Webs and skew Howe duality

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Abstract

Contents

1 Introduction

1.1 Generators and relations for $\mathcal{R}ep(\mathrm{SL}_n)$

The representation theory of SL_n is a pivotal tensor category, and it is natural to ask for a presentation by generators and relations, as a pivotal tensor category.

There are two main choices one needs to make before looking for such a presentation. First, it would be reasonable to pass to any full subcategory, whose idempotent completion recovers the entire representation theory. In particular, in this paper we look at the full subcategory (denoted $\mathcal{R}ep(\mathrm{SL}_n)$) whose objects are isomorphic to tensor products of the fundamental representations $\bigwedge^k \mathbb{C}^n$ of $\mathfrak{sl}_n$. Second, we need to decide which generators to use. We take the maps $\bigwedge^a \mathbb{C}^n \otimes \bigwedge^b \mathbb{C}^n \otimes \bigwedge^c \mathbb{C}^n \to \mathbb{C}$. (The space of such maps is one-dimensional if $a+b+c$ is a multiple of n, or zero-dimensional otherwise.) It is relatively easy to show that these are indeed generators, i.e. that every $\mathfrak{sl}_n$-linear map between tensor products of fundamental representations can be written as tensor products and compositions of these maps, along with the duality pairing and copairing maps [**?**, Proposition 3.5.8]. The question then, is to identify the relations holding between such planar compositions.

Said another way, we have a pivotal category (the "free spider category" $\mathcal{FS}p(\mathrm{SL}_n)$) of trivalent webs, with oriented edges labelled by $\{1, \ldots, n-1\}$, and at each vertex the labels summing to a multiple of n, and a full and dominant functor $\mathcal{FS}p(\mathrm{SL}_n) \to \mathcal{R}ep(\mathrm{SL}_n)$. The question is to identify the pivotal ideal which is the kernel of this functor.

This problem has been studied previously. For $n = 2$, there are no trivalent vertices, and the category of webs is essentially just the category of embedded 1-manifolds up to isotopy. The kernel of the functor to representation theory is the ideal generated by the difference $\bigcirc - 2$.

For $n = 3$, generators for the kernel were determined by Kuperberg [**?**]. The relations allow one to remove circles, bigons, and squares. He introduced the term "SL_3 spider" for the resulting diagrammatic category.

For $n \geq 4$, generators for the kernel have been proposed, by [**?**] (for $n = 4$) and by [**?**] (generally), but without proving that their lists of relations were complete.

This paper answers the question, in particular showing that the relations of [**?**] are complete. In fact, those relations are over complete; just the bigon collapse, $I = H$ and 'square-switch' relations suffice (and in fact only one of the square-switch relations implies the others). The relevant relations are reproduced in §2. The main theorem, stating the isomorphism between a combinatorially defined web category $\mathcal{S}p(SL_n)$ (the SL_n-spider) and the representation theory of SL_n, appears in §3.

1.2 Skew Howe duality and webs

The core idea of our proof is to use skew Howe duality. In fact we give a very succinct recipe for the relations, as certain truncations of relations holding in $\mathfrak{gl}_m$. We now give a quick overview of the argument.

We consider the commuting actions of SL_n and $\mathfrak{gl}_m$ on $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$. Skew Howe duality tells us that the resulting map

$$\dot{\mathcal{U}}\mathfrak{gl}_m \to \mathrm{Hom}_{\mathrm{SL}_n}\left(\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)\right) \tag{1.1}$$

is surjective. Moreover, we prove that its kernel is the ideal generated by those weight space idempotents falling outside the weight support of $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$. This result is proved in §**??**. The quotient of $\dot{\mathcal{U}}\mathfrak{gl}_m$ by this ideal is denote $\dot{\mathcal{U}}^n\mathfrak{gl}_m$.

As SL_n-representations, we have

$$\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m) = \wedge^L(\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n) = \bigoplus_{\underline{l}:\sum \underline{l} = K} \wedge^{l_1}\mathbb{C}^n \otimes \cdots \otimes \wedge^{l_m}\mathbb{C}^n$$

Thus combining the last two equations, we obtain an isomorphism,

$$\dot{\mathcal{U}}^n(\mathfrak{gl}_m) \to \oplus_{\underline{l},\underline{k}:\sum \underline{l} = \sum \underline{k}} \mathrm{Hom}_{\mathrm{SL}_n}(\wedge^{l_1}\mathbb{C}^n \otimes \cdots \otimes \wedge^{l_m}\mathbb{C}^n, \wedge^{k_1}\mathbb{C}^n \otimes \cdots \otimes \wedge^{k_m}\mathbb{C}^n)$$

Under this map, elements of $\mathfrak{gl}_m$ acts by particular webs, which we call ladders. This allows us to write the generating relations of $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ in a diagrammatic form. By the above isomorphism, we see that these diagrammatic relations become the generating relations in the SL_n-spider.

(1.2)

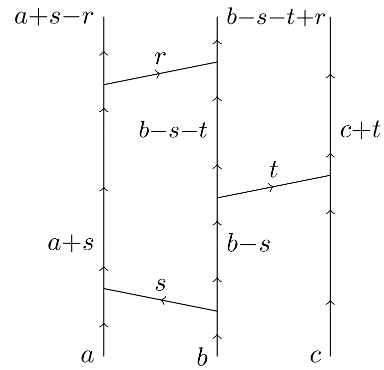


Figure 1: An example of a ladder; this one corresponds to the word $F_1^{(r)}F_2^{(t)}E_1^{(s)}1_{abc}$ (reading bottom to top, right to left) in $\dot{\mathcal{U}}(\mathfrak{gl}_3)$

2 The categories $\mathcal{R}ep(\mathrm{SL}_n)$ and $\mathcal{S}p(\mathrm{SL}_n)$

We denote by $[n]_q$ the quantum integer $q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$. More generally,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q \dots [1]_q}{([n-k]_q \dots [1]_q)([k]_q \dots [1]_q)}.$$

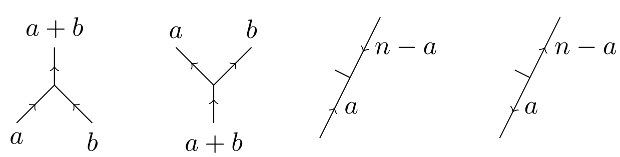
We will denote by $U_q(\mathfrak{g})$ the quantum group associated to a Lie algebra $\mathfrak{g}$.

