Chapter 2.12: Partial-Sum Processes
Chapter 2.13: Other Donsker Classes

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Reading group *Weak Convergence and Empirical Processes*

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Partial-Sum Processes

Donsker considered the partial-sum process:

$$\mathbb{Z}_n(s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} Y_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Y_i, \qquad \frac{k}{n} \leq s < \frac{k+1}{n}.$$

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If $\mathbb{E}\, Y_i = 0$ and $\operatorname{var} Y_i = 1$: $\mathbb{Z}_n$ converges to Brownian motion.

Partial-Sum Processes

Consider

$$\mathbb{Z}_n(s,f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor ns \rfloor} (f(X_i) - Pf) = \sqrt{\frac{\lfloor ns \rfloor}{n}} \mathbb{G}_{\lfloor ns \rfloor}(f).$$

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The covariance function of $\mathbb{Z}_n$ is

$$\operatorname{cov}(\mathbb{Z}_n(s,f), \mathbb{Z}_n(t,g)) = \frac{\lfloor ns \rfloor \wedge \lfloor nt \rfloor}{n}(Pfg - PfPg).$$

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The marginals converge to the *Kiefer-Müller* process $\mathbb{Z}$ with covariance function

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Partial-Sum Processes

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$$\operatorname{cov}(\mathbb{Z}(s,f), \mathbb{Z}(t,g)) = (s \wedge t)(Pfg - PfPg).$$

Definition

$\mathcal{F}$ is functionally Donsker if $\mathbb{Z}_n$ converges to a tight limit in $\ell^\infty([0,1] \times \mathcal{F})$.

Partial-Sum Processes

Consider the (square of) the semimetric

$$\begin{aligned}&\mathbb{E}(\mathbb{Z}(s,f)-\mathbb{Z}(t,g))^2\\&=|s-t|[\rho_P^2(f)1_{s>t}+\rho_P^2(g)1_{s\le t}]+(s\wedge t)\rho_P^2(f-g).\end{aligned}$$

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If $\mathcal{F}$ is bounded for ρ_P, then it is bounded up to a constant by

$$|s-t|+\rho_P(f-g).$$

Partial-Sum Processes

Theorem (2.12.1)

A class of measurable functions is functionally Donsker if and only if it is Donsker.

Partial-Sum Processes

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A class of measurable functions is functionally Donsker if and only if it is Donsker.

Proof.

If functionally Donsker: since $\mathbb{G}_n(f) = \mathbb{Z}_n(1, f)$, and a restriction map is continuous, $\mathcal{F}$ is Donsker.

Partial-Sum Processes

Proof.

We show asymptotic equicontinuity by showing

$$\lim_{\delta \searrow 0} \lim_{n \to \infty} \sup_{|t-s|+\rho_P(f,g)<\delta} \mathbb{P}^*(|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,g)| > \epsilon) = 0.$$

Partial-Sum Processes

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Triangle inequality gives

$$\begin{aligned} |\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,g)| \leq & \sup_{|s-t|<\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,f)\|_{\mathcal{F}} \\ & + \sup_{0 \leq t \leq 1} \|\mathbb{Z}_n(t,f)\|_{\mathcal{F}_\delta}. \end{aligned}$$

Partial-Sum Processes

Proof.

We can write

$$\sup_{0 \le t \le 1} \|\mathbb{Z}_n(t, f)\|_{\mathcal{F}_\delta} = \max_{k \le n} \sqrt{k/n} \|\mathbb{G}_k\|_{\mathcal{F}_\delta}.$$

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Ottaviani's inequality bounds this by

$$\mathbb{P}^*\left(\max_{k\leq n} \sqrt{k/n}\|\mathbb{G}_k\|_{\mathcal{F}_\delta} > 2\epsilon\right) \leq \frac{P^*(\|\mathbb{G}_n\|_{\mathcal{F}_\delta} > \epsilon)}{1-\max_{k\leq n} P^*\left(\sqrt{k/n}\|\mathbb{G}_k\|_{\mathcal{F}_\delta} > \epsilon\right)}.$$

Partial-Sum Processes

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$\mathcal{F}$ is Donsker, so by equicontinuity the numerator goes to zero.

