Chapter 2.9: Uniformity in the underlying distribution

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March 2, 2019

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Previously in...

... Reading group - Mathematical statistics

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- In Chapter 2.5, we have seen
  - the conditions on $\mathcal{F}$ so that it is P-Glivenko-Cantelli
  - the conditions on $\mathcal{F}$ so that it is P-Donsker

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- In Chapter 2.5, we have seen
  - the conditions on $\mathcal{F}$ so that it is P-Glivenko-Cantelli
  - the conditions on $\mathcal{F}$ so that it is P-Donsker
- In Chapter 2.3, we have seen
  - the symmetrization lemma
  - Hoefding's inequality

Overview

Intentions

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2. finding conditions on $\mathcal{F}$ so that it is uniformly Donsker

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1. finding conditions on $\mathcal{F}$ so that it is uniformly Glivenko-Cantelli
2. finding conditions on $\mathcal{F}$ so that it is uniformly Donsker
3. applying these results to a (hopefully) more concrete case

Notations

- $\mathcal{F}$ is a class of P-measurable functions on a measurable space (for all $P \in \mathcal{P}$)
- F is a measurable envelope function of $\mathcal{F}$
- $\mathcal{Q}_n$ is the set of all discrete probability measures with atom sized k/n
- $\mathbb{P}_n := 1/n \sum_{i=1}^{n} \delta_{X_i}$ $(\mathbb{P}_n \in \mathcal{Q}_n)$
- $\mathbb{G}_{n,P} := \sqrt{n}(\mathbb{P}_n - P)$ and $\mathbb{G}_P$ Brownian bridge

Glivenko-Cantelli

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What is a Glivenko-Cantelli class of functions?

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Definition

We say that $\mathcal{F}$ is P-Glivenko Cantelli iff $\limsup_{n\to+\infty} \|\mathbb{P}_n - P\|^*_{\mathcal{F}} = 0$ almost surely.

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Glivenko-Cantelli Theorems

Theorem

Let $\mathcal{F}$ such that:

- $\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F>M}$

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then $\mathcal{F}$ is Glivenko-Cantelli uniformly in $P \in \mathcal{P}$.

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Since showing almost sure convergence directly is almost impossible, we have to find another way

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Since showing almost sure convergence directly is almost impossible, we have to find another way

- $X_n \to 0$ a.s is equivalent to $\sup_{m\geq n} |X_m| \to 0$ in probability
- Therefore $\mathcal{F}$ is uniformly Glivenko-Cantelli in $P \in \mathcal{P}$
  $\sup_{P\in\mathcal{P}} P_P^*(\sup_{m\geq n} \|\mathbb{P}_n - P\|_{\mathcal{F}} > \epsilon) \to 0$

Glivenko-Cantelli Theorems

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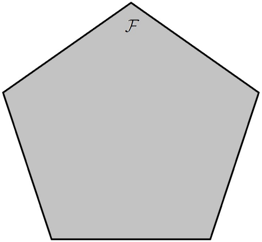

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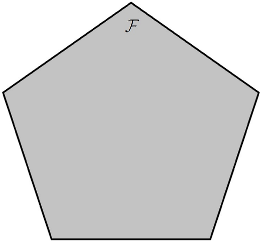

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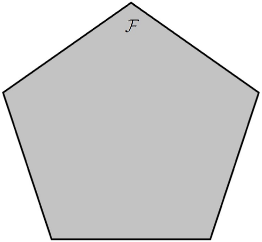

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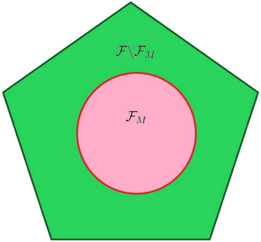

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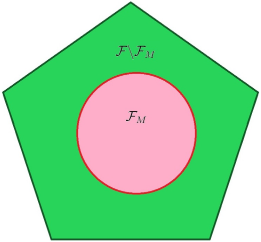