2.1 Definition of the representation category $\mathcal{R}ep(\mathrm{SL}_n)$

The objects in $\mathcal{R}ep(\mathrm{SL}_n)$ are tensor products of the fundamental representations $\wedge^k\mathbb{C}^n$ of $U_q(\mathfrak{sl}_n)$. These are clearly in bijection with tuples $\underline{k} = (k_1, \dots, k_m)$ with $0 \le k_i \le n$.

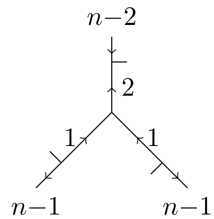
2.2 The free spider category $\mathcal{FS}p(\mathrm{SL}_n)$

The free spider category $\mathcal{FS}p(\mathrm{SL}_n)$ has as objects sequences $\underline{k}$ in $\{1^\pm, \dots, (n-1)^\pm\}$, and as morphisms (linear combinations of) oriented planar graphs locally modeled on the following four types of vertices:



with all labels drawn from the set $\{1, \dots, n-1\}$. The third and fourth graphs depict bivalent vertices, called 'tags', which are not rotationally symmetric, meaning that the tag provides a distinguished side. The bottom boundary of any planar graph in $\mathrm{Hom}(\underline{k}, \underline{k'})$ is $\underline{k}$ with the strand oriented up for each positive entry, and the strand oriented down for each negative entry. Similary, the top boundary is determined by $\underline{k'}$ in the same way.

Example 1. *We can build a trivalent vertex with one incoming edge labelled by $n-2$ and two outgoing edges* *labelled by $n-1$, for example as*

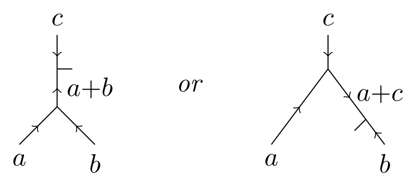


(2.1)

Once we impose the relations in the spider category, the various choices of which direction each tag points will all result in the same diagram, up to a sign, via Equation (2.3).

There are many ways to build a trivalent vertex with all edges oriented inwards, and boundary labels summing to n. For example,

(2.2)



Again, these will all become equal (possibly up to a sign), via Equations (2.3) and (2.9).

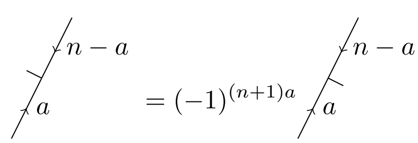
We will often draw diagrams with edges also labelled by 0 or n. This is a notational convenience, to be interpreted as follows. Edges labelled by 0 and n are to be deleted; trivalent vertices involving a 0 edge become simple strands, and trivalent vertices involving a n edge are replaced by tags:

Any trivalent vertices with all edges labelled either 0 or n can be deleted. Any diagram with an edge labeled less than 0 or greater than n is zero.

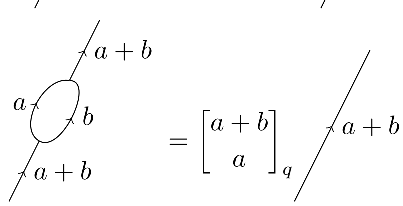
2.3 Definition of the spider category $\mathcal{S}p(\mathrm{SL}_n)$

The spider category $\mathcal{S}p(\mathrm{SL}_n)$ is the quotient of $\mathcal{FS}p(\mathrm{SL}_n)$ by the following relations

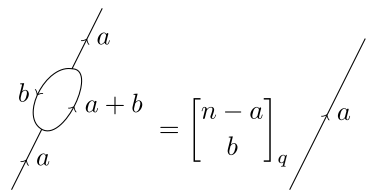
(2.3)



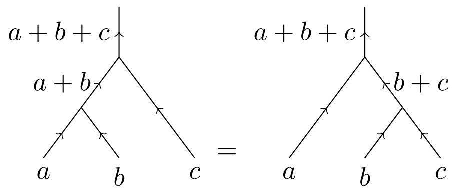
(2.4)



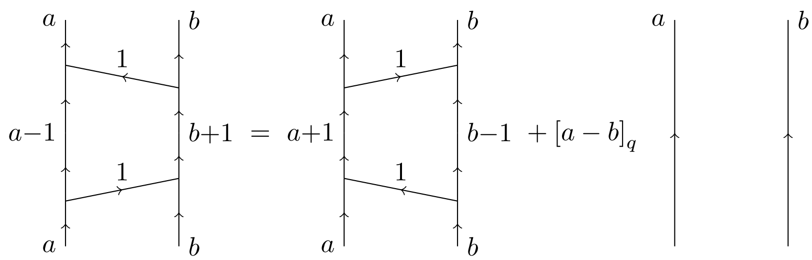
(2.5)



(2.6)



(2.7)

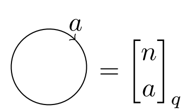


together with the mirror reflections and the arrow reversals of these. These relations will be refered to as the 'switching a tag' (2.3), 'removing a bigon' (2.4) and (2.5), '$I = H$' (2.6) and 'commutation' (2.7).

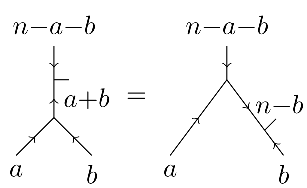
Here are a couple of easy consequences of these relations.