Partial-Sum Processes

Proof.

Fix $n_0 \in \mathbb{N}$. Consider

$$\max_{k \leq n} P^* \left(\sqrt{k/n} \|\mathbb{G}_k\|_{\mathcal{F}_\delta} > \epsilon \right)$$

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For $k \leq n_0$:

$$\sqrt{k} \|\mathbb{G}_k\|_{\mathcal{F}_\delta} = \sup_{h \in \mathcal{F}_\delta} \left| \sum_{i=1}^{k} h(X_i) - Ph(X_i) \right|$$

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For $k > n_0$:

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For $k > n_0$:

$$P^* \left(\sqrt{k/n} \|\mathbb{G}_k\|_{\mathcal{F}_\delta} > \epsilon \right) \leq P^* \left(\|\mathbb{G}_k\|_{\mathcal{F}_\delta} > \epsilon \right) < 1.$$

Partial-Sum Processes

Proof.

$$\mathbb{P}^*\big(\sup_{0\le t\le 1}\|\mathbb{Z}_n(t,f)\|_{\mathcal{F}_\delta} > \epsilon\big) = \mathbb{P}^*\left(\max_{k\le n}\sqrt{k/n}\|\mathbb{G}_k\|_{\mathcal{F}_\delta} > \epsilon\right)$$

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$$\le\frac{P^*(\|\mathbb{G}_n\|_{\mathcal{F}_\delta}>\epsilon)}{1-\max_{k\le n}P^*\left(\sqrt{k/n}\|\mathbb{G}_k\|_{\mathcal{F}_\delta}>\epsilon\right)}$$

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which goes to 0 as $n \to \infty$ and then $\delta \searrow 0$.

Partial-Sum Processes

Proof.

We consider

$$\sup_{|s-t|<\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,f)\|_{\mathcal{F}}$$
$$= \max_{0\le j\delta\le 1} \sup_{j\delta\le s\le (j+1)\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(j\delta,f)\|_{\mathcal{F}}.$$

Partial-Sum Processes

Proof.

We consider

$$\sup_{|s-t|<\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,f)\|_{\mathcal{F}}$$
$$= \max_{0\le j\delta\le 1} \sup_{j\delta\le s\le (j+1)\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(j\delta,f)\|_{\mathcal{F}}.$$

For each $1 \le j \le \lceil \frac{1}{\delta} \rceil$, the supremum is i.i.d., so

$$P^*\left(\sup_{|s-t|<\delta} \|\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,f)\|_{\mathcal{F}} > 2\epsilon\right)$$
$$\le \left\lceil \frac{1}{\delta} \right\rceil P^*\left(\sup_{0\le s\le\delta} \|\mathbb{Z}_n(s,f)\|_{\mathcal{F}} > 2\epsilon\right).$$

Proof.

By Ottaviani's inequality:

$$\left\lceil \frac{1}{\delta} \right\rceil P^* \left(\sup_{0 \le s \le \delta} \|\mathbb{Z}_n(s,f)\|_{\mathcal{F}} > 2\epsilon \right) \le \frac{\lceil 1/\delta \rceil P^* \left(\sqrt{\lfloor n\delta \rfloor / n} \|\mathbb{G}_{\lfloor n\delta \rfloor}\|_{\mathcal{F}} > \epsilon \right)}{1 - \max_{k \le n\delta} P^* \left(\sqrt{k/n} \|\mathbb{G}_k\|_{\mathcal{F}} > \epsilon \right)}.$$

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As $n \to \infty$,

$$\limsup_{n\to\infty} P^* \left(\sqrt{\lfloor n\delta \rfloor / n} \|\mathbb{G}_{\lfloor n\delta \rfloor}\|_{\mathcal{F}} > \epsilon \right) \le P \left(\|\mathbb{G}\|_{\mathcal{F}} \ge \epsilon / \delta^{1/2} \right).$$

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$\|\mathbb{G}\|_{\mathcal{F}}$ has moments of all order, so this converges faster to zero than any power of δ as $\delta \searrow 0$.

