Predicate Logic

- **OUTLINE**
  - **Introduction**
  - **Basic Terms and Concepts**
    - Open propositions
    - Quantifiers
  - **Syntax & Semantics**
  - Translating of English Statements
  - **Inference Rules / Laws of Equivalence**
  - Methods of Proof
    - Chain of Reasoning
    - Proof of Resolution

Rules of Inference for Quantified Statements

- **Universal Instantiation (UI)**

  $(\forall x)P(x)$

  $\therefore P(a)$

  where *a* is an arbitrary individual object in the universe of discourse

  - ***Example***:

    All birds have feathers.
    A penguin is a bird.
    Therefore a penguin has feathers.

Rules of Inference for Quantified Statements

- **Universal Generalization (UG)**

  $P(a)$

  $\therefore (\forall x)P(x)$

  provided we know P(a) is true for each object in the universe of discourse

  - ***Example***:

    A shark has gills.
    (A shark is a fish.)
    Therefore all fish have gills.

Rules of Inference for Quantified Statements

- **Existential Instantiation (EI)**

  $(\exists x)P(x)$

  $\therefore P(a)$

  where *a* is an individual object satisfying P and has no previous occurence.

  - ***Example***:

    Some dogs bite.
    Therefore Fluffy bites. (Assuming Fluffy is a dog.)

Rules of Inference for Quantified Statements

- **Existential Generalization (EG)**

  $P(a)$

  $\therefore (\exists x)P(x)$

  - ***Example***:

    Roger is a Swiss who plays tennis.
    (Therefore) Some Swiss play tennis.

Laws of Equivalence for Quantified Statements

- **First four laws...**
  1. $(\forall x)P(x) \vee R \leftrightarrow (\forall x)[\, P(x) \vee R \,]$
  2. $(\exists x)P(x) \vee R \leftrightarrow (\exists x)[\, P(x) \vee R \,]$
  3. $(\forall x)P(x) \wedge R \leftrightarrow (\forall x)[\, P(x) \wedge R \,]$
  4. $(\exists x)P(x) \wedge R \leftrightarrow (\exists x)[\, P(x) \wedge R \,]$
- **Rules of Negation**
  5. $\sim(\forall x)P(x) \leftrightarrow (\exists x) \sim P(x)$
  6. $\sim(\exists x)P(x) \leftrightarrow (\forall x) \sim P(x)$

  Examples:
  - *Not* all people can read.
    $\leftrightarrow$ Some people can*not* read.
  - *It is not true* that there is a person that has horns.
    $\leftrightarrow$ All people do *not* have horns.

Laws of Equivalence
for Quantified Statements

- **Distributive Laws**
  1. $(\forall x)P(x) \wedge (\forall x)Q(x) \leftrightarrow (\forall x)[\ P(x) \wedge Q(x)\ ]$
     Ex: All things are beautiful and all things are free.
     $\leftrightarrow$ All things are both beautiful and free.

  2. $(\exists x)P(x) \vee (\exists x)Q(x) \leftrightarrow (\exists x)[\ P(x) \vee Q(x)\ ]$
     Ex: Some things are good or some things do not last.
     $\leftrightarrow$ Some things are either good or do not last.

Laws of Equivalence
for Quantified Statements

- **Distributive Laws**
  3. $(\forall x)P(x) \vee (\forall x)Q(x) \leftrightarrow (\forall x)P(x) \vee (\forall y)Q(y)$
     $\leftrightarrow (\forall x)(\forall y)[\ P(x) \vee Q(y)\ ]$
  4. $(\exists x)P(x) \wedge (\exists x)Q(x) \leftrightarrow (\exists x)P(x) \wedge (\exists y)Q(y)$
     $\leftrightarrow (\exists x)(\exists y)[\ P(x) \wedge Q(y)\ ]$

Examples:
- Some integers are even and some integers are odd.
  $\neq$ Some integers are both even and odd.
- Everything is good or everything are perfect.
  $\neq$ Everything is (either) good or perfect.

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  - **Methods of Proof**
    - Chain of Reasoning
    - Proof of Resolution

Methods of Proof

- **CHAIN of REASONING**
  - Same procedure as in propositional logic:
    - Obtain a sequence of statements
      - Use Rules of Inference and Laws of Equivalence in propositional logic
      - Apply Quantificational Rules of Inference to drop/add quantifiers.
      - Apply Laws of Equivalence *for Quantified Statements* to replace open propositions involving quantifiers.
    - Arrive at the given conclusion.