Lemma 2.3.1. *We have:*

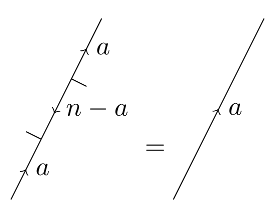
(2.8)



(2.9)



(2.10)

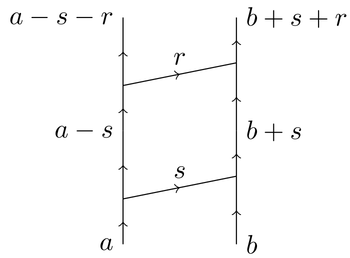


Proof. The first identity follows from relation (2.5) with $a = 0$ after deleting the 0-strings. The second follows from (2.7) when $a + b + c = n$, after replacing n-strands with tags. The third also follows from (2.5) with $b = n - a$ after replacing the n-strand with a matching pair of tags. $\square$

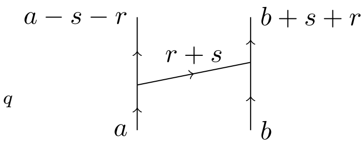
We also have the following relations.

Lemma 2.3.2. *The following identities hold in $\mathcal{S}p(\mathrm{SL}_n)$*

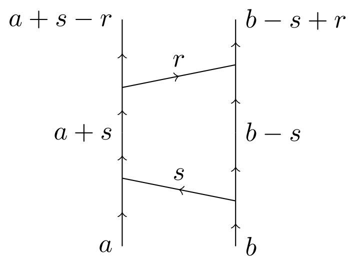
(2.11)



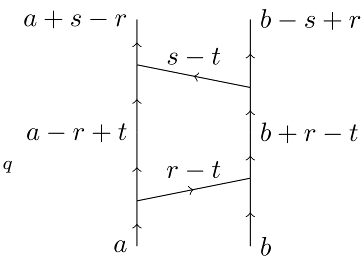
$$= \begin{bmatrix} r+s \\ r \end{bmatrix}_q$$



(2.12)



$$= \sum_t (-1)^t \begin{bmatrix} t+s-r-1+a-b \\ t \end{bmatrix}_q$$



Remark. The summation in Equation (2.12) is over the range $\max(b+r-n, r-a, 0) \leq t \leq \min(s, r)$.

Proof. □

Remark. If we worked over $\mathbb{Z}$, then these relations would not hold and we would need to add them as extra relations.

2.3.1 The upwards subcategory

We will use $\mathcal{FS}p(\mathrm{SL}_n)^+$ and $Sp(\mathrm{SL}_n)^+$ to denote the full subcategories of the above where we restrict to objects which are words in $\{1^+, \ldots, (n-1)^+\}$. This means that strands at the boundary are always oriented upwards. These are tensor subcategories but not pivotal subcategories because the pairing and copairing morphisms for an object k do not lie in this subcategory.

Lemma 2.3.3. *Every morphism in $\mathcal{S}p(\mathrm{SL}_n)$ is of the form $\alpha D \beta$, where D is a morphism in $\mathcal{S}p(\mathrm{SL}_n)^+$ while α and β are isomorphisms built out of tags, sending k^- to $(n-k)^+$. This decomposition is canonical up to a sign.*

Proof. For any object $\underline{k} \in \mathcal{S}p(\mathrm{SL}_n)$, define $|\underline{k}|$ to be the object of $\mathcal{S}p(\mathrm{SL}_n)^+$ with every k^- in $\underline{k}$ replaced with $(n-k)^+$. There are isomorphisms $\underline{k} \to |\underline{k}|$ given by a tensor product of identity factors and tags, canonical up to a choice of the direction of each tag, which by Equation (2.3) is just an overall sign ambiguity.

For a morphism $C \in \mathcal{S}p(\mathrm{SL}_n)$ between objects $\underline{k}$ and $\underline{k'}$, let α be such an isomorphism from $\underline{k}$ to $|\underline{k}|$, and β be such an isomorphism from $|\underline{k'}|$ to $\underline{k'}$. Define $D = \alpha^{-1} C \beta^{-1}$. □

In fact $\mathcal{S}p(\mathrm{SL}_n)^+$ is just a skeletonization of $\mathcal{S}p(\mathrm{SL}_n)$, and the lemma above is just describing how to see that every object of $Sp(\mathrm{SL}_n)$ is isomorphic to one of $\mathcal{S}p(SL_n)^+$.

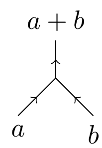
3 Statement and proof of the main theorem

3.1 Definition of the functor $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n)^+ \to \mathcal{R}ep(\mathrm{SL}_n)$

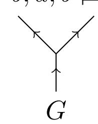
At the level of objects we take

$$(k_1, \ldots, k_m) \mapsto \wedge^{k_1}\mathbb{C}^n \otimes \cdots \otimes \wedge^{k_m}\mathbb{C}^n.$$

For morphisms we take



$\mapsto F_{a,b}$ and $a+b, a, b \mapsto$



$_{a,b}$

where $F_{a,b}$ and $G_{a,b}$ are defined by

$$F_{a,b} : \bigwedge^{a} \mathbb{C}^n \otimes \bigwedge^{b} \mathbb{C}^n \to \bigwedge^{a+b} \mathbb{C}^n$$
$$y_{i_1} \wedge \cdots \wedge y_{i_a} \otimes y_{j_1} \wedge \cdots \wedge y_{j_b} \mapsto y_{i_1} \wedge \cdots \wedge y_{i_a} \wedge y_{j_1} \wedge \cdots \wedge y_{j_b}$$

$$G_{a,b} : \bigwedge^{a+b} \mathbb{C}^n \to \bigwedge^{a} \mathbb{C}^n \otimes \bigwedge^{b} \mathbb{C}^n$$
$$y_{i_1} \wedge \cdots \wedge y_{i_{a+b}} \mapsto \sum_S (-1)^S y_S \otimes y_{S^c}$$

