Reading group mathematical foundations of statistics

Chapter 2.10: Permanence of the Glivenko-Cantelli and Donsker Properties

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May 18, 2020

Recap

Last week I talked about chapter 2.9, the (un)conditional multiplier central limit theorems, i.e., under some regularity conditions the following two are "equivalent"

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- $\mathcal{F}$ is Donsker
- $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(\delta_{X_i} - P)$ converges weakly to a Gaussian (un)conditionally on $X_1, X_2, \cdots$

Difference between version 1 and 2 of the book

This is one of the chapters that got an extension, notably a lot more results on the permanence of the Glivenko-Cantelli properties has been added.

Overview

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Notation

Given a class $\mathcal{F}$ of measurable functions, let $\bar{\mathcal{F}}$ denote the set of all $f : \mathcal{X} \to \mathbb{R}$ for which there exists a sequence f_m in $\mathcal{F}$, with $f_m \to f$ both pointwise and in $L_2(P)$.

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Let $\mathrm{sconv}\mathcal{F}$ denote the set of convex combinations $\sum_{i=1}^{\infty} \lambda_i f_i$ of functions f_i in $\mathcal{F}$, where $\sum_i |\lambda_i| \leq 1$ and the series converges both pointwise and in $L_2(P)$.

Closures and Convex Hulls of Donsker classes are Donsker

Theorem (2.10.3)

If $\mathcal{F}$ is Donsker and $\mathcal{G} \subset \mathcal{F}$, then $\mathcal{G}$ is Donsker.

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Continuous transformations of classes

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$$\phi \circ (\mathcal{F}_1, \cdots, \mathcal{F}_k) = \{x \mapsto \phi(f_1(x), \cdots, f_k(x)) | f_i \in \mathcal{F}_i, i = 1, \cdots, k\}.$$

Continuous transformations of Glivenko-Cantelli Classes

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Let $\mathcal{F}_i$ be P-Glivenko-Cantelli classes with integrable envelopes. If $\phi : \mathbb{R}^k \to \mathbb{R}$ is continuous, then $\phi \circ (\mathcal{F}_1, \cdots, \mathcal{F}_k)$ is P-Glivenko-Cantelli provided that it has an integrable envelope function.

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We omit the proof.

Lipschitz Condition

We say ϕ satisfies the Lipschitz condition if

$$|\phi \circ f(x) - \phi \circ g(x)| \leq \sum_{l=1}^{k} (f_l(x) - g_l(x))^2.$$

Lipschitz transformations of Donsker Classes

Theorem (20.10.8)

Let $\mathcal{F}_1, \cdots, \mathcal{F}_k$ be Donsker classes with $\|P\|_{F_i} < \infty$ for each i. Let $\phi : \mathbb{R}^k \to \mathbb{R}$ satisfy the Lipschitz condition. Then the class $\phi \circ (\mathcal{F}_1, \cdots, \mathcal{F}_k)$ is Donsker, provided that there exists $(f_1, \cdots, f_k) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ such that $\phi \circ (f_1, \cdots, f_k)$ is square integrable.

Consequences

Example (2.10.9)

If $\mathcal{F}$ and $\mathcal{G}$ are Donsker classes and $\|P\|_{\mathcal{F}\cup\mathcal{G}} < \infty$, then the pairwise infima $\mathcal{F} \wedge \mathcal{G}$, the pairwise suprema $\mathcal{F} \vee \mathcal{G}$, and pairwise sums $\mathcal{F} + \mathcal{G}$ are Donsker classes.

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Example (2.10.9)

Under the same assumptions as the previous example it follows that the union of $\mathcal{F}$ and $\mathcal{G}$.

Consequences part 2

Example (2.10.10)

Let $\mathcal{F}, \mathcal{G}$ be uniformly bounded Donsker classes, then the pairwise products $\mathcal{F} \cdot \mathcal{G}$ form a Donsker class.

Consequences part 3

Example (2.10.12)

If $\mathcal{F}$ is a Donsker class with $\|P\|_{\mathcal{F}} < \infty$ and g a function so that $|g|_\infty < \infty$, then $\mathcal{F} \cdot g$ is Donsker.

