Weak Convergence and Empirical Processes

Chapter 2.2: Maximal Inequalities and Covering Numbers

Bart Eggen

Leiden University

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What are we going to discuss?

1. Introduction
2. Orlicz norm
3. Finite Maximal Inequalities
4. Maximal Inequalities

What are maximal inequalities?

$$\left\| \max_{1 \le i \le m} X_i \right\| \le K \max_{1 \le i \le m} \|X_i\|$$

What are maximal inequalities?

$$\left\| \max_{1 \le i \le m} X_i \right\| \le K \max_{1 \le i \le m} \|X_i\|$$

Can we do the same for an infinite amount of random variables?

Orlicz norm

Definition (Orlicz norm)

Let ψ be a nondecreasing, convex function with $\psi(0) = 0$ and X a random variable. Then the *Orlicz norm* $\|X\|_\psi$ is defined as

$$\|X\|_\psi = \inf\left\{C > 0 : E\psi\left(\frac{|X|}{C}\right) \leq 1\right\}.$$

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Definition (Norm)

Given a vector space V over $\mathbb{R}$, a *norm* on V is a function $\|\cdot\| : V \to \mathbb{R}_+$ with the following properties:
For all $a \in \mathbb{R}$ and all $\mathbf{u}, \mathbf{v} \in V$,

1) $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$.
2) $\|a\mathbf{v}\| = |a| \cdot \|\mathbf{v}\|$.
3) If $\|\mathbf{v}\| = 0$, then $\mathbf{v} = 0$.

Orlicz norm

Definition (Orlicz norm)

Let ψ be a nonzero, nondecreasing, convex function with $\psi(0) = 0$ and X a random variable. Then the *Orlicz norm* $\|X\|_\psi$ is defined as

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Examples

Example

For $p \geq 1$, let $\psi(x) = x^p$. Then for any random variable X

$$\|X\|_\psi = \inf\left\{C > 0 : E\left(\frac{|X|}{C}\right)^p \leq 1\right\}$$

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For $p \geq 1$, let $\psi_p(x) = e^{x^p} - 1$

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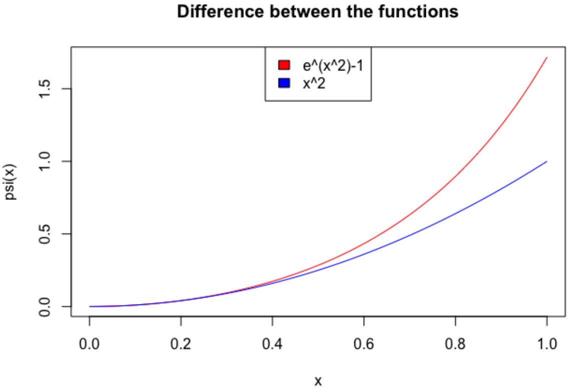
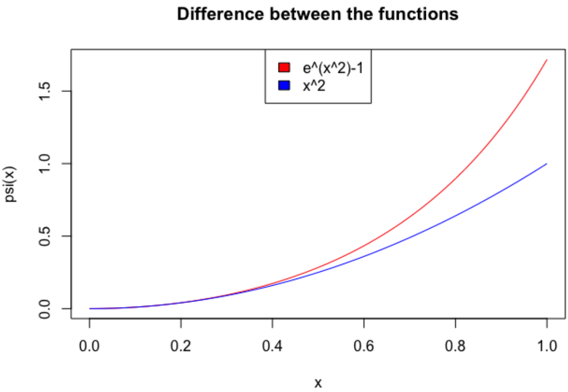

Difference between the functions
e^(x^2)-1
x^2
psi(x)
0.0
0.5
1.0
1.5
0.0
0.2
0.4
0.6
0.8
1.0
x

Examples



Inequality

For all $p \geq 1$ we have

$$\|X\|_p \leq \|X\|_{\psi_p}$$

Tails of distributions

Tail bound

Let X be a random variable and suppose $\|X\|_\psi$ exists. Then for any $x \in \mathbb{R}$:

$$
\begin{aligned}
P(|X| > x) &\leq P\left(\psi(|X|/\|X\|_\psi) \geq \psi(x/\|X\|_\psi)\right) \\
&\leq \frac{E\psi(|X|/\|X\|_\psi)}{\psi(x/\|X\|_\psi)} \\
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\end{aligned}
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Taking ψ_p

$$P(|X| > x) \lesssim e^{-Cx^p}$$

Tails of distributions

Lemma 2.2.1.

