Chapter 2.8

Amine Hadj

Notations & Intentions

Glivenko-Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under

Chapter 2.9: Uniformity in the underlying distribution

Amine Hadji

University of Leiden

March 2, 2019



Previously in...

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Glivenko-Cantelli

Donsker

Central Limi Theorem (Under \dots Reading group - Mathematical statistics

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Glivenko Cantelli Theorem

Donsker Theorem

Central Limi Theorem (Under Seguences) ... Reading group - Mathematical statistics

- In Chapter 2.5, we have seen
 - the conditions on \mathcal{F} so that it is P-Glivenko-Cantelli
 - ullet the conditions on ${\mathcal F}$ so that it is $P ext{-}\mathsf{Donsker}$

Previously in...

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Central Limi Theorem (Under Seguences) ... Reading group - Mathematical statistics

- In Chapter 2.5, we have seen
 - the conditions on \mathcal{F} so that it is P-Glivenko-Cantelli
 - ullet the conditions on ${\mathcal F}$ so that it is $P ext{-}\mathsf{Donsker}$
- In Chapter 2.3, we have seen
 - the symmetrization lemma
 - Hoefding's inequality

Overview

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- Notations & Intentions
- Quality of the Control of the Con
- Onsker Theorem
- Central Limite Theorem (under Sequences)

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Chapter 2.8 is mostly about:

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Central Lim Theorem (Under Sequences)

Chapter 2.8 is mostly about:

lacktriangled finding conditions on $\mathcal F$ so that it is uniformly Glivenko-Cantelli

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Central Limi Theorem (Under Sequences)

Chapter 2.8 is mostly about:

- floor finding conditions on ${\cal F}$ so that it is uniformly Glivenko-Cantelli
- ${\bf 2}$ finding conditions on ${\cal F}$ so that it is uniformly Donsker

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Chapter 2.8 is mostly about:

- $\textbf{ 0} \ \, \text{finding conditions on } \mathcal{F} \text{ so that it is uniformly } \\ \text{Glivenko-Cantelli}$
- $oldsymbol{0}$ finding conditions on $\mathcal F$ so that it is uniformly Donsker
- applying these results to a (hopefully) more concrete case

Notations

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Central Limi[,] Theorem (Under Seguences)

- \mathcal{F} is a class of P-measurable functions on a measurable space (for all $P \in \mathcal{P}$)
- ullet F is a measurable envelope function of ${\cal F}$
- Q_n is the set of all discrete probability measures with atom sized k/n
- $\mathbb{P}_n := 1/n \sum_{i=1}^n \delta_{X_i} \ (\mathbb{P}_n \in \mathcal{Q}_n)$
- ullet $\mathbb{G}_{n,P}:=\sqrt{n}(\mathbb{P}_n-P)$ and \mathbb{G}_P Brownian bridge

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Central Limi Theorem (Under First, little reminder:

What is a Glivenko-Cantelli class of functions?

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Central Limi Theorem (Under Seguences) First, little reminder:

What is a Glivenko-Cantelli class of functions?

Definition

We say that \mathcal{F} is P-Glivenko Cantelli iff $\limsup_{n\to+\infty} \|\mathbb{P}_n - P\|_{\mathcal{F}}^* = 0$ almost surely.

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We say that \mathcal{F} is Glivenko Cantelli uniformly in $P \in \mathcal{P}$ iff $\limsup_{n \to +\infty} \|\mathbb{P}_n - P\|_{\mathcal{F}}^* = 0$ almost surely for all $P \in \mathcal{P}$

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Theorem

Let \mathcal{F} such that:

• $\lim_{M\to+\infty} \sup_{P\in\mathcal{P}} PF\mathbf{1}_{F>M}$

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Theorem

Let \mathcal{F} such that:

- $\lim_{M\to+\infty} \sup_{P\in\mathcal{P}} PF\mathbf{1}_{F>M}$
- $\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon ||F||_{Q,1}, \mathcal{F}, L_1(Q)) = o(n), \ \forall \epsilon > 0$

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then \mathcal{F} is Glivenko-Cantelli uniformly in $P \in \mathcal{P}$.