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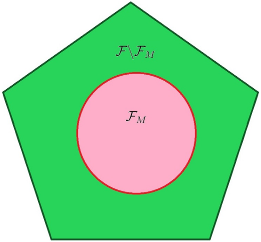

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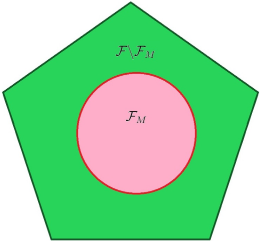

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3. $\|\mathbb{P}_n^o\|_{\mathcal{F}_M} \leq \|\mathbb{P}_n^o\|_{\mathcal{F}_{nX}} + \eta M$

Flashback

Glivenko-Cantelli Theorems

Lemma (Symmetrization)

For every nondecreasing, convex $\Phi : \mathbb{R} \to \mathbb{R}$ *and* $\mathcal{F}$

$$E^* \Phi\big(\|\mathbb{P}_n - P\|_{\mathcal{F}}\big) \leq E^* \Phi\big(2\|\mathbb{P}_n^o\|_{\mathcal{F}}\big).$$

Lemma (Hoeffding's inequality)

Let $a_1, \ldots, a_n$ *constants and* $r_1, \ldots, r_n$ *i.i.d Rademacher. Then*

$$P\Big(\Big|\sum r_i a_i\Big| > x\Big) \leq 2e^{-x^2/(2\|a\|^2)}.$$

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4. $P_P(\|\mathbb{P}_n^o\|_{\mathcal{F}_M} > \epsilon) = E_{P,X}P_R(\|\mathbb{P}_n^o\|_{\mathcal{F}_M} > \epsilon) \le E_{P,X}N_n(\eta)2e^{-n(\epsilon-\eta M)^2/(2M^2)} \le 2\exp\{-n\epsilon^2/(4M^2)\}$

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5. $\sum_{m\geq n} P_P(\|\mathbb{P}_m^o\|_{\mathcal{F}_M} > \epsilon) \to 0$

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3. Then, $Q|f\mathbf{1}_{F\leq M}| = k/nQ_k|f|$ for any f

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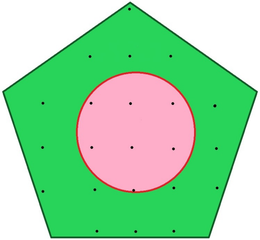

1. Thus $N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq N(\epsilon\|F\|_{Q_k}, \mathcal{F}, L_1(Q_{)}))$
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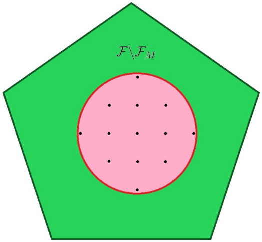

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$\mathbb{G}_{n,P} \rightsquigarrow \mathbb{G}_P$ in $\ell^\infty(\mathcal{F})$ uniformly in $P \in \mathcal{P}$.
$\sup_{P \in \mathbb{P}} \sup_{h \in \mathcal{H}} |E_P^* h(\mathbb{G}_{n,P}) - Eh(\mathbb{G}_P)| \to 0$.

$\mathcal{H} := \{h : \ell^\infty(\mathcal{F}) \to \mathbb{R} \ : |h(z_1) - h(z_2)| \leq \|z_1 - z_2\|_{\mathcal{F}} \& |h(z_1)| \leq 1\}$

Slide-ception

In order to better rembember understand the following procedure, we have to remember better Chapter 2.5