Consequences part 4

Example (2.10.11)

If $\mathcal{F}$ is Donsker with $\|P\|_{\mathcal{F}} < \infty$ and $f \geq \delta$ for some constant $\delta > 0$ for every $f \in \mathcal{F}$, then $1/\mathcal{F} = \{\frac{1}{f} : f \in \mathcal{F}\}$ is Donsker as well.

Consequences part 5

Example (2.10.13)

Suppose $\mathcal{F}$ is Donsker with integrable envelope function F, then the class

$$\mathcal{F}_{\leq M} = \{f\mathbb{1}_{F\leq M} | f \in \mathcal{F}\}$$

Is Donsker.

Consequences part 5

Example (2.10.13)

Suppose $\mathcal{F}$ is Donsker with integrable envelope function F, then the class

$$\mathcal{F}_{\leq M} = \{f\mathbb{1}_{F\leq M} | f \in \mathcal{F}\}$$

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$$\mathcal{F}_{\geq M} = \{f\mathbb{1}_{F\geq M} | f \in \mathcal{F}\}$$

is Donsker as well.

Consequences part 6

Example

Let A be a measurable set, then $\mathcal{F}_A = \{f\mathbb{1}_A | f \in \mathcal{F}\}$ is Donsker if $\mathcal{F}$ is Donkser.

Extending the theorem

Theorem (2.10.15)

Let $L_{\alpha,i}$ be measurable functions $1 \leq i \leq k$. Denote $L_{\alpha,i}\mathcal{F}$ the class of functions $\{L_{\alpha,i}f | f \in \mathcal{F}\}$. Suppose that

- *(Lipschitz condition) for every $f, g \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ we have*

$$|\phi \circ f(x) - \phi \circ g(x)|^2 \leq \sum_{l=1}^{k} L_{\alpha,l}^2(x)(f_l(x) - g_l(x))^2.$$

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- *(Lipschitz condition) for every $f, g \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ we have*

$$|\phi \circ f(x) - \phi \circ g(x)|^2 \leq \sum_{l=1}^{k} L^2_{\alpha,l}(x)(f_l(x) - g_l(x))^2.$$

- *For every i the class $L_{\alpha,i}\mathcal{F}$ is Donsker and $\|P\|_{L_{\alpha,i}\mathcal{F}} < \infty$.*

Then the class $\phi \circ (\mathcal{F}_1, \cdots, \mathcal{F}_k)$ is Donsker provided that there exists $f_i \in \mathcal{F}_i$ so that $\phi(f_1, \cdots, f_k)$ is square integrable.

Donsker implies square is Glivenko-Cantelli

Lemma (2.10.16)

Let $\mathcal{F}$ be a Donsker class with $\|P\|_{\mathcal{F}} < \infty$. Then the class $\mathcal{F}^2 = \{f^2 | f \in \mathcal{F}\}$ is Glivenko-Cantelli in probability:
$\|\mathbb{P}_n - P\|^*_{\mathcal{F}^2} \xrightarrow{P} 0$.

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Lemma (2.10.16)

Let $\mathcal{F}$ be a Donsker class with $\|P\|_{\mathcal{F}} < \infty$. Then the class $\mathcal{F}^2 = \{f^2 | f \in \mathcal{F}\}$ is Glivenko-Cantelli in probability: $\|\mathbb{P}_n - P\|^*_{\mathcal{F}^2} \xrightarrow{P} 0$. If inaddition $P^* F^2 < \infty$ for some envelope function F, then $\mathcal{F}^2$ is also Glivenko-Cantelli in mean and almost surely.

A few words on the proofs

The proofs are based on Gaussianification, in essence, making use of the multiplier central limit theorem and use as multipliers standard Gaussian random variables. Then a whole bunch of analysis for this "Gaussian" case allows you to derive bounds and get the results we want.

Permanence of the Uniform entropy bounds

Suppose that, in the same notation as before, the map ϕ is Lipschitz of orders $\alpha_1, \cdots, \alpha_k \in (0, 1]$, i.e.

$$|\phi \circ (f_1, \cdots, f_k)(x) - \phi \circ (g_1, \cdots, g_k)(x)|^2 \leq \sum_{i=1}^{k} L^2_{\alpha,i}(x) |f_i - g_i|^{2\alpha_i}(x).$$

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If we also suppose that the Donkser class $\mathcal{F}_i$ have measurable envelopes F_i, and we define

$$L_\alpha \cdot \mathcal{F}^\alpha = (\sum_{i=1}^{k} L_{\alpha,i}^2 F_i^{2\alpha_i})^{1/2}.$$

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If we also suppose that the Donkser class $\mathcal{F}_i$ have measurable envelopes F_i, and we define

$$L_\alpha \cdot \mathcal{F}^\alpha = (\sum_{i=1}^{k} L_{\alpha,i}^2 F_i^{2\alpha_i})^{1/2}.$$

Then $L_\alpha \cdot \mathcal{F}^\alpha$ is an envelope function.