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$$\begin{split} & \lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ & \sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{split}$$

Since showing almost sure convergence directly is almost impossible, we have to find another way

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ullet $X_n o 0$ a.s is equivalent to $\sup_{m \geq n} |X_m| o 0$ in probability

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Since showing almost sure convergence directly is almost impossible, we have to find another way

- $X_n o 0$ a.s is equivalent to $\sup_{m \geq n} |X_m| o 0$ in probability
- Therefore \mathcal{F} is uniformly Glivenko-Cantelli in $P \in \mathcal{P}$ $\sup_{P \in \mathcal{P}} P_P^*(\sup_{m \geq n} \|\mathbb{P}_n P\|_{\mathcal{F}} > \epsilon) \to 0$

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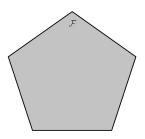
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Glivenko-Cantelli Theorem

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• Let $\mathcal{F}_M := \{ f \mathbf{1}_{f \le M} : f \in \mathcal{F} \}$ for fixed M > 0

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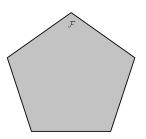
Glivenko-Cantelli Theorem

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Central Limit Theorem (Under

$$\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F>M}$$

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- Let $\mathcal{F}_M := \{ f \mathbf{1}_{f \le M} : f \in \mathcal{F} \}$ for fixed M > 0
- $\|\mathbb{P}_n P\|_{\mathcal{F}} \leq \|\mathbb{P}_n P\|_{\mathcal{F}_M} + \mathbb{P}_n F \mathbf{1}_{F>M} + PF \mathbf{1}_{F>M}$

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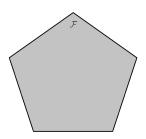
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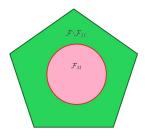
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 $\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} \frac{PF1}{F > M}$ $\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon || F ||_{Q,1}, \mathcal{F}, L_1(Q)) = o(n)$



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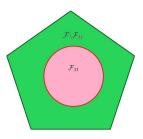
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• $\mathbb{P}_n F \mathbf{1}_{F>M} \to P F \mathbf{1}_{F>M}$ almost surely uniformly

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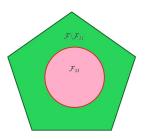
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- $\mathbb{P}_n F \mathbf{1}_{F>M} \to P F \mathbf{1}_{F>M}$ almost surely uniformly
- $\lim_{M\to+\infty} PF\mathbf{1}_{F>M}$ uniformly

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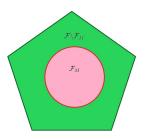
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Central Limit Theorem (Under Sequences)

$$\begin{aligned} &\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ &\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{aligned}$$



- Let $\mathcal{F}_M := \{ f \mathbf{1}_{f \le M} : f \in \mathcal{F} \}$ for fixed M > 0
- $\|\mathbb{P}_n P\|_{\mathcal{F}} \leq \|\mathbb{P}_n P\|_{\mathcal{F}_M} + \mathbb{P}_n F \mathbf{1}_{F>M} + PF \mathbf{1}_{F>M}$

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$$\begin{split} & \lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ & \sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{split}$$

$$\sup_{Q\in\mathcal{Q}_n} N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq \sup_{n\leq k\leq 2n} \sup_{Q\in\mathcal{Q}_k} N(\epsilon \|F\|_{Q,1}, \mathcal{F}, L_1(Q))$$

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1 Let \mathcal{F}_{nX} a minimal ηM -net for $L_1(\mathbb{P}_n)$

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Central Limi Theorem (Under Seguences)

$$\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M}$$

$$\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon || F ||_{Q,1}, \mathcal{F}, L_1(Q)) = o(n)$$

$$\sup_{Q\in\mathcal{Q}_n} N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq \sup_{n\leq k\leq 2n} \sup_{Q\in\mathcal{Q}_k} N(\epsilon \|F\|_{Q,1}, \mathcal{F}, L_1(Q))$$

- **1** Let \mathcal{F}_{nX} a minimal ηM -net for $L_1(\mathbb{P}_n)$
- $N(\eta M, \mathcal{F}_M, L_1(\mathbb{P}_n)) \leq N_n(\eta)$ with $\log N_n(\eta) = o(n)$

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Central Limit Theorem (Under $\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M}$ $\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon || F ||_{Q,1}, \mathcal{F}, L_1(Q)) = o(n)$

$$\sup_{Q\in\mathcal{Q}_n} N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq \sup_{n\leq k\leq 2n} \sup_{Q\in\mathcal{Q}_k} N(\epsilon \|F\|_{Q,1}, \mathcal{F}, L_1(Q))$$

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Flashback

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Lemma (Symmetrization)