Partial-Sum Processes

Proof.

Since the map $\ell^\infty(\mathcal{F}) \to \mathbb{R}$:

$$g \mapsto 1\left(\sup_{f \in \mathcal{F}} |g(f)| > \epsilon\right)$$

is bounded continuous, $P^*(\|\mathbb{G}_n\|_{\mathcal{F}} > \epsilon) \to P(\|\mathbb{G}\|_{\mathcal{F}} > \epsilon) < 1$.

Partial-Sum Processes

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$$1 - \max_{k \leq n\delta} P^* \left(\sqrt{k/n} \|\mathbb{G}_k\|_{\mathcal{F}} > \epsilon \right)$$

is bounded away from 0. Hence for all $\epsilon > 0$

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} P^* \left(\sup_{|s-t| + \rho_P(f,g) < \delta} |\mathbb{Z}_n(s,f) - \mathbb{Z}_n(t,g)| > 2\epsilon \right) = 0.$$

□

Partial-Sum Processes on Lattices

Consider independent random variables $Y_{n1}, \ldots, Y_{nm_n}$, and deterministic probability measures $Q_{n1}, \ldots, Q_{nm_n}$ on a space $(\mathcal{X}, \mathcal{A})$.

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Definition

For a collection $\mathcal{C}$ of measurable subsets of $\mathcal{X}$, define $\mathbb{S}_n$ by

$$\mathbb{S}_n(C) = \sum_{i=1}^{m_n} Y_{ni} Q_{ni}(C)$$

for $C \in \mathcal{C}$.

Partial-Sum Processes on Lattices

Example (2.12.3)

Let $\mathcal{C} = \{[0, s] : 0 \leq s \leq 1\}$, $Q_{ni} = \delta_{i/n}$. Then

$$\mathbb{S}_n([0, s]) = \sum_{i/n \leq s} Y_{ni} = \sum_{i=1}^{k} Y_{ni} \qquad \text{if } \frac{k}{n} \leq s < \frac{k+1}{n}.$$

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We recover the normal partial-sum process by taking $Y_{ni} = Y_i/\sqrt{n}$.

Partial-Sum Processes on Lattices

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Example (2.12.4)

Let $\mathcal{C} = \{[0, s] : 0 \leq s \leq 1\}$, $Q_{ni} = U([i/n, (i+1)/n))$. Then

$$\mathbb{S}_n([0, s]) = \sum_{i=1}^{k} Y_{ni} + \frac{s - k/n}{1/n} Y_{n,k+1} \qquad \text{if } \frac{k}{n} \leq s < \frac{k+1}{n}.$$

Partial-Sum Processes on Lattices

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This is a linear interpolation of the partial-sum process.

Partial-Sum Processes on Lattices

Definition

Z_{ni} is measurelike with respect to a (random) measure μ_{ni} if

$$(Z_{ni}(f) - Z_{ni}(g))^2 \leq \int (f-g)^2 d\mu_{ni}.$$

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Define $Z_{ni} = Y_{ni}\ Q_{ni}$, for which

$$(Z_{ni}(C) - Z_{ni}(D))^2 \leq Y_{ni}^2\ Q_{ni}(C \triangle D).$$

Hence they are measurelike with respect to $\mu_{ni} = Y_{ni}^2 Q_{ni}$.

