1 > \cdots > k_a\}$ and $S_2 = \{l_1 > \cdots > l_b\}$. Then we have

$$E_1(z_{S_1,1} \wedge_q z_{S_2,2}) = z_{S_1,1} \wedge_q (z_{l_1,1} \wedge_q K_1(z_{l_2,2}) \wedge_q \cdots \wedge_q K_1(z_{l_b,2})$$

$$+ z_{l_1,2} \wedge_q z_{l_2,1} \wedge_q K_1(z_{l_3,2}) \wedge_q \cdots \wedge_q K_1(z_{l_b,2}) + \cdots)$$

$$= q^{-b} z_{S_1,1} \wedge_q (q z_{l_1,1} \wedge_q z_{l_2,2} \wedge_q \cdots \wedge_q z_{l_b,2}$$

$$+ q^2 z_{l_1,2} \wedge_q z_{l_2,1} \wedge_q z_{l_3,2} \wedge_q \cdots \wedge_q z_{l_b,2} + \cdots)$$

$$= q^{-b} z_{S_1,1} \wedge_q (q z_{l_1,1} \wedge_q z_{S_2 \smallsetminus l_1, 2} - q^2 z_{l_2,1} \wedge z_{S_2 \smallsetminus l_2, 2}$$

$$+ q^3 z_{l_3,1} \wedge_q z_{S_2 \smallsetminus l_3, 2} - \cdots).$$

The result follows after some simplification. $\qquad\square$

4.3 Definition of the functor Φ_m

We will now use quantum skew Howe duality to define the functor Φ_m. By Theorem 4.2(4), we have a $U_q(\mathfrak{gl}_m)$ action with the weight spaces $\bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n$ and commuting with the $U_q(\mathfrak{sl}_n)$ action. Thus we get a map

$$1_{\underline{l}} \dot{\mathcal{U}}_q(\mathfrak{gl}_m) 1_{\underline{k}} \to \operatorname{Hom}_{U_q(\mathfrak{sl}_n)} \left(\bigwedge_q^{k_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}^n, \bigwedge_q^{l_1} \mathbb{C}^n \otimes \cdots \otimes \bigwedge_q^{l_m} \mathbb{C}^n \right)$$

$$(4.7)$$

for any two n-bounded weights $\underline{k}, \underline{l}$ with $\sum_i k_i = \sum_i l_i$.

Since the action of $U_q(\mathfrak{gl}_m)$ on $\bigwedge_q^K(\mathbb{C}^n \otimes \mathbb{C}^m)$ generates the commutant of the $U_q(\mathfrak{sl}_n)$ action, the map (4.7) is surjective.

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Thus we may define a functor $\Phi_m : \dot{\mathcal{U}}_q(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ as follows:

- On objects $\underline{k} \mapsto \begin{cases} \bigwedge_q^{k_1} \mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m} \mathbb{C}_q^n & \text{if } \underline{k} \text{ is } n\text{-bounded} \\ 0 & \text{otherwise.} \end{cases}$

- On morphisms Φ_m is given by (4.7).

Since (4.7) was surjective the functor Φ_m is full.

4.4 Fully-faithfulness of Φ_m^n

Since all weights of $\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)$ are n-bounded, the functor $\Phi_m : \dot{\mathcal{U}}_q(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ factors through $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$. We denote this induced functor $\Phi_m^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$.

Theorem 4.4 *The functor $\Phi_m^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ is fully faithful (meaning that it induces an isomorphisms between* Hom-*spaces).*

To prove this result, we need a general fact about reductive Lie algebras (and quantum groups). This result was not previously known to us and we thank the referee for pointing out the reference [4, Thm. 4.2]. We include here a proof for completeness.

For simplicity, we state this result in the $\mathfrak{gl}_m$ case. We will need to consider the algebra version (instead of the category version) of Lusztig's idempotent form,

$$\dot{U}_q(\mathfrak{gl}_m) := \bigoplus_{\underline{k},\underline{l}} 1_{\underline{l}} \dot{\mathcal{U}}_q(\mathfrak{gl}_m) 1_{\underline{k}}.$$

In a similar fashion we define $\dot{U}_q^n(\mathfrak{gl}_m)$ as a quotient of $\dot{U}_q(\mathfrak{gl}_m)$.

For any dominant weight λ, let $V(\lambda)$ the corresponding highest weight representation of $U_q(\mathfrak{gl}_m)$.

We have the usual dominance order on dominant weights of $\mathfrak{gl}_m$ where $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of the simple roots α_i. We extend this notion as follows. We say that a dominant weight λ **dominates** a weight ν, if ν lies in the Weyl group orbit of a dominant weight $\mu \leq \lambda$.

Let I_λ be the 2-sided ideal in $\dot{U}_q(\mathfrak{gl}_m)$ generated by all 1_ν such that λ does not dominate ν. If μ is a dominant weight of $\mathfrak{gl}_m$ with $\mu \leq \lambda$, then for each ν as above, ν is not a weight of $V(\mu)$. Thus I_λ acts trivially on $V(\mu)$ and we get a representation $\dot{U}_q(\mathfrak{gl}_m)/I_\lambda \to \mathrm{End}\,(V(\mu))$.