For every nondecreasing, convex $\Phi:\mathbb{R}\to\mathbb{R}$ and $\mathcal F$

$$E^*\Phi(\|\mathbb{P}_n - P\|_{\mathcal{F}}) \le E^*\Phi(2\|\mathbb{P}_n^o\|_{\mathcal{F}}).$$

Lemma (Hoeffding's inequality)

Let $a_1, ..., a_n$ constants and $r_1, ..., r_n$ i.i.d Rademacher. Then

$$P(|\sum r_i a_i| > x) \le 2e^{-x^2/(2||a||^2)}.$$

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$$\begin{split} & \lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ & \sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{split}$$

$$\sup_{Q\in\mathcal{Q}_n} N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq \sup_{n\leq k\leq 2n} \sup_{Q\in\mathcal{Q}_k} N(\epsilon \|F\|_{Q,1}, \mathcal{F}, L_1(Q))$$

- **1** Let \mathcal{F}_{nX} a minimal ηM -net for $L_1(\mathbb{P}_n)$
- $N(\eta M, \mathcal{F}_M, L_1(\mathbb{P}_n)) \leq N_n(\eta) \text{ with log } N_n(\eta) = o(n)$
- $\|\mathbb{P}_n^o\|_{\mathcal{F}_M} \leq \|\mathbb{P}_n^o\|_{\mathcal{F}_{nX}} + \eta M$
- $P_{P}(\|\mathbb{P}_{n}^{o}\|_{\mathcal{F}_{M}} > \epsilon) = E_{P,X}P_{R}(\|\mathbb{P}_{n}^{o}\|_{\mathcal{F}_{M}} > \epsilon) \le E_{P,X}N_{n}(\eta)2e^{-n(\epsilon-\eta M)^{2}/(2M^{2})} \le 2\exp\{-n\epsilon^{2}/(4M^{2})\}$

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$$\begin{split} &\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ &\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{split}$$

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- $P_P(\|\mathbb{P}_n^o\|_{\mathcal{F}_M} > \epsilon) = E_{P,X} P_R(\|\mathbb{P}_n^o\|_{\mathcal{F}_M} > \epsilon) \le$ $E_{P,X} N_n(\eta) 2e^{-n(\epsilon \eta M)^2/(2M^2)} \le 2 \exp\{-n\epsilon^2/(4M^2)\}$

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Central Limit Theorem (Under Seguences)

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1 Let
$$Q \in \mathcal{Q}_n$$
 and $|I| = k$, where $I := x_i : F(x_i) \le M$

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- **1** Let $Q \in \mathcal{Q}_n$ and |I| = k, where $I := x_i : F(x_i) \leq M$
- **2** $Q_k \in \mathcal{Q}_k$ the discrete measure on I

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- **1** Let $Q \in \mathcal{Q}_n$ and |I| = k, where $I := x_i : F(x_i) \leq M$
- 2 $Q_k \in Q_k$ the discrete measure on I
- **3** Then, $Q|f\mathbf{1}_{F\leq M}|=k/nQ_k|f|$ for any f

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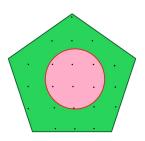
Notations & Intentions

Glivenko-Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under

$$\begin{aligned} &\lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ &\sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{aligned}$$



- Thus $N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq N(\epsilon ||F||_{Q_k}, \mathcal{F}, L_1(Q))$
- **2** Moreover $Q_k \subseteq Q_{2k} \subseteq ...$

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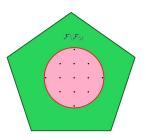
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Central Limit Theorem (Under

$$\begin{split} & \lim_{M \to +\infty} \sup_{P \in \mathcal{P}} PF\mathbf{1}_{F > M} \\ & \sup_{Q \in \mathcal{Q}_n} \log N(\epsilon \| F \|_{Q,1}, \mathcal{F}, L_1(Q)) = o(n) \end{split}$$



- Thus $N(\epsilon M, \mathcal{F}_M, L_1(Q)) \leq N(\epsilon ||F||_{Q_k}, \mathcal{F}, L_1(Q))$
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Central Lim
Theorem
(Under

Again, little reminder: What is a Donsker class of functions?

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Central Limi Theorem (Under Seguences) Again, little reminder:

What is a Donsker class of functions?

Definition

We say that \mathcal{F} is P-Donsker iff $\mathbb{G}_n \rightsquigarrow \mathbb{G}$ in $\ell^{\infty}(\mathcal{F})$.