3.2 The main result

Theorem 3.2.1. *The functor $\Gamma_n : \mathcal{S}p(\mathrm{SL}_n)^+ \to \mathcal{R}ep(\mathrm{SL}_n)$ is an equivalence of categories.*

Proof. We will use the following commutative diagram

(3.1)
$$\begin{tikzcd}
 & \mathcal{L}ad_n^m \arrow[r] \arrow[d] & \dot{\mathcal{U}}^n(\mathfrak{gl}_m) \arrow[d, "\Psi_m^n"'] \arrow[dr, "\Phi_m^n"] & \\
\mathcal{FS}p(\mathrm{SL}_n)^+ \arrow[rr] & & \mathcal{S}p(\mathrm{SL}_n)^+ \arrow[r, "\Gamma_n"] & \mathcal{R}ep(\mathrm{SL}_n)
\end{tikzcd}$$

where the three categories in the bottom row were defined in section 2, while Φ and $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ and defined in section 4 and $\mathcal{L}ad_n^m$ and Ψ are defined in section 5.

We now explain why Γ_n is an equivalence of categories. Since it is clearly an isomorphism on objects we must show that it is fully faithful.

Surjectivity (fullness) of Γ_n on Hom spaces follows from the fullness of the functor Φ_n^m, which is proven in Theorem 4.3.1[1] More precisely, given any two objects V, W in $\mathcal{R}ep(\mathrm{SL}_n)$ we can find some m such that there exist n-bounded weights $\underline{k}, \underline{l}$ of GL_m such that $\Phi\_n^m(\underline{k}) = V$ and $\Phi_n^m(\underline{l}) = W$. The fullness of Φ_n^m tells us that the map

$$\Phi_n^m : 1_{\underline{l}} \dot{\mathcal{U}}(\mathfrak{gl}_m) 1_{\underline{k}} \to \mathrm{Hom}_{SL_n}(V, W)$$

is surjective (i.e. all the morphisms come from ladders with m uprights). The commutativity of the right triangle shows us that these morphisms all come from webs in $\mathcal{S}p(\mathrm{SL}_n)$.

Now we sill show that Γ_n is faithful, i.e. that it is injective on Hom spaces. Let w be a morphism in $\mathcal{S}p(\mathrm{SL}_n)^+$ such that $\Gamma_n(w) = 0$. By Theorem 5.3.1 and the commutativity of the left square, we can find some m and some $\tilde{(w)} \in \dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ such that $\Psi_n^m(\tilde{w}) = w$ (i.e. we find a ladder $\tilde{w}$ equivalent to the web w). Then by the commutativity of the right triangle, we see that $\Phi_n^m(\tilde{w}) = 0$. However, by Theorem 4.3.1, Φ_m^n is faithful which means $\tilde{w} = 0$ and hence $w = 0$ as desired. ∎

[1]The fullness of Γ_n was proven in Proposition 3.5.8 of [?] using Schur-Weyl duality instead of Skew-Howe duality, but the argument is essentially the same.

4 The functor $\Phi_m^n : \dot{\mathcal{U}}(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$

4.1 The idempotent form $\dot{\mathcal{U}}(\mathfrak{gl}_m)$

In this paper, we use the Lusztig's idempotent form $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ of the universal envelopping algebra of $\mathfrak{gl}_m$. We regard $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ as a $\mathbb{C}(q)$-linear category with objects $\underline{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$. The identity morphism of the object $\underline{k}$ is denoted $1_{\underline{k}}$ and we write $1_{\underline{k'}}\dot{\mathcal{U}}^n(\mathfrak{gl}_m)1_{\underline{k}}$ for the space of morphisms.

The morphisms are generated by $E_i 1_{\underline{k}} \in 1_{\underline{k}+\alpha_i}\dot{\mathcal{U}}(\mathfrak{gl}_m)1_{\underline{k}}$ and $F_i 1_{\underline{k}} \in 1_{\underline{k}-\alpha_i}\dot{\mathcal{U}}(\mathfrak{gl}_m)1_{\underline{k}}$, for $i = 1, \dots, m-1$ (here $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0)$ where the 1 appears in position i). Notice that $\mathrm{Hom}(\underline{k}, \underline{k'}) = 0$ unless $\sum k_i = \sum k'_i$.

Remark. When the specific weight space is not important (or is obvious from the context) we will write E_i instead of $E_i 1_{\underline{k}}$, F_i instead of $F_i 1_{\underline{k}}$ etc.

As a category,$\dot{\mathcal{U}}(\mathfrak{gl}_m)$ is defined by the above objects, the above generating morphism and the following relations:

$$E_iE_j1_{\underline{k}} = E_jE_i1_{\underline{k}} \text{ and } F_iF_j1_{\underline{k}} = F_jF_i1_{\underline{k}}, \text{ if } |i-j| > 1 \tag{4.1}$$

$$(E_iF_i - F_iE_i)1_{\underline{k}} = [\langle \underline{k}, \alpha_i \rangle]_q 1_{\underline{k}}, \text{ while if } i \neq j \text{ then } E_iF_j1_{\underline{k}} = F_jE_i1_{\underline{k}} \tag{4.2}$$

$$\text{If } |i-j| = 1 \text{ then} [2]_q E_iE_jE_i1_{\underline{k}} = (E_i^2E_j + E_jE_i^2)1_{\underline{k}} \text{and likewise with } F\text{'s.} \tag{4.3}$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{Z}^m$.

It is also convenient to include the "divided powers" morphisms

$$E_i^{(r)} := \frac{E_i^r}{[r]_q \dots [1]_q} \quad \text{and} \quad F_i^{(r)} := \frac{F_i^r}{[r]_q \dots [1]_q}.$$

These satisfy a series of relations such as $E_iE_i^{(r)} = [r+1]_q E_i^{(r+1)}$, but all of these follow from the relations (4.1)–(4.3) above.

Notice that using this notation, relation (4.3) takes on the nice form $E_iE_jE_i = E_i^{(2)}E_j + E_jE_i^{(2)}$.

