# Webs and skew Howe duality

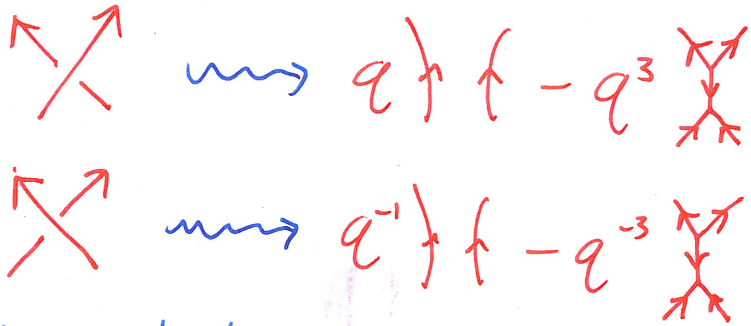
(Joint with Sabin Cautis and Joel Kamnitzer
arXiv: 1210.6437 )

In quantum topology we've come across many beautiful examples of skein theory

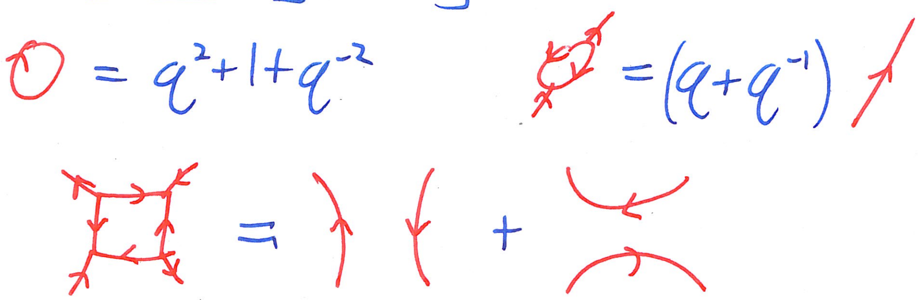
Skein theory 'is' the study of pivotal categories from the perspective of generators and relations

## Examples

- Compute the $U_q\mathfrak{sl}_3$ quantum knot invariants by replacing




and evaluating using



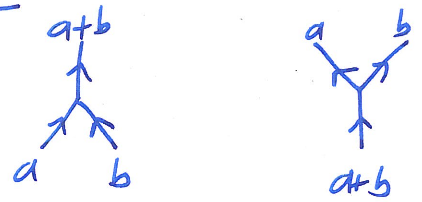
- Construct exotic subfactors
  (e.g. Extended Haagerup, arXiv:0909.4099)
  by guessing generators and relations for
  an associated pivotal category
  and then using skein theory (in particular, a nonlocal
  evaluation algorithm called the 'jellyfish algorithm')
  to show it is non-trivial and has the right properties.

Today: we have a skein theory for $\mathrm{Rep}\, U_q\mathfrak{sl}_n$ and the construction uses some nice representation theory, 'skew Howe duality'

Our skein theory consists of:

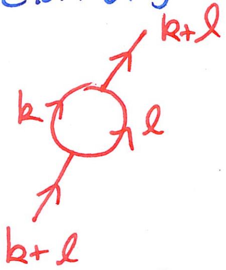
edges:– oriented, labelled $0, \ldots, n$ (corresponding to $\Lambda^k \mathbb{C}^n$)

vertices:–

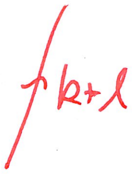



(corresponding to the maps $\Lambda^a \otimes \Lambda^b \to \Lambda^{a+b}$
and $\Lambda^{a+b} \to \Lambda^a \otimes \Lambda^b$)

relations:–




$= \begin{bmatrix} k+l \\ l \end{bmatrix}_q$

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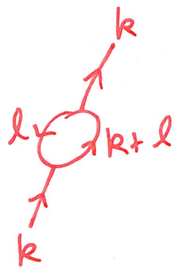
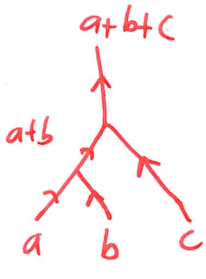



$= \begin{bmatrix} n-k \\ l \end{bmatrix}_q$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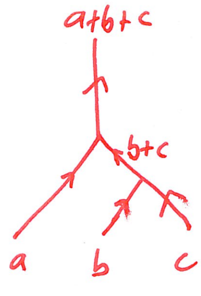
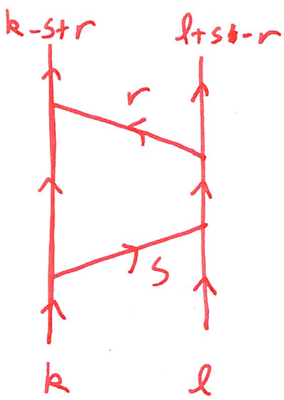
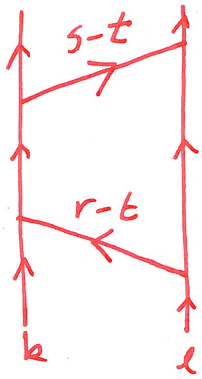