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> Central Limit Theorem (Under Seguences)

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We say that \mathcal{F} is Donsker uniformly in $P \in \mathcal{P}$ iff $\mathbb{G}_{n,P} \leadsto \mathbb{G}_P$ in $\ell^{\infty}(\mathcal{F})$ uniformly in $P \in \mathcal{P}$.

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We say that \mathcal{F} is Donsker uniformly in $P \in \mathcal{P}$ iff $\mathbb{G}_{n,P} \leadsto \mathbb{G}_P$ in $\ell^{\infty}(\mathcal{F})$ uniformly in $P \in \mathcal{P}$. $\sup_{P \in \mathbb{P}} \sup_{h \in \mathcal{H}} |E_P^* h(\mathbb{G}_{n,P}) - Eh(\mathbb{G}_P)| \to 0$.

$$\mathcal{H} := \{ h : \ell^{\infty}(\mathcal{F}) \to \mathbb{R} : |h(z_1) - h(z_2)| \le \|z_1 - z_2\|_{\mathcal{F}} \& |h(z_1)| \le 1 \}$$

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In order to better rembember understand the following procedure, we have to remember better Chapter 2.5

2.4: Glivenko-Cantelli Theorems

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Central Limi Theorem (Under Sequences) In order to better rembember understand the following procedure, we have to remember better Chapter 2.5

Weak convergence in $\ell^{\infty}(\mathscr{F})$

Recall: X_{α} converges weakly to a tight limit taking values in $\ell^{\infty}(\mathscr{F})$ if and only if X_{α} is asymptotically tight and its maringals converge weakly.

By the CLT, we already have marginal convergence, so asymptotic tightness of \mathbb{G}_n is what we are after.

Lasse Vuursteen Weak Convergence and Empirical Processes

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Central Limi Theorem (Under Sequences) In order to better rembember understand the following procedure, we have to remember better Chapter 2.5

Asymptotic tightness

2.4: Glivenko-Cantelli Theorems

Recall that a net X_{α} is asymptotically tight if for every $\varepsilon > 0$ there exists a compact set K such that

$$\liminf P_*(X_\alpha \in \mathcal{K}^\delta) \geq 1 - \epsilon \ \ \text{for every } \delta > 0.$$

This is hard to show directly for \mathbb{G}_n , but we can use a characterization: *asymptotic equicontinuity*.

Lasse Vuursteen

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Central Limit Theorem (Under Sequences) In order to better rembember understand the following procedure, we have to remember better Chapter 2.5

2.4: Glivenko-Cantelli Theorems

Asymptotic equicontinuity

A net $X_\alpha:\Omega\to\ell^\infty(T)$ is asymptotically uniformly ρ -equicontinuous in probability if for every $\varepsilon,\eta>0$ there exists a $\delta>0$ such that

$$\limsup_{lpha} P^* \left(\sup_{
ho(s,t) < \delta} |X_lpha(s) - X_lpha(t)| > arepsilon
ight) < \eta.$$

Theorem 1.5.7: X_{α} is asymptotically tight if and only if $X_{\alpha}(t)$ is asympotically tight in $\mathbb R$ for every t and there exists a semimetric ρ on $\mathcal T$ such that $(\mathcal T, \rho)$ is totally bounded and X_{α} is asymptotically uniformly ρ -equicontinuous in probability.

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2.4: Glivenko-Cantelli Theorems

The Donsker theorems come in two flavours (two types of conditions on \mathscr{F}):

Based on the uniform entropy condition

$$\int_0^\infty \sup_Q \sqrt{\log N(\varepsilon||F||_{Q,2}, \mathscr{F}, L_2(Q))} d\varepsilon < \infty.$$

Based on bracketing entropy

$$\int_0^\infty \sqrt{\log N_{[]}(\varepsilon,\mathscr{F},L_2(P))}d\varepsilon < \infty.$$

These conditions are generally not comparable. Examples of function classes satisfying either or both are given in Chapter 2.7.



Uniform conditions?

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Central Limit Theorem (Under However, a uniform version of those conditions is stronger So, just for curiosity what is a pre-Gaussian class of functions?

Uniform conditions?

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Central Limi[,] Theorem (Under Sequences) However, a uniform version of those conditions is stronger So, just for curiosity what is a pre-Gaussian class of functions?

Definition

We say that $\mathcal F$ is pre-Gaussian with respect to P iff

- $E\|\mathbb{G}\|_{\mathcal{F}} < +\infty$
- $\lim_{\delta \to 0} E \sup_{\rho(f,g) < \delta} |\mathbb{G}(f) \mathbb{G}(g)| = 0$

Uniform conditions?

