Methods of Proof

- **OUTLINE**
  - Propositions and Logical Connectives
  - Syntax and Semantics of Propositional Logic
  - Rules of Inference
  - Laws of Equivalence
  - **Methods of Proof**
    - Truth Tables
    - Chain of Equivalence
    - Chain of Reasoning
    - Proof by Contradiction
    - Proof by Resolution

Methods of Proof

- **TRUTH TABLES**

Procedure:

Construct truth tables to show that

- **Method 1:** whenever all the premises are true, the conclusion will also be true and that therefore the argument is valid

  OR
- **Method 2:** that the (entire) argument is a tautology (always true).

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- **TRUTH TABLES**

Example:

Show that [(**M** → **~N**)   **N**]   **~M** is valid.

Method 1:

M	N	~N	M → ~N	N	~M
T	T				
T	F				
F	T				
F	F				

Methods of Proof

- **TRUTH TABLES**

Example:

Show that [(**M** → **~N**)   **N**]   **~M** is valid.

Method 2:

M	N	~N	M → ~N	N	~M	~M
T	T					
T	F					
F	T					
F	F					

Methods of Proof

- **CHAIN OF EQUIVALENCE**

Procedure:

To prove that a sentence

A: (F_1 F_2 … F_n)   Q

is valid, provide a sequence of statements A_1, A_2, …, A_n such that A A_1, A_1 A_2, … , $A_{n\text{-}1}$ A_n

- A_n is known to be a **tautology**; and
- each step in the proof should be **justified** by the **rules of logic** (laws of equivalence) only.

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- **CHAIN OF EQUIVALENCE**

Examples:

- Prove the validity of the following arguments:
  - ~A
∴ A → B
  - ~P   Q
P
∴ Q
  - ~(E → S)
~S
∴ E

Methods of Proof

- **CHAIN of REASONING (Direct Proof)**

Procedure:

To prove that the argument

$$(F_1 \quad F_2 \quad \ldots \quad F_n) \quad Q$$

is valid, provide a **sequence of statements** such that

- each statement can be **inferred from** or **is equivalent to** the given premises or previously inferred statements;
- each statement **must be justified** by a rule of inference or rule of logic; and
- the sequence **arrives at the given conclusion**.

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- **CHAIN of REASONING (Direct Proof)**

A few remarks:

- Chain of Reasoning is used to prove the validity of an argument and cannot be used to prove the invalidity of an argument.
- A proof of validity is not unique for a given argument.
- A short proof is not necessarily a "better" proof than a longer one.

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- ***Example 1***:
  - A student does not miss a long exam if and only if he is exempted from taking the final exam. If a student is exempted from taking the final exam, then he received a pre-final grade equal to or better than 3.0. The student did not receive a pre-final grade equal to or better than 3.0. Therefore the student missed a long exam.

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- ***Example 2*:**
  - If the band was not available to perform on 14 Nov or the organization was not able to raise enough funds, then the concert would have been cancelled and the manager would have been angry. If the concert were cancelled, then refunds would have to be made. No refunds were made. Therefore the band was available to perform on 14 Nov.

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- ***Example 3*:**
  - If Tina will watch a movie this weekend, she will eat lunch with her friends at Pizza Hut. Else, she will rent a DVD. Either she will not rent a DVD or she will order a Zinger from KFC for delivery. Therefore, if Tina does not eat lunch with her friends at Pizza Hut, she will order a Zinger from KFC for delivery.

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- **PROOF by CONTRADICTION (Indirect)**

Rationale:

The argument $(F_1 \quad F_2 \quad \ldots \quad F_n) \quad Q$ is valid iff $(F_1 \quad F_2 \quad \ldots \quad F_n) \quad \sim Q$ is *false*.

Procedure:

To prove that the argument

$$(F_1 \quad F_2 \quad \ldots \quad F_n) \quad Q$$

is valid

- **negate** the conclusion Q
- use ~Q along with the other premises and produce a sequence of statements; and
- arrive at a **contradiction**.