Let X be a random variable with $P(|X| > x) \leq Ke^{-Cx^p}$ for every x, for constants K and C, and for $p \geq 1$. Then its Orlicz norm satisfies $\|X\|_{\psi_p} \leq ((1+K)/C)^{1/p}$.

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Proof Idea

Our goal is to find the smallest D' such that

$$E\psi_p\left(\frac{|X|}{D'}\right) = E\left(e^{D|X|^p} - 1\right) \leq 1.$$

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Idea: write

$$e^{D|X|^p} - 1 = \int_0^{|X|^p} De^{Ds} ds$$

and use Fubini's theorem.

Finite maximal inequality

$$\left\| \max_{1 \leq i \leq m} X_i \right\|_{\psi}$$

Finite maximal inequality

$$\begin{aligned}
\left\| \max_{1\le i\le m} X_i \right\|_p &= \left(E \max_{1\le i\le m} |X_i|^p \right)^{1/p} \\
&\le \left(E \sum_{i=1}^{m} |X_i|^p \right)^{1/p} \\
&\le \left(m \cdot \max_{1\le i\le m} E|X_i|^p \right)^{1/p} \\
&= m^{1/p} \cdot \max_{1\le i\le m} \|X_i\|_p
\end{aligned}$$

Finite maximal inequality

$$\left\| \max_{1 \le i \le m} X_i \right\|_p \le m^{1/p} \cdot \max_{1 \le i \le m} \|X_i\|_p$$

Finite maximal inequality

$$\left\| \max_{1\le i\le m} X_i \right\|_p \le \psi^{-1}(m) \cdot \max_{1\le i\le m} \|X_i\|_p$$

Finite maximal inequality

Lemma 2.2.2.

Let ψ be a convex, nondecreasing, nonzero function with $\psi(0) = 0$ and

$$\limsup_{x,y\to\infty} \frac{\psi(x)\psi(y)}{\psi(cxy)} < \infty$$

for some constant c . Then, for any random variables $X_1, \dots, X_m$,

$$\left\| \max_{1\leq i\leq m} X_i \right\|_\psi \leq K\psi^{-1}(m) \max_{1\leq i\leq m} \|X_i\|_\psi,$$

for a constant K depending only on ψ.

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for a constant K depending only on ψ.

Proof Lemma 2.2.2.

$$\left\|\max X_i\right\|_\psi \leq K\psi^{-1}(m) \max \|X_i\|_\psi$$

Assumption

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1) Suppose there exists a constant c such that

$$\psi(x)\psi(y) \leq \psi(cxy) \text{ for all } x, y \geq 1$$

Proof Lemma 2.2.2.

$\|\max X_i\|_\psi \leq K\psi^{-1}(m) \max \|X_i\|_\psi$

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$$\psi(x)\psi(y) \leq \psi(cxy) \text{ for all } x, y \geq 1$$

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$$\psi\left(\frac{\max |X_i|}{Cy}\right) \leq \max \psi\left(\frac{|X_i|}{Cy}\right)$$

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$$\leq \max\left[\frac{\psi(c|X_i|/C)}{\psi(y)} \mathbb{1}\left\{\frac{|X_i|}{Cy} \geq 1\right\} + \psi\left(\frac{|X_i|}{Cy}\right) \mathbb{1}\left\{\frac{|X_i|}{Cy} < 1\right\}\right]$$

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$$C = c \max \|X_i\|_\psi$$

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$$E\left[\psi\left(\frac{\max |X_i|}{cy \max \|X_i\|_\psi}\right)\right] \leq \sum \frac{E\psi(|X_i|/\max \|X_i\|_\psi)}{\psi(y)} + \psi(1)$$

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$\psi(1) \leq \frac{1}{2}$ $\qquad$ $y = \psi^{-1}(2m)$