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Definition

We say that ${\mathcal F}$ is pre-Gaussian uniformly in $P\in {\mathcal P}$ iff

- $\sup_{P\in\mathbb{P}} E \|\mathbb{G}_P\|_{\mathcal{F}} < +\infty$
- $\lim_{\delta \to 0} \sup_{P \in \mathbb{P}} E \sup_{\rho_P(f,g) < \delta} |\mathbb{G}(f) \mathbb{G}(g)| = 0$

Donsker & pre-Gaussian Theorems

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Central Limi Theorem (Under Sequences)

Theorem 1

Let $\mathcal F$ then the following statements are equivalent:

ullet ${\mathcal F}$ is Donsker and pre-Gaussian uniformly in $P\in {\mathcal P}$

Donsker & pre-Gaussian Theorems

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Theorem

Let $\mathcal F$ then the following statements are equivalent:

- ullet ${\mathcal F}$ is Donsker and pre-Gaussian uniformly in $P\in {\mathcal P}$
- the sequence $\mathbb{G}_{n,P}$ is
 - asymptotcally \rightarrow_P -equicontinuous uniformly in $P \in \mathcal{P}$
 - $\sup_{P\in\mathcal{P}} N(\epsilon, \mathcal{F}, \rho_P) < +\infty$ for every $\epsilon > 0$

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Central Limit Theorem (Under Sequences) $\bullet \quad \text{Fix } \delta > 0 \text{ and } P$

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Donsker Theorem

Central Lim Theorem (Under Sequences) **1** Fix $\delta > 0$ and P

$$2 \mapsto h_P(z) = \sup_{\rho_P(f,g) < \delta} |z(f) - z(g)| \land 1 \in 2\mathcal{H}$$

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Donsker Theorem

Central Limi Theorem (Under Sequences) **1** Fix $\delta > 0$ and P

- **③** Since \mathcal{F} is Donsker, $E_P^*h_P(\mathbb{G}_{n,P})$ → $Eh_P(\mathbb{G}_P)$ when $n \to +\infty$

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Donsker Theorem

Central Limi Theorem (Under Sequences) • Fix $\delta > 0$ and P

- **③** Since \mathcal{F} is Donsker, $E_P^*h_P(\mathbb{G}_{n,P})$ → $Eh_P(\mathbb{G}_P)$ when $n \to +\infty$
- **1** Since $\mathcal F$ is pre-Gaussian, $Eh_P(\mathbb G_P) o 0$ when $\delta o 0$

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Central Limi Theorem (Under Sequences) • Fix $\delta > 0$ and P

 $2 \mapsto h_P(z) = \sup_{\rho_P(f,g) < \delta} |z(f) - z(g)| \land 1 \in 2\mathcal{H}$

§ Since $\mathcal F$ is Donsker, $E_P^*h_P(\mathbb G_{n,P}) o Eh_P(\mathbb G_P)$ when $n o +\infty$

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Assertion \Leftarrow

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Central Limi Theorem (Under $\textbf{ 1 Let } \mathcal{G} \subseteq \mathcal{F} \text{ with } \mathcal{G} \text{ finite}$

Assertion ←

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Central Lim Theorem (Under Sequences) $\bullet \ \, \mathsf{Let} \,\, \mathcal{G} \subseteq \mathcal{F} \,\, \mathsf{with} \,\, \mathcal{G} \,\, \mathsf{finite}$

Assertion \Leftarrow

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Central Limit Theorem (Under Sequences) **1** Let $\mathcal{G} \subseteq \mathcal{F}$ with \mathcal{G} finite

Assertion \leftarrow

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Donsker Theorem

Central Limi Theorem (Under Sequences) **1** Let $\mathcal{G} \subseteq \mathcal{F}$ with \mathcal{G} finite

- lacksquare in \mathcal{G} , $P(\sup_{
 ho(f,g)<\delta}|\mathbb{G}_P(f)-\mathbb{G}_P(g)|<\epsilon) o 0$

Assertion \leftarrow

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Donsker Theorem

Central Limi Theorem (Under Sequences)

- **1** Let $\mathcal{G} \subseteq \mathcal{F}$ with \mathcal{G} finite

- $lack {f 0}$ in ${\cal G}$, $P(\sup_{
 ho(f,g)<\delta}|\mathbb{G}_P(f)-\mathbb{G}_P(g)|<\epsilon) o 0$
- **1** Letting \mathcal{G} increasing to \mathcal{F} leads to $\lim_{\delta \to 0} \sup_{P \in \mathbb{P}} E \sup_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$

