Lecture Notes on Complex Analysis

N.B.: This Lecture Note is freely distributed study material prepared as per CBCS Syllabus (UG) of UGC. For maximum benefit, it should be read along with the non-proprietary and proprietary resources (links given) available in Internet (web site, video, etc).

Course prepared by: *Surajit Sen*
Contact: *ssen55@yahoo.com*

Instruction to Student/Tutor: If any part of this note appears difficult, a student may avoid that topic in the first-reading. We prefer to keep such advanced topics for the students preparing for competitive exams like JAM, JEST, NET, etc.

Contents

1 Course Overview

1.1 Course Structure

- **Unit 1**:
  Complex Analysis: Brief Revision of Complex Numbers and their Graphical Representation. Euler's formula, De Moivre's theorem, Roots of Complex Numbers. Functions of Complex Variables. Analyticity and Cauchy-Riemann Conditions. Examples of analytic functions.

- **Unit 2**:
  Singular functions: poles and branch points, order of singularity, branch cuts. Integration of a function of a complex variable. Cauchy Inequality. Cauchy Integral formula. Simply and multiply connected region. Laurent and Taylor expansion.

- **Unit 3**:
  Residues at simple pole, Residue at a pole of order greater than unity, Residue at infinity, Cauchy Residue Theorem, Application in solving Definite Integrals.

2 Complex Number & its properties

2.1 Some examples of different types of numbers

- Natural Number: $\mathbb{N} = \{1, 2, 3, 4 \dots\}$
- Whole Number: $\mathbb{W} = \{0, 1, 2, 3, 4 \dots\}$, i.e., $\mathbb{W} = \{0, \mathbb{N}\}$
- Integers: $\mathbb{Z} = \{\cdots - 2, -1, 0, 1, 2, 3, 4 \dots\}$, i.e., $\mathbb{Z} = \{-\mathbb{N}, \mathbb{W}\}$
- Rational Number ($\mathbb{Q} = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$): $\mathbb{Q} = \{\frac{6}{7}, \frac{2}{3}, 2.25, \pi \dots, \mathbb{Z}\}$
- Irrational Number ($\mathbb{Q}' \neq \frac{p}{q}$): $\mathbb{Q}' = \{\sqrt{2}, \sqrt{7}, 3\pi, \frac{1+\sqrt{5}}{2} \dots\}$
- Pure Real Number ($\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$): $\mathbb{R} = \{1, 2.5, 3/2, \sqrt{2}, 4.07, \pi, e, 7e \dots\}$
- Complex Number: $\mathbb{C} = \{3 + 4i, -3 + 4i, -3 - 4i, 3 - 4i\}$

2.2 Complex Numbers and their Properties

The complex number (z) and its conjugate ($\bar{z}$) is defined as,

$$\begin{aligned} z &= a + ib \\ \bar{z} &= a - ib, \end{aligned} \tag{2.1}$$

where

$$i = \sqrt{-1} \tag{2.2}$$

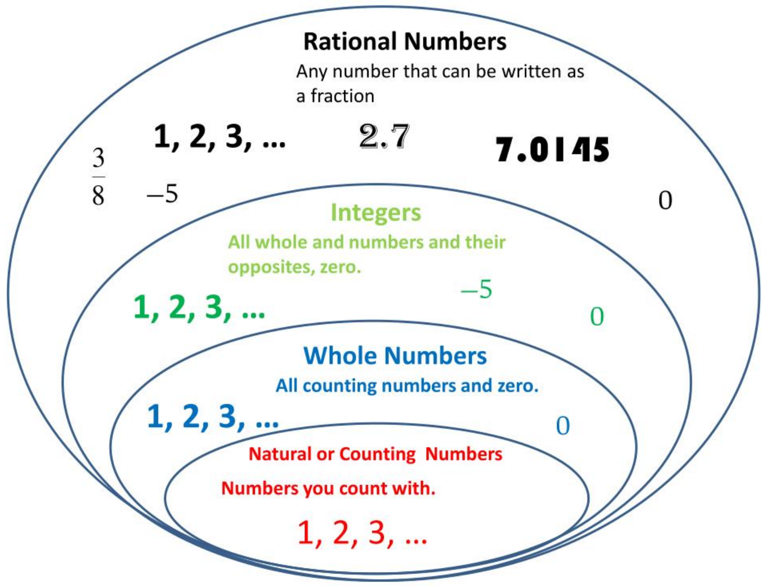

Figure 1: Set of different types of numbers: $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ (with $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$, Not shown)

with and $a, b \in \mathbb{R}$, respectively. Then it follows that,

$$\begin{aligned} a &= \Re\{z\} = \frac{1}{2}(z + \bar{z}) \\ b &= \Im\{z\} = \frac{1}{2i}(z - \bar{z}) \end{aligned} \tag{2.3}$$

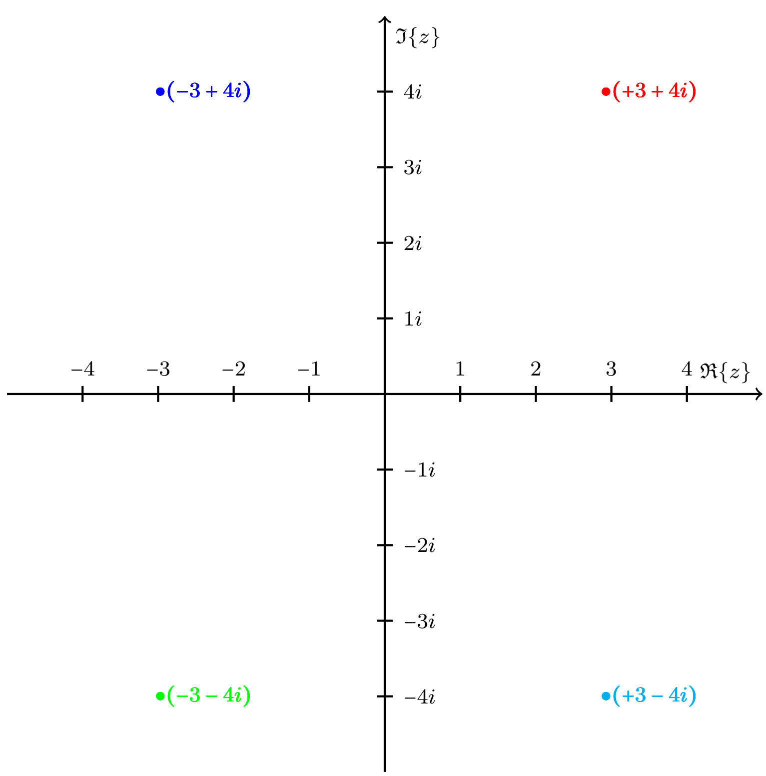

Figure 2: Location of a complex number in the rectangular (Cartesian) representation

- The algebraic operation of complex number gives a complex number. Let $z_1 = a_1 + ib_1, z_2 = a_2 + ib_2$ ($z_1, z_2 \in \mathbb{C}$ with $\in a_1, a_2, b_1, b_2 \in \mathbb{R}$)
  - i) Addition :

$$\begin{aligned} z_1 + z_2 &= (a_1 + ib_1) + (a_2 + ib_2) \\ &= (a_1 + a_2) + i(b_1 + b_2) \in \mathbb{Z} \end{aligned} \tag{2.4a}$$

  - ii) Substraction:

$$\begin{aligned} z_1 - z_2 &= (a_1 + ib_1) - (a_2 + ib_2) \\ &= (a_1 - a_2) - i(b_2 - b_1) \in \mathbb{Z} \end{aligned} \tag{2.4b}$$

  - iii) Multiplication:

$$\begin{aligned} z_1 z_2 &= (a_1 + ib_1)(a_2 + ib_2) \\ &= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2) \in \mathbb{Z} \end{aligned} \tag{2.4c}$$

  - iv) Division:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} \\ &= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} - i\frac{a_1 b_2 - b_1 a_2}{a_2^2 + b_2^2} \in \mathbb{Z} \end{aligned} \tag{2.4d}$$