Lemma 4.5 *For any dominant weight λ, the map $\dot{U}_q(\mathfrak{gl}_m)/I_\lambda \to \bigoplus_{\mu \leq \lambda} \mathrm{End}\,(V(\mu))$ is an isomorphism (where the sum is over dominant μ).*

Proof First note that $\dot{U}_q(\mathfrak{gl}_m)/I_\lambda$ is finite-dimensional. By Wedderburn's theorem, it suffices to show that the category of finite-dimensional $\dot{U}_q(\mathfrak{gl}_m)/I_\lambda$-modules is semisimple with simple objects the $V(\mu)$, for $\mu \leq \lambda$.

Now a $\dot{U}_q(\mathfrak{gl}_m)/I_\lambda$ module is the same thing as a $\dot{U}_q(\mathfrak{gl}_m)$ module in which I_λ acts trivially. Since the category of finite-dimensional $\dot{U}_q(\mathfrak{gl}_m)$ modules is semisimple and the ones where I_λ acts trivially are precisely the $V(\mu)$ for $\mu \leq \lambda$, the result follows. $\qquad\square$

670 *Example* Suppose $m = 2$ and $\lambda = (2, 0)$. Then the weights dominated by λ are
671 $(2, 0), (1, 1), (0, 2)$. Subsequently, the morphisms in $\dot{U}_q(\mathfrak{gl}_2)/I_\lambda$ are spanned by

672 $$1_{(2,0)}, 1_{(1,1)}, 1_{(0,2)}, E1_{(0,2)}, E1_{(1,1)}, E^2 1_{(0,2)}, F1_{(2,0)}, F1_{(1,1)}, F^2 1_{(2,0)},$$
673 $$EF1_{(1,1)} = FE1_{(1,1)}.$$

674 On the other hand, the *dominant* weights dominated by $(2, 0)$ are $(2, 0)$ and $(1, 1)$.
675 Note that $V((2, 0)) = S^2 \mathbb{C}^2$, which is three-dimensional, and $V((1, 1)) = \bigwedge^2 \mathbb{C}^2$,
676 which is one-dimensional. Thus $\bigoplus_{\mu \leq \lambda} \text{End}(V(\mu)) \cong \text{End}(\mathbb{C}^3) \oplus \text{End}(\mathbb{C})$. Notice
677 that these spaces have the same dimension (they are both 10-dimensional).

678 *Proof* We now return to proving Theorem 4.4. Recall that by quantum skew Howe
679 duality, we have a decomposition

680 $$\bigwedge_q^\bullet (\mathbb{C}_q^n \otimes \mathbb{C}_q^m) = \bigoplus_\mu V(\mu^t) \otimes V(\mu)$$

681 as $U_q(\mathfrak{sl}_n) \otimes U_q(\mathfrak{gl}_m)$-representations, where μ varies over all n-bounded weights of
682 $U_q(\mathfrak{gl}_m)$. Thus for any $0 \leq K \leq mn$,

683 $$\text{Hom}_{U_q(\mathfrak{sl}_n)} \left(\bigwedge_q^K (\mathbb{C}_q^n \otimes \mathbb{C}_q^m), \bigwedge_q^K (\mathbb{C}_q^n \otimes \mathbb{C}_q^m) \right) = \bigoplus_\mu \text{End}(V(\mu))$$

684 where μ ranges over n-bounded weights with $\sum \mu_i = K$. Note that these μ are exactly
685 the set of dominant weights of $\mathfrak{gl}_m$ which satisfy $\mu \leq \lambda(K)$, where $\lambda(K)$ is the unique
686 weight of the form $(n, \ldots, n, r, 0, \ldots, 0)$ where the terms sum to K. Applying the
687 previous lemma, we see that the map

688 $$\dot{U}_q(\mathfrak{gl}_m)/I_{\lambda(K)} \rightarrow \text{Hom}_{U_q(\mathfrak{sl}_n)} \left(\bigwedge_q^K (\mathbb{C}_q^n \otimes \mathbb{C}_q^m), \bigwedge_q^K (\mathbb{C}_q^n \otimes \mathbb{C}_q^m) \right)$$

689 is an isomorphism. Since a weight μ is n-bounded if and only if μ is dominated by
690 $\lambda(K)$ where $K = \sum \mu_i$ we get

691 $$\dot{U}_q^n(\mathfrak{gl}_m) = \bigoplus_{K=0}^{nm} \dot{U}_q(\mathfrak{gl}_m)/I_{\lambda(K)}$$

692 and the result follows. $\qquad\square$

693 *Remark* When $K = n$, the algebra $\dot{U}_q(\mathfrak{gl}_m)/I_{\lambda(K)}$ appearing above is known as the
694 q-Schur algebra. The algebras $\dot{U}_q(\mathfrak{gl}_m)/I_{\lambda(K)}$ for general K are called generalized
695 q-Schur algebras by Doty [4].

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5 Ladders

5.1 Ladders and $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$

We will now introduce a diagrammatic notation for morphisms in $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$. We begin by formalizing the notion of a ladder.

Definition 5.1 An *n*-**ladder** with *m* uprights is a diagram drawn in a rectangle, with

- *m* parallel vertical lines running from the bottom edge to the top edge of the rectangle, oriented upwards,
- some number of oriented horizontal **rungs** connecting adjacent uprights,
- a labelling of each interval (rungs or segments of uprights) by an integer between 0 and *n* inclusive,

such that the sum of labels (taken with signs according to the orientations of the intervals) at each trivalent vertex is zero.