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$$\|X\|_\psi \leq \frac{\|X\|_\phi}{\sigma\tau} \leq \frac{\|X\|_\psi}{\sigma}$$

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Maximal Inequality

$$\left\| \max_{1 \le i \le m} X_i \right\|_{\psi} \le K \psi^{-1}(m) \max_{1 \le i \le m} \|X_i\|_{\psi}$$

Maximal Inequality

$$\left\| \sup_{t \in T} |X_t| \right\|_{\psi} \leq ?$$

Covering numbers

Definition

Let (T, d) be an arbitrary semi-metric space. Then the covering number $N(\epsilon, d)$ is the minimal number of balls of radius ϵ needed to cover T. Call a collection of points ϵ-*separated* if the distance between each pair of points is strictly larger than ϵ. The packing number $D(\epsilon, d)$ is the maximum number of ϵ-separated points in T.

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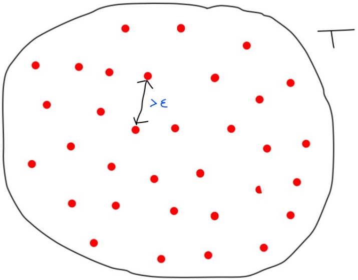


Maximal Inequalities

Theorem

Let ψ be a convex, nondecreasing, nonzero function with $\psi(0) = 0$ and $\limsup_{x,y\to\infty} \psi(x)\psi(y)/\psi(cxy) < \infty$, for some constant c. Let $\{X_t : t \in T\}$ be a separable stochastic process with

$$\|X_s - X_t\|_\psi \leq C\, d(s,t), \qquad \text{for every } s,t,$$

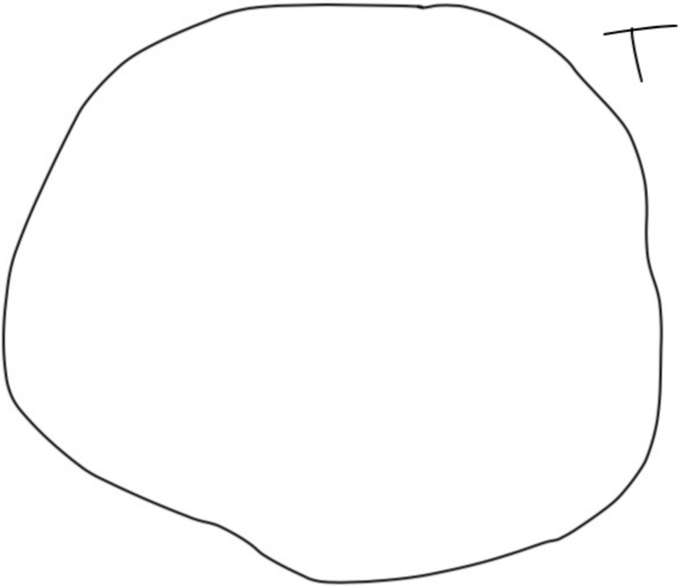
for some semimetric d on T and a constant C. Then, for any $\eta, \delta > 0$,

$$\Big\| \sup_{d(s,t)\leq\delta} |X_s - X_t| \Big\|_\psi \leq K \left[\int_0^\eta \psi^{-1}\left(D(\epsilon, d)\right) d\epsilon + \delta \psi^{-1}\left(D^2(\eta, d)\right)\right],$$

for a constant K depending on ψ and C only.

Separable may be understood in the sense that $\sup_{d(s,t)<\delta} |X_s - X_t|$ remains almost surely the same if the index set T is replaced by a suitable countable subset.