- Property VII: Apart from above properties, it holds following properties:
  - i) Commutative law for addition (multiplication): $z_1 + z_2 = z_2 + z_1$ $(z_1 \cdot z_2 = z_2 \cdot z_1)$
  - ii) Associative law for addition (multiplication): $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ $(z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3)$
  - iii) Distributive law for multiplication: $z_1 \cdot (z_2 + z_3) = (z_1 \cdot z_2 + z_1 \cdot z_3)$
  - iv) Existence of Additive Inverse (Multiplicative Inverse): $z + (-z) = 0$ $(z \cdot z^{-1} = 1)$
  - vi) Existence of Additive Identity (Multiplicative Identity): $z + 0 = z$ $(z \cdot 1 = z)$

2.3 Polar Representation of Complex Number

- Modulus and Argument: The rectangular or cartesian form of a complex number can be also expressed in the polar (trigonometric) form shown in Fig.4:,

$$z = re^{\theta} \tag{2.5}$$

The *Modulus* and *Argument* of the complex number are given by

$$Mod(z) = \mid z \mid = r \tag{2.6a}$$

$$arg(z) = \theta = \tan^{-1} \frac{y}{x} \tag{2.6b}$$

It is important to note that by replacing $\theta \to \theta + 2\pi k$ $(k = 0, 1, 2, \ldots, n-1)$, Eq.((2.6b)) retains its value. This indicates that *a complex number is indeed a multi-valued function.*

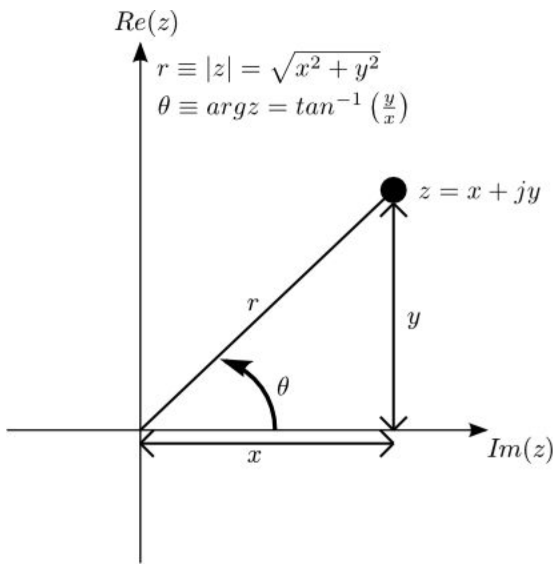

Figure 3: Polar representation of a complex number

- Property of Modulus (Triangular Inequality):

$$\begin{aligned} i) &\quad |z_1 \cdot z_2| = |z_1| \cdot |z_2| \\ ii) &\quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \\ iii) &\quad |z_1 + z_2| \leq |z_1| + |z_2| \\ iv) &\quad |z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2 \\ v) &\quad ||z_1| - |z_2|| \leq |z_1| + |z_2| \\ vi) &\quad |z_1 - z_2| \geq ||z_1| - |z_2|| \end{aligned} \tag{2.7}$$

- Property of argument:

$$\begin{aligned} i) &\quad arg(z_1 z_2) = arg(z_1) + arg(z_2) \\ ii) &\quad arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) \end{aligned} \tag{2.8}$$

2.4 n-th Root, n-th Power and logarithm of a complex number

The cartesian representation of a complex number can be also expressed in planer polar form,

$$z = re^{i\frac{\theta}{n}} \tag{2.9}$$

Below we discuss i) n-th root, ii) Power and iii) logarithm of a complex number:

- **n-th root of a complex number**:

The n-th root of a complex number is given by,

$$\begin{aligned}
z^{\frac{1}{n}} &= r^{\frac{1}{n}} e^{i\frac{\theta}{n}} \quad \forall n = \pm 1, \pm 2, \dots \\
&= r^{\frac{1}{n}} e^{\frac{\Theta + 2\pi k}{n}}, \quad \forall k = 0, \pm 1, \pm 2, \dots, \pm n - 1 \\
&= r^{\frac{1}{n}} \Big(\cos\frac{\Theta \pm 2\pi k}{n} + i \sin\frac{\Theta + 2\pi k}{n} \Big) \\
&= r^{\frac{1}{n}} \Big(\cos\frac{Arg(z) \pm 2\pi k}{n} + i \sin\frac{Arg(z) + 2\pi k}{n} \Big) \\
&= r^{\frac{1}{n}} \Big(\cos\frac{arg(z)}{n} + i \sin\frac{arg(z)}{n} \Big)
\end{aligned} \tag{2.13}$$

Thus we note that the the complex number is a multi-valued function because of different values of $\{k, n\}$.

- **Relation between 'Principal Argument' and 'argument' of the complex number:**

  From above we obtain the *argument* and its relation with Principal Argument defined by,

$$\begin{aligned}
\theta &= \Theta \pm 2\pi k \quad i.e., \\
arg(z) &= Arg(z) \pm 2\pi k
\end{aligned} \tag{2.10}$$

  Here, $arg(z)$ is measured from the positive direction of x-axis and therefore have the range: $0 \leq arg(z) \leq 2\pi$, while $Arg(z)$ is measured from the minimum distance from the positive direction of x-axis and hence the range: $-\pi \leq Arg(z) \leq \pi$. Figure illustrated below the key difference of *argument* and *Argument*.

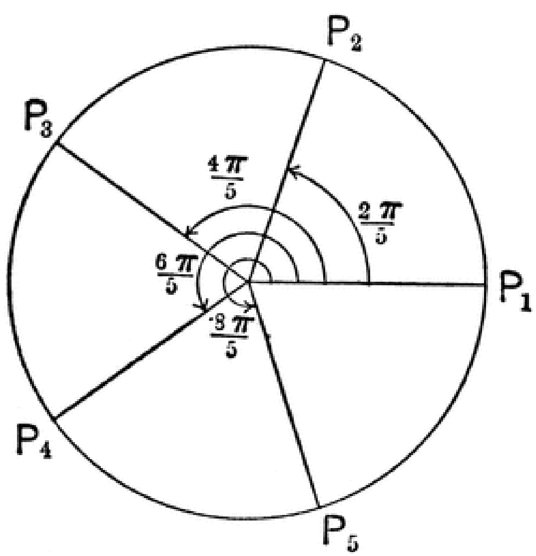

Figure 4: Representation of $arg(z)$ which lies between $0 \leq arg(z) \leq 2\pi$

- **Primitive of the complex number:**

  The Primitive of a complex number is the n-th root defined on a unit circle i.e., $z^n = 1$ with $r = 1$

Quadrant	Sign of x & y	Arg(z)
I	x, y>0	$\tan^{-1}\frac{y}{x}$
II	x<0, y>0	$\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$
III	x, y < 0	$-\pi + \tan^{-1}\left\|\frac{y}{x}\right\|$
IV	x>0, y<0	$-\tan^{-1}\left\|\frac{y}{x}\right\|$

Figure 5: Table of $Arg(z)$ which lies between $0 \le Arg(z) \le \pi$

and $\theta = 2\pi$. Thus we have,

$$\begin{aligned} z^n &= 1 \\ \Rightarrow z &= e^{i\frac{2\pi}{n}} \\ \Rightarrow z &= \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \end{aligned} \tag{2.11}$$

Thus it follows that for $n = 1, 2, 3, \ldots$, we have $z^1 = 1$, $z^2 = -1$, $z^3 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ etc. This establishes that a complex function is a muti-valued function.

- **n-th power of the complex number:**

  From above, the n-th power of the complex number is given by,

$$\begin{aligned} z^n &= r^n e^{n\theta}, \\ &= r^n\Big(\cos n\theta + i\sin n\theta\Big) \quad 0 \neq \theta \neq 2\pi \end{aligned} \tag{2.12}$$

  where the De Moivre's theorem [1] is used. Thus in planer polar coordinate, we have

$$\begin{aligned} \cos\theta &= \Re\{z\} = \frac{1}{2}(1 + \frac{1}{z}) \\ \sin\theta &= \Im\{z\} = \frac{1}{2i}(1 - \frac{1}{z}) \end{aligned} \tag{2.13}$$

2.5 Logarithm of a Complex Number:

- **Principal Value of Logarithm:**

  The *Principal Value of Logarithm log*(z) is given by,

$$\begin{aligned} z &= re^{i\theta} \\ \Rightarrow a + ib &=\mid z \mid e^{i\theta} \\ \Rightarrow log(a + ib) &= log \mid z \mid + i\theta \\ &= log[\sqrt{a^2 + b^2}] + i\tan^{-1}\frac{x}{y} \\ &= log[Mod(z)] + iarg(z) \\ &= log[Mod(z)] + iArg(z) + 2\pi ik \quad \Big(0 \le arg(z) \le 2\pi\Big) \end{aligned} \tag{2.14}$$

[1]For example, De Moivre's theorem gives: i) $Arg(i) = \frac{\pi}{2}$, ii) $Arg(-i) = -\frac{\pi}{2}$, ii) $Arg(-1) = \pi$ etc.

- **Generalized Value of Logarithm**

  The *Generalized Value of Logarithm* ($Log(z)$ is defined by $n = 0$ in Eq.(2.16),

$$\begin{aligned} a + ib &= re^{i\theta} \\ \Rightarrow Log(a+ib) &= Log[Mod(z)] + i\Theta \quad \text{where} \quad k = 0 \\ &= Log[Mod(z)] + iArg(z) \quad -\pi \leq Arg(z) \leq \pi \end{aligned} \tag{2.15}$$

  From (2.13) and (2.14) it follows that $Log|z| = log|z|$, then the Principal and Generalized values of logarithm are related by,

$$log(z) = Log(z) + i2\pi n \quad \forall \quad n = \pm1, \pm2, \ldots \tag{2.16}$$

- **Properties of logarithm of a complex number:**

$$i)\,\log(z_1 z_2) = \log z_1 + \log z_2 \tag{2.17a}$$

$$ii)\,\log(\frac{z_1}{z_2}) = \log z_1 - \log z_2 \tag{2.17b}$$

$$iii)\,\log z_1 \log z_2 \neq Logz_1 Logz_2 \tag{2.17c}$$

- Complex Number Primer: Some properties of complex numbers

3 Differential Calculus on a Complex Plane

3.1 Analytic Function - Cauchy-Riemann (CR) relation

Consider the following mapping from z-plane to w-plane, i.e., $w = f(z)$, A complex function defined on a

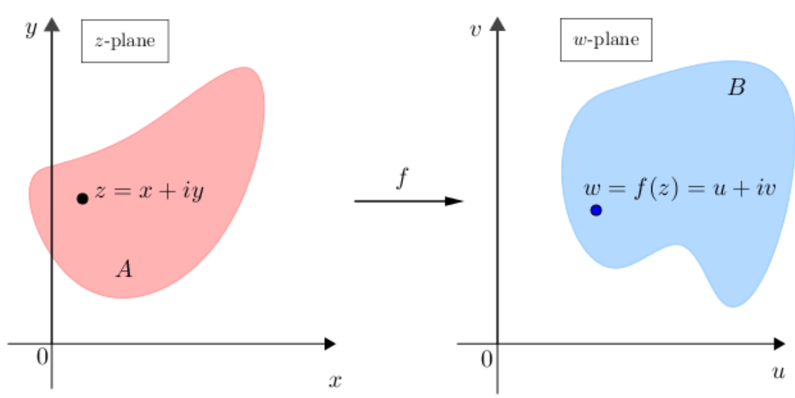

Figure 6: :
Mapping of a complex function

complex plane (w-plane) is said to be analytic if it is differentiable, i.e.,

$$\begin{aligned} w' &= \frac{\partial f(z)}{\partial z}, \\ &= \lim_{\Delta z \to \infty} \frac{f(z + \Delta z) - f(z)}{\Delta z}. \end{aligned} \tag{3.1}$$

Now recalling that,

$$\begin{aligned} f(z) &= \Delta u + i\Delta v, \\ \Delta f &= \Delta u + i\Delta v, \\ z &= x + iy, \\ \Delta z &= \Delta x + i\Delta y, \end{aligned} \tag{3.2}$$

Eq.(**??**) can be written as as

$$\begin{aligned} \frac{\partial f(z)}{\partial z} &= \lim_{\Delta x\to 0, \Delta y \to 0} \frac{\Delta u(x+\Delta x, y+\Delta y) + i\Delta v(x+\Delta x, y+\Delta y) - (\Delta u(x,y) - i\Delta v(x,y))}{\Delta x + i\Delta y} \\ &= \lim_{\Delta x\to 0, \Delta y \to 0} \frac{\Delta u(x+\Delta x, y+\Delta y) - \Delta u(x,y)) - i(\Delta v(x+\Delta x, y+\Delta y - \Delta v(x,y))}{\Delta x + i\Delta y} \end{aligned} \tag{3.3}$$

Now the limiting value can be achieved in two possible routes: Firstly, if we approaching along real axis (X-axis), i.e.,
Route I: $\Delta x \to 0, \Delta y = 0$

$$\begin{aligned} \frac{\partial f(z)}{\partial z} &= \lim_{\Delta x\to 0} \frac{\Delta u(x+\Delta x, y) - \Delta u(x,y)) - i(\Delta v(x+\delta x, y - \Delta v(x,y))}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{(\Delta u(x+\Delta x, y - \Delta u(x,y))}{\Delta x} + i \lim_{\Delta x\to \infty} \frac{(\Delta v(x+\Delta x, y - \Delta v(x,y))}{\Delta x} \\ &= \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} \end{aligned} \tag{3.4}$$

Or otherwise, i.e., if we move along the imaginary axis (Y-axis):
Route II: For $\Delta x = 0, \Delta y \to 0$, proceeding in the similar manner we obtain,

$$\frac{\partial f(z)}{\partial z} = -i\frac{\partial u(x,y)}{\partial y} + i\frac{\partial v(x,y)}{\partial y} \tag{3.5}$$

Comparing the real and imaginary part of Eqs.(3.4) and (3.5), we obtain well known Cauchy-Riemann (CR) relation,

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} &= -\frac{\partial v(x,y)}{\partial x}. \end{aligned} \tag{3.6}$$

which gives sufficient condition of analyticity of a function $f(z) \in \mathbb{C}$. [2] The above two relation can be combined to write as,

$$\frac{\partial f(z,\bar{z})}{\partial \bar{z}} = 0 = \frac{\partial f(z,\bar{z})}{\partial z}. \tag{3.7}$$

[2] Using the fact that, $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z - i\bar{z})$,

$$\begin{aligned} \frac{\partial}{\partial z} &= \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{i}{2}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial \bar{z}} &= \frac{\partial x}{\partial \bar{z}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar{z}}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y} \end{aligned}$$

In the plane polar coordinate, by noting the fact that, $x = r\cos\theta$ and $x = r\sin\theta$ and the chain rules, $\frac{\partial u(x,y)}{\partial r} = \frac{\partial u(x,y)}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y}{\partial r}$ and $\frac{\partial u(x,y)}{\partial \theta} = \frac{\partial u(x,y)}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u(x,y)}{\partial y}\frac{\partial y}{\partial \theta}$, the C-R relation can be expressed as,

$$\begin{aligned}\frac{\partial u(x,y)}{\partial r} &= \frac{1}{r}\frac{\partial v(x,y)}{\partial \theta}\\ \frac{\partial v(x,y)}{\partial r} &= -\frac{1}{r}\frac{\partial u(x,y)}{\partial \theta}.\end{aligned} \tag{3.8}$$

Furthermore, from Eq.(**??**) it follows that any complex function $\phi(x,y)$ $(= u(x,y)+iv(x,y))$ which satisfies the second order Laplace equation, i.e.,

$$\frac{\partial^2\phi(x,y)}{\partial x^2} + \frac{\partial^2\phi(x,y)}{\partial y^2} = 0, \tag{3.9}$$

which is called *Harmonic function*. In complex notation it can be equivalently written as

$$\frac{\partial^2\phi(z,\bar{z})}{\partial z\partial\bar{z}} = 0. \tag{3.10}$$

Finally we mention that if a function $f(z)$ in a domain $\mathfrak{D}$ is said to be a *holomorphic function* if it is analytic in that domain.

4 Series and sequence in a Complex Plane

4.1 Series and its convergence

A series is an infinite sum of complex number of definite order in a given domain $\mathfrak{D}$. For example,

$$\begin{aligned}S(z) &= \sum_{n=0}^{\infty} z^n + \dots\\ &= 1 + z + z^2 + z^3 + \dots + z^n\end{aligned} \tag{4.1}$$

In more general footing, it can be written as,

$$\begin{aligned}S(z) &= a_1(z) + a_2(z) + \dots a_n(z) + \dots\\ &= \sum_{n=1}^{\infty} a_n(z)\end{aligned} \tag{4.2}$$

The *convergence* of a *monotonically decreasing series* requires $a_{n+1}(z) < a_n(z)$ for which,

$$\lim_{n\to\infty} a_n(z) = 0 \tag{4.3}$$

and there exists an upper bound M such that $|a_n(z)| < M$. The necessary and sufficient condition for convergence defined as

$$|a_m(z) - a_n(z)| < \epsilon \quad \text{for} \quad m,n \in N. \tag{4.4}$$

There exists several tests for the convergence of a complex series:

Ratio Test:

The *Ratio Test* is a very popular test by which the convergence of a complex series can be tested very conclusively. To do that we define,

$$\left|\frac{z_m}{z_n}\right| = R, \tag{4.5}$$

- If $R > 1$, the series id said to *Divergent*.
- If $R < 1$, the series id said to *Convergent*.
- If $R = 1$, no consequence can be drawn.

This condition is often referred as *Cauchy Convergence Condition*. Apart from that there exists other tests like, Root Test, Raabe Test, Weierstrass M-test etc.

4.2 Taylor and Maclaurin Series

If a function $f(z)$ is analytic on $|z - z_0| < R$, then the complex Taylor series is defined as,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (z - z_0)^n a_n \quad \left(where a_n = \frac{f^{(n)}(z)}{n!} \right) \\ &= f(z_0) + (z - z_0) f'(z_0) + (z - z_0)^2 \frac{f''(z_0)}{2!} + \cdots + (z - z_0)^n \frac{f^{(n)}(z_0)}{n!} \end{aligned} \tag{4.6}$$

If the series is defined at $z = 0$, i.e., $f(z_0) = f(0)$ in every term, then it is called complex Maclaurin Series.

4.3 Laurent Series

If a function $w = f(z)$ is analytic for all $R_1 \leq |z - z_0| \leq R_2$, then

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} + \sum_{k=0}^{\infty} b_k (z - z_0)^k. \end{aligned} \tag{4.7}$$

The singular part $\{a_{-k}\}$ of the series is called *Principal part*, while remaining one is called *Analytic Part*. The coefficients of the series is given by

$$a_{-k} = \frac{1}{2\pi i} \oint_{\Gamma_1} dz \frac{f(z)}{(z - z_0)^{k+1}} \quad \text{for} \quad k = 1, 2, \ldots n \tag{4.8a}$$

$$b_k = \frac{1}{2\pi i} \oint_{\Gamma_2} dz f(z)(z - z_0)^k \quad \text{for} \quad k = 0, 1, 2, \ldots n \tag{4.8b}$$

A *Meromorphic Function* is a function which is analytic everywhere in the complex plane except at a finite number of poles, while an *entire function* is analytic throughout the entire complex plane.

5 Integral Calculus on a Complex Plane

5.1 Jordan and Non-Jordan Curve

The *Contour Integration* of a complex function $w = f(z)$ is a one dimensional line integral carried out along a fixed path having definite orientation. Broadly speaking the curves can be classified into two categories:

- Closed Contour: A *Closed Contour* is a curve which is parameterized as $t \subset \mathbb{R}$ such that $b \leq t \leq a$ and $t(a) = t(b)$. A closed curve is said to be a *Simple Contour* if it has different value inside and outside. Simple Closed Contour is also referred as *Jordan curve.*

- Open Contour: A *Open Contour* is a curve which is parameterized as $t \subset \mathbb{R}$ such that $b \leq t \leq a$ and $t(a) \neq t(b)$.

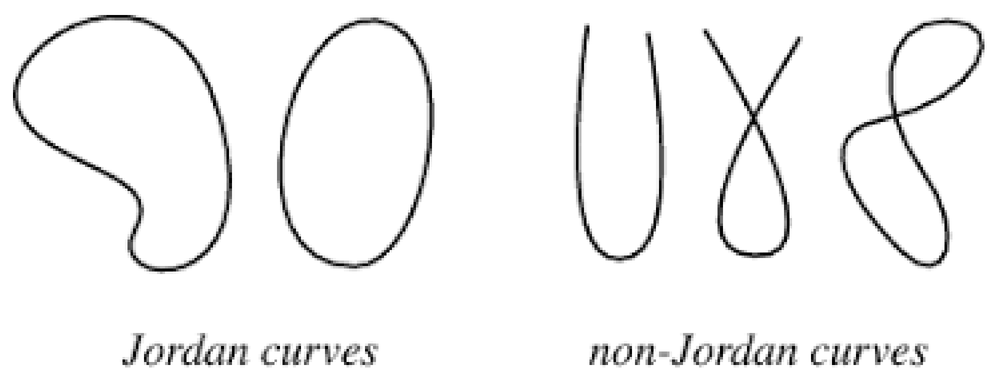

Figure 7: Jordan and Non-Jardon curves

The Jordan Curves may be either *Simply Connected Curve* or *Multiply Connected Curve* shown below:

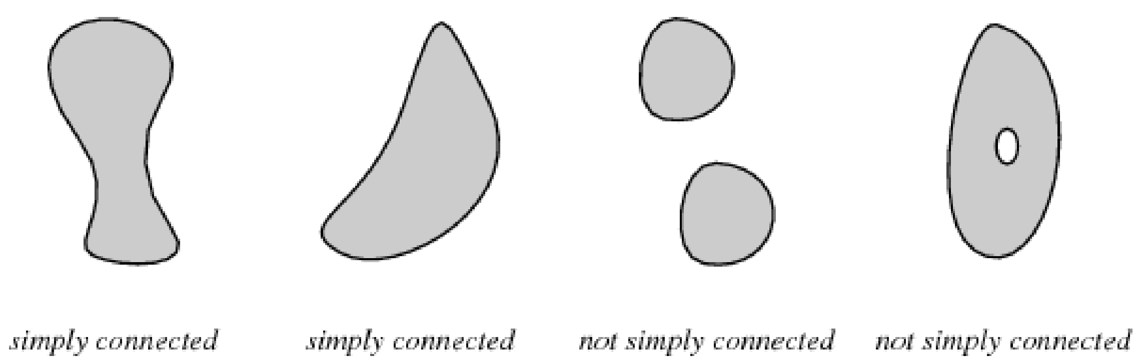

Figure 8: Simply Connected and Non-Simply (Multiply) Connected Curve

5.2 Various curves and their representations

Different types of Jodan (Simple) curves:

a) Circle with centre at origin ($z = 0$): $z(t) := re^{it}$ with $[t \in [0, 2\pi]$.

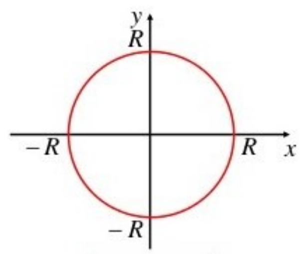

Figure 9: Circle with centre at $z = 0$ with radius R

b) Circle with centre at shifted origin ($z = z_0$): $z(t) := z_0 re^{it}$ with $[t \in [0, 2\pi]$.

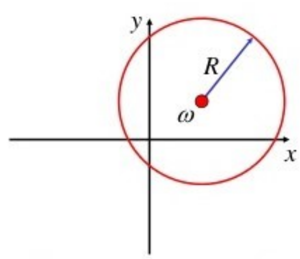

Figure 10: Circle with centre at $z = 0$ with radius R

c) Semi-circle with centre at origin ($z = 0$): $\gamma(t) := re^{it}$ with $([t \in [0, \pi])$.

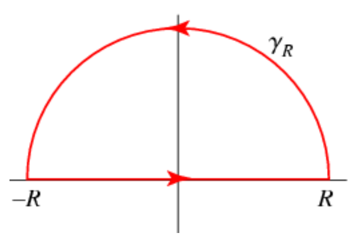

Figure 11: Half-circle with centre at $z = z_0$

d) Circle with centre at origin ($z = 0$): $\gamma(t) := re^{it/2}$ with $([t \in [0, 2\pi])$ with branch cut $(0, \infty]$.

5.3 Zeros, Singularity, Branch Point and Branch Cut

A complex function $f(z)$ by construction is a multi-valued function. If such a function is *not analytic* (i.e., not differentiable) at any point (or multiple number of points), but blows to infinity, then the point is said to be the *Singular Point* and the function is called *Singular Function*.

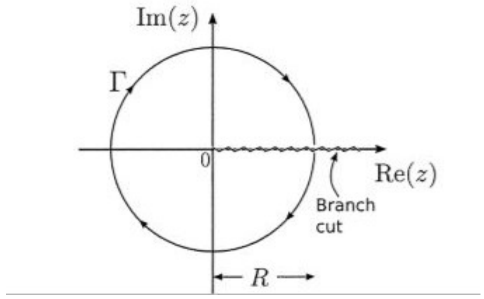

Figure 12: Circle with centre at $z = z_0$ with branch cut along positive x axis

- *Zero and Order of a Function*: A function $f(z)$ is said to have a *Zero* value if it vanishes at any point (or points). Furthermore, if the derivative of the function is not equal to zero at a given order (say, $f^{(n)} \neq 0$), it is called *order of zero* of that function. Example: The function $f(z) = \frac{z-2}{7} \sin \frac{1}{z-1}$ vanishes at $z = 2$ and $z = 1 + \frac{1}{n\pi}$ with $n = \pm 1, \pm 2, \ldots$. So it has multiple number of zeros.

- *Regular Point*: A function $f(z)$ is said to have an *Regular Point* if the function is analytic at that point. Example: $f(z) = x^7$

- *Isolated Singularity*: A function $f(z)$ is said to have an *Isolated Singularity* (also called *Singular Point*), if there exists a point $z = z_0$ such that $z_0 - \epsilon \leq z_0 \leq z_0 + \epsilon$ where ϵ be a small real number. Example: The function $f(z) = 1/(z - z_0)$ has a isolated singularity a $z = z_0$

- *Non-isolated Singularity*: A function $f(z)$ is said to have *Non-isolated Singularity* if any function nearby of it within range $|z - z_0| < \epsilon$ is also singular.
  Example: $f(z) = L(z)$

- *Removable Singularity*: If the limit of a function exists at a point and the function has a finite value at that point, then the point is said to be *Removal Singularity.*
  Example: $f(z) = \frac{\sin z}{z}$ has removal singularity at $z = z_0$.

- *Simple Pole*: The type of singularity is called *Simple Pole* which has order 0
  , if $\lim_{z \to z_0} f(z)(z - z_0)^n$ has a non-zero and finite value.
  Example: The function $f(z) = 1/(z - z_0)$ has simple pole (pole of order $n = 0$) at $z = z_0$.

- *Pole of order n*: The type of singularity at any point $z = z_0$ is a *pole* of order n, if $\lim_{z \to z_0} f(z)(z - z_0)^n$ has a non-zero and finite value.
  Example: The function $f(z) = 1/(z - z_0)^4$ has pole of order $n = 4$ at $z = z_0$.

- *Meromorphic Function*: A Meromorphic function is analytic everywhere in the complex plane except at a finite number of distinct poles.
  Example: $f(z) = 1/(z - 1)(z + 2)$, which has a poles at $n = 1$ and $z = -2$.

- *Essential Singularity*: A singular point $z = z_0$ for which $f(z)(z - z_0)^n$ is not differentiable for any integer $n > 0$ is called Essential Singularity. Other than removable, pole and branch point is known as essential singularity.

- *Branch Point and Branch Cut*: A *Branch Point* is a point at $z = z_0$ about which the function $f(z)$ changes its value after each complete rotation (i.e., $0 \leq \theta \leq 2\pi$). The set of branch points constitutes a barrier (along which the analytic function is discontinuous) is called *Branch Cut*. The function on the *Branch Cut* is non-analytic which has to be avoided by drawing suitable contour avoiding this cut.

Example: $f(z) = \sqrt{z}$ has single Branch point about which the function changes its sign as, $f(z) = \{+1, -1\}$.
Example: $f(z) = \sqrt[3]{z}$ has single Branch point about which the function changes its sign, $f(z) = \{+1, e^{2\pi i/3}, e^{-2\pi i/3}\}$.
Example: $f(z) = ln(z)$ has infinite number of branch points $z = 0$ and $z = \infty$ (i.e., range $(\infty, 1]$) forming a branch cut.
Example: $f(z) = \sqrt{z-1}\sqrt{z-1}$ has branch cuts between $z = -1$ and $z = +1$ (i.e., range $[-1, 1]$) forming a branch cut.

(Advanced Topic: Riemann Surface)

6 Contour Integration

6.1 Cauchy-Goursat Theorem

- The integration over a complex function $f(z)$ parameterized by $z(t)$ along the path γ within range $a \le t \le b$ is given by,

$$\begin{aligned} I &= \oint_{\gamma} f(z)dz \\ &= \int_a^b f(z)dz \\ &= \int_a^b f(z)z'(t)dt \end{aligned} \tag{6.1}$$

In the limit $a \to b$, the path of integration is a closed path, i.e., $\gamma \to \Gamma$. Then it can be shown,

$$\oint_{\Gamma} f(z)dz = 0 \tag{6.2}$$

Proof:

$$\begin{aligned}
\oint_{\Gamma} f(z)dz &= \oint_{\Gamma} f(z(t))z'(t)dt \\
&= \oint_{\Gamma} [u(z(t)) + iv[z(t)]][x'(t) + iy'[t]]dt \\
&= \oint_{\Gamma} [u(z(t))x'(t) - iv[z(t)]y'[t]]dt + i\oint_{\Gamma} [v(z(t))x'(t) + u[z(t)]y'[t]]dt \\
&= \oint_{\Gamma} [ux'(t) - vy']dt + i\int_a^b [vx' + uy']dt \\
&= \oint_{\Gamma} [udx(t) - vdy] + i\oint_{\Gamma} [vdx + udy] \\
&= \oiint_{\partial S} dxdy(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}) + i\oiint_{\partial S} dxdy(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}) \\
&= 0, \qquad (6.3)
\end{aligned}$$

where in the last step Cauchy-Riemann condition is used. We are now in position to define *Cauchy-Goursat Theorem*:
Let $f(z)$ be a continuous, complex-valued function in the complex plane. Suppose that

$$\oint_{\Gamma} f(z)dz = 0, \qquad (6.4)$$

for all closed Jordan curves Γ, then it follows that $f(z)$ is analytic and there is no isolated singularity (or simply connected region) within that domain. Equivalently, for a simply Connected curve $\oint_{\Gamma} f(z)dz = 0$, while for multiply connected region $\oint_{\Gamma} f(z)dz \neq 0$.

- **Properties of Contour Integral:**

  If $f(z)$ and $g(z)$ are two complex functions are in the same domain D, then

  i) $\oint_{\Gamma} dz[f(z) + g(z)] = \oint_{\Gamma} dzf(z) + \oint_{\Gamma} dzg(z)$

  ii) If Γ_1 and Γ_2 are two closed curve joined end-to-end to create resulting curve Γ, than $\oint_{\Gamma} dzf(z) = \oint_{\Gamma_1} dzf(z) + \oint_{\Gamma_2} dzf(z)$

  iii) If a curve $-\Gamma$ is drawn by tracing out a curve C with a opposite orientation, then $\oint_{-\Gamma} dzf(z) = -\oint_{\Gamma} dzf(z)$.

