Reading group mathematical foundations of statistics

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Recap

We want to study the empirical distribution

$$\mathbb{P}_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}$$

Recap: Outer measure

Definition (p. 6)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an probability space and let $T : \Omega \to \bar{\mathbb{R}}$ be any (possibly nonmeasureable) function. The **outer integral** *of T with respect to P is defined as:*

$$\mathbb{E}^* [T] = \inf \left\{ \mathbb{E} U : U \geq T, U : \Omega \to \bar{\mathbb{R}} \textit{ integrable} \right\}$$

Recap: Outer probability

Definition (p. 6)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an probability space and let B be any (possibly nonmeasureable) subset of Ω. The **outer probability** *of B with respect to P is defined as:*

$$\mathbb{P}^* (B) = \inf \{\mathbb{P}(A) : B \subset A, A \in \mathcal{A}\}$$

Recap: Weak convergence of random variables

Definition (1.3.3)

Let $(\Omega_\alpha, \mathcal{A}_\alpha P_\alpha)$ be a net of probability spaces and let $X_\alpha : \Omega_\alpha \to \mathbb{D}$ be arbitrary maps. The net X_α **converges weakly** *to a borel measure L if*

$$\mathbb{E}^* f(X_\alpha) \to \int f \, dL, \qquad \forall f \in C_b(\mathbb{D}).$$

We denote this convergence by $X_\alpha \rightsquigarrow L$.

Functional analysis perspective

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Understanding weak convergence of stochastic processes

Lemma (Lemma 1.5.2)

Let $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ be asymptotically tight.

Understanding weak convergence of stochastic processes

Lemma (Lemma 1.5.2)

Let $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ be asymptotically tight. Then it is asymptotically measurable if and only if $X_\alpha(t)$ is asymptotically measurable for every $t \in T$.

Testing equality for stochastic processes

Lemma (Lemma 1.5.3)

Let X and Y be tight Borel measurable maps into $\ell^\infty(T)$.

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Let X and Y be tight Borel measurable maps into $\ell^\infty(T)$. Then X and Y are equal in Borel law if and only if all corresponding marginals of X and Y are equal in law.

A Kolmogorov extension theorem for limits

Theorem (Theorem 1.5.4)

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Let $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ be arbitrary. Then X_α converges weakly to a tight limit if and only if X_α is asymptotically tight and the marginals $(X_\alpha(t_1), \cdots, X_\alpha(t_k))$ converge weakly to a limit for every finite subset $t_1, \cdots, t_k$ of T.

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Strategies for showing weak convergence

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The previous theorem gives a strategy for showing weak convergence. We need to show the weak convergence of the finite dimensional marginals and the asymptotic tightness. Since the former is easily done with the classical theory we only need to start to understand the latter part.

uniformly ρ-equicontinuous in probability

Definition (Definition Page 37)

A net $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ *is* **asymptotically uniformly ρ-equicontinuous** *if for every* $\epsilon, \nu > 0$ *there exists a* $\delta > 0$ *such that*

$$\limsup_{\alpha} \mathbb{P}^* \left(\sup_{\rho(s,t)<\delta} |X_\alpha(s) - X_\alpha(t)| > \epsilon \right) < \nu.$$

Understanding asymptotic tightness

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A net $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ is asymptotically tight if and only if $X_\alpha(t)$ is asymptotically tight for all $t \in T$ and

Understanding asymptotic tightness

Theorem (Theorem 1.5.7)

A net $X_\alpha : \Omega_\alpha \to \ell^\infty(T)$ is asymptotically tight if and only if $X_\alpha(t)$ is asymptotically tight for all $t \in T$ and there exists a semimetric ρ on T such that (T, ρ) is totally bounded and X_α is asymptotically uniformly ρ-equicontinuous.

addendum to the theorem

Theorem (Addendum 1.5.8)

If, moreover, $X_\alpha \rightsquigarrow X$, then almost all paths $t \mapsto X(t,\omega)$ are uniformly ρ continuous; and the semimetric ρ can without loss of generality be taken equal to any semimetric ρ for which this is true and (T,ρ) is totally bounded.

Which semimetric?

Lemma (Problem 1.5.2)

Let (T, ρ) be a totally bounded semimetric space and X a map into $\ell^\infty(T)$ with uniformly ρ-continuous paths. Then T is totally bounded for the semimetric $\rho_0(s,t) = \mathbb{E}\arctan|X(s) - X(t)|$. Furthermore, almost all paths of X are uniformly continuous with respect to ρ_0.