Slide-ception

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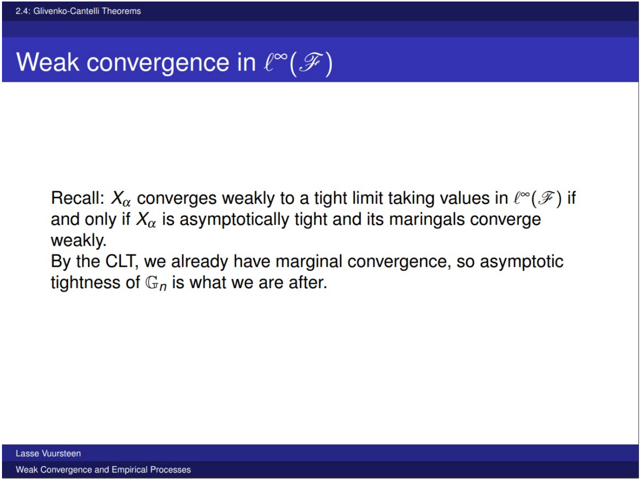

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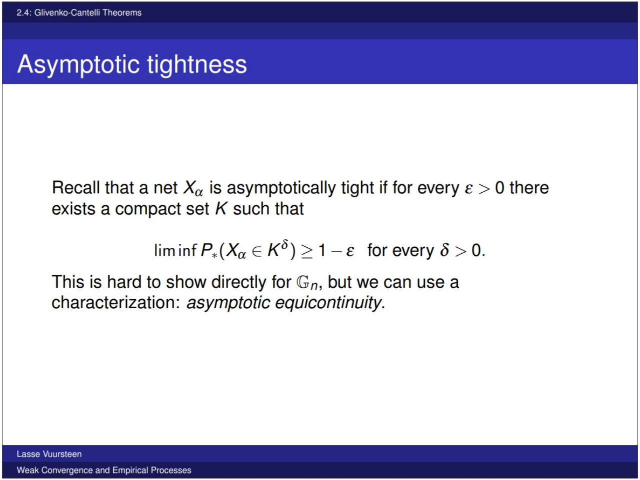

Slide-ception

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Asymptotic equicontinuity

A net $X_\alpha : \Omega \to \ell^\infty(T)$ is *asymptotically uniformly ρ-equicontinuous in probability* if for every $\varepsilon, \eta > 0$ there exists a $\delta > 0$ such that

$$\limsup_\alpha P^* \left(\sup_{\rho(s,t)<\delta} |X_\alpha(s) - X_\alpha(t)| > \varepsilon \right) < \eta.$$

Theorem 1.5.7: X_α is asymptotically tight if and only if $X_\alpha(t)$ is asympotically tight in $\mathbb{R}$ for every t and there exists a semimetric ρ on T such that (T, ρ) is totally bounded and X_α is asymptotically uniformly ρ-equicontinuous in probability.

Slide-ception

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The Donsker theorems come in two flavours (two types of conditions on $\mathscr{F}$):

- Based on the uniform entropy condition

$$\int_0^\infty \sup_Q \sqrt{\log N(\varepsilon \|F\|_{Q,2}, \mathscr{F}, L_2(Q))}\, d\varepsilon < \infty.$$

- Based on bracketing entropy

$$\int_0^\infty \sqrt{\log N_{[]}(\varepsilon, \mathscr{F}, L_2(P))}\, d\varepsilon < \infty.$$

These conditions are generally not comparable. Examples of function classes satisfying either or both are given in Chapter 2.7.

Uniform conditions?

However, a uniform version of those conditions is stronger
So, just for curiosity what is a pre-Gaussian class of functions?

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Definition

We say that $\mathcal{F}$ is pre-Gaussian with respect to P iff

- $E\|\mathbb{G}\|_{\mathcal{F}} < +\infty$
- $\lim_{\delta\to 0} E \sup_{\rho(f,g)<\delta} |\mathbb{G}(f) - \mathbb{G}(g)| = 0$

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Definition

We say that $\mathcal{F}$ is pre-Gaussian uniformly in $P \in \mathcal{P}$ iff

- $\sup_{P\in\mathbb{P}} E\|\mathbb{G}_P\|_{\mathcal{F}} < +\infty$
- $\lim_{\delta\to 0} \sup_{P\in\mathbb{P}} E\sup_{\rho_P(f,g)<\delta} |\mathbb{G}(f) - \mathbb{G}(g)| = 0$