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- **PROOF by RESOLUTION**
  - Essentially a proof by contradiction:

    The argument $(F_1 \quad F_2 \quad \ldots \quad F_n) \quad Q$ is valid

    iff $(F_1 \quad F_2 \quad \ldots \quad F_n) \quad \sim Q$ is *false*.
  - Each sentence in the sequence of statements to be provided can be justified only by the resolution rule:

    Resolution Rule:

    [(P Q) (~Q R)] P R
  - The argument $(F_1 \quad F_2 \quad \ldots \quad F_n) \quad \sim Q$ must be converted to its **conjunctive normal form (CNF)**.

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- **PROOF by RESOLUTION (cont'd)**

Procedure:

To prove that the argument

$$(F_1 \quad F_2 \quad \ldots \quad F_n) \quad Q$$

is valid

- **convert argument** to its **CNF**—i.e., all premises must be disjunctions only;
- **negate** the conclusion Q, convert it into a disjunction and use this along with the other normalized premises; and
- **derive a contradiction** using the resolution rule only.

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- **PROOF by RESOLUTION (cont'd)**

What is conjunctive normal form?

- A sentence written in its conjunctive normal form (CNF) is a sentence which is
  - a *conjunction* of several terms
  - where each term is a *disjunction* of literals.
- ***Example***:

  (A ~B) (B C ~D) (~A ~D)

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- **PROOF by RESOLUTION (cont'd)**

Normalizing Premises

- Replace all instances of and
  - Replace **(P Q)** with **(P Q) (Q P)**
  - Replace **(P Q)** with **~P Q**
- Reduce the scope of ~
  - Replace **~(P Q)** with **~P ~Q**
  - Replace **~(P Q)** with **~P ~Q**
  - Replace **P (Q R)** with **(P Q) (P R)**

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- **PROOF by RESOLUTION (cont'd)**

Normalizing Premises (cont'd)

- Treat disjunctions joined by as separate premises.
  - ***Example***

    (P Q) (~P R S) ~Q

    should be split into three premises:

    (P Q)

    (~P R S)

    ~Q

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Logic in Action

- Prove that **if $x^2 - 4 = 0$ then either $x = -2$ or $x = 2$.**

Proof:

Given: $\mathbf{x^2 - 4 = 0}$

Since $x^2 - 4 = (x - 2)(x + 2)$

then $\mathbf{(x - 2)(x + 2) = 0}$.

If $(x - 2)(x + 2) = 0$,

then either $\mathbf{x - 2 = 0}$ **or** $\mathbf{x + 2 = 0}$.

Therefore, either $\mathbf{x = -2}$ **or** $\mathbf{x = 2}$.

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Logic in Action

- Prove that **if $x^2 - 4 = 0$ then either $x = -2$ or $x = 2$.**

  Let

  P: $x^2 - 4 = 0$

  Q: $(x - 2)(x + 2) = 0$

  R: $x - 2 = 0$ or $x + 2 = 0$

  S: $x = -2$ or $x = 2$

  Argument:

  P Q

  Q R

  R S

  $\therefore$ **P S**

Logic in Action

- Prove that **√2 is irrational**.

  Proof: (**Proof by contradiction**)

  Suppose: **√2 is rational**

  Then **√2 = a/b** *where a/b is in its lowest terms*.

  Now

  $(\sqrt{2})b = a$

  $2b^2 = a^2$

  This means that **a^2 is even** and that **a is even**.

  So we can write **a = 2n**. It follows that

  $2b^2 = (2n)^2$

  $2b^2 = 4n^2$

  $b^2 = 2n^2$

  This also means that **b is even**.

Logic in Action

- Prove that **√2 is irrational.**

  Proof: (**Proof by contradiction**)

  We have arrived at **a is even** and **b is even**

  But we said that

  √2 = a/b where *a/b is in its lowest terms*.

  We have a contradiction and so our assumption that

  √2 is rational

  is not correct.