Which semimetric: an alternative semimetric

Definition

$\rho_p(s,t) = (\mathbb{E}|X(s) - X(t)|^p)^{1/(p \vee 1)}$

Alternative semimetrics

Lemma (Lemma 1.5.9)

Let X be a tight, Borel measurable map into $\ell^\infty(T)$. Then there is a semimetric on T which which almost all paths of X are uniformly continuous and T is totally bounded. Moreover, for any fixed $p > 0$, the following statements are equivalent:

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- *for the semimetric ρ_p, the set T is totally bounded and almost all paths of X are uniformly-ρ_p continuous;*
- *for every semimetric ρ for which almost all paths of X are uniformly ρ continuous, the map $t \mapsto X(T)$ is uniformly ρ-continuous in pTh mean;*

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- *for the semimetric ρ_p, the set T is totally bounded and almost all paths of X are uniformly-ρ_p continuous;*
- *for every semimetric ρ for which almost all paths of X are uniformly ρ continuous, the map $t \mapsto X(T)$ is uniformly ρ-continuous in pTh mean;*
- *there is a semimetric ρ for which T is totally bounded, the map $t \mapsto X(t)$ is uniformly ρcontinuous in pTh mean, and almost all paths of X are uniformly ρ continuous.*

Alternative semimetrics

Corollary (lemma 1.5.9)

In particular, if T is compact for a semimetric ρ such that the map $t \mapsto \mathbb{E}|X(t)|^p$ and almost all sample maths are ρ-continuous, then the first statement of lemma 1.5.9 holds.

Application to Gaussian processes

Lemma

Let X be a Gaussian process with "intrinsic" semimetrics ρ_p, and let X_α be a net of random elements with values in $\ell^\infty(T)$. Then there exists a version of X which is a tight Borel measurable map into $\ell^\infty(T)$, and X_α converges weakly to X if and only if

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- *The marginals of X_α converge weakly to the corresponding marginals of X;*

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- *The marginals of X_α converge weakly to the corresponding marginals of X;*
- *X_α is asymptotically equicontinuous in probability with respect to ρ_p;*

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- *The marginals of X_α converge weakly to the corresponding marginals of X;*
- *X_α is asymptotically equicontinuous in probability with respect to ρ_p;*
- *T is totally bounded for ρ_p.*

Hilbert spaces

Hilbert spaces are often natural spaces where objects of interest live. We can use the existence of the inner product on Hilbert space.

asymptotically finite dimensional

Definition (P 49)

Call a net $X_\alpha : \Omega_\alpha \to \mathbb{H}$ **asymptotically finite-dimensional** *if, for all* $\delta, \epsilon > 0$*, there exists a finite subset* $\{E_i : i \in I\}$ *of the orthonormal basis such that*

$$\limsup_\alpha \mathbb{P}^* \left(\sum_{j \notin I} \langle X_\alpha, e_j \rangle^2 \right) < \epsilon.$$

Characterisation of asymptotic tightness

Lemma (Lemma 1.8.1)

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Characterisation of asymptotic measurability

Lemma (Lemma 1.8.2)

Let $X_\alpha : \Omega_\alpha \to \mathbb{H}$ *be asymptotically tight. Then it is asymptotically measurable if and only if* $\langle X_\alpha, e_j \rangle$ *is asymptotically measurable for every* j.

Equality testing for functionals on Hilbert spaces

Lemma (Lemma 1.8.3)

Tight Borel measurable random elements X and Y in $\mathbb{H}$ are equal in distribution if and only if

Equality testing for functionals on Hilbert spaces

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Tight Borel measurable random elements X and Y in $\mathbb{H}$ are equal in distribution if and only if the random variables $\langle X, h\rangle$ and $\langle Y, h\rangle$ are equal in distribution for all $h \in \mathbb{H}$.

Characterisation of convergence

Theorem (Theorem 1.8.4)

A net of random maps $X_\alpha : \Omega_\alpha \to \mathbb{H}$ converges in distribution to a tight Borel measurable random variable X if and only if

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Characterisation of convergence

Theorem (Theorem 1.8.4)

A net of random maps $X_\alpha : \Omega_\alpha \to \mathbb{H}$ converges in distribution to a tight Borel measurable random variable X if and only if it is asymptotically finite-dimensional and the net $\langle X_\alpha, h\rangle$ converges in distribution to $\langle X, h\rangle$ for all $h \in \mathbb{H}$.

Convergence: Almost surely and in probability

Definition (Definition 1.9.1)

Let $X_\alpha, X : \Omega \to \mathbb{D}$ *be arbitrary maps.*

- X_α *converges in outer probability to* X *if* $d(X_\alpha, X)^* \to 0$ *in probability; This means that* $\mathbb{P}(d(X_\alpha, X)^* > \epsilon) = \mathbb{P}^*(d(X_\alpha, X) > \epsilon) \to 0$*, for every* $\epsilon > 0$*, and is denoted by* $X_\alpha \overset{\mathbb{P}^*}{\to} X$.

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- X_α *converges almost uniformly to* X *if, for every* $\epsilon > 0$*, there exists a measurable set* A *with* $P(A) \geq 1 - \epsilon$ *and* $d(X_\alpha, X) \to 0$ *uniformly on* A*; this is denoted* $X_\alpha \xrightarrow{au} X$*.*

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- X_α *converges outer almost surely to* X *if* $d(X_\alpha, X)^* \to 0$ *almost surely for some versions of* $d(X_\alpha, X)^*$*; this is denoted* $X_\alpha \overset{as*}{\to} X$*.*