Remark. We can also define Lusztig's $\mathbb{Z}$-form of $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ as the $\mathbb{Z}$-algebra generated by all $E_i^{(r)}, F_i^{(r)}$ with appropriate relations.

4.2 Definition of the functor Φ_m^n

The vector space $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$ carries commuting actions of GL_m and SL_n. The weight decomposition of $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$ with respect to the maximal torus of GL_m is given by

$$\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_{\underline{k}} \wedge^{k_1}\mathbb{C}^n \otimes \dots \otimes \wedge^{k_m}\mathbb{C}^n \tag{4.4}$$

where the sum ranges over $\underline{k} = (k_1, \dots, k_m)$ with $0 \le k_i \le n$. We will refer to such $\underline{k}$ as the n**-bounded weights of** GL_m.

Since we have a GL_m action with these weight spaces and commuting with the SL_n action, we get a map

$$1_{\underline{k'}}\dot{\mathcal{U}}(\mathfrak{gl}_m)1_{\underline{k}} \to \mathrm{Hom}_{SL_n}\left(\wedge^{k_1}\mathbb{C}^n \otimes \dots \otimes \wedge^{k_m}\mathbb{C}^n, \wedge^{k'_1}\mathbb{C}^n \otimes \dots \otimes \wedge^{k'_m}\mathbb{C}^n\right) \tag{4.5}$$

for any two n-bounded weights $\underline{k}, \underline{k'}$ with $\sum_i k_i = \sum_i k'_i$.

Since the action of GL_m on $\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)$ generates the commutant of the SL_n action, the map (4.5) is surjective.

Thus we may define a functor $\Phi_m^n : \dot{\mathcal{U}}(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ as follows:

- On objects $\underline{k} \mapsto \begin{cases} \wedge^{k_1}\mathbb{C}^n \otimes \cdots \otimes \wedge^{k_m}\mathbb{C}^n & \text{if } \underline{k} \text{ is } n\text{-bounded} \\ 0 & \text{otherwise.} \end{cases}$

- On morphisms Φ^n_m is given by (4.5).

Since (4.5) was surjective the functor Φ^n_m is full.

4.3 The quotient category $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$

We denote by $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ the quotient of $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ where we set to zero all objects which are not an n-bounded weight. In other words, we quotient by the 2-sided ideal of morphisms generated by all $1_{\underline{k}}$ such that $\underline{k}$ is not n-bounded.

Since all weights of $\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)$ are n-bounded, the functor $\Phi^n_m : \dot{\mathcal{U}}(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ factors through $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ (we abuse notation slightly and use Φ^n_m as notation for this functor as well).

Theorem 4.3.1. *The functor $\Phi^n_m : \dot{\mathcal{U}}^n(\mathfrak{gl}_m) \to \mathcal{R}ep(\mathrm{SL}_n)$ is fully faithful (meaning that it induces an isomorphisms between* Hom*-spaces).*

To prove this result, we need a general result about reductive Lie algebras which may be well-known, but was not previously known to us. For simplicity, we state this result in the $\mathfrak{gl}_m$ case. For any dominant weight λ, let $V(\lambda)$ the corresponding highest weight representation of $\mathfrak{gl}_m$.

Recall that we have the usual dominance order of the set of dominant weights of $\mathfrak{gl}_m$ where $\mu \leq \lambda$ if $\lambda - \mu$ is a sum of the simple roots α_i. We extend this notion as follows. We say that a dominant weight λ **dominates** a weight ν if ν lies in the Weyl group orbit of a dominant weight $\mu \leq \lambda$.

Let I_λ be the 2-sided ideal in $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ generated by all 1_ν such that λ does not dominate ν.

If μ is a dominant weight of $\mathfrak{g}$ with $\mu \leq \lambda$, then for each ν as above, ν is not a weight of $V(\mu)$. Thus I_λ acts trivially on $V(\mu)$ and we get a representation $\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_\lambda \to \mathrm{End}\,(V(\mu))$.

Lemma 4.3.2. *For any λ, the resulting map $\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_\lambda \to \bigoplus_{\mu\leq\lambda} \mathrm{End}\,(V(\mu))$ is an isomorphism.*

Proof. First note that $\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_\lambda$ is finite-dimensional. By Wedderburn's theorem, it suffices to show that the category of finite-dimensional $\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_\lambda$-modules is semisimple with simple objects the $V(\mu)$, for $\mu \leq \lambda$.

Now a $\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_\lambda$ module is the same thing as a $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ module in which I_λ acts trivially. Since the category of finite-dimensional $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ modules is semisimple and the ones where I_λ acts trivially are precisely the $V(\mu)$ for $\mu \leq \lambda$, the result follows. □

Proof. We now return to proving Theorem 4.3.1. Recall that by skew-Howe duality, we have a decomposition

$$\bigwedge^{\bullet}(\mathbb{C}^n \otimes \mathbb{C}^m) = \bigoplus_\mu V(\mu^t) \otimes V(\mu)$$

as $\mathrm{SL}_n \times \mathrm{GL}_m$-representations, where μ varies over all n-bounded weights of GL_m (μ^t denotes the transpose of μ which is a weight of SL_n). Thus for any $0 \leq K \leq mn$,

$$\mathrm{Hom}_{\mathrm{SL}_n}\left(\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m), \bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)\right) = \bigoplus_\mu \mathrm{End}\,(V(\mu))$$

where μ ranges over n-bounded weights with $\sum \mu_i = K$. Note that these μ are exactly the set of dominant weights of $\mathfrak{gl}_m$ which satisfy $\mu \leq \lambda(K)$, where $\lambda(K)$ is the unique weight of the form $(n, \dots, n, r, 0, \dots, 0)$ where the terms sum to K. Applying the previous lemma, we see that the map

$$\dot{\mathcal{U}}(\mathfrak{gl}_m)/I_{\lambda(K)} \to \mathrm{Hom}_{\mathrm{SL}_n}\left(\bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m), \bigwedge^K(\mathbb{C}^n \otimes \mathbb{C}^m)\right)$$

is an isomorphism.