Now we introduce the category $\mathcal{L}ad_m^n$ of ladders. The objects are sequences of length *m* of integers between 0 and *n* inclusive (that is, *n*-bounded weights of $\dot{\mathcal{U}}_q(\mathfrak{gl}_m)$). The morphisms are linear combinations of ladders. The source of a ladder is the sequence of labels appearing on the lowest segments of the uprights, and the target is the sequence of labels appearing on the highest segments. Composition of morphisms is given by vertical concatenation of ladders.

Notice that, as *m* varies, the categories $\mathcal{L}ad_m^n$ fit together as a tensor category $\mathcal{L}ad^n$, with tensor product given by horizontal juxtaposition. In this tensor category the morphisms are generated by the single rung ladders.

Next, we define a functor from $\mathcal{L}ad_m^n \to \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ which on objects is just the identity. On morphisms we send rungs between the *i*-th and $(i+1)$-th uprights to divided powers as follows:

and

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Proposition 5.2 *Under the functor above, $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ is the quotient of $\mathcal{L}ad_m^n$ by the following relations:*

$$
\begin{array}{c}
k_2+r+s \quad s \quad k_3-s \\
k_1-r \qquad k_2+r \\
r \\
k_1 \qquad k_2 \qquad k_3
\end{array}
=
\begin{array}{c}
k_1-r \quad r \quad k_2+r+s \\
k_2+s \qquad k_3-s \\
s \\
k_1 \qquad k_2 \qquad k_3
\end{array}
\tag{5.1}
$$

$$
\begin{array}{c}
k_2-r-s \quad s \quad k_3+s \\
k_1+r \qquad k_2-r \\
r \\
k_1 \qquad k_2 \qquad k_3
\end{array}
=
\begin{array}{c}
k_1+r \quad r \quad k_2-r-s \\
k_2-s \qquad k_3+s \\
s \\
k_1 \qquad k_2 \qquad k_3
\end{array}
\tag{5.2}
$$

$$
\begin{array}{c}
k-s-r \quad r \quad l+s+r \\
k-s \qquad l+s \\
s \\
k \qquad l
\end{array}
=
\begin{bmatrix} r+s \\ r \end{bmatrix}_q
\begin{array}{c}
k-s-r \quad r+s \quad l+s+r \\
k \qquad l
\end{array}
\tag{5.3}
$$

$$
\begin{array}{c}
k-s+r \quad r \quad l+s-r \\
k-s \qquad l+s \\
s \\
k \qquad l
\end{array}
=
\sum_t \begin{bmatrix} k-l+r-s \\ t \end{bmatrix}_q
\begin{array}{c}
k-s+r \quad s-t \quad l+s-r \\
k+r-t \qquad l-r+t \\
r-t \\
k \qquad l
\end{array}
\tag{5.4}
$$

736

$$- [2]_q \qquad + \qquad = 0$$

$$k_1 \qquad k_2 \qquad k_3 \qquad k_1 \qquad k_2 \qquad k_3 \qquad k_1 \qquad k_2 \qquad k_3$$

737

(5.5)

738 together with the mirror reflection of (5.3) and (5.5) (note that nothing happens to
739 the coefficients of these equations under these operations). These relations are to be
740 understood as containing arbitrarily many vertical strands on either side. Moreover,
741 the horizontal rungs in (5.5) are all labelled 1.

742 *Proof* Since all E_i, F_i are in the image of the functor, we see that $\mathcal{L}ad_m^n \to \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$
743 is full (it is obviously dominant). It remains to see that the above relations generate
744 the kernel. To see this we need to check Eqs. (4.1, 4.2, 4.3, 4.4, 4.5), along with the
745 relation that $1_{\underline{k}} = 0$ if $\underline{k}$ is not an n-bounded weight (although this last thing is clear
746 from the definition of $\mathcal{L}ad_m^n$).
747 Equation (4.1) becomes (5.4) in diagrammatic form. When $|i - j| > 1$, Eq. (4.2) is
748 reflected in the isotopy invariance of ladders, while when $|i - j| = 1$, (4.2) is (5.1) and
749 (5.2) in diagrammatic form. Relation (4.3) corresponds to (5.5) while (4.4) is again
750 isotopy invariance. Finally, (4.5) corresponds to (5.3). □

751 5.2 Ladders as webs

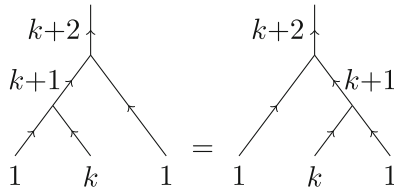752 There is a functor from $\mathcal{L}ad_m^n \to \mathcal{FS}p(\mathrm{SL}_n)$ by forgetting the ladder structure of a
753 ladder and thinking of it as a web. However, there is a slight discrepancy at the level
754 of objects. More precisely, in $\mathcal{L}ad_m^n$ the objects $\underline{k}$ are sequences in $\{0, \ldots, n\}$, while
755 in $\mathcal{FS}p(\mathrm{SL}_n)$ the objects are sequences in $\{1^+, \ldots, (n-1)^+\}$. The functor deletes 0s
756 and ns from the sequences, and sends k to k^+.

757 **Proposition 5.3** *The composition* $\mathcal{L}ad_m^n \to \mathcal{FS}p(\mathrm{SL}_n) \to \mathcal{S}p(\mathrm{SL}_n)$ *can be factored*
758 *through the functor* $\mathcal{L}ad_m^n \to \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ *from the previous section, giving rise to a*
759 *functor* $\Psi_m^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \to \mathcal{S}p(\mathrm{SL}_n)$.

760 *Proof* To see that Ψ_m^n exists, we need only show that the diagrammatic relations of
761 $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ from Proposition 5.2 are taken to the kernel of the functor $\mathcal{FS}p(\mathrm{SL}_n) \to$
762 $\mathcal{S}p(\mathrm{SL}_n)$.
763 Relations (5.3) and (5.4) hold in $\mathcal{S}p(\mathrm{SL}_n)$ since they are exactly (2.7) and (2.8).
764 To show that (5.1) holds in $\mathcal{S}p(\mathrm{SL}_n)$, we first observe that

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is a special case of (2.6). This establishes (5.1) in the special case $r = s = 1$. For all other cases, we first use the previously established (5.3) to replace the r and s rungs each with a collection of parallel 1 rungs, and then repeatedly apply the special case. (Similarly for (5.2) using the arrow reversal of (2.6).)

Finally, relation (5.5) is exactly (2.12) from Lemma 2.1. □

We have now reached the situation described in the proof of the main result. We have the diagram

$$\begin{array}{ccc}
\mathcal{L}ad_m^n & \longrightarrow & \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) \\
\downarrow & {\scriptstyle \Psi_m^n}\downarrow & \quad\searrow^{\Phi_m^n} \\
\mathcal{FS}p(\mathrm{SL}_n) & \longrightarrow \mathcal{S}p(\mathrm{SL}_n) \xrightarrow{\Gamma_n} & \mathcal{R}ep(\mathrm{SL}_n)
\end{array} \tag{5.6}$$

Note that the left square of this diagram commutes by definition of Ψ_m^n.

Proposition 5.4 *The right triangle of (5.6) commutes.*

Proof We proceed by an explicit calculation. Consider $E_j 1_{\underline{k}}$. Via the map (4.7), we see that

$$\Phi_m^n(E_j 1_{\underline{k}}) : \textstyle\bigwedge_q^{k_1}\mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{k_m}\mathbb{C}_q^n \to \bigwedge_q^{l_1}\mathbb{C}_q^n \otimes \cdots \otimes \bigwedge_q^{l_m}\mathbb{C}_q^n$$

where $\underline{l} = \underline{k} + \alpha_j$.

From Lemma 4.3, we see that

$$\Phi_m^n(E_j 1_{\underline{k}}) = I^{\otimes j-1} \otimes M_{k_j,1} \otimes I^{\otimes m-j} \circ I^{\otimes j} \otimes M'_{1,k_{j+1}-1} \otimes I^{\otimes m-j-1}$$

On the other hand, consider the web

From the definition of Γ_n in Sect. 3.2 we see that

$$\Gamma_n(w) = I^{\otimes j-1} \otimes M_{k_j,1} \otimes I^{\otimes m-j} \circ I^{\otimes j} \otimes M'_{1,k_{j+1}-1} \otimes I^{\otimes m-j-1}.$$

Thus $\Gamma_n(\Psi_m^n(E_j 1_{\underline{k}})) = \Phi_m^n(E_j 1_{\underline{k}})$. A similar argument also holds for F_j and since these generate $\mathcal{U}_q(\mathfrak{gl}_m)$ the result follows. $\qquad\square$

5.3 Surjectivity

We would like to show that any web can be written, possibly using some relations, into ladder form. This is not quite the case, simply because the boundary points of ladders are always oriented upwards. However, this is not a problem, because every object in $\mathcal{S}p(\mathrm{SL}_n)$ is isomorphic, via a map made out of tags, to an upwards oriented one.

Theorem 5.5 *Let D be a morphism in $\mathcal{S}p(\mathrm{SL}_n)$ between upwards oriented objects. Then there exists $m \in \mathbb{N}$ and a morphism $E \in \mathcal{L}ad_m^n$ such that $\Psi_m^n(E) = D$.*

Proof Given any such diagrammatic morphism $D \in \mathcal{S}p(\mathrm{SL}_n)$, we first write express every tag in the diagram as the end of a strand labelled by n connecting that tag to the edge of the diagram; anywhere this strand crosses an existing strand we interpret the crossing as a pair of trivalent vertices via Eq. (2.11). Then, if the total number of incoming and outgoing strands are not equal, we introduce new strands labelled by 0 as needed to balance.

Now, just by a planar isotopy, we can write D as
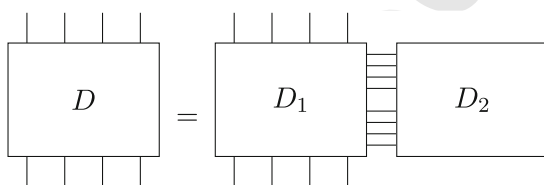
where
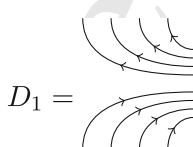
and D_2 is in Morse position relative to the x-coordinate (that is, no two critical x-values or x-coordinates of vertices coincide) and further each trivalent vertex has two edges pointing to the left and one to the right. This can always be achieved at the expense of extra critical x-values in the strings.