$= \sum_t \begin{bmatrix} k-l+r-s \\ t \end{bmatrix}_q$

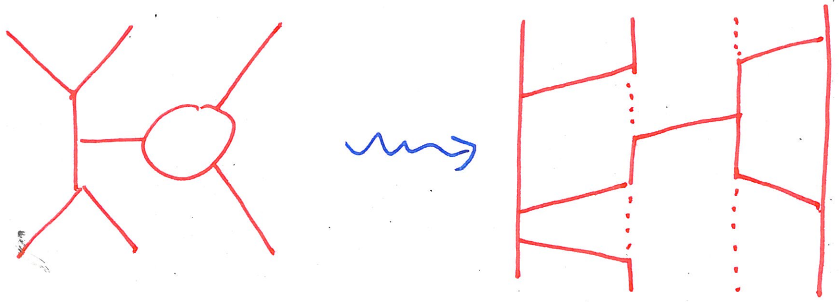


It's relatively easy to see that all maps in $\mathrm{Rep}\, U_q \mathfrak{sl}_n$ can be built in this way, and further that the above relations hold.

But is it a <u>presentation</u> of the category?

(i.e. do two diagrams correspond to the same morphism only if they are related by a sequence of diagrams?)

The essential idea is that using <u>isotopy</u> and <u>relations</u> we can rewrite any diagram in <u>ladder form</u>.



The rungs of these ladders behave exactly like the $E_i$'s and $F_i$'s in $U_q \mathfrak{gl}_\infty$.

Now some representation theory:

Skew Howe duality says that

$$GL(m) \circlearrowright \Lambda^\bullet(\mathbb{C}^m \otimes \mathbb{C}^n) \circlearrowleft SL(n)$$

generate each others commutants.

(And, with work, there's a quantum version.)

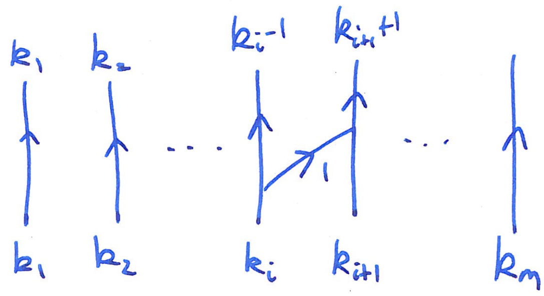
Now $\Lambda^K(\mathbb{C}^m \otimes \mathbb{C}^n) = \Lambda^K(\underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{m \text{ times}})$

$$= \bigoplus_{\sum k_i = K} \Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n$$

So in particular we have

$$U_q\mathfrak{gl}_m \twoheadrightarrow \mathrm{Hom}_{SL(n)}\left(\Lambda^{k_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{k_m}\mathbb{C}^n \to \Lambda^{l_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{l_m}\mathbb{C}^n\right)$$

with

$E_i \longmapsto$ 

Further, we can explicitly identify the kernel of

$$U\mathfrak{gl}_m \twoheadrightarrow \mathrm{End}_{SL(n)}(\Lambda^k(\mathbb{C}^m \otimes \mathbb{C}^n))$$

as

$$I_{\lambda(k)} = \langle 1_\nu \mid \nu \text{ not dominated by } \lambda(k) \rangle$$

where

$$\lambda(k) = (n, n, \dots, n, r, 0, \dots, 0), \quad \text{summing to } k.$$

(This follows from something more general:

Skew Howe duality tells us

$$\Lambda^k(\mathbb{C}^m \otimes \mathbb{C}^n) \cong \bigoplus_{\mu \le \lambda(k)} V(\mu) \otimes V(\mu^t)$$

(as $GL(m)$-$SL(n)$ bimodules)

so $$\mathrm{End}_{SL(n)} \Lambda^k(\mathbb{C}^m \otimes \mathbb{C}^n) = \bigoplus_{\mu \le \lambda(k)} \mathrm{End}(V(\mu))$$

and $$\dot{U}\mathfrak{gl}_m / I_\lambda \cong \bigoplus_{\mu \le \lambda} \mathrm{End}(V(\mu)).\ )$$

Here's the diagram which proves everything:

$$\begin{tikzcd}
\mathrm{Ladders}^n_m \arrow[r] \arrow[d] & \dot{U}\mathfrak{gl}_m/I_{\lambda(k)} \arrow[d] \arrow[dr] & \\
\mathrm{FreeSpider}(SL(n)) \arrow[r] & \mathrm{Spider}(SL(n)) \arrow[r, "\Gamma"'] \arrow[ul, dashed, red] & \mathrm{Rep}(SL(n))
\end{tikzcd}$$

Note that we pick different $m$ and $k$ for different elements!

$\Gamma$ is surjective — for any morphism in $\mathrm{Rep}(SL(n))$, it comes from something in some $U\mathfrak{gl}_m/I_{\lambda(k)}$

$\Gamma$ is injective — for $m \in \mathrm{Spider}(SL(n))$ with $\Gamma(m)=0$, lift it to a ladder for some $m$. This goes to zero in $\mathrm{Rep}(SL(n))$, so also zero in $\dot{U}\mathfrak{gl}_m/I_{\lambda(k)}$, so also zero in $\mathrm{Spider}(SL(n))$.

From the standard relations in $U_q\mathfrak{sl}_n$ and this ideal, we can immediately read off the desired relations.

Curiously, we don't need a relation corresponding to the Serre relation; once we allow *planar isotopy* it follows from the others: