Exercise 1: (Elves and Vampires)

Leading Thoughts: We know, that any given person A either a vampire, always lying, or an elf, always telling the truth, is. As convention, we will give persons truth-values, just like atomic formulae, where elfs will be true ("1") and vampires will be false ("0").
Any statement F given by person A is either true and A is an elf, or it is false and A is an vampire. Thus, we can establish a new "meta"-statement G, which must always be true: $G = (A \wedge F) \oplus (\neg A \wedge \neg F)$.
For parts D and E we will use $G = G_1 \wedge G_2$, where $G_{1,2} = (A \wedge F_{1,2}) \oplus (\neg A \wedge \neg F_{1,2})$, and $F_{1,2}$ are the statements of A and B respectively.

1.A

We model F as follows:
$F = \neg A \vee \neg B$
So the truth table looks like this:

A B	F	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	1	0	0	0
0 1	1	0	0	0
1 0	1	1	0	1
1 1	0	0	0	0

Thus our meta-statement G holds only true for $A = 1$ (elf), $B = 0$ (vampire).

1.B

We model F as follows:
$F = \neg A \oplus B$
So the truth table looks like this:

A B	F	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	1	0	0	0
0 1	0	0	1	1
1 0	0	0	0	0
1 1	1	1	0	1

Thus our meta-statement G holds true for $B = 1$ (elf). A is ambig, meaning it could either be an elf or a vampire.

1.C

This is a special case of 1.B, where we could just assume $B = 0$. Our meta-statement G is not satisfiable in this case, meaning the statement cannot be made by elfs or vampires.

1.D

We model our F's as follows:
$F_1 = \neg A \wedge \neg B \wedge \neg C$ (A's statement)
$F_2 = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)$ (B's statement)
So the truth table looks like this:

A B C	F_1	$A \wedge F_1$	$\neg A \wedge \neg F_1$	G_1	F_2	$B \wedge F_2$	$\neg B \wedge \neg F_2$	G_2	G
0 0 0	1	0	0	0	0	0	1	1	0
0 1 0	0	0	1	1	1	1	0	1	1
1 0 0	0	0	0	0	1	0	0	0	0
1 1 0	0	0	0	0	0	0	0	0	0
0 0 1	0	0	1	1	1	0	0	0	0
0 1 1	0	0	1	1	0	0	0	0	0
1 0 1	0	0	0	0	0	0	1	1	0
1 1 1	0	0	0	0	0	0	0	0	0

Thus our meta-statement G holds true for $A = C = 0$ (vampire), $B = 1$ (elf).

1.E

We model our F's as follows:
$F_1 = \neg A \wedge \neg B \wedge \neg C$ (A's statement)
$F_2 = (\neg A \wedge \neg B \wedge C) \vee (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C)$ (B's statement)
So the truth table looks like this: (note, that G_1 is equivalent to part D)

A B C	F_1	$A \wedge F_1$	$\neg A \wedge \neg F_1$	G_1	F_2	$B \wedge F_2$	$\neg B \wedge \neg F_2$	G_2	G
0 0 0	1	0	0	0					
0 1 0	0	0	1	1					
1 0 0	0	0	0	0					
1 1 0	0	0	0	0					
0 0 1	0	0	1	1					
0 1 1	0	0	1	1					
1 0 1	0	0	0	0					
1 1 1	0	0	0	0					

Thus our meta-statement G holds true for $A = 0$ (vampire), $B = C = 1$ (elf).

1.F

We model F as follows:
$F = \neg A \wedge B$
So the truth table looks like this:

A B	F	$A \wedge F$	$\neg A \wedge \neg F$	G
0 0	0	0	1	1
0 1	1	0	0	0
1 0	0	0	0	0
1 1	0	0	0	0

Thus our meta-statement G holds true for $A = B = 0$ (vampire).

Exercise 2: (Borromean formulas)

Choose
$A_1 = A \wedge B$
$A_2 = B \wedge C$
$A_3 = \neg C \vee \neg A$

- $A_1 \wedge A_2 = A \wedge B \wedge B \wedge C = A \wedge B \wedge C$ satisfiable for A = B = C = 1
- $A_1 \wedge A_3 = (A \wedge B) \wedge (\neg C \vee \neg A) = B \wedge ((A \wedge \neg C) \vee \underbrace{(A \wedge \neg A)}_{unsatisfiable}) = B \wedge A \wedge \neg C$ satisfiable for A = B = 1, C = 0
- $A_2 \wedge A_3 =$ similar to $A_1 \wedge A2$, now choose B = C = 1, A = 0 to satisfy formula
- $A_1 \wedge A_2 \wedge A_3 = A \wedge B \wedge B \wedge C \wedge (\neg C \vee \neg A) = A \wedge B \wedge (\underbrace{(\neg C \wedge C)}_{unsatisfiable} \vee (\neg A \wedge \neg C)) = \underbrace{A \wedge \neg A}_{unsatisfiable} \wedge B \wedge C = 0$, for all possible inputs

Exercise 3: (XOR)

A B	$A \oplus B = 1$ iff $A \neq B$	$(A \vee B) \wedge \neg(A \vee B)$
0 0	0	0
0 1	1	1
1 0	1	1
1 1	0	0

The truth table shows that both formulas evaluate to the same value for all possible inputs, hence they are equivalent.

Exercise 4: (Truth value tables)

$F_1 = \neg(A \implies B) = \neg(\neg A \vee B) = A \wedge \neg B$

$F_2 = \neg(\neg A \vee \neg(\neg B \vee \neg A)) = A \wedge (\neg B \vee \neg A) = (A \wedge \neg B) \vee \underbrace{(A \wedge \neg A)}_{=unsatisfiable} = A \wedge \neg B$

$F_3 = (A \wedge B) \wedge (\neg B \vee C) = \underbrace{(A \wedge B \wedge \neg B)}_{unsatisfiable} \vee (A \wedge B \wedge C) = A \wedge B \wedge C$

$F_4 = A \Leftrightarrow (B \Leftrightarrow C)$

Evaluated in a truth table:

C A B	F_1	F_2	F_3	F_4
0 0 0	0	0	0	1
0 0 1	0	0	0	0
0 1 0	1	1	0	0
0 1 1	0	0	0	0
1 0 0	-	-	0	0
1 0 1	-	-	0	0
1 1 0	-	-	0	0
1 1 1	-	-	1	1

From the lecture we know that two Formulas are equivalent if their truth tables are equivalent, hence F_1 and F_2 are equivalent, F_3 and F_4 are not, F_1 and F_3 aren't either.