Now replace D_1 with
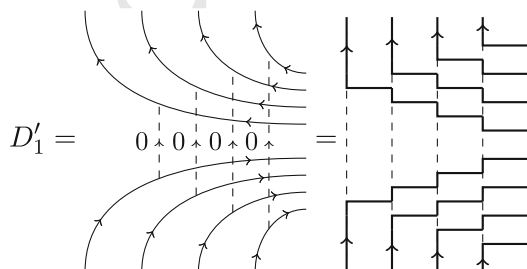
811 Next, to the right of each elementary piece of the Morse decomposition of D_2,
812 superimpose either a vertical 0 strand or a vertical n strand from the bottom to the
813 top of the diagram and then make one of the following local replacements in the
814 neighbourhood of that elementary piece:

815

816 $$(5.7)$$
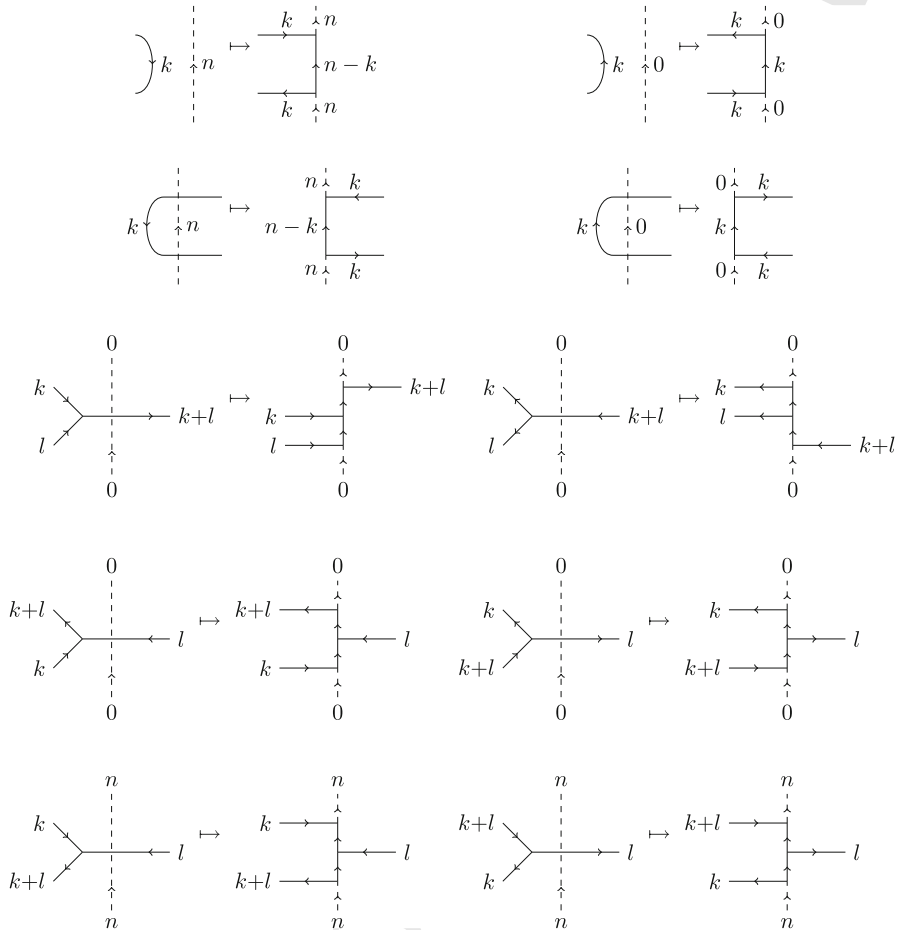
817 and then finally replacing each other instance where a superimposed strand crosses a
818 horizontal strand of D_2 as follows

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819
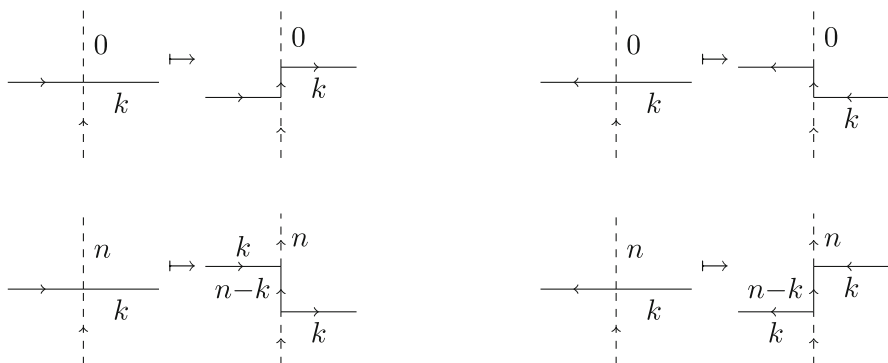
820 to obtain D_2'. Now one can check that D_2' is in fact equal to D_2, using only a few
821 relations from the spider. In particular, for each of the replacements above involving a
822 0 strand, when we delete the 0 strands we see that nothing has changed. In the replace-
823 ments involving an n strand but no trivalent vertices, after removing the n-strands and
824 replacing their endpoints with tags, we find we can cancel the tags according to Eq.
825 (2.11). Finally, in the replacements involving an n strand to the right of a trivalent
826 vertex (those in (5.7)), we need to use Eq. (2.10) to move one tag past the trivalent
827 vertex, and then Eq. (2.11) to cancel them.