Chaining



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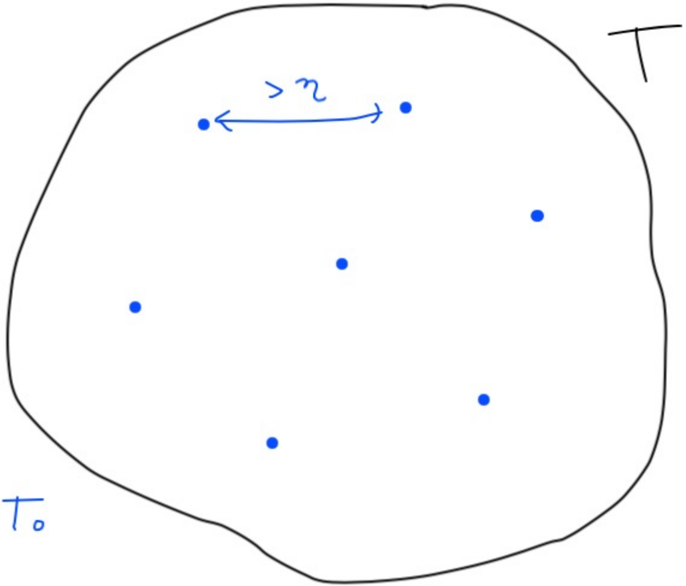
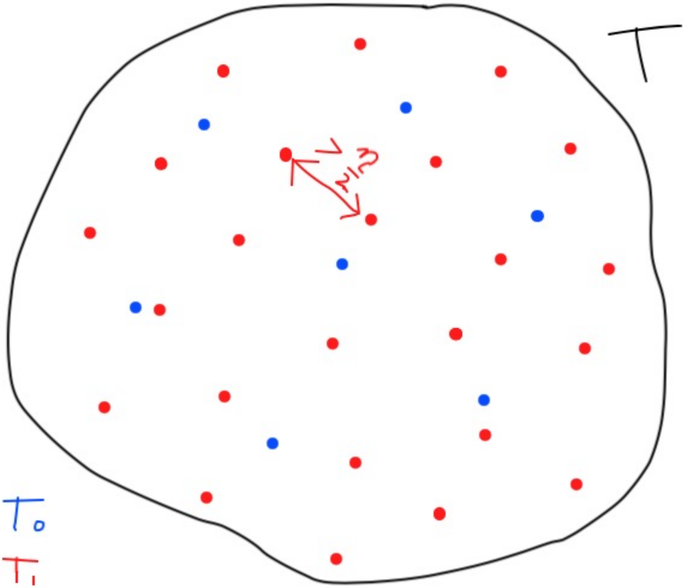
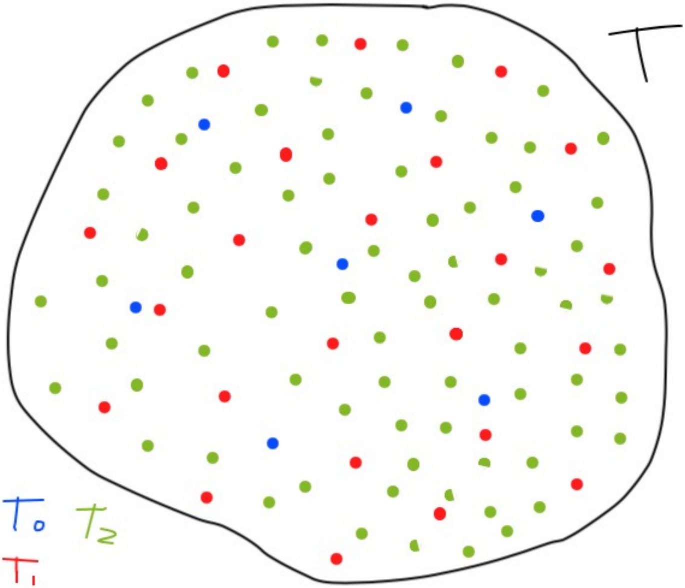