Donsker & pre-Gaussian Theorems

Theorem

Let $\mathcal{F}$ then the following statements are equivalent:

- *$\mathcal{F}$ is Donsker and pre-Gaussian uniformly in $P \in \mathcal{P}$*

Donsker & pre-Gaussian Theorems

Theorem

Let $\mathcal{F}$ then the following statements are equivalent:

- *$\mathcal{F}$ is Donsker and pre-Gaussian uniformly in $P \in \mathcal{P}$*
- *the sequence $\mathbb{G}_{n,P}$ is*
  - *asymptotcally $\rightarrow_P$-equicontinuous uniformly in $P \in \mathcal{P}$*
  - $\sup_{P \in \mathcal{P}} N(\epsilon, \mathcal{F}, \rho_P) < +\infty$ *for every $\epsilon > 0$*

Assertion $\Rightarrow$

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4. Since $\mathcal{F}$ is pre-Gaussian, $E h_P(\mathbb{G}_P) \to 0$ when $\delta \to 0$
5. $\lim_{\delta \to 0} \limsup_{n \to +\infty} E_P^* h_P(\mathbb{G}_{n,P}) = 0$

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3. in $\mathcal{G}$, $P(\sup_{\rho(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}_P(g)| < \epsilon) \leq \liminf_{n\to\infty} \sup_{P\in\mathcal{P}} P_P^*(\sup_{\rho(f,g)<\delta} |\mathbb{G}_{n,P}(f) - \mathbb{G}_{n,P}(g)| < \epsilon)$

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4. in $\mathcal{G}$, $P(\sup_{\rho(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}_P(g)| < \epsilon) \to 0$
5. Letting $\mathcal{G}$ increasing to $\mathcal{F}$ leads to $\lim_{\delta\to 0} \sup_{P\in\mathbb{P}} E \sup_{\rho(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}_P(g)| = 0$

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- $\lim_{\delta \to 0} \sup_{P \in \mathbb{P}} E \sup_{\rho(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}_P(g)| = 0$

1. Fix $\delta > 0$ and $\mathcal{G}_P$ a δ-net for ρ_P over $\mathcal{F}$ and $|\mathcal{G}_P| \leq k$ uniformly

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2. $\|\mathbb{G}_P\|_{\mathcal{F}} \leq \|\mathbb{G}_P\|_{\mathcal{G}_P} + \sup_{\rho_P(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}(g)|$

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2. $\|\mathbb{G}_P\|_{\mathcal{F}} \leq \|\mathbb{G}_P\|_{\mathcal{G}_P} + \sup_{\rho_P(f,g)<\delta} |\mathbb{G}_P(f) - \mathbb{G}(g)|$
3. Thus $E\|\mathbb{G}_P\|_{\mathcal{F}} < +\infty$ uniformly in $P \in \mathcal{P}$

Assertion $\Leftarrow$

- $\sup_{P \in \mathcal{P}} E \|\mathbb{G}_P\|_{\mathcal{F}} < +\infty$
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3. $\sup_{h\in\mathcal{H}}|E_P^* h(\mathbb{G}_{n,P}\circ\Pi_P) - Eh(\mathbb{G}_P\circ\Pi_P)| \to 0$

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3. $\sup_{h\in\mathcal{H}}|E_P^* h(\mathbb{G}_{n,P}\circ\Pi_P) - Eh(\mathbb{G}_P\circ\Pi_P)| \to 0$
4. Since $h \in \mathcal{H}$, $\sup_{h\in\mathcal{H}}|E_P^* h(\mathbb{G}_{n,P}\circ\Pi_P) - E^* h(\mathbb{G}_{n,P})| \leq \epsilon + 2P_P^*(\|\mathbb{G}_{n,P}\circ\Pi_P - \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon)$

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5. By construction of $\mathcal{G}_P$, $\limsup P_P^*(\|\mathbb{G}_{n,P} \circ \Pi_P - \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon) \to 0$