Assertion ←

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Central Limi Theorem (Under Sequences)

- $\bullet \ \operatorname{lim}_{\delta \to 0} \operatorname{sup}_{P \in \mathbb{P}} E \operatorname{sup}_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- ① Fix $\delta>0$ and \mathcal{G}_P a δ -net for ρ_P over \mathcal{F} and $|\mathcal{G}_P|\leq k$ uniformly

Assertion ←

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Central Limir Theorem (Under Sequences)

- $\bullet \ \operatorname{lim}_{\delta \to 0} \operatorname{sup}_{P \in \mathbb{P}} E \operatorname{sup}_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- Fix $\delta>0$ and \mathcal{G}_P a δ -net for ρ_P over \mathcal{F} and $|\mathcal{G}_P|\leq k$ uniformly

Assertion \leftarrow

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Central Limi[.] Theorem (Under Sequences)

- $\lim_{\delta \to 0} \sup_{P \in \mathbb{P}} E \sup_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- ① Fix $\delta>0$ and \mathcal{G}_P a δ -net for ρ_P over \mathcal{F} and $|\mathcal{G}_P|\leq k$ uniformly
- **3** Thus $E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$ uniformly in $P \in \mathcal{P}$

Assertion ←

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Glivenko Cantelli Theorem

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Central Limi Theorem (Under Sequences) • $\sup_{P \in \mathcal{P}} E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$

ullet $\lim_{\delta o 0} \sup_{P \in \mathbb{P}} E \sup_{
ho(f,g) < \delta} |\mathbb{G}_P(f) - \mathbb{G}_P(g)| = 0$

1 Let $\Pi_P : \mathcal{F} \to \mathcal{G}_P$ a (sort of) projection

Assertion ←

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Central Limi Theorem (Under Sequences)

- $\sup_{P \in \mathcal{P}} E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$
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Assertion \leftarrow

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Central Limit Theorem (Under Sequences)

- $\sup_{P \in \mathcal{P}} E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$
- $\bullet \ \operatorname{lim}_{\delta \to 0} \operatorname{sup}_{P \in \mathbb{P}} E \operatorname{sup}_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
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Assertion \leftarrow

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Central Limit Theorem (Under Sequences)

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- $\bullet \ \operatorname{lim}_{\delta \to 0} \operatorname{sup}_{P \in \mathbb{P}} E \operatorname{sup}_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- **1** Let $\Pi_P: \mathcal{F} \to \mathcal{G}_P$ a (sort of) projection

- **③** Since $h \in \mathcal{H}$, $\sup_{h \in \mathcal{H}} |E_P^* h(\mathbb{G}_{n,P} \circ \Pi_P) E^* h(\mathbb{G}_{n,P})| \le \epsilon + 2P_P^* (\|\mathbb{G}_{n,P} \circ \Pi_P \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon)$

Assertion \leftarrow

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Central Limit Theorem (Under Sequences)

- $\sup_{P \in \mathcal{P}} E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$
- $\bullet \ \operatorname{lim}_{\delta \to 0} \operatorname{sup}_{P \in \mathbb{P}} E \operatorname{sup}_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- **1** Let $\Pi_P: \mathcal{F} \to \mathcal{G}_P$ a (sort of) projection

- **③** Since $h \in \mathcal{H}$, $\sup_{h \in \mathcal{H}} |E_P^* h(\mathbb{G}_{n,P} \circ \Pi_P) E^* h(\mathbb{G}_{n,P})| \le \epsilon + 2P_P^* (\|\mathbb{G}_{n,P} \circ \Pi_P \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon)$
- **5** By construction of \mathcal{G}_P , $\limsup P_P^*(\|\mathbb{G}_{n,P} \circ \Pi_P \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon) \to 0$

Assertion \leftarrow

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Central Limit Theorem (Under Sequences)

- $\sup_{P \in \mathcal{P}} E \| \mathbb{G}_P \|_{\mathcal{F}} < +\infty$
- $\lim_{\delta \to 0} \sup_{P \in \mathbb{P}} E \sup_{\rho(f,g) < \delta} |\mathbb{G}_P(f) \mathbb{G}_P(g)| = 0$
- **1** Let $\Pi_P: \mathcal{F} \to \mathcal{G}_P$ a (sort of) projection