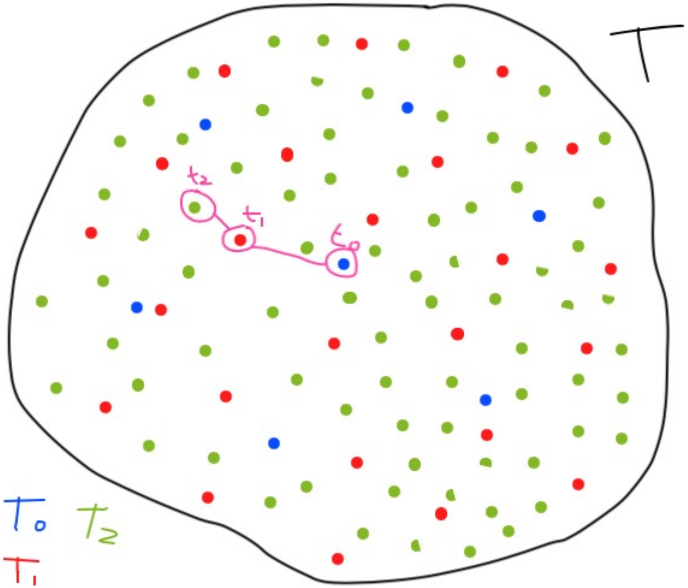
Chaining



Chaining



Chaining



Chaining

- For every $s, t \in T_j$, we have $d(s,t) > \eta 2^{-j}$
- The T_j are maximal, meaning no point can be added without destroying the validity of the inequality
- $|T_j| \leq D(\eta 2^{-j}, d)$
- Every $t_{j+1} \in T_{j+1}$ is linked to a unique $t_j \in T_j$ such that $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

Chaining

For every t_{j+1}, does there exist a t_j with $d(t_j, t_{j+1}) \leq \eta 2^{-j}$?

Chaining

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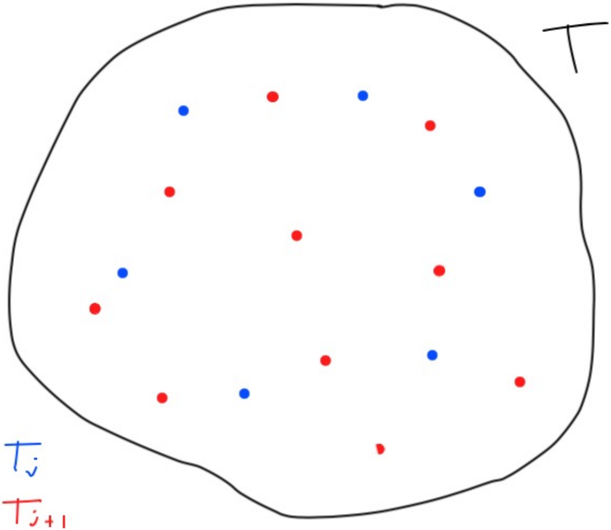
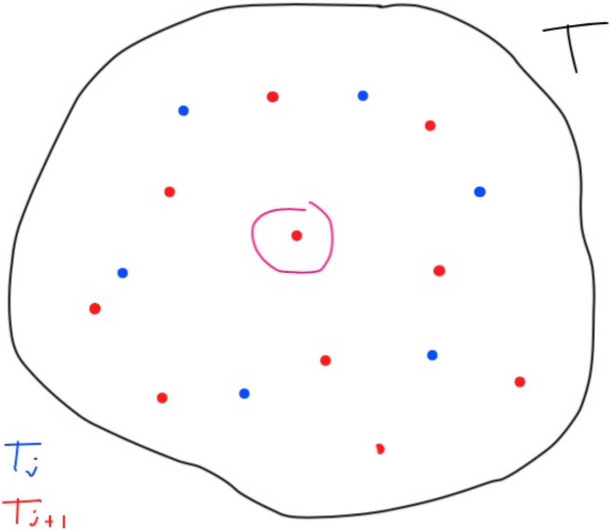


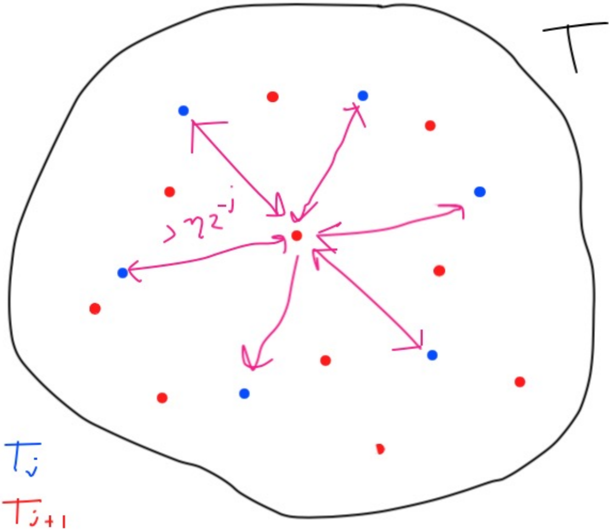
Chaining

For every t_{j+1}, does there exist a t_j with $d(t_j, t_{j+1}) \leq \eta 2^{-j}$?