828 In each of the local replacements used to form D_2' the new diagram consists of part
829 of an upright of the ladder, along with several 'half-rungs'. It is easy to see that all
830 of these half-rungs come in matching pairs forming complete rungs, except at the left
831 margin of D_2'. Similarly, D_1' is a ladder except that it has half-rungs along its right
832 margin. The horizontal juxtaposition $D_1' D_2'$ is then a ladder. Since D is equivalent in
833 $\mathcal{S}p(\mathrm{SL}_n)$ to $D_1' D_2'$, we are finished. □

834 6 Braidings

835 Let $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet) = \bigoplus_{m=1}^\infty \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$. This is a category whose objects are n-bounded
836 weights $\underline{k} = (k_1, \ldots, k_m)$ and whose morphisms are given by

837
$$\mathrm{Hom}((k_1, \ldots, k_m), (l_1, \ldots, l_p)) = \begin{cases} 1_{(l_1,\ldots,l_p)} \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) 1_{(k_1,\ldots,k_m)} & \text{if } m = p \\ 0 & \text{otherwise.} \end{cases}$$

838 The functors Φ_m^n combine together to a functor $\Phi^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet) \to \mathcal{R}ep(\mathrm{SL}_n)$ which
839 factors through $\mathcal{S}p(\mathrm{SL}_n)$. Our goal in this section is to define a braided monoidal
840 category structure on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$ and to show that Φ^n preserves the braiding.

841 6.1 Braided monoidal category structure on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$

842 First, we define a monoidal category structure on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$. The tensor product of objects
843 is given by concatenation $(k_1, \ldots, k_m) \otimes (l_1, \ldots, l_p) = (k_1, \ldots, k_m, l_1, \ldots, l_p)$.
844 The tensor product of morphisms comes from the obvious embedding $U_q(\mathfrak{gl}_m) \otimes$

845 $U_q(\mathfrak{gl}_p) \to U_q(\mathfrak{gl}_{m+p})$. From the perspective of ladders, the tensor product is given
846 by horizontal juxtaposition. This monoidal structure is associative with trivial associ-
847 ator.

848 Recall that a braiding β on a monoidal category $\mathcal{C}$ is system of natural isomorphisms
849 $\beta_{V,W} : V \otimes W \to W \otimes V$ satisfying the "hexagon equations"

$$\beta_{U \otimes V, W} = \beta_{U,W} \otimes I_V \circ I_U \otimes \beta_{V,W} \tag{6.1}$$

851 and

$$\beta_{U,V \otimes W} = I_V \otimes \beta_{U,W} \circ \beta_{U,V} \otimes I_W$$

853 We will now define a braiding on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$.

854 For each n-bounded weight $\underline{k}$ and for $1 \leq i < m$, we set $s_i(\underline{k}) =$
855 $(k_1, \ldots, k_{i+1}, k_i, \ldots, k_m)$. Following Lusztig [14, 5.2.1], we define $T''_{i,-1} 1_{\underline{k}} \in$
856 $1_{s_i(\underline{k})} \dot{\mathcal{U}}_q^n(\mathfrak{gl}_m) 1_{\underline{k}}$ by the formula

$$T''_{i,-1} 1_{\underline{k}} = \sum_{a,b,c \geq 0, -a+b-c=k_i-k_{i+1}} (-1)^b q^{ac-b} E_i^{(a)} F_i^{(b)} E_i^{(c)} 1_{\underline{k}} \tag{6.2}$$

858 Note that these sums are actually finite since we are working in the truncation $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$.

859 Lusztig's elements admit the following simplified form, a fact which seems to have
860 been first observed by Chuang–Rouquier [3] when $q = 1$.

861 **Lemma 6.1** $T''_{i,-1} 1_{\underline{k}} = \sum_{a,b \geq 0, -a+b=k_i-k_{i+1}} (-q)^{-b} E_i^{(a)} F_i^{(b)} 1_{\underline{k}}$

862 *Proof* By the definition of $T''_{i,-1}$, we must show that

$$\sum_{a,b,c \geq 0, -a+b-c=k_i-k_{i+1}, c>0} (-1)^b q^{ac-b} E_i^{(a)} F_i^{(b)} E_i^{(c)} 1_{\underline{k}} = 0$$

864 This is easily proven by direct calculation starting with the usual commutation relation
865 $(E_i F_i - F_i E_i) 1_{\underline{k}} = [k_i - k_{i+1}]_q$.   □

866 We will need the following modification of Lusztig's definition in order to later
867 match with the braiding on $\mathcal{R}ep(\mathrm{SL}_n)$,

$$T_i 1_{\underline{k}} := (-1)^{k_i + k_i k_{i+1}} q^{k_i - \frac{k_i k_{i+1}}{n}} T''_{i,-1} 1_{\underline{k}}.$$

869 **Lemma 6.2** *The elements T_i are invertible. Moreover they satisfy the braid relations*

$$T_{i+1} T_i T_{i+1} 1_{\underline{k}} = T_i T_{i+1} T_i 1_{\underline{k}} \quad and \quad T_i T_j 1_{\underline{k}} = T_j T_i 1_{\underline{k}}, \ if \ |i - j| \geq 2.$$

871 *Proof* Lusztig proved that the $T''_{i,-1}$ are invertible (section 5.2.3 in [14]) and that they
872 satisfy the braid relations (section 39.4.1 in [14]). The corresponding result for the T_i
873 follows immediately.   □

874　　From the lemma, we can define $T_w 1_{\underline{k}}$ for any $w \in S_m$ by using the usual lift of S_m
875　into the braid group B_m.