Continuation

Theorem (2.10.22)

In the notation of the previous slide, for all $\delta > 0$*, we have*

Continuation

Theorem (2.10.22)

In the notation of the previous slide, for all $\delta > 0$, we have

$$\int_0^\delta \sqrt{\log N\left(\epsilon \|L_\alpha \cdot F^\alpha\|_{Q,2}, \phi(\mathcal{F}), L_2(Q)\right)}\, d\epsilon$$
$$\leq \sum_{i=1}^k \int_0^{\delta^{1/\alpha_i}} \sup_Q \sqrt{N\left(\epsilon \|F_i\|_{Q,2\alpha_i}, \mathcal{F}_i, L_{2\alpha_i}(Q)\right)} \frac{d\epsilon}{\epsilon^{1-\alpha_i}}$$

Continuation

Theorem (2.10.22)

In the notation of the previous slide, for all $\delta > 0$, we have

$$\int_0^{\delta} \sqrt{\log N\left(\epsilon \| L_\alpha \cdot F^\alpha \|_{Q,2}, \phi(\mathcal{F}), L_2(Q)\right)}\, d\epsilon$$

$$\leq \sum_{i=1}^{k} \int_0^{\delta^{1/\alpha_i}} \sup_Q \sqrt{N\left(\epsilon \| F_i \|_{Q,2\alpha_i}, \mathcal{F}_i, L_{2\alpha_i}(Q)\right)} \frac{d\epsilon}{\epsilon^{1-\alpha_i}}$$

In particular, if the right hand side is finite and $P^*(L_\alpha \cdot \mathcal{F}^\alpha)^2 < \infty$, then $\phi(\mathcal{F})$ is Donsker.

Partitions of the sample space

We have seen that if $\mathcal{F}$ is Donsker, then the restrictions of the functions to any set are again Donsker. Can we turn this around?

The Glivenko-Cantelli case

Theorem (2.10.26)

If the restriction $\mathcal{F}_j$ of $\mathcal{F}$ to $\mathcal{X}_j$ is Glivenko-Cantelli for every j and a partition $\mathcal{X}_j, j = 1, \cdots, \infty$ and $\mathcal{F}$ has an integrable envelope, then $\mathcal{F}$ is Glivenko-Cantelli.

The Donsker case

Theorem (2.10.27)

For each j, let the classes $\mathcal{F}_j$ of $\mathcal{F}$ to $\mathcal{X}_j$ be Donsker for a partition $\mathcal{X}_j, j = 1, \cdots, \infty$ and satisfy

$$\mathbb{E}[G_n]_{\mathcal{F}_j} < Cc_j$$

for a constant C not depending on n and j. If $\sum_{j=1}^{\infty} c_j < \infty$ and $P^*F < \infty$ for some envelope function $\mathcal{F}$, then the class $\mathcal{F}$ is Donsker.

Smooth functions

Example (2.10.29)

Let $\mathbb{R}^d = \cup_{j=1}^{\infty} \mathcal{X}_j$ be a partition of $\mathbb{R}^d$ into uniformly bounded, convex sets with nonempty interior. Consider the class $\mathcal{F}$ of functions such that the class $\mathcal{F}_j$ of restrictions is contained in $C_{M_j}^{\alpha}$ for each j, for given constants M_j.

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Suppose that $\alpha > d/2$ and $\sum_{j=1}^{\infty} m_j P(\mathcal{X}_j)^{1/2} < \infty$, then the class $\mathcal{F}$ is P Donsker.

Monotone functions

Example (2.10.30)

Consider the class $\mathcal{F}$ of all nondecreasing functions $f : \mathbb{R} \to RR$, such that $0 \leq f \leq F$, for some nondecreasing function F. This class is Donsker provided that $\|F\|_{2,1} < \infty$.

Questions and Discussion

- Any questions so far,
- Plan next shedule for the talks.