

## Exercise 1: (Elves and Vampires)

Leading Thoughts: We know, that any given person A either a vampire, always lying, or an elf, always telling the truth, is. As convention, we will give persons truth-values, just like atomic formulae, where elves will be true ("1") and vampires will be false ("0").

Any statement F given by person A is either true and A is an elf, or it is false and A is a vampire. Thus, we can establish a new "meta"-statement G, which must always be true:  $G = (A \wedge F) \oplus (\neg A \wedge \neg F)$ .

For parts D and E we will use  $G = G_1 \wedge G_2$ , where  $G_{1,2} = (A \wedge F_{1,2}) \oplus (\neg A \wedge \neg F_{1,2})$ , and  $F_{1,2}$  are the statements of A and B respectively.

### 1.A

We model F as follows:

$$F = \neg A \vee \neg B$$

So the truth table looks like this:

A B	F	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	1	0	0	0
0 1	1	0	0	0
1 0	1	1	0	1
1 1	0	0	0	0

Thus our meta-statement G holds only true for  $A = 1$  (elf),  $B = 0$  (vampire).

### 1.B

We model F as follows:

$$F = \neg A \oplus B$$

So the truth table looks like this:

A B	F	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	1	0	0	0
0 1	0	0	1	1
1 0	0	0	0	0
1 1	1	1	0	1

Thus our meta-statement G holds true for  $B = 1$  (elf). A is ambig, meaning it could either be an elf or a vampire.

### 1.C

This is a special case of 1.B, where we could just assume  $B = 0$ . Our meta-statement G is not satisfiable in this case, meaning the statement cannot be made by elves or vampires.

**1.D**

We model our F's as follows:

$$F_1 = \neg A \wedge \neg B \wedge \neg C$$

$$F_2 = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)$$

So the truth table looks like this:

A B C	$F_1$	$A \wedge F_1$	$\neg A \wedge \neg F_1$	$G_1$	$F_2$	$A \wedge F_2$	$\neg A \wedge \neg F_2$	$G_2$	G
0 0 0	1	0	0	0	0	0	1	1	0
0 1 0	0	0	1	1	1	0	0	0	0
1 0 0	0	0	0	0	1	1	0	1	0
1 1 0	0	0	0	0	0	0	0	0	0
0 0 1	0	0	1	1	1	0	0	0	0
0 1 1	0	0	1	1	0	0	1	1	1
1 0 1	0	0	0	0	0	0	0	0	0
1 1 1	0	0	0	0	0	0	0	0	0

Thus our meta-statement G holds true for  $A = 0$  (vampire),  $B = C = 1$  (elf).

**1.E****1.F**

We model F as follows:

$$F = \neg A \wedge B$$

So the truth table looks like this:

A B	$F$	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	0	0	1	1
0 1	1	0	0	0
1 0	0	0	0	0
1 1	0	0	0	0

Thus our meta-statement G holds true for  $A = B = 0$  (vampire).

**Exercise 2: (Borromean formulas)**

Choose

$$A_1 = A \wedge B$$

$$A_2 = B \wedge C$$

$$A_3 = \neg C \vee \neg A$$

-  $A_1 \wedge A_2 = A \wedge B \wedge B \wedge C = A \wedge B \wedge C$  satisfiable for  $A = B = C = 1$

-  $A_1 \wedge A_3 = (A \wedge B) \wedge (\neg C \vee \neg A) = B \wedge ((A \wedge \neg C) \vee (A \wedge \neg A)) = B \wedge A \wedge \neg C$   
*unsatisfiable*

satisfiable for  $A = B = 1, C = 0$

-  $A_2 \wedge A_3 =$  similar to  $A_1 \wedge A_2$ , now choose  $B = C = 1, A = 0$  to satisfy formula

$$- A_1 \wedge A_2 \wedge A_3 = A \wedge B \wedge B \wedge C \wedge (\neg C \vee \neg A) = A \wedge B \wedge ((\neg C \wedge C) \vee (\neg A \wedge \neg C)) =$$

*unsatisfiable*

$$\underset{\text{unsatisfiable}}{A \wedge \neg A} \wedge B \wedge C = 0, \text{ for all possible inputs}$$

### Exercise 3: (XOR)

A B	$A \oplus B = 1 \text{ iff } A \neq B$	$(A \vee B) \wedge \neg(A \vee B)$
0 0	0	0
0 1	1	1
1 0	1	1
1 1	0	0

The truth table shows that both formulas evaluate to the same value for all possible inputs, hence they are equivalent.

### Exercise 4: (Truth value tables)

$$F_1 = \neg(A \implies B) = \neg(\neg A \vee B) = A \wedge \neg B$$

$$F_2 = \neg(\neg A \vee \neg(\neg B \vee \neg A)) = A \wedge (\neg B \vee \neg A) = (A \wedge \neg B) \vee \underset{= \text{unsatisfiable}}{(A \wedge \neg A)} = A \wedge \neg B$$

$$F_3 = (A \wedge B) \wedge (\neg B \vee C) = \underset{\text{unsatisfiable}}{(A \wedge B \wedge \neg B)} \vee (A \wedge B \wedge C) = A \wedge B \wedge C$$

$$F_4 = A \Leftrightarrow (B \Leftrightarrow C)$$

Evaluated in a truth table:

C A B	$F_1$	$F_2$	$F_3$	$F_4$
0 0 0	0	0	0	1
0 0 1	0	0	0	0
0 1 0	1	1	0	0
0 1 1	0	0	0	0
1 0 0	-	-	0	0
1 0 1	-	-	0	0
1 1 0	-	-	0	0
1 1 1	-	-	1	1

From the lecture we know that two Formulas are equivalent if their truth tables are equivalent, hence  $F_1$  and  $F_2$  are equivalent,  $F_3$  and  $F_4$  are not,  $F_1$  and  $F_3$  aren't either.