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- *X_α converges almost uniformly to X if, for every $\epsilon > 0$, there exists a measurable set A with $P(A) \geq 1 - \epsilon$ and $d(X_\alpha, X) \to 0$ uniformly on A; this is denoted $X_\alpha \stackrel{au}{\to} X$.*
- *X_α converges outer almost surely to X if $d(X_\alpha, X)^* \to 0$ almost surely for some versions of $d(X_\alpha, X)^*$; this is denoted $X_\alpha \stackrel{as*}{\to} X$.*
- *X_α converges almost surely to X if $P_*(\lim d(X_\alpha, X) = 0) = 1$; this is denoted $X_\alpha \stackrel{as}{\to} X$.*

Relations between these types of convergence

Lemma (Lemma 1.9.2)

Let X be Borel measurable. Then

- $X_n \stackrel{as*}{\rightarrow} X$ *implies* $X_n \stackrel{\mathbb{P}*}{\rightarrow} X$;

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- $X_n \overset{\mathbb{P}*}{\to} X$ *if and only if every subsequence* $X_{n'}$ *has a further subsequence* $X_{n''}$ *with* $X_{n''} \overset{as*}{\to} X$;

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Lemma (Lemma 1.9.3)

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- $X_\alpha \stackrel{au}{\to} X$ *if and only if* $\sup_{\beta \geq \alpha} d(X_\beta, X)* \stackrel{\mathbb{P}}{\to} 0$ *if and only if*

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- $X_\alpha \stackrel{au}{\to} X$ *implies* $X_\alpha \stackrel{\mathbb{P}*}{X}$.

Continuous mapping theorem

Theorem

Let $g : \mathbb{D} \to \mathbb{E}$ *be continuous at every point of a Borel set* $\mathbb{D}_0 \subset \mathbb{D}$. *Let* X *be Borel measurable with* $\mathbb{P}(X \in \mathbb{D}_0) = 1$. *Then*

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- $X_\alpha \stackrel{\mathbb{P}*}{\to} X$ *implies that* $g(X_\alpha) \stackrel{\mathbb{P}*}{\to} g(X)$

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Refined continuous mapping

Theorem (Theorem 1.11.1)

Let $\mathbb{D}_0, \mathbb{D}_n \subset \mathbb{D}$ and $g_n : \mathbb{D}_n \to \mathbb{E}$, $g : \mathbb{D}_0 \to \mathbb{E}$ satisfy the following statements: if $x_n \to x$ with $x_n \in \mathbb{D}_n$ for every n and $x \in \mathbb{D}_0 \subset \mathbb{D}$ then $g_n(x_n) \to g(x)$. Let X_n be maps in $\mathbb{D}_n$, let X be Borel measurable and separable and take values in $\mathbb{D}_0$. Then

- $X_n \rightsquigarrow X$ *implies* $g(X_n) \rightsquigarrow g(X)$;

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- $X_n \rightsquigarrow X$ *implies* $g(X_n) \rightsquigarrow g(X)$;
- $X_n \stackrel{\mathbb{P}*}{\to} X$ *implies* $g(X_n) \stackrel{\mathbb{P}*}{\to} g(X)$;

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- $X_n \rightsquigarrow X$ *implies* $g(X_n) \rightsquigarrow g(X)$;
- $X_n \stackrel{\mathbb{P}*}{\to} X$ *implies* $g(X_n) \stackrel{\mathbb{P}*}{\to} g(X)$;
- $X_n \stackrel{as*}{\to} X$ *implies* $g(X_n) \stackrel{as*}{\to} g(X)$

Asymptotic uniform integrability

Definition (p 69)

We say a net of real valued maps X_α is **asymptotically uniformly integrable** *if*

$$\lim_{m\to\infty} \limsup_{\alpha} \mathbb{E}^*|X_\alpha|\{|X_\alpha| > M\} = 0.$$

Application of asymptotic uniform integrability

Theorem (1.11.3)

Let $f : \mathbb{D} \to \mathbb{R}$ be continuous at every point in a set $\mathbb{D}_0$. Let $X_\alpha \rightsquigarrow X$, where X takes values in $\mathbb{D}_0$.

Application of asymptotic uniform integrability

Theorem (1.11.3)

Let $f : \mathbb{D} \to \mathbb{R}$ *be continuous at every point in a set* $\mathbb{D}_0$*. Let* $X_\alpha \rightsquigarrow X$*, where* X *takes values in* $\mathbb{D}_0$*.*

- *If* $f(X_\alpha)$ *is asymptotically uniformly integrable, then* $\mathbb{E}^* f(X_\alpha) \to \mathbb{E} f(X)$*.*

Application of asymptotic uniform integrability

Theorem (1.11.3)

Let $f : \mathbb{D} \to \mathbb{R}$ be continuous at every point in a set $\mathbb{D}_0$. Let $X_\alpha \rightsquigarrow X$, where X takes values in $\mathbb{D}_0$.

- *If $f(X_\alpha)$ is asymptotically uniformly integrable, then $\mathbb{E}^* f(X_\alpha) \to \mathbb{E} f(X)$.*
- *If $\limsup_\alpha \mathbb{E}^* |f(X_\alpha)| \leq \mathbb{E}|f(X)| < \infty$, then $\mathbb{E}^* f(X_\alpha) \to \mathbb{E} f(X)$.*

The end

Any questions?