6.2 Cauchy's Integral Formula

Let $f(z)$ be analytic on a simple closed contour Γ and suppose that $f(z)$ is also analytic everywhere on its interior. If the point z_0 is enclosed by the contour Γ, then

$$\oint_{\Gamma} dz\frac{f(z)}{(z - z_0)} = 2\pi i f(z_0) \qquad (6.5)$$

Proof:

To prove *Cauchy's Integral Formula* we shall redraw the path of integration in the following way, where we have split the contour γ into two parts, namely, γ_0 (moving anticlockwise) and γ_0 (moving

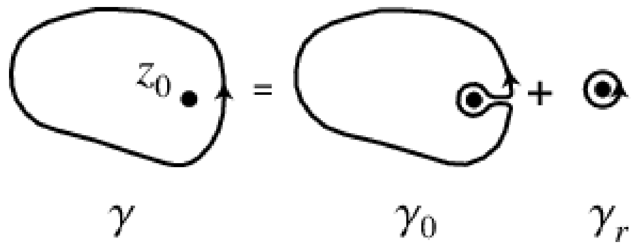

Figure 13: Integration along the path Γ is amounts to the sum of integration along γ_0 and γ_r

clockwise) shown in Fig.15.

$$\oint_{\gamma} dz \frac{f(z)}{(z-z_0)} = \oint_{\gamma_0} dz \frac{f(z)}{(z-z_0)} + \oint_{\gamma_r} dz \frac{f(z)}{(z-z_0)} \tag{6.6}$$

Since the first integration γ_0 of right hand side does not enclose any pole (i.e., analytic), therefore it vanishes and Eq.(7.2) becomes,

$$\oint_{\gamma} dz \frac{f(z)}{(z-z_0)} = \oint_{\gamma_r} dz \frac{f(z)}{(z-z_0)} \tag{6.7}$$

The pole at $z = z_0$ appearing in the infinitesimally small circle γ_r can be removed by substituting $z = z_0 + re^{i\theta}$ (and $dz = ire^{i\theta}d\theta$) and this integral becomes,

$$\oint_{\gamma_r} dz \frac{f(z)}{(z-z_0)} = \oint_{\gamma_r} dz f(z_0 + re^{i\theta}) i d\theta \tag{6.8}$$

In the limit $r \to 0$, i.e., when the circle γ_r becomes infinitesimally small, then

$$\begin{aligned} \oint_{\gamma_r} dz \frac{f(z)}{(z-z_0)} &= \oint_{\gamma_r} f(z_0) i d\theta \\ &= i f(z_0) \oint_{\gamma_r} d\theta \\ &= 2\pi i f(z_0) \end{aligned} \tag{6.9}$$

Thus plucking back Eq.((6.9)) into Eq.((6.7)) we obtain,

$$\oint_{\gamma} dz \frac{f(z)}{(z-z_0)} = 2\pi i f(z_0) \tag{6.10}$$

which is the precise statement of *Cauchy's Integral Formula.* Often Cauchy's theorem is written as,

$$f(z_0) = \frac{1}{2\pi i} \oint_{\gamma} dz \frac{f(z)}{(z-z_0)} \tag{6.11}$$

NOTE: Cauchy's Integral Formula is used to evaluate the integrals for which the poles are inside the contour.

6.3 Inequalities Associated with Cauchy's Integral Theorem

(i) **Cauchy Inequality**:

Let M be a positive integer such that $|f(z)| \leq M$ inside the region $|z-a| \leq f(z)$, the

$$|f^{(n)}(a)| \leq \frac{Mn!}{r^n} \tag{6.12}$$

This relation is known as *Cauchy Inequality.*
Proof:
To prove this let us recall that *Cauchy's Integral Formula* can be written as,

$$f(z_0) = \frac{1}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)} \tag{6.13}$$

Taking derivative with respect to z_0, we obtain,

$$\begin{aligned} f'(z_0) &= \frac{d}{dz}\Big[\frac{1}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)}\Big] \\ &= \frac{1}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)^2} \end{aligned} \tag{6.14}$$

Iterating the differentiation again we find,

$$f''(z_0) = \frac{2}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)^3} \tag{6.15}$$

Proceeding this way, finally taking the n-th derivative we obtain,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)^{(n+1)}} \quad \text{for} \quad n = 0, 1, 2, \dots n \tag{6.16}$$

Thus the function $f(z)$ is analytic indicating that it exists along with its derivative exists inside the contour Γ. [3] Now if we take $|z-z_0| = r$ the right hand side of Eq.((6.16)) becomes,

$$\left|\frac{n!}{2\pi i}\frac{f_{max}(z)}{r^{(n+1)}}\oint_\Gamma dz\right| = \frac{n!}{2\pi}\frac{M}{r^{(n+1)}}2\pi r = \frac{Mn!}{r^n} \tag{6.17}$$

where, $f_{max} = M$. Thus from Eqs.(6.16) and (6.17) we obtain,

$$\left|f^{(n)}(z)\right| \leq \frac{Mn!}{r^n} \tag{6.18}$$

[3] Finally to prove the inequality we substitute $z - z_0 = re^{i\theta}$ in Eq.((6.16)),

$$\left|f^{(n)}(z)\right| = \left|\frac{n!}{2\pi i}\oint_\Gamma dz \frac{f(z)}{(z-z_0)^{(n+1)}}\right|$$

(ii) **Theorem: ML Inequality**:

If $f(z) = u(x,y) + iv(x,y)$ is a continuous variable on a contour C, then

$$\left| \oint_C f(z)dz \right| \leq ML, \tag{6.19}$$

where L is the length of contour, M is the upper bound for the modulus of the function $|f(z)|$, i.e., $|f(z)| \leq M$ for all C. *Proof*:

$$\begin{aligned} L.H.S. &= \left| \oint_C f(z)dz \right| \\ &= \left| \int_a^b f[z(t)]dz \right| \\ &= \left| \int_a^b f[z(t)]\frac{dz(t)}{dt}dt \right| \end{aligned} \tag{6.20}$$

Since $|\oint_C dz f(z)| \leq \oint_C dz|f(z)|$, therefore we have,

$$\begin{aligned} \left| \int_a^b f[z(t)]\frac{dz(t)}{dt}dt \right| &\leq \int_a^b \left| f[z(t)]\frac{dz(t)}{dt} \right| dt \\ &\equiv \int_a^b \left| f[z(t)] \right| \times \left| \frac{dz(t)}{dt} \right| dt \\ &\equiv \int_a^b M \times \left| \frac{dz(t)}{dt} \right| dt \quad \because \left| f[z(t)] \right| = \left| f_{max}[z] \right| = M \\ &\equiv M \times \oint_C \left| dz \right| \quad \because \left| z_{max} \right| = L \\ &\equiv ML \end{aligned} \tag{6.21}$$

Thus from Eqs.(6.21) and (6.22), we have

$$\left| \oint_C f[z]dz \right| \leq ML \quad \text{Q.E.D.} \tag{6.22}$$

6.4 Cauchy's Residue Theorem

If $f(z)$ is a analytic function inside a simple closed curve C except the finite number of singular points z_1, z_2, ..., z_n inside C, then

$$\oint\limits_{\Gamma} f(z)dz = 2\pi i \sum_i R_i \tag{6.23}$$

where $\sum_i R$ be the summation of the residues of $f(z)$ at the poles z_i within C.

Proof:

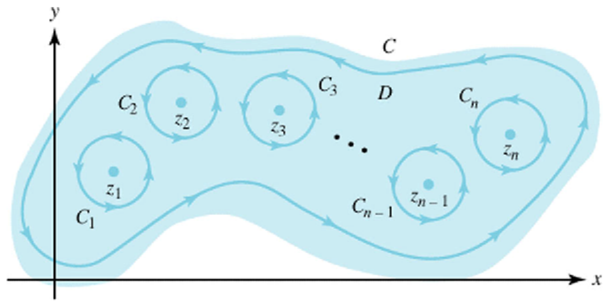

Figure 14: Multiple poles $\{C_i\}$ inside a contour C

Let C_{-i} $(i = 1, 2, \ldots n)$ be the set of circles around the poles at centres $z = z_i$ within the closed contour C as shown in Fig.10. The using the theorem [4]

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \ldots \tag{6.24}$$

Then recalling that,

$$\begin{aligned}
\oint_{C_1} f(z)dz &= 2i\pi Res[f, z_1], \\
\oint_{C_1} f(z)dz &= 2i\pi Res[f, z_2], \\
\oint_{C_2} f(z)dz &= 2i\pi Res[f, z_3], \\
&\vdots \\
\oint_{C_2} f(z)dz &= 2i\pi Res[f, z_n],
\end{aligned} \tag{6.25}$$

Plucking back Eq.((6.24)) into Eq.((6.25)) we obtain,

$$\begin{aligned}
\oint_\Gamma f(z)dz &= 2\pi i(Res[f, z_1] + Res[f, z_2] + \cdots + Res[f, z_n]) \\
&= 2i\pi \sum_{i=1}^{n} Res[f, z_i]
\end{aligned} \tag{6.26}$$

NOTE: Cauchy's Residue Theorem is used to evaluate the integrals for which the poles are outside the contour.

[4] Let $f(z)$ be analytic in a region bounded by the non-overlapping simple closed curves $C_1, C_2, C_3 \ldots, C_n$ inside the closed contour C as shown in Fig.10, then,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \oint_{C_3} f(z)dz + \cdots + \oint_{C_n} f(z)dz$$

6.5 Evaluation of Residue

Let $f(z)$ is a holomorphic function in domain D with its Analytic (or Regular) and Principal (or Singular) parts with poles $z = z_0$ shown in figure. Then the Laurent series having pole at $z = z_0$ is given by,

$$
\begin{aligned}
f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\
&= \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} + \sum_{p=0}^{\infty} a_p (z - z_0)^p \\
&= \cdots + \frac{a_{-3}}{(z - z_0)^3} + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + (z - z_0)^1 a_1 + (z - z_0)^2 a_2 + \ldots \\
&= \phi_P(z) + \frac{a_{-1}}{z - z_0} + \phi_A(z)
\end{aligned} \tag{6.27}
$$

where $\phi_P(z)$ and $\phi_A(z)$ be the *Principal part* and *Analytic Part*, respectively with residue a_{-1} which is to be evaluated for different order of n.

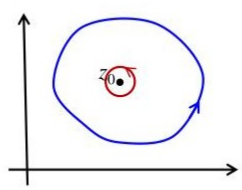

Figure 15: Simple Pole at $z = z_0$ inside the contour C

Case I: Evaluation of Simple Pole (Pole of order $n = 0$):

To find the residue in this case, we consider the series,

$$
\begin{aligned}
& f(z) = \frac{a_{-1}}{z - z_0} + a_0 + (z - z_0)^1 a_1 + (z - z_0)^2 a_2 + \ldots \\
& \Rightarrow (z - z_0) f(x) = a_{-1} + (z - z_0)^1 a_0 + (z - z_0)^2 a_1 + (z - z_0)^3 a_1 + \ldots \\
& \Rightarrow \lim_{z \to z_0} (z - z_0) f(x) = \lim_{z \to z_0} \left[a_{-1} + (z - z_0) a_0 + (z - z_0)^2 a_1 + (z - z_0)^3 a_1 + \ldots \right] \\
& \Rightarrow \lim_{z \to z_0} (z - z_0) f(x) \\
& = a_{-1} \\
& \equiv Res[f; z_0],
\end{aligned} \tag{6.28}
$$

where we note that the second and successive terms vanish in the limit $z \to z_0$. Thus if $f(z)$ has a *Simple Pole* (Pole of order $n = 0$), then the residue is given by,

$$
Res[f; z_0] = (z - z_0) f(z). \tag{6.29}
$$

Case II: Evaluation of Pole of order $n = 1$:

Taking $k = -2, -1, 0, 1, 2, \ldots$ we have,

$$
\begin{aligned}
f(z) =& \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + (z-z_0)^1 a_1 + (z-z_0)^2 a_2 \ldots \\
(z-z_0)^2 f(z) =& a_{-2} + (z-z_0)^1 a_{-1} + (z-z_0)^2 a_0 + (z-z_0)^3 a_1 + (z-z_0)^4 a_2 \ldots \\
\Rightarrow \frac{d}{dz}(z-z_0)^2 f(z) =& \; a_{-1} + 2(z-z_0)a_0 + 3(z-z_0)^2 a_2 \ldots \\
\Rightarrow \lim_{z\to z_0} \frac{d}{dz}(z-z_0)^2 f(z) =& \lim_{z\to z_0} \left[a_{-1} + 2(z-z_0)a_0 + 3(z-z_0)^2 a_2 \ldots\right] \\
=& a_{-1} \\
=& Res[f; z_0]
\end{aligned}
\tag{6.30}
$$

Thus if $f(z)$ has a pole of order 2, then the residue is given by,

$$
Res[f; z_0] = \lim_{z\to z_0} \frac{d}{dz}[(z-z_0)^2 f(z)
\tag{6.31}
$$

Case III: Evaluation of Pole of order $n = 2$:

For $k = -3, -2, 0, 1, 2, \ldots$, proceeding in similar way,

$$
\begin{aligned}
f(z) &= \frac{a_{-3}}{(z-z_0)^3} + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + (z-z_0)^1 a_1 + (z-z_0)^2 a_2 + \ldots \\
(z-z_0)^3 f(z) &= a_{-3} + (z-z_0)a_{-2} + (z-z_0)^2 a_{-1} + (z-z_0)^3 a_0 + (z-z_0)^4 a_1 + \ldots \\
\Rightarrow \frac{d}{dz}(z-z_0)^3 f(z) &= a_{-2} + 2(z-z_0)a_{-1} + 3(z-z_0)^2 a_0 + 4(z-z_0)^3 a_1 + \ldots \\
\Rightarrow \frac{d^2}{dz^2}(z-z_0)^3 f(z) &= 2a_{-1} + 6(z-z_0)a_0 + 12(z-z_0)^2 a_1 + \ldots \\
\Rightarrow \frac{d^2}{dz^2}(z-z_0)^3 f(z) &= 2a_{-1} \\
&= Res[f; z_0]
\end{aligned}
\tag{6.32}
$$

Thus, if $f(z)$ has a pole of order 3, then the residue is given by,

$$
Res[f; z_0] = \lim_{z\to z_0} \frac{1}{2} \frac{d^2}{dz^2}[(z-z_0)^3 f(z)]
\tag{6.33}
$$

Thus, by the method of iteration, the **residue of a pole of arbitrary n-th order** can be given by most general formula,

$$
Res[f; z_0] = \lim_{z\to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(x)\right]
\tag{6.34}
$$

6.6 Jordan Lemma

This lemma can be used to calculate the improper complex integrals having poles in the upper half of the semi-circle. The *Jordan Lemma* for a complex valued function $f(z)$ states that,

$$
\lim_{R\to\infty} \int_{C_1} f(z) e^{mz} = 0
\tag{6.35}
$$

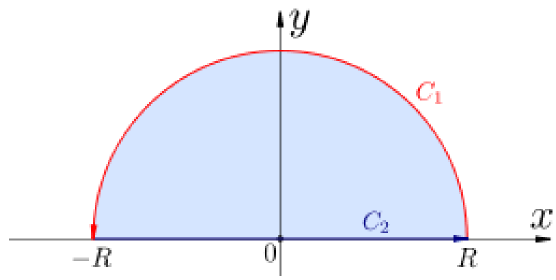

Figure 16: Poles in the upper-half of semicircle

with $m > 0$ and $|f(z)| \to 0$ and $R \to \infty$. This theorem can also be applied as,

$$\lim_{R\to\infty} \int_{C_1} f(z) = 0 \tag{6.36}$$

provided that $|f(z) \to \infty$ faster than $1/z$ as $R \to \infty$.

Proof: (To be added)

7 Evaluation of Contour Integrals

There exists different types of integrals with complex integrand [5] and below we have given some exercise to understand these complex integrals based on the theorems, inequality, lemma etc discussed in the previous section:

[5] a) Improper Integral of First kind:
An integral is said to be improper integral over the range $[0, \infty)$,

$$\int_0^\infty dz f(z) = \lim_{R\to\infty} \int_0^R dz f(z)$$

provided the limit exists. Similarly we can write

$$\int_{-\infty}^\infty f(z) = \lim_{R\to\infty} \int_{-R}^R dz f(z)$$

b) Cauchy's Principal Value:
If $f(x)$ is a real and continuous function for all x, then **Cauchy's Principal Value** of integral $\int_{-\infty}^\infty dx f(x)$ is defined as

$$P.V. \int_{-\infty}^\infty dx f(x) = \lim_{R\to\infty} \int_{-R}^R dx f(x).$$

provided the limit exists.
c) Improper integral with trigonometric function:
Let $P(x)$ and $Q(x)$ be two polynomial of degree m and n, respectively where $n \geq m+1$. Show that if $Q(x) \neq 0$ for all real x. If $\alpha > 0$ and $f(z) = \frac{e^{i\alpha z} P(z)}{Q(z)} \cos z$, then,

$$P.V. \int_{-\infty}^\infty \frac{P(x)}{Q(x)} \cos x = -2\pi \sum_{i=1}^k Im(Res[f, z_i])$$

$$P.V. \int_{-\infty}^\infty \frac{P(x)}{Q(x)} \sin x = 2\pi \sum_{i=1}^k Re(Res[f, z_i])$$

provided the limit exists with $\{z_k\}$ be the set of poles in the upper half of the plane $Im[Res[f(z), z_k]]$ and $Re[Res[f(z), z_k]]$ be the imaginary and real part of residues $Res[f(z), z_k]$, respectively.