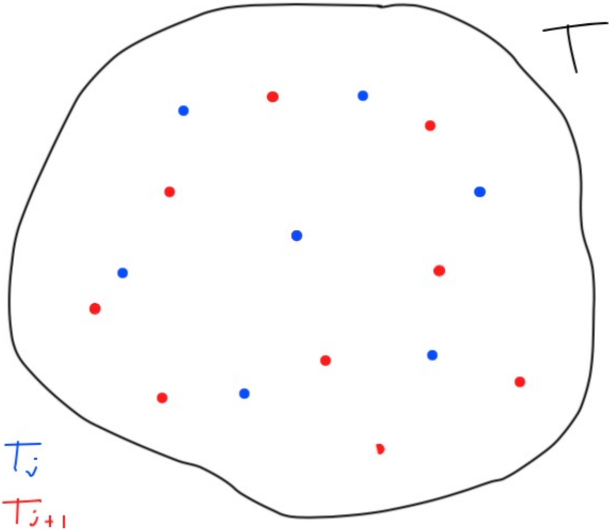
Chaining

For every t_{j+1}, does there exist a t_j with $d(t_j, t_{j+1}) \leq \eta 2^{-j}$?



Chaining

For every t_{j+1}, does there exist a t_j with $d(t_j, t_{j+1}) \leq \eta 2^{-j}$?



Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

Let $s_{k+1}, t_{k+1} \in T_{k+1}$, then we have two chains $t_{k+1}, t_k, \ldots, t_0$ and $s_{k+1}, s_k, \ldots, s_0$.

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

Let $s_{k+1}, t_{k+1} \in T_{k+1}$, then we have two chains $t_{k+1}, t_k, \ldots, t_0$ and $s_{k+1}, s_k, \ldots, s_0$.

$$\begin{aligned}
|(X_{s_{k+1}} - X_{s_0}) - (X_{t_{k+1}} - X_{t_0})| &= \left| \sum_{j=0}^{k} (X_{s_{j+1}} - X_{s_j}) - \sum_{j=0}^{k} (X_{t_{j+1}} - X_{t_j}) \right| \\
&\leq \sum_{j=0}^{k} |X_{s_{j+1}} - X_{s_j}| + |X_{t_{j+1}} - X_{t_j}| \\
&\leq 2 \sum_{j=0}^{k} \max |X_u - X_v|,
\end{aligned}$$

where for every j the maximum is taken over all links (u, v) from T_{j+1} to T_j. At most $|T_{j+1}| \leq D(\eta 2^{-j-1}, d)$ links!

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$|(X_{s_{k+1}} - X_{s_0}) - (X_{t_{k+1}} - X_{t_0})| \leq 2 \sum_{j=0}^{k} \max |X_u - X_v|$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \leq 2 \sum_{j=0}^{k} \max |X_u - X_v|$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_{\psi} \leq 2 \sum_{j=0}^{k} \| \max |X_u - X_v| \|_{\psi}$$

Chaining

- $|T_j| \le D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \le \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \le 2 \sum_{j=0}^{k} \|\max |X_u - X_v|\|_\psi$$

$$\left\| \max_{1 \le i \le m} X_i \right\|_\psi \le K \psi^{-1}(m) \max_{1 \le i \le m} \|X_i\|_\psi$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq$$

$$2 \sum_{j=0}^{k} K \psi^{-1} \left(D(\eta 2^{-j-1}, d) \right) \max \|X_u - X_v\|_\psi$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq$$

$$2 \sum_{j=0}^{k} K \psi^{-1} \left(D(\eta 2^{-j-1}, d) \right) \max \|X_u - X_v\|_\psi$$

$$\|X_s - X_t\|_\psi \leq C\, d(s,t), \text{ for every } s, t$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq$$

$$2KC \sum_{j=0}^{k} \psi^{-1} \Big(D(\eta 2^{-j-1}, d) \Big) \max d(u, v)$$

Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
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$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq$$