Partial-Sum Processes on Lattices

Theorem (2.12.6 1/2)

For each n, let $Q_{n1}, \dots, Q_{nm_n}$ be deterministic probability measures on some measurable space, and let $Y_{n1}, \dots, Y_{nm_n}$ be independent, real-valued random variables with mean zero, satisfying

$$\sum_{i=1}^{m_n} EY_{ni}^2 = O(1),$$

$$\sum_{i=1}^{m_n} EY_{ni}^2\{|Y_{ni}| > \eta\} \to 0, \qquad \text{for every } \eta > 0.$$

Partial-Sum Processes on Lattices

Theorem (2.12.6 2/2)

Let $\mathcal{C}$ be a class of measurable sets that satisfies the uniform-entropy condition, and assume that for some probability measure Q,

$$\sup_{Q(C\triangle D)<\delta_n} \sum_{i=1}^{m_n} EY_{ni}^2 (Q_{ni}(C) - Q_{ni}(D))^2 \to 0, \qquad \text{for every } \delta_n \searrow 0.$$

Suppose that the covariance function $E\mathbb{S}_n(C)\mathbb{S}_n(D)$ converges pointwise on $\mathcal{C} \times \mathcal{C}$. Then the sequence $\mathbb{S}_n = \sum_{i=1}^{m_n} Y_{ni} Q_{ni}$ converges weakly in $\ell^\infty(\mathcal{C})$ to a tight Gaussian process with uniformly continuous sample paths with respect to the semimetric $Q(C\triangle D)$.

Proof.

Use Theorem 2.11.1 with $Z_{ni} = Y_{ni} Q_{ni}$ and ρ the $L_1(Q)$-semimetric.

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$$\sum_{i=1}^{m_n} E^* \|Z_{ni}\|_{\mathcal{C}}^2 \{ \|Z_{ni}\|_{\mathcal{C}} > \eta \} \to 0, \qquad \forall \eta > 0;$$

Partial-Sum Processes on Lattices

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Partial-Sum Processes on Lattices

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$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{C}, d_n)}\, d\epsilon \xrightarrow{P_*} 0, \qquad \forall \delta_n \downarrow 0;$$

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$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{C}, d_n)} d\epsilon \xrightarrow{P_*} 0, \qquad \forall \delta_n \downarrow 0;$$

$$(x_1, \ldots, x_{m_n}) \mapsto \sup_{Q(C,D)<\delta} \left| \sum_{i=1}^{m_n} e_i (Z_{ni}(C) - Z_{ni}(D)) \right|$$

Proof.

Use Theorem 2.11.1 with $Z_{ni} = Y_{ni}Q_{ni}$ and ρ the $L_1(Q)$-semimetric.We need to check:

$$\sum_{i=1}^{m_n} E^* \|Z_{ni}\|_{\mathcal{C}}^2 \{\|Z_{ni}\|_{\mathcal{C}} > \eta\} \to 0, \qquad \forall \eta > 0;$$

$$\sup_{Q(C,D)<\delta_n} \sum_{i=1}^{m_n} E(Z_{ni}(C) - Z_{ni}(D))^2 \to 0, \qquad \forall \delta_n \downarrow 0;$$

$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{C}, d_n)}\, d\epsilon \xrightarrow{P_*} 0, \qquad \forall \delta_n \downarrow 0;$$

$$(x_1, \ldots, x_{m_n}) \mapsto \sup_{Q(C,D)<\delta} \left| \sum_{i=1}^{m_n} e_i (Z_{ni}(C) - Z_{ni}(D)) \right|$$

is measurable for all $\delta > 0$, $(e_1, \ldots, e_{m_n}) \in \{-1, 0, 1\}^{m_n}$ and $n \in \mathbb{N}$.

Proof.