876　　Now we are in a position to define the braiding on our category. For any two
877　objects $\underline{k} = (k_1, \ldots, k_m), \underline{l} = (l_1, \ldots, l_p)$, we define $\beta_{\underline{k},\underline{l}} = T_w 1_{(k_1, \ldots, k_m, l_1, \ldots, l_p)}$
878　where $w \in S_{m+p}$ is defined by

$$
879 \qquad w(i) = \begin{cases} p + i & \text{if } i \leq m \\ i - m & \text{if } i > m. \end{cases}
$$

880　**Lemma 6.3** *The map β is a natural transformation from the bifunctor $- \bigotimes -$ to the*
881　*bifunctor $- \bigotimes^{op} -$.*

882　*Proof* We must prove that for any morphism $\phi : \underline{k} \to \underline{k'}$ in $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_m)$ we have $\beta_{\underline{k'},\underline{l}} \circ$
883　$(\phi \otimes I) = (I \otimes \phi) \circ \beta_{\underline{k},\underline{l}}$.

884　　Since the morphisms are generated by the E_i, F_i, it suffices to prove the result when
885　ϕ is an E_i or F_i. Because of the definition of T_w, it suffices to prove that

886 $\qquad T_i T_j E_i = E_j T_i T_j \quad \text{when } |i - j| = 1 \text{ and } T_j E_i = E_i T_j \text{ when } |i - j| \geq 2$

887　along with the same equations when E_i is replaced by F_i. The second equation follows
888　from the definition of T_j and the commutativity of E_i, E_j when $|i - j| \geq 2$.

889　　Thus it suffices to prove the first equation. Note that this equation is equivalent to

$$
890 \qquad T_j E_i T_j^{-1} = T_i^{-1} E_j T_i
$$

891　The analogous equation with T_i replaced by $T_{i,-1}''$ was proven by Lusztig [14, section
892　39.2.4]. The result for T_i follows immediately. □

893　**Theorem 6.4** *This defines a braided monoidal category structure on $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$.*

894　*Proof* First, by Lemma 6.3, β is a natural transformation and by Lemma 6.2, β is a
895　natural isomorphism. The hexagon equations hold by the definition of β and thus we
896　conclude that β gives a braided monoidal category structure. □

897　6.2 Comparison of braidings

898　Recall that $\mathcal{R}ep(\mathrm{SL}_n)$ carries the structure of a braided monoidal category using the
899　usual R-matrix. We will use β to denote the braiding. Our goal now is to prove that

900　**Theorem 6.5** *The functor $\Phi^n : \dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet) \to \mathcal{R}ep(\mathrm{SL}_n)$ is a functor of braided*
901　*monoidal categories.*

902　*Proof* Clearly, Φ^n is a tensor functor. So we need to show that Φ^n carries the braiding
903　in $\dot{\mathcal{U}}_q^n(\mathfrak{gl}_\bullet)$ to the braiding in $\mathcal{R}ep(\mathrm{SL}_n)$. By the hexagon equations (6.1), it suffices to
904　prove that if $0 \leq k, l \leq m$ then

$$
905 \qquad \Phi^n(\beta_{k,l}) = \beta_{\wedge_q^k \mathbb{C}_q^n, \wedge_q^l \mathbb{C}_q^n}.
$$

First we claim that when $k_i = 0$ or $k_{i+1} = 0$, then $\Phi^n(T_i 1_{\underline{k}})$ is the identity. To see this recall that Lusztig proved (Proposition 5.2.2(b) in [14]) that if $F_i v = 0$ and $H_i v = -rv$, then $T''_{i,-1}(v) = E^{(r)}v$. On the other hand, Lusztig proved (in the same Proposition together with Proposition 5.2.3(b)), that if $E_i v = 0$ and $H_i v = rv$, then $T''_{i,-1}(v) = (-1)^r q^{-r} F_i^{(r)} v$. From this it follows that $T_i 1_{\underline{k}} = E_i^{(k_{i+1})} 1_{\underline{k}}$ if $k_i = 0$ and $T_i 1_{\underline{k}} = F_i^{(k_i)}$ if $k_{i+1} = 0$. Thus the claim follows.

From the above claim, we see that if we consider the weights $(k, 0^{k-1}) = (k, 0, \ldots, 0)$ and similarly $(l, 0^{l-1})$ then we have $\Phi^n(\beta_{(k,0^{k-1}),(l,0^{l-1})}) = \Phi^n(\beta_{k,l})$.

Now consider weights $1^k = (1, \ldots, 1)$ where there are k 1s and similarly 1^l. We define morphisms $\phi : (k, 0^{k-1}) \to 1^k$ and $\psi : (l, 0^{l-1}) \to 1^l$ by

$$\phi = F_1 F_2 F_1 \cdots F_{k-1} \cdots F_1 1_{(k,0^{k-1})}$$

and similarly for ψ.