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Chaining

- $|T_j| \leq D(\eta 2^{-j}, d)$
- $d(t_j, t_{j+1}) \leq \eta 2^{-j}$

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq K \sum_{j=0}^{k} \psi^{-1} \left(D(\eta 2^{-j-1}, d) \right) \eta 2^{-j}$$

Chaining

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq 2K \sum_{j=0}^{k} \psi^{-1}\left(D(\eta 2^{-j-1}, d) \right) \eta 2^{-j-1}$$

Chaining

$$\left\| \max_{s,t \in T_{k+1}} |(X_s - X_{s_0}) - (X_t - X_{t_0})| \right\|_\psi \leq 2K \sum_{j=0}^{k} \psi^{-1}\left(D(\eta 2^{-j-1}, d)\right) \eta 2^{-j-1}$$
$$\leq 2K \int_0^{k+1} \psi^{-1}\left(D(\eta 2^{-j-1}, d)\right) \eta 2^{-j-}$$

Chaining

$$
\begin{aligned}
\left\| \max_{s,t \in T_{k+1}} \left| (X_s - X_{s_0}) - (X_t - X_{t_0}) \right| \right\|_{\psi} &\leq 2K \sum_{j=0}^{k} \psi^{-1}\left(D(\eta 2^{-j-1}, d) \right) \eta 2^{-j-1} \\
&\leq 2K \int_0^{\infty} \psi^{-1}\left(D(\eta 2^{-j-1}, d) \right) \eta 2^{-j-1} \\
&\leq \frac{2}{\log 2} K \int_0^{\frac{\eta}{2}} \psi^{-1}\left(D(\epsilon, d) \right) d\epsilon
\end{aligned}
$$

Chaining

$$\left\| \max_{s,t \in T_{k+1}} \left| (X_s - X_{s_0}) - (X_t - X_{t_0}) \right| \right\|_\psi \leq 2K \sum_{j=0}^{k} \psi^{-1} \Big(D(\eta 2^{-j-1}, d) \Big) \eta 2^{-j-1}$$

$$\leq 2K \int_0^\infty \psi^{-1} \Big(D(\eta 2^{-j-1}, d) \Big) \eta 2^{-j-1}$$

$$\leq 4K \int_0^\eta \psi^{-1} \left(D(\epsilon, d) \right) d\epsilon$$

Chaining

We want to bound

$$\left\| \sup_{d(s,t)\leq\delta} |X_s - X_t| \right\|_\psi$$

Chaining

By $\{X_t : t \in T\}$ being separable, we can bring it to

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi ,$$

letting $k \to \infty$.

Chaining

By $\{X_t : t \in T\}$ being separable, we can bring it to

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi ,$$

letting $k \to \infty$. Note that we have

$$\begin{aligned} |X_s - X_t| &= |(X_s - X_{s_0}) - (X_t - X_{t_0}) + X_{s_0} - X_{t_0}| \\ &\leq |(X_s - X_{s_0}) - (X_t - X_{t_0})| + |X_{s_0} - X_{t_0}| \end{aligned}$$

Chaining

Two terms to bound

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} \left|(X_s - X_{s_0}) - (X_t - X_{t_0})\right| \right\|_{\psi}, \qquad \left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} \left|X_{s_0} - X_{t_0}\right| \right\|_{\psi}$$

Chaining

Two terms to bound

$$4K\int_0^{\eta}\psi^{-1}(D(\epsilon,d))d\epsilon \qquad \left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_{s_0}-X_{t_0}| \right\|_{\psi}$$

Chaining

Two terms to bound

$$4K\int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon \qquad\qquad \left\| \max_{\substack{s,t,\in T_{k+1}\\ d(s,t)\le\delta}} |X_{s_0} - X_{t_0}| \right\|_\psi$$

Circular argument: For every pair of endpoints s_0, t_0 of chains starting at two points $s, t \in T_{k+1}$ with $d(s,t) \le \delta$, choose exactly one pair $s_{k+1}, t_{k+1} \in T_{k+1}$ with $d(s_{k+1}, t_{k+1}) \le \delta$, whose chains end at s_0, t_0. At most $D(\eta, d)^2$ pairs!