Methods of Proof

- **CHAIN of REASONING**
  Examples/Exercises
  1) Every dog barks. Poochie does not bark. Therefore, Poochie is not a dog.
  2) Every dog barks and bites. Poochie does not bark. Therefore, Poochie is not a dog.
  3) No fish can walk. Some things that can walk have two legs. Therefore, some things that have two legs are not fish.
  4) All geeks are thinkers. All thinkers are night owls. Therefore, anyone who is not a night owl is not a geek.

Methods of Proof

- **CHAIN of REASONING**
  Examples/Exercises
  5) Everyone in this class owns a cell phone. Someone in this class has not used MMS. Therefore, someone who owns a cell phone has not used MMS.
  6) A check is void if it has not been cashed for 30 days. You cannot cash a check which is void. This check (is a check that) has not been cashed for 30 days. Therefore, we have a check that cannot be cashed.
  7) Some good people admire all honest people. All good people do not admire cheaters. Therefore, no honest person is a cheater.

Methods of Proof

- **CHAIN of REASONING**
  - **Remarks on applying Rules of Inference**
    - Use EI before UI.
    - When using EI, the individual object must be one that has *not* been used previously in the proof.
    - Don't apply UG to an object originally derived by EI.

Methods of Proof

- **CHAIN of REASONING**
  - **Remarks on applying Rules of Inference**
    - ***Example***:

1. $(\exists x)[B(x) \wedge E(x)]$	(*Some bats have big ears*.)	
2. $(\exists x)[P(x) \wedge \sim S(x)]$	(*Some people cannot swim*.)	
3. $B(a) \wedge E(a)$	1	EI
4. $P(a) \wedge \sim S(a)$	2	EI
5. $B(a)$	3	Simplication
6. $P(a)$	4	Simplication
7. $B(a) \wedge P(a)$	5, 6	Conjunction
8. $(\exists x)[B(x) \wedge P(x)]$	7	EG

Methods of Proof

- **CHAIN of REASONING**
  - **Remarks on applying Rules of Inference**
    - ***Example***:

1. $(\exists x)[B(x) \wedge \sim F(x)]$	(*Some birds cannot fly.*)	
2. $B(a) \wedge \sim F(a)$	1	EI
3. $(\forall x)[B(x) \wedge \sim F(x)]$	2	UG

Methods of Proof

- **PROOF by RESOLUTION**
  - Same procedure as in propositional logic:
    - Convert argument to its conjunctive normal form.
    - Negate conclusion and convert to CNF.
    - Arrive at a contradiction using the Resolution Rule.
      - Involves **unification** (more later)

Methods of Proof

- **PROOF by RESOLUTION**
  - **Conversion to Clausal Form (CNF)**
    - Replace $\rightarrow$.
    - Reduce the scope of $\sim$.
      - Replace $\sim[\sim P(x)]$ with $P(x)$
      - Replace $\sim[P(x) \wedge Q(x)]$ with $[\sim P(x) \vee \sim Q(x)]$
      - Replace $\sim[P(x) \vee Q(x)]$ with $[\sim P(x) \wedge \sim Q(x)]$
      - Replace $\sim(\forall x)P(x)$ with $(\exists x) \sim P(x)$
      - Replace $\sim(\exists x)P(x)$ with $(\forall x) \sim P(x)$

Methods of Proof

- **PROOF by RESOLUTION**
  - **Conversion to Clausal Form (CNF)**
    - Remove quantifiers

      1) Standardize variables so that each quantifier binds a unique variable.

      Example:

      change $(\forall x)P(x) \vee (\forall \mathbf{x})Q(\mathbf{x})$

      to $(\forall x)P(x) \vee (\forall \mathbf{y})Q(\mathbf{y})$

      2) Move all quantifiers to the left of the scope without changing their relative order.

      Example:

      change $(\forall x)[\, P(x) \vee \mathbf{(\forall y)}Q(y)\,]$

      to $(\forall x)\mathbf{(\forall y)}[\, P(x) \vee Q(y)\,]$

Methods of Proof

- **PROOF by RESOLUTION**
  - **Conversion to Clausal Form (CNF)**
    - Remove quantifiers *(cont'd)*

      3) Eliminate existential quantifiers by substituting a constant.

      Example:

      convert $(\exists x)P(\mathbf{x})$ to simply $P(\mathbf{s_1})$

      If the existential quantifiers *occurs within the scope of universal quantifiers*, substitute a skolem function for each variable.