Note that a weight μ is n-bounded if and only if μ is dominated by $\lambda(K)$ where $K = \sum \mu_i$. Thus

$$\dot{\mathcal{U}}^n(\mathfrak{gl}_m) = \bigoplus_{K=0}^{nm} \dot{\mathcal{U}}(\mathfrak{gl}_m)/I_{\lambda(K)}$$

and so the result follows. $\square$

5 Ladders

5.1 Ladders and $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$

We will now introduce a diagrammatic notation for morphisms in $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$. We will picture E_i as a ladder with a single rung connecting strands i and $i+1$ angling up diagonally to the left.

We begin by formalizing the notion of a ladder.

Definition 5.1.1. *A ladder with m uprights is ...*

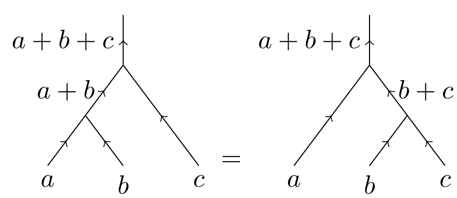
We now introduce the category $\mathcal{L}ad_n^m$ of ladders. The objects are sequences of length m of integers between 0 and n inclusive (that is, n-bounded weights of GL_m). The morphisms are linear combinations of ladders. Composition of morphisms is given by vertical concatenation of ladders.

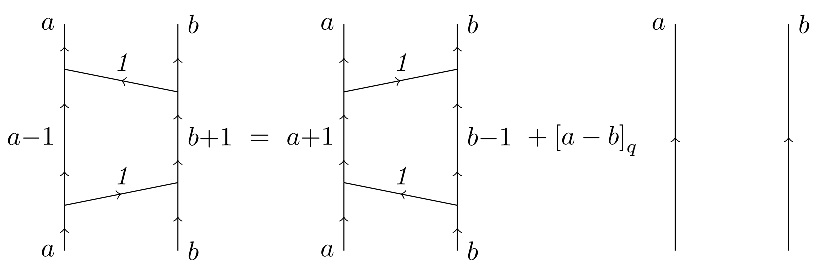
Remark. Notice that the categories $\mathcal{L}ad_n^m$ for all m fit together as a tensor category $\mathcal{L}ad_n$, with tensor product given by horizontal juxtaposition. In this tensor category the morphisms are generated by the single rung ladders.

We define a functor from $\mathcal{L}ad_n^m$ to $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ which on objects is just the identity. On morphisms we send the rungs

...

Proposition 5.1.2. *The category $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ is the quotient of $\mathcal{L}ad_n^m$ by the diagrammatic relations* ***TODO: The first of these diagrams needs to be changed to a two rung ladder equation equivalent to $E_iF_{i+1} = F_{i+1}E_i$. Both of these diagrams should be drawn so there are lots of vertical strands on either side of the action.***

(5.1)


(5.2)


Proof. Since all E_i, F_i are in the image of the functor, we see that $\mathcal{L}ad_n^m \to \dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ is full (it is obviously dominant). It remains to see that the above relations generate the kernel. To see this, just note that the defining relations of $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ are given by equations (4.1) - (4.3) along with the relation that $1_{\underline{k}} = 0$ if $\underline{k}$ is not an n-bounded weight. When $|i-j| > 1$, equation (4.1) is reflected in the isotopy invariance of ladders, while when $|i-j| = 1$, (4.1) is (5.1) in diagrammatic form. Equation (4.2) becomes (5.2) in diagrammatic form. Finally, (4.3) is not actually needed as a defining relation of $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ because it is a formal consequence of working with bounded weight space (in particular we work with only n-bounded weights). $\square$

5.2 Ladders as webs

There is a functor from $\mathcal{L}ad_n^m \to \mathcal{FS}p(SL_n)$ by forgetting the ladder structure of a ladder and thinking of it as a web. There is a slightly discrepancy at the level of objects: in $\mathcal{L}ad_n^m$ the objects $\underline{k}$ are sequences in $\{0,\dots,n\}$, while in $\mathcal{FS}p(SL_n)$ the objects are sequences in $\{1^\pm,\dots,(n-1)^\pm\}$. The functor deletes 0s and ns from the sequences, and sends k to k^+.

Proposition 5.2.1. *The functor $\mathcal{L}ad_n^m \to \mathcal{FS}p(SL_n) \to \mathcal{S}p(SL_n)$ can be factored through the functor $\mathcal{L}ad_n^m \to \dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ of the previous section, giving rise to a functor $\Psi_m^n : \dot{\mathcal{U}}^n(\mathfrak{gl}_m) \to \mathcal{S}p(SL_n)$.*

Proof. To see that Ψ_m^n exists, we must just show that the diagrammatic relations of $\dot{\mathcal{U}}^n(\mathfrak{gl}_m)$ from Proposition 5.1.2 are taken to the kernel of the functor $\mathcal{FS}p(\mathrm{SL}_n) \to \mathcal{S}p(\mathrm{SL}_n)$.

To show that (5.1) holds in $\mathcal{S}p(\mathrm{SL}_n)$, it suffices by pivotal composition to prove the relation which we see is a special case of (2.6).

Finally (5.2) holds in $\mathcal{S}p(\mathrm{SL}_n)$ since it is exactly (2.7). □

Remark. The above results show that relation (**??**) holds in $\mathcal{S}p(SL_n)$. A diagrammatic argument for this appears in Appendix §A.

We have now reached the situation described in the proof of the main result. We have the diagram

$$\begin{tikzcd}
\mathcal{L}ad_n^m \arrow[r] \arrow[d] & \dot{\mathcal{U}}^n(\mathfrak{gl}_m) \arrow[d, "\Psi_m^n"'] \arrow[dr, "\Phi_m^n"] & \\
\mathcal{FS}p(\mathrm{SL}_n)^+ \arrow[r] & \mathcal{S}p(\mathrm{SL}_n)^+ \arrow[r, "\Gamma_n"] & \mathcal{R}ep(\mathrm{SL}_n)
\end{tikzcd} \tag{5.3}$$

Note that the left square of this diagram commutes by definition of Ψ_m^n.