$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{F}, d_n)}\, d\epsilon \xrightarrow{P_*} 0, \qquad \forall \delta_n \downarrow 0 \qquad (*)$$

This follows from Lemma 2.11.6:

Proof.

$$\int_0^{\delta_n} \sqrt{\log N(\epsilon, \mathcal{F}, d_n)} d\epsilon \xrightarrow{P_*} 0, \qquad \forall \delta_n \downarrow 0 \qquad (*)$$

This follows from Lemma 2.11.6:

Lemma (2.11.6, Shortened)

If $\mathcal{C}$ satisfies (...) and

$$\int_0^{\infty} \sup_{Q \in \mathcal{Q}} \sqrt{\log N(\epsilon \|F\|_{Q,2}, \mathcal{C}, L_2(Q))} d\epsilon < \infty$$

for $\mathcal{Q}$ that contains μ_{ni}, and $\sum_{i=1}^{m_n} \mu_{ni} F^2 = O_P^*(1)$, then Equation (*) holds.

Partial-Sum Processes on Lattices

Proof.

Measurability of

$$(x_1, \ldots, x_{m_n}) \mapsto \sup_{\rho(f,g)<\delta} \left| \sum_{i=1}^{m_n} e_i(Z_{ni}(f) - Z_{ni}(g)) \right|.$$

Proof.

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$$(x_1, \ldots, x_{m_n}) \mapsto \sup_{\rho(f,g)<\delta} \left| \sum_{i=1}^{m_n} e_i(Z_{ni}(f) - Z_{ni}(g)) \right|.$$

By the uniform entropy condition, $\mathcal{C}$ is totally bounded and thus seperable in $L_1(\sum_{i=1}^{m_n} Q_{ni} + Q)$.

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$$\lim_{k\to\infty} (\sum_{i=1}^{m_n} Q_{ni} + Q)(C_k \Delta C) = 0.$$

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Partial-Sum Processes on Lattices

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Thus $\lim_{k\to\infty} \mathbb{S}_n(C_k) = \mathbb{S}_n(C)$. Thus the supremum can be changed into countably supremum. $\square$

Partial-Sum Processes on Lattices

Example

The covariance function of $\mathbb{S}_n$ is

$$E\mathbb{S}_n(C)\mathbb{S}_n(D) = \sum_{i=1}^{m_n} EY_{ni}^2 Q_{ni}(C)Q_{ni}(D).$$

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If each Q_{ni} is the dirac measure, then

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Thus pointwise convergence of Q_n on $\{C \cap D : C, D \in \mathcal{C}\}$ suffices for convergence of the covariance-function.

Partial-Sum Processes on Lattices

Example

For all measures Q_{ni}, we have

$$\sup_{Q(C\triangle D)<\delta_n} \sum_{i=1}^{m_n} EY_{ni}^2 (Q_{ni}(C) - Q_{ni}(D))^2 \leq \sup_{Q(C\triangle D)<\delta_n} Q_n(C\triangle D).$$

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If $\mathcal{C}$ has this property, then $\mathcal{C} \cap \mathcal{C}$ and $\mathcal{C} \triangle \mathcal{C}$ have this property as well. Assume $C_l \subset C \subset C^u$ and $C_k \subset D \subset C^v$. Then

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So w.l.o.g. we check the uniform convergence on $\mathcal{C}$. □

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Note that

$$(Q_n - Q)(C) \leq (Q_n - Q)(C^u) + Q(C^u - C)$$

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Thus $\sup_{C \in \mathcal{C}} (Q_n - Q)(C) \leq \max_{C^u} (Q_n - Q)(C^u) + \epsilon \to \epsilon$. This holds for all $\epsilon > 0$, so

$$\lim_{n \to \infty} \sup_{C \in \mathcal{C}} |(Q_n - Q)(C)| = 0.$$

Partial-Sum Processes on Lattices

Example

Let $\mathcal{X} = [0,1]^d$ and $\mathcal{C} = \{[0,t] : 0 \leq t \leq 1\}$. Let Q_{ni} be the n^d Dirac measures at the points $\{1/n, 2/n, \ldots, 1\}^d$. For a mean-zero, unit variance sequence Y_1, $Y_2, \ldots$, let $Y_{ni} = Y_i / n^{d/2}$.