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5. By construction of $\mathcal{G}_P$, $\limsup P_P^*(\|\mathbb{G}_{n,P} \circ \Pi_P - \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon) \to 0$
6. Besides, $\sup_{h\in\mathcal{H}} |Eh(\mathbb{G}_P \circ \Pi_P) - Eh(\mathbb{G}_P \circ \Pi_P)| \to 0$ since $\mathcal{F}$ uniformly pre-Gaussian

Donsker & pre-Gaussian Theorems

Theorem

Let $\mathcal{F}$ *such that* $\mathcal{F}_{\delta,P} = \{f - g : f, g \in \mathcal{F}, \|f - g\|_{P,2} < \delta\}$ *and* $\mathcal{F}^2_\infty = (f - g)^2 : f, g \in \mathcal{F}$ *are* P*-measurable for every* $\delta > 0$ *and* $P \in \mathcal{P}$.

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- $\lim_{M\to+\infty} \sup_{P\in\mathcal{P}} PF^2\mathbf{1}_{F>M} \to 0,$
- $\int_0^{+\infty} \sup_{Q\in\mathcal{Q}_n} \sqrt{\log N(\epsilon\|F\|_{Q,2}, \mathcal{F}, L_2(Q))}d\epsilon < +\infty,$

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Application of the results

For each $n \in \mathbb{N}$, let $X_{n1}, ..., X_{nn}$ be i.i.d according to P_n and $\mathbb{P}_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{X_{ni}}$.

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If $P_n \to P_0$ in a suitable sense, then hopefully

$$\mathbb{G}_{n,P_n} := \sqrt{n}(\mathbb{P}_n - P_n) \rightsquigarrow \mathbb{G}_{P_0} \text{ in } \ell^\infty(\mathcal{F})$$

Lemmas

Lemma

Let $\mathcal{F}$ be Donsker and pre-Gaussian uniformly in $\{P_m, m \leq n\}$ and

- $\sup_{f,g\in\mathcal{F}} |\rho_{P_n}(f,g) - \rho_{P_0}(f,g)| \to 0$,
- $\limsup_{n\to+\infty} P_n F^2 \mathbf{1}_{F\geq\epsilon\sqrt{n}} = 0$ *for all* $\epsilon > 0$.

Then $\mathbb{G}_{n,P_n} \rightsquigarrow$ *in* $\ell^\infty(\mathcal{F})$.

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Let $\mathcal{F}$ be pre-Gaussian uniformly in $\{P_m, m \leq n\}$ and

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Final Theorems

Theorem

Let $\mathcal{F}$ such that $\mathcal{F}_{\delta,P_n} = \{f - g : f, g \in \mathcal{F}, \|f - g\|_{P_n,2} < \delta\}$ and $\mathcal{F}^0_\infty = (f - g)^2 : f, g \in \mathcal{F}$ are P_n-measurable for every $\delta > 0$ and n.

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- $P_n F^2 = O(1)$,
- $\int_0^{+\infty} \sup_{Q \in \mathcal{Q}_n} \sqrt{\log N(\epsilon \|F\|_{Q,2}, \mathcal{F}, L_2(Q))} d\epsilon < +\infty$,
- $\sup_{f,g \in \mathcal{F}} |\rho_{P_n}(f, g) - \rho_{P_0}(f, g)| \to 0$,
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Final Theorems

Theorem

Let $\mathcal{F}$ such that

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In the next...

... session of the Reading group - Mathematical statistics, we'll see

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- Chapter 2.9, with
  - the multiplier central limit theorem
  - the conditional central limit theorem

MLCT

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$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \rightsquigarrow \mathbb{G},$$

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if the ξ_i's satisfy some condition and $\mathcal{F}$ is Donsker (why?).

MLCT

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$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \rightsquigarrow \mathbb{G},$$

but also,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i Z_i \rightsquigarrow \mathbb{G},$$

if the ξ_i's satisfy some condition and $\mathcal{F}$ is Donsker (why?). Moreover the previous assertion is true given almost every sequence $Z_1, Z_2, \dots$ (why?)