- **③** Since $h \in \mathcal{H}$, $\sup_{h \in \mathcal{H}} |E_P^* h(\mathbb{G}_{n,P} \circ \Pi_P) E^* h(\mathbb{G}_{n,P})| \le \epsilon + 2P_P^* (\|\mathbb{G}_{n,P} \circ \Pi_P \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon)$
- **5** By construction of \mathcal{G}_P , $\limsup P_P^*(\|\mathbb{G}_{n,P} \circ \Pi_P \mathbb{G}_{n,P}\|_{\mathcal{F}} > \epsilon) \to 0$
- **Besides, sup**_{h∈H} $|Eh(\mathbb{G}_P \circ \Pi_P) Eh(\mathbb{G}_P \circ \Pi_P)|$ → 0 since \mathcal{F} uniformly pre-Gaussian

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Central Limit Theorem (Under Sequences)

Theorem

Let \mathcal{F} such that $\mathcal{F}_{\delta,P}=\{f-g:f,g\in\mathcal{F},\|f-g\|_{P,2}<\delta\}$ and $\mathcal{F}_{\infty}^2=(f-g)^2:f,g\in\mathcal{F}$ are P-measurable for every $\delta>0$ and $P\in\mathcal{P}$.

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Central Limi Theorem (Under Seguences)

Theorem

Let \mathcal{F} such that $\mathcal{F}_{\delta,P}=\{f-g:f,g\in\mathcal{F},\|f-g\|_{P,2}<\delta\}$ and $\mathcal{F}_{\infty}^2=(f-g)^2:f,g\in\mathcal{F}$ are P-measurable for every $\delta>0$ and $P\in\mathcal{P}$. Furthermore, suppose

- $\lim_{M\to +\infty} \sup_{P\in\mathcal{P}} PF^2\mathbf{1}_{F>M} \to 0$,
- $\int_0^{+\infty} \sup_{Q \in \mathcal{Q}_n} \sqrt{\log N(\epsilon \|F\|_{Q,2}, \mathcal{F}, L_2(Q))} d\epsilon < +\infty,$

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Theorem

Let \mathcal{F} such that $\mathcal{F}_{\delta,P}=\{f-g:f,g\in\mathcal{F},\|f-g\|_{P,2}<\delta\}$ and $\mathcal{F}_{\infty}^2=(f-g)^2:f,g\in\mathcal{F}$ are P-measurable for every $\delta>0$ and $P\in\mathcal{P}$. Furthermore, suppose

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- $\int_0^{+\infty} \sup_{Q \in \mathcal{Q}_n} \sqrt{\log N(\epsilon ||F||_{Q,2}, \mathcal{F}, L_2(Q))} d\epsilon < +\infty,$

then ${\mathcal F}$ is Donsker and pre-Gaussian uniformly in $P\in {\mathcal P}$

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Application of the results

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Donsker

Central Limit Theorem (Under Sequences) For each $n \in \mathbb{N}$, let $X_{n1},...,X_{nn}$ be i.i.d according to P_n and $\mathbb{P}_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_{ni}}$.

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Central Limit Theorem (Under Sequences) For each $n \in \mathbb{N}$, let $X_{n1},...,X_{nn}$ be i.i.d according to P_n and $\mathbb{P}_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_{ni}}$. If $P_n \to P_0$ in a suitable sense, then hopefully

$$\mathbb{G}_{n,P_n}:=\sqrt{n}(\mathbb{P}_n-P_n) \rightsquigarrow \mathbb{G}_{P_0} \text{ in } \ell^\infty(\mathcal{F})$$

Lemmas

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Central Limit Theorem (Under Sequences)

Lemma

Let ${\mathcal F}$ be Donsker and pre-Gaussian uniformly in $\{P_m, m \le n\}$ and

- $\sup_{f,g\in\mathcal{F}} |\rho_{P_n}(f,g) \rho_{P_0}(f,g)| \to 0$,
- $\limsup_{n\to+\infty} P_n F^2 \mathbf{1}_{F>\epsilon\sqrt{n}} = 0$ for all $\epsilon>0$.