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$$Q_n = \sum_{i=1}^{m_n} EY_{ni}^2 Q_{ni}$$

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Q_n converges weakly to the uniform measure on $[0,1]^d$.

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Q_n converges weakly to the uniform measure on $[0, 1]^d$. This satisfies the condition of the theorem, and the limiting Gaussian process has covariance function $E\mathbb{S}(C)\mathbb{S}(D) = \lambda(C \cap D)$.

Other Donsker Classes

Theorem (2.13.1)

Any sequence $\{f_i\}$ of square-integrable, measurable functions with the property $\sum_{i=1}^{\infty} P(f_i - Pf_i)^2 < \infty$ is P-Donsker.

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For fixed m, define the partition $\mathcal{F}_i = \{f_i\}$, $i \leq m$, and $\mathcal{F}_{m+1} = \{f_{m+1}, f_{m+2}, \ldots\}$.

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$$\leq \frac{4}{\epsilon^2} \sum_{i=m+1}^{\infty} P(f_i - Pf_i)^2.$$

This is arbitrarily small for large enough m. □

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Example (Dudley, 1984)

The requirement $\sum_{i=1}^{\infty} P(f_i - Pf_i)^2 < \infty$ is in some sense sharp: if for $a_i > 0$, $\sum_{i=}^{\infty} a_i = \infty$ then there exists a class $\{f_i\}$ on $L^2([0,1], \mathcal{B}, \lambda)$ with $P(f_i - Pf_i)^2 \leq a_i$ that is not Donsker.

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Example

The class

$$\left\{ \sum_{i=1}^{\infty} c_i f_i : \sum |c_i| \leq 1, \text{ and the series converges pointwise} \right\}$$

is Donsker for any sequence f_i with $\sum_{i=1}^{\infty} Pf_i^2 < \infty$.

Other Donsker Classes

Theorem (2.13.2)

Let $\{f_i\}$ be a sequence of measurable functions such that $Pf_if_j = 0$ for every $i \neq j$ and $\sum_{i=1}^{\infty} Pf_i^2 < \infty$. Then the class of all pointwise converging series $\sum_{i=1}^{\infty} c_i f_i$, such that $\sum_{i=1}^{\infty} c_i^2 \leq 1$, is P-Donsker.

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The series $\sum_{i=1}^{\infty} c_i f_i$ converges pointwise and in $L_2(P)$:

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&= \lim_{n\to\infty} \sum_{i=n+1}^{\infty} c_i^2 Pf_i^2 \\
&= 0.
\end{aligned}$$

Other Donsker Classes

Proof.

It suffices to show that $\mathbb{G}_n$ is asymptotically equicontinuous with respect to the $L_2(P)$-seminorm. Let $f = \sum c_i f_i$, $g = \sum d_i f_i$.

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\begin{aligned}
&\left|\mathbb{G}_n(f) - \mathbb{G}_n(g)\right|^2 \\
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&\leq 2\sum_{i=1}^{k}(c_i - d_i)^2 Pf_i^2 \sum_{i=1}^{k}\frac{\mathbb{G}_n^2(f_i)}{Pf_i^2} + 2\sum_{i=k+1}^{\infty}(c_i - d_i)^2 \sum_{i=k+1}^{\infty}\mathbb{G}_n^2(f_i).
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Note that $\mathbb{E}\mathbb{G}_n^2 f_i \leq Pf_i^2$.