Chaining

Two terms to bound

$$4K\int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon \qquad \left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_{s_0} - X_{t_0}| \right\|_\psi$$

Circular argument: For every pair of endpoints s_0, t_0 of chains starting at two points $s, t \in T_{k+1}$ with $d(s,t) \leq \delta$, choose exactly one pair $s_{k+1}, t_{k+1} \in T_{k+1}$ with $d(s_{k+1}, t_{k+1}) \leq \delta$, whose chains end at s_0, t_0. At most $D(\eta, d)^2$ pairs! Then we have

$$|X_{s_0} - X_{t_0}| \leq |(X_{s_{k+1}} - X_{s_0}) - (X_{t_{k+1}} - X_{t_0})| + |X_{s_{k+1}} - X_{t_{k+1}}|$$

Chaining

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi \leq 8K \int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon + \left\| \max |X_{s_{k+1}} - X_{t_{k+1}}| \right\|_\psi ,$$

where the maximum is taken over the pairs $s_{k+1}, t_{k+1} \in T_{k+1}$ uniquely attached to the pairs s_0, t_0 as before (at most $D(\eta, d)^2$).

Chaining

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi \leq 8K \int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon + \left\| \max |X_{s_{k+1}} - X_{t_{k+1}}| \right\|_\psi ,$$

where the maximum is taken over the pairs $s_{k+1}, t_{k+1} \in T_{k+1}$ uniquely attached to the pairs s_0, t_0 as before (at most $D(\eta, d)^2$).

Lemma 2.2.2

$$\left\| \max_{1\leq i\leq m} X_i \right\| \leq K\psi^{-1}(m) \max_i \|X_i\|_\psi$$

Chaining

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi \leq 8K \int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon + K\psi^{-1}(D(\eta, d)^2)$$

$$\cdot \max \|X_{s_{k+1}} - X_{t_{k+1}}\|_\psi$$

Chaining

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi \leq 8K \int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon + K\psi^{-1}(D(\eta, d)^2) \cdot C\delta$$

Chaining

$$\left\| \max_{\substack{s,t,\in T_{k+1} \\ d(s,t)\leq\delta}} |X_s - X_t| \right\|_\psi \leq K \left[\int_0^\eta \psi^{-1}(D(\epsilon, d))d\epsilon + \delta\psi^{-1}(D(\eta, d)^2) \right].$$

Let $k \to \infty$ to finish the proof.

General bound

Corollary 2.2.5

The constant K can be chosen such that

$$\left\| \sup_{s,t} |X_s - X_t| \right\|_{\psi} \leq K \int_0^{\operatorname{diam} T} \psi^{-1}(D(\epsilon, d))d\epsilon$$

General bound

Corollary 2.2.5

The constant K can be chosen such that

$$\left\| \sup_{s,t} |X_s - X_t| \right\|_\psi \leq K \int_0^{\operatorname{diam} T} \psi^{-1}(D(\epsilon, d))\,d\epsilon$$

Proof

$$\left\| \sup_{d(s,t)\leq\delta} |X_s - X_t| \right\|_\psi \leq K \left[\int_0^{\eta} \psi^{-1}(D(\epsilon, d))\,d\epsilon + \delta\psi^{-1}(D(\eta, d)^2) \right].$$

Take $\eta = \operatorname{diam} T$.

Corollary of the corrolary

$$\left\| \sup_t |X_t| \right\|_\psi \leq \|X_{t_0}\|_\psi + K \int_0^{\text{diam } T} \psi^{-1}(D(\epsilon, d))d\epsilon$$

Assumptions

Assummption Theorem 2.2.4

- Standard assumptions on ψ
- $\{X_t : t \in T\}$ separable
- $\|X_s - X_t\|_\psi \leq Cd(s, t)$, for every s, t