Then $\mathbb{G}_{n,P_n} \rightsquigarrow \text{ in } \ell^{\infty}(\mathcal{F}).$

Lemma

Let \mathcal{F} be pre-Gaussian uniformly in $\{P_m, m \leq n\}$ and

$$ullet$$
 $\sup_{f,g\in\mathcal{F}}|
ho_{P_n}(f,g)-
ho_{P_0}(f,g)| o 0$,

Then $\mathbb{G}_{P_n} \rightsquigarrow \text{ in } \ell^{\infty}(\mathcal{F}).$

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Notations of Intentions

Glivenko-Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under Sequences)

Theorem

Let \mathcal{F} such that $\mathcal{F}_{\delta,P_n} = \{f - g : f, g \in \mathcal{F}, \|f - g\|_{P_n,2} < \delta\}$ and $\mathcal{F}_{\infty}^0 = (f - g)^2 : f, g \in \mathcal{F}$ are P_n -measurable for every $\delta > 0$ and n.

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- $P_n F^2 = O(1)$,
- $\int_0^{+\infty} \sup_{Q \in \mathcal{Q}_n} \sqrt{\log N(\epsilon ||F||_{Q,2}, \mathcal{F}, L_2(Q))} d\epsilon < +\infty,$
- $\sup_{f,g\in\mathcal{F}} |\rho_{P_n}(f,g) \rho_{P_0}(f,g)| \to 0$,
- $\limsup_{n\to+\infty} P_n F^2 \mathbf{1}_{F\geq\epsilon\sqrt{n}} = 0$ for all $\epsilon>0$.

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- $P_n F^2 = O(1)$,
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Then $\mathbb{G}_{n,P_n} \rightsquigarrow \text{ in } \ell^{\infty}(\mathcal{F}).$

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Theorem

Let ${\mathcal F}$ such that

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Central Limit Theorem (Under Sequences)

Theorem

Let \mathcal{F} such that

- \mathcal{F} totally bounded for ρ_{P_0}
- $\int_0^{+\infty} \sup_{P \in \mathcal{P}} \sqrt{\log N_{[]}(\epsilon ||F||_{P,2}, \mathcal{F}, L_2(P))} d\epsilon < +\infty,$
- $\sup_{f,g\in\mathcal{F}} |\rho_{P_n}(f,g) \rho_{P_0}(f,g)| \to 0$,
- $\limsup_{n\to+\infty} P_n F^2 \mathbf{1}_{F>\epsilon\sqrt{n}} = 0$ for all $\epsilon > 0$.

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Central Limit Theorem (Under Sequences)

Theorem

Let \mathcal{F} such that

- \mathcal{F} totally bounded for ρ_{P_0}
- $\bullet \ \int_0^{+\infty} \sup\nolimits_{P \in \mathcal{P}} \sqrt{\log \textit{N}_{[]}(\epsilon \|F\|_{P,2}, \mathcal{F}, \textit{L}_2(P))} d\epsilon < +\infty,$
- ullet $\sup_{f,g\in\mathcal{F}}|
 ho_{P_n}(f,g)ho_{P_0}(f,g)| o 0$,
- $\limsup_{n\to+\infty} P_n F^2 \mathbf{1}_{F>\epsilon\sqrt{n}} = 0$ for all $\epsilon > 0$.

Then $\mathbb{G}_{n,P_n} \leadsto \text{ in } \ell^{\infty}(\mathcal{F}).$

In the next...

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Glivenko-Cantelli Theorem

Donsker

Central Limit Theorem (Under Sequences) \dots session of the Reading group - Mathematical statistics, we'll see

In the next...

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Glivenko-Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under Sequences) ... session of the Reading group - Mathematical statistics, we'll see

- Chapter 2.9, with
 - the multiplier central limit theorem
 - the conditional central limit theorem

MLCT

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Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under Sequences) Let $Z_i := \delta_{X_i} - P$, then we have directly

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i \leadsto \mathbb{G},$$

but also,

Glivenko Cantelli Theorem

Donsker Theorem

Central Limit Theorem (Under Sequences) Let $Z_i := \delta_{X_i} - P$, then we have directly

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i \leadsto \mathbb{G},$$

but also,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i Z_i \leadsto \mathbb{G},$$

if the ξ_i 's satisfy some condition and \mathcal{F} is Donsker (why?).

MLCT

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Notations & Intentions

Glivenko Cantelli Theorem

Donsker Theorem

Central Limit
Theorem
(Under

Sequences)

Let $Z_i := \delta_{X_i} - P$, then we have directly

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n Z_i \leadsto \mathbb{G},$$

but also,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i Z_i \leadsto \mathbb{G},$$

if the ξ_i 's satisfy some condition and \mathcal{F} is Donsker (why?). Moreover the previous assertion is true given almost every sequence Z_1, Z_2, \ldots (why?)