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Note that $\mathbb{E}\mathbb{G}_n^2 f_i \leq Pf_i^2$. Now take the supremum over all f, g with $\|f-g\|_{P,2} < \delta$ and then the expectation:

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$$2\sum_{i=1}^{k}(c_i-d_i)^2 Pf_i^2 \sum_{i=1}^{k}\frac{\mathbb{G}_n^2(f_i)}{Pf_i^2} + 2\sum_{i=k+1}^{\infty}(c_i-d_i)^2\sum_{i=k+1}^{\infty}\mathbb{G}_n^2(f_i)$$
$$\le 2\|f-g\|_{P,2}^2\sum_{i=1}^{k}\frac{\mathbb{G}_n^2(f_i)}{Pf_i^2} + 8\sum_{i=k+1}^{\infty}\mathbb{G}_n^2(f_i).$$

Note that $\mathbb{E}\mathbb{G}_n^2 f_i \le Pf_i^2$. Now take the supremum over all f, g with $\|f-g\|_{P,2} < \delta$ and then the expectation:

$$\mathbb{E}\sup_{\|f-g\|_{P,2}<\delta}|\mathbb{G}_n(f)-\mathbb{G}_n(g)|^2 \le 2\delta^2 k + 8\sum_{i=k+1}^{\infty} Pf_i^2.$$

Other Donsker Classes

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For k large and then δ small enough this is arbitrarily small. $\square$

Other Donsker Classes

Example

For $\{\psi_i\}$ be orthonormal in $\mathcal{L}_2(P)$, and $\{b_i\}$ a fixed sequence, define

$$\mathcal{F} = \left\{ \sum_{i=1}^{\infty} c_i \psi_i : \sum \frac{c_i^2}{b_i^2} \leq 1 \text{ and the series converges pointwise} \right\}.$$

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For $\leq$ use Cauchy-Schwarz inequality; for $\geq$ take $c_i = \frac{b_i^2 \mathbb{G}_n \psi_i}{\sqrt{\sum_{j=1}^{\infty} b_j^2 \mathbb{G}_n^2 \psi_j}}$.

Other Donsker Classes

Example (2.13.3)

Write

$$\mathbb{G}_n(t) = \sqrt{n}(\mathbb{P}_n - P)[0,t] = \sqrt{n}\left(\frac{1}{n}\sum_i 1(0 \le X_i \le t) - t\right).$$

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Then the Cramér-von Mises statistic equals the square of the Kolmogorov-Smirnov statistic:

$$\int_0^1 \mathbb{G}_n^2(t)\,dt = \|\mathbb{G}_n\|_{\mathcal{F}}^2.$$

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Example (2.13.3)

Let $f_j(t) = \frac{\sqrt{2}}{\pi j}\cos(\pi j t)$ and $f_j'(t) := -\sqrt{2}\sin(\pi j t)$. Parseval's formula gives

$$\int_0^1 \mathbb{G}_n(t)^2 dt = \sum_{j=1}^{\infty}\left[\int_0^1 \mathbb{G}_n(t) f_j'(t) dt\right]^2$$

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(remember $\|\mathbb{G}_n\|_{\mathcal{F}}^2 = \sum_{i=1}^{\infty} b_i^2 \mathbb{G}_n^2(\psi_i)$)

Classes of sets

Definition

For a collection of sets $\mathcal{C}$ and points $X_1, \ldots, X_n$, $\Delta_n(\mathcal{C}, X_1, \ldots, X_n)$ denotes the number of subsets of $\{X_1, \ldots, X_n\}$ picked out by $\mathcal{C}$.

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Let $K_n(\mathcal{C}, X_1, \ldots, X_n)$ be the cardinality of a maximal subset of $\{X_1, \ldots, X_n\}$ shattered by $\mathcal{C}$:

$$K_n(\mathcal{C}, X_1, \ldots, X_n) = \max\left\{\#A : \Delta_n(\mathcal{C}, A) = 2^{\#A}\right\}.$$

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4. $\log N(\epsilon n^{-1/2}, \mathcal{C}, L_1(\mathbb{P}_n)) = o_P^*(\sqrt{n})$ *for every* $\epsilon > 0$ *and* $\mathcal{C}$ *is* P*-pre-Gaussian.*