Note that $\Phi^n(\phi)$ is the injective map $\bigwedge_q^k \mathbb{C}_q^n \to (\mathbb{C}_q^n)^{\otimes k}$. By the naturality of β, we have $\beta_{1^k,1^l} \circ \phi \otimes \psi = \psi \otimes \phi \circ \beta_{(k,0^{k-1}),(l,0^{l-1})}$. Thus we have

$$\Phi^n(\beta_{1^k,1^l}) \circ \Phi^n(\phi) \otimes \Phi^n(\psi) = \Phi^n(\psi) \otimes \Phi^n(\phi) \circ \Phi^n(\beta_{k,l})$$

From Lemma 6.6 below, we see that $\Phi^n(\beta_{1,1}) = \beta_{\mathbb{C}_q^n, \mathbb{C}_q^n}$ and thus by the hexagon equations (6.1), $\Phi^n(\beta_{1^k,1^l}) = \beta_{(\mathbb{C}_q^n)^{\otimes k}, (\mathbb{C}_q^n)^{\otimes l}}$. So the above equation becomes

$$\beta_{(\mathbb{C}_q^n)^{\otimes k}, (\mathbb{C}_q^n)^{\otimes l}} \circ \Phi^n(\phi) \otimes \Phi^n(\psi) = \Phi^n(\psi) \otimes \Phi^n(\phi) \circ \Phi^n(\beta_{k,l})$$

On the other hand, by the naturality of the braiding in $\mathcal{R}ep(\mathrm{SL}_n)$, we have

$$\beta_{(\mathbb{C}_q^n)^{\otimes k}, (\mathbb{C}_q^n)^{\otimes l}} \circ \Phi^n(\phi) \otimes \Phi^n(\psi) = \Phi^n(\psi) \otimes \Phi^n(\phi) \circ \beta_{\bigwedge_q^k \mathbb{C}_q^n, \bigwedge_q^l \mathbb{C}_q^n}.$$

Thus the injectivity of $\Phi^n(\psi) \otimes \Phi^n(\phi)$ implies the desired result. $\qquad\square$

Lemma 6.6 $\Phi^n(\beta_{1,1}) = \beta_{\mathbb{C}_q^n, \mathbb{C}_q^n}$

Proof This follows by a direct computation. First, it is a standard fact that $\beta_{\mathbb{C}_q^n, \mathbb{C}_q^n}$ acts by $q^{1-1/n}$ on $S_q^2(\mathbb{C}_q^n)$ and by $-q^{-1-1/n}$ on $\bigwedge_q^2(\mathbb{C}_q^n)$. (To see this, we use the fact that the eigenvalue of the square of the braiding on the Y summand of $X \otimes X$ is $\theta(Y)\theta(X)^{-2}$, where $\theta(V_\lambda) = q^{\langle \lambda, \lambda+2\rho \rangle}$ is the twist factor. See [17, §1.1.4 and §3.5].) So it suffices to check that $\Phi^n(\beta_{1,1})$ does the same thing.

Now we claim that $T''_{1,-1}$ acts by 1 on $S_q^2(\mathbb{C}_q^n)$ and acts by $-q^{-2}$ on $\bigwedge_q^2(\mathbb{C}_q^n)$. This is because we have the following action of $U_q(\mathfrak{gl}_2)$

$$\bigwedge_q^2(\mathbb{C}_q^n) \underset{F}{\overset{E}{\underset{\longleftarrow}{\longrightarrow}}} \mathbb{C}_q^n \otimes \mathbb{C}_q^n \underset{F}{\overset{E}{\underset{\longleftarrow}{\longrightarrow}}} \bigwedge_q^2(\mathbb{C}_q^n)$$

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where $T''_{1,-1}1_{(1,1)} = 1_{(1,1)} - q^{-1}EF1_{(1,1)}$. Thus F acts by zero on the summand $S^2_q(\mathbb{C}^n_q) \subset \mathbb{C}^n_q \otimes \mathbb{C}^n_q$ which means $T''_{1,-1}$ acts by 1. Also,

$$T''_{1,-1}(E) = (E - q^{-1}EFE) = E - q^{-1}(q + q^{-1})E = -q^{-2}E$$

which means that $T''_{1,-1}$ acts by multiplication by $-q^{-2}$ on the summand $\bigwedge^2_q(\mathbb{C}^n_q) \subset \mathbb{C}^n_q \otimes \mathbb{C}^n_q$.

From the above computation of $T''_{1,-1}$ we see that $\Phi^n(\beta_{1,1})$ acts by $q^{1-1/n}$ on $S^2_q(\mathbb{C}^n_q)$ and by $-q^{-1-1/n}$ on $\bigwedge^2_q(\mathbb{C}^n_q)$ as desired. □

Using the form for $T''_{1,-1}$ given in Lemma 6.1, we can translate Theorem 6.5 to the language of webs as follows.

Corollary 6.7 *The braiding $\beta_{\wedge^k_q\mathbb{C}^n_q, \wedge^l_q\mathbb{C}^n_q}$ is given by the following sum of webs*

$$(-1)^{k+kl}q^{k-\frac{kl}{n}} \sum_{\substack{a,b\geq 0 \\ b-a=k-l}} (-q)^{-b} \quad$$

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