      Example:

      replace $(\forall x)(\exists y)$ likes(x, **y**)

      with $(\forall x)$ likes(x, **s(x)**)

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Methods of Proof

- **PROOF by RESOLUTION**
  - **Conversion to Clausal Form (CNF)**
    - Remove quantifiers *(cont'd)*

      4) Drop remaining universal quantifiers.

      5) Convert the matrix into a conjunction of disjuncts.
    - Treat each term in a conjunct as a separate clause (premise).
    - Standardize apart variables in the set of clauses generated in the previous step. That is, rename the variables so that no two clauses refer to the same variable.

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Methods of Proof

- **PROOF by RESOLUTION**
  - **Unification**
    - The process of **finding a substitution** (a set of replacements of variables by terms) for predicate parameters.
    - ***Rules***:
      - different constants cannot match.

        Example: **P**(**b**) and **P**(**c**) are *not* unifiable
      - a variable may be replaced by a constant.

        Examples:

        P(**x**) and P(**c**) are unifiable

        P(f(**x**)) and **P**(f(**a**)) are unifiable

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Methods of Proof

- **PROOF by RESOLUTION**
  - **Unification** *(cont'n)*
    - a variable may be replaced by another variable.

      Example: **P**(**x**) and **P**(**y**) are unifiable
    - a variable may be replaced by a function as long as the function does not contain an instance of the variable.

      Examples:

      P(**x**) and P(**f**(**x**)) are *not* unifiable

      P(**x**) and P(**f**(**a**)) are unifiable

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Methods of Proof

- **PROOF by RESOLUTION**
  - ***Another example*:**

    Dialogue from <u>*Monty Python and the Holy Grail*</u> *(1975)*

Sir Bedevere: There are ways of telling whether she is a witch.
Peasant 1: Are there? Oh well, tell us.
Sir Bedevere: Tell me. What do you do with witches?
Peasant 1: Burn them.
Sir Bedevere: And what do you burn, apart from witches?
Peasant 1: More witches.
Peasant 2: Wood.

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Methods of Proof

- **PROOF by RESOLUTION**
  - ***Example 2*:** (cont'n)

Sir Bedevere: Good. Now, why do witches burn?
Peasant 3: Because they're made of ... wood?
Sir Bedevere: Good. So how do you tell whether she is made of wood?
Peasant 1: Build a bridge out of her.
Sir Bedevere: But can you not also build bridges out of stone?
Peasant 1: Oh yeah.
Sir Bedevere: Does wood sink in water?
Peasant 1: No, no, it floats! ... It floats! Throw her into the pond!

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Methods of Proof

- **PROOF by RESOLUTION**
  - ***Example 2*:** (cont'n)

    Sir Bedevere: No, no. What else floats in water?
    King Arthur: A duck.
    Sir Bedevere: Exactly. So, logically...
    Peasant 1: If she weighed the same as a duck ... she's made of wood.
    Sir Bedevere: And therefore ...
    Peasant 2: A witch!

Methods of Proof

- **PROOF by RESOLUTION**
  - ***Example 2*:** (cont'n)

    <u>*Translations*</u>:

    1) $(\forall x)$ [**B**urns (x) $\wedge$ **W**oman (x) $\rightarrow$ wi**T**ch(x)]
    2) **W**oman(girl)
    3) $(\forall x)$ [woo**D**(x) $\rightarrow$ **B**urns(x)]
    4) $(\forall x)$ [**F**loats(x) $\rightarrow$ woo**D**(x)]
    5) **F**loats(duck)
    6) $(\forall x)(\forall y)$ [**F**loats(x) $\wedge$ **S**ameweight(x,y) $\rightarrow$ **F**loats(y)]
    7) **S**ameweight(duck,girl)
    8) *Therefore*: wi**T**ch(girl)

Methods of Proof

- **PROOF by RESOLUTION**
  - ***Example 2*:** (cont'n)

    <u>*Conjunction normal form*</u> (verify!!):

    1) $\sim\mathbf{B}(x_1) \vee \sim\mathbf{W}(x_1) \vee \mathbf{T}(x_1)$
    2) $\mathbf{W}(\text{girl})$
    3) $\sim\mathbf{D}(x_2) \vee \mathbf{B}(x_2)$
    4) $\sim\mathbf{F}(x_3) \vee \mathbf{D}(x_3)$
    5) $\mathbf{F}(\text{duck})$
    6) $\sim\mathbf{F}(x_4) \vee \sim\mathbf{S}(x_4,y) \vee \mathbf{F}(y)$
    7) $\mathbf{S}(\text{duck,girl})$
    8) $\sim\mathbf{T}(\text{girl})$