Proposition 5.2.2. *The right triangle of (5.3) commutes.*

Proof. We proceed by an explicit calculation. Consider some $E_i 1_{\underline{k}}$. Via the map (4.5), we see that $\Phi_n^m(E_i 1_{\underline{k}})$ is a map

$$\bigwedge^{k_1}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k_m}\mathbb{C}^n \to \bigwedge^{k'_1}\mathbb{C}^n \otimes \cdots \otimes \bigwedge^{k'_m}\mathbb{C}^n$$

where $\underline{k'} = \underline{k} + \alpha_i$.

For any $0 \le k, l \le n$, define

$$T_{k,l} : \bigwedge^{k}\mathbb{C}^n \otimes \bigwedge^{l}\mathbb{C}^n \to \bigwedge^{k+1}\mathbb{C}^n \otimes \bigwedge^{l-1}\mathbb{C}^n$$

$$y_{i_1} \wedge \cdots \wedge y_{i_k} \otimes y_{j_1} \wedge \cdots \wedge y_{j_l} \mapsto \sum_{p=1}^{k} (-1)^{p+1} y_{i_1} \wedge \cdots \widehat{(y_{i_p})} \cdot y_{i_k} \otimes y_{i_p} \wedge y_{j_1} \wedge \cdots \wedge y_{j_l}$$

for $y_1,\dots,y_n$ a basis for $\mathbb{C}^n$. So in the notation from section 3.1, $T_{k,l} = G_{k,1} \otimes I \circ I \otimes F_{1,l-1}$.

From examining the action of $\mathfrak{gl}_m$ on $\bigwedge^\bullet(\mathbb{C}^n \otimes \mathbb{C}^m)$, we see that $\Phi_m^n(E_i 1_{\underline{k}}) = I \otimes T_{k_i,k_{i+1}} \otimes I$.

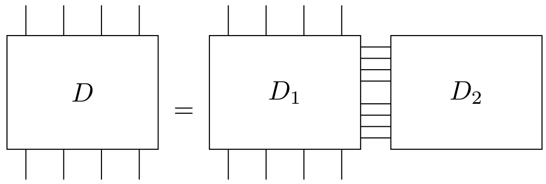
On the other hand, let $w = \Psi_m^n(E_i 1_{\underline{k}})$ be the web From the definition of Γ_n in section 3.1 we see that $\Gamma_n(w) = I \otimes T_{k_i,k_{i+1}} \otimes I$. A similar argument holds for F_i and since these generate $\dot{\mathcal{U}}(\mathfrak{gl}_m)$, the result follows. □

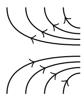
5.3 Surjectivity

Finally, we show that any web can be modified, possibly using some relations, into ladder form.

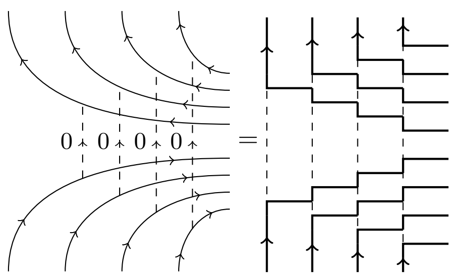
Theorem 5.3.1. *Let D be a morphism in $\mathcal{S}p(\mathrm{SL}_n)^+$. Then there exists m and a morphism $E \in \mathcal{L}ad_n^m$ such that $\Psi_n^m(E) = D$.*

Proof. Given any diagrammatic morphism $D \in \mathcal{S}p(SL_n)^+$, just by a planar isotopy we can write it as

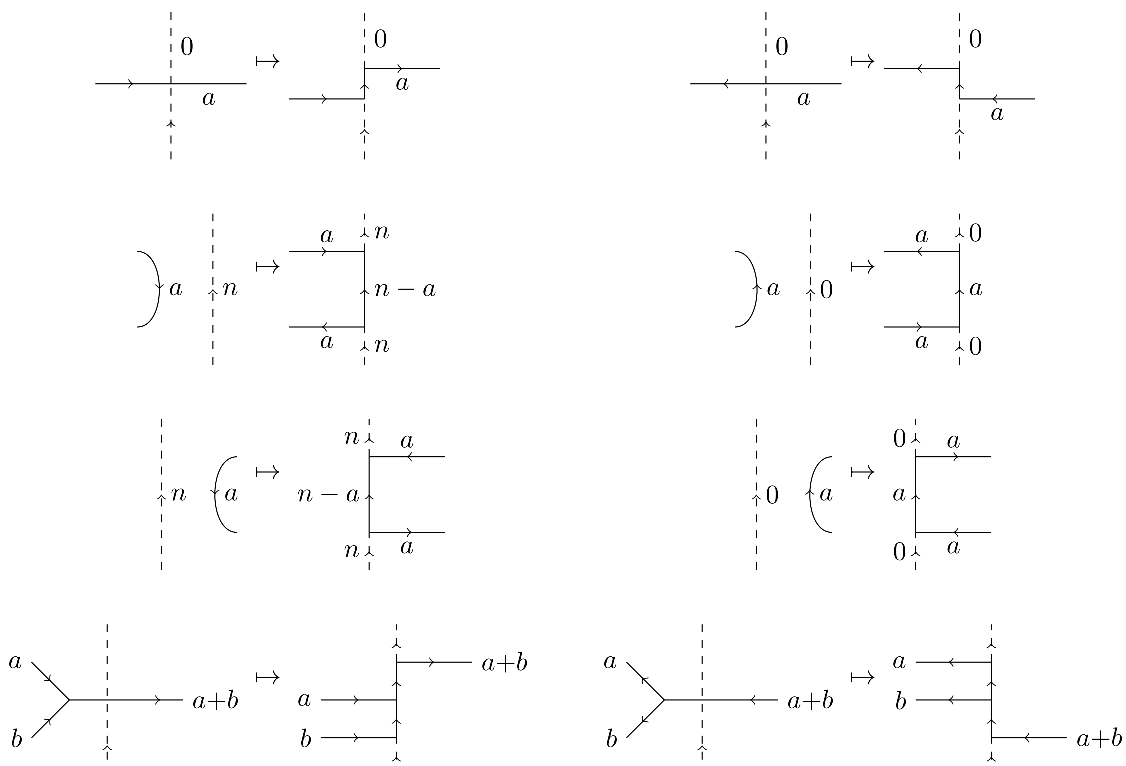


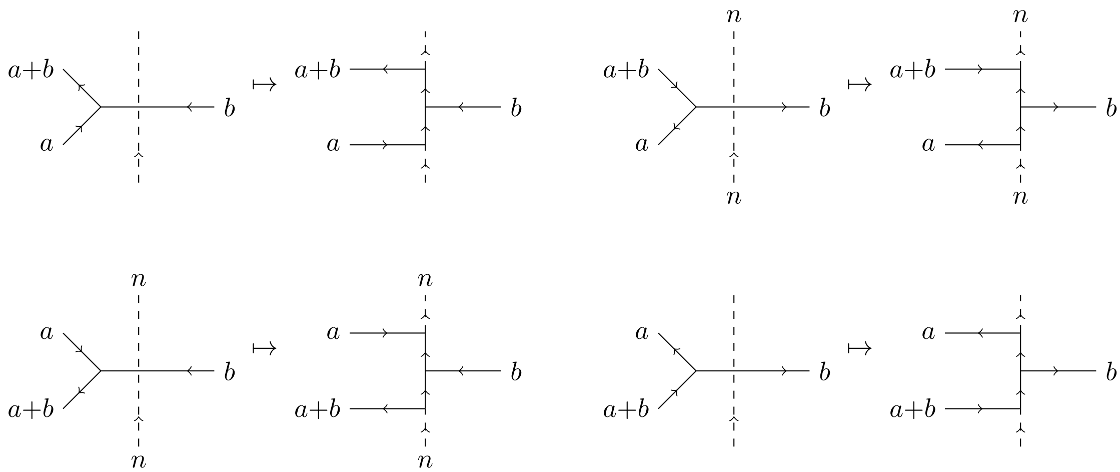
where $D_1 =$  and D_2 is in Morse position relative to the x-coordinate (that is, no two critical x-values or x-coordinates of vertices coincide) and further each trivalent vertex has two edges pointing to the left and one to the right (this can always be achieved at the expense of extra critical x-values in the strings).

Now replace D_1 with

$E_1 =$ 

Next, to the right of each elementary piece of the Morse decomposition of D_2, superimpose either a vertical 0 strand or a vertical n strand and then make the following local replacements





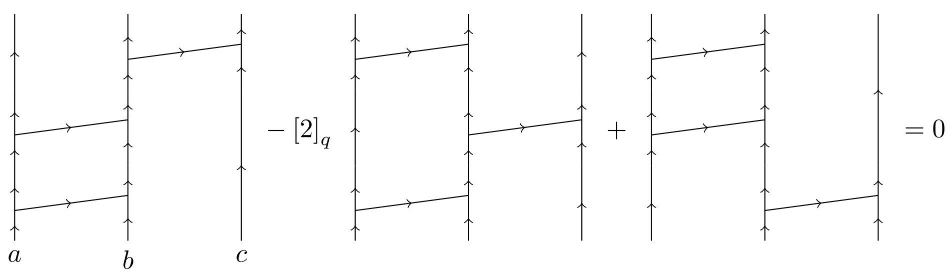
to obtain E_2.

In each of the local replacements used to form E_2 the new diagram consists of part of an upright of the ladder, along with several 'half-rungs'. It is easy to see that all of these half-rungs come in matching pairs forming complete rungs, except at the left margin of E_2. Similarly, E_1 is a ladder except that it has half-rungs along its right margin. The horizontal juxtaposition E_1E_2 is then a ladder. It is clear that D is equivalent (via deleting 0 and n strands, relation 2.10 and $\square$

A The Serre relation is a pivotal consequence of the spider relations

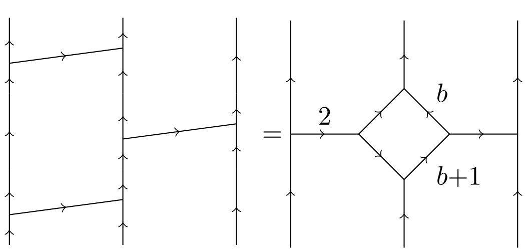
Lemma A.0.1.

(A.1)



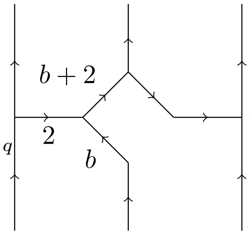
where we use the convention that any non-vertical unlabeled strand carries a 1*, while the vertical strands have arbitrary compatible labels.*

Proof. Applying the $I = H$ relation along the leftmost upright, we obtain

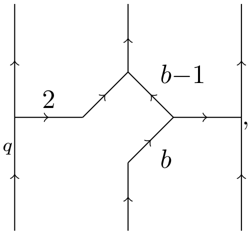


and now apply the mirror reflection of Equation (2.12) (with the variables a, b, r and s there specialized to $b, 2, b, 1$) to the central square, obtaining terms for $t = 0$ and $t = 1$, to obtain

$$= (-1)^0 \begin{bmatrix} -2 \\ 0 \end{bmatrix}_q$$



$$+ (-1)^1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}_q$$



,

(here both coefficients are just $+1$). Finally an application of Equation (2.11) on each 2-strand gives the desired identity. $\square$