7.1 Some Examples

i) Show that for a unit circle ($C : z(t) = e^{it}$),

$$\int_0^{2\pi} \frac{dz}{z} = 2\pi i \tag{7.1}$$

Proof:

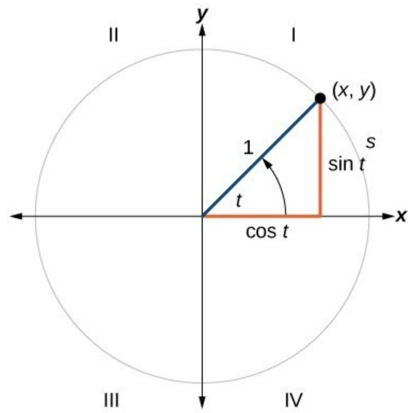

Figure 17: Circular contour of unit radius ($r = |z| = 1$) with centre at origin $z = 0$.

$$\begin{aligned}\oint_C \frac{dz}{z} &= \int_0^{2\pi} dt i e^{it} e^{-it} \quad \because z = e^{it}, dz = idt \\ &= i\int_0^{2\pi} dt \\ &= 2\pi i \quad \text{Q.E.D.} \end{aligned} \tag{7.2}$$

ii) Show that:

$$\int_{-i}^{i} \frac{dz}{z} = 2\pi i \tag{7.3}$$

Proof:

$$\begin{aligned}\int_{-i}^{i} \frac{dz}{z} &= log(z)\Big|_{-i}^{i} \\ &= log|z| + iarg(z)\Big|_{-i}^{i} \\ &= (log|i| + iarg(i)) - (log|-i| + iarg(-i)) \\ &= (log1 + i\frac{\pi}{2}) - (log1 + i\frac{-3\pi}{2}) \\ &= 2\pi i \quad \text{Q.E.D.}\end{aligned} \tag{7.4}$$

(Note: The problem can be done with parametrization $z(t) = re^{i\theta}$ with $-\pi/2 \le \theta \le \pi/2$.)

iii) Show that

$$\int_0^{2\pi} dz(z-a)^m = \begin{cases} 2\pi i \quad \text{for} \quad m = -1 & (7.5a) \\ 0 \quad \text{for} \quad m \neq= -1, \text{integer} & (7.5b)\end{cases}$$

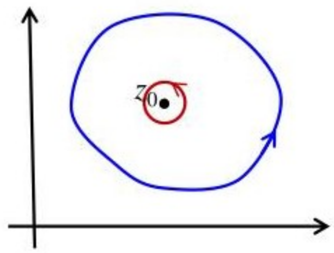

Figure 18: Circular contour of unit radius ($r = |z| = 1$) with shifted origin at $z = z_0$.

Proof:

By noting the fact that $z(t) = z_0 + re^{it}$, we have,

$$(z(t) - z_0)^m = r^m e^{imt}, \quad \text{with} \quad dz = izdt.$$

Thus we obtain,

$$\begin{aligned}\int_0^{2\pi} (z-a)^m &= ir^{m+1}\int_0^{2\pi} dte^{i(m+1)t} \\ &= ir^{m+1}\Big[\int_0^{2\pi} dt\cos(m+1)t + i\int_0^{2\pi} dt\sin(m+1)t\Big] \\ &= 2i\pi \quad \text{for} \quad m = -1 \\ &= 0 \quad \text{for} \quad m \neq -1 \qquad \text{Q.E.D.}\end{aligned} \tag{7.6}$$

iv) Show that

$$\int_0^{2\pi} \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & \text{for} \quad n = -1 & (7.7a) \\ 0 & \text{for} \quad n \neq = -1, \text{integer} & (7.7b) \end{cases}$$

Proof:

Once again by noting the fact that $z(t) = z_0 + re^{it}$, we have,

$$\frac{1}{(z(t) - z_0)^n} = r^{-n}e^{-int}, \quad \text{with} \quad dz = izdt.$$

We therefore obtain,

$$\begin{aligned}\int_0^{2\pi} \frac{dz}{(z-a)^n} &= ir^{1-n}\int_0^{2\pi} dte^{i(1-n)t} \\ &= ir^{1-n}\Big[\int_0^{2\pi} dt\cos(1-n)t + i\int_0^{2\pi} dt\sin(1-n)t\Big] \\ &= 2i\pi \quad \text{for} \quad n = 1 \\ &= 0 \quad \text{for} \quad n \neq 1 \qquad \text{Q.E.D.}\end{aligned} \tag{7.8}$$

v) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}} \tag{7.9}$$

Proof: By Cauchy theorem, this integral is the sum of the residue of the upper part of half-plane, i.e.,

$$\oint_0^{\pi} \frac{dz}{z^4+1} = 2\pi \sum_i Res_i[f(z), z_i] \tag{7.10}$$

and we have following set of four poles, namely,

$$\begin{aligned}\{z_i\} &= (-i)^4 \\ &= e^{\frac{1}{4}(i\pi+2\pi k)} \\ &= \{e^{i\pi/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}\}\end{aligned} \tag{7.11}$$

The residue corresponding to $z_0 = e^{\pi/4}$ is,

$$\begin{aligned}Res[f(z), e^{i\pi/4}] &= \lim_{z\to e^{i\pi/4}} (z-e^{i\pi/4}) \frac{1}{(z-e^{i\pi/4})(z-e^{3i\pi/4})(z-e^{5i\pi/4})(z-e^{7i\pi/4})} \\ &= -\frac{1}{4}(-1)^{i/4} \\ &= -\frac{1}{4}e^{\pi/4} \\ &= -\frac{1}{4}(\cos\pi/4 + i\sin\pi/4) \\ &= -\frac{1}{4\sqrt{2}}(1+i)\end{aligned} \tag{7.12}$$

Similarly,

$$Res[f(z), e^{3i\pi/4}] = \frac{1}{4\sqrt{2}}(1-i) \tag{7.13a}$$

$$Res[f(z), e^{5i\pi/4}] = \frac{1}{4\sqrt{2}}(1+i) \tag{7.13b}$$

$$Res[f(z), e^{7i\pi/4}] = -\frac{1}{4\sqrt{2}}(1-i) \tag{7.13c}$$

Out of 4 poles, third and fourth poles are outsides of the upper-half of the domain D (i.e., $0 \le \theta \le \pi$) and therefore are outside the contour. Thus the required integral becomes

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{dx}{x^4+1} \\ &= \int_0^{\pi} \frac{dz}{z^4+1} \\ &= 2\pi i\Big(-\frac{1}{4\sqrt{2}}(1-i) + \frac{1}{4\sqrt{2}}(1-i)\Big) \\ &= \frac{\pi}{\sqrt{2}} \quad \text{Q.E.D.}\end{aligned} \tag{7.14}$$

vi) Show that

$$\int_0^{2\pi} \frac{dz}{(z-i)} = 2\pi i. \tag{7.15}$$

Proof:

Here the function $f(z)$ has a *Simple Pole* (i.e., Pole of order zero) at $z = i$ and the corresponding residue is given by,

$$\begin{aligned} Res[f,i] &= 2\pi i(z-i)\cdot\frac{1}{z-i} \\ &= 2\pi i \end{aligned}$$

Thus we obtain

$$\int_0^{2\pi}\frac{dz}{(z-i)} = 2\pi i \quad \text{Q.E.D.} \tag{7.16}$$

vii) Find the residues from the following complex integral

$$\int_0^{2\pi}\frac{z^3}{(z-1)^4(z-2)(z-3)} \tag{7.17}$$

Answer:

The above function has one pole of order $n = 4$ at $z = -1$ and two simple poles at $z_0 = 2, 3$, respectively. The residue corresponding to pole of 4-th order is calculated using formula:

$$\begin{aligned} Res[f,4] &= \lim_{z\to 4}\frac{1}{(4-1)!}\frac{d^{4-1}}{dz^{4-1}}\Big[(z-1)^4\frac{1}{(z-1)^4(z-2)(z-3)}\Big] \\ &= \frac{101}{16}, \end{aligned} \tag{7.18}$$

where L'Hospital's Rule is used. On the other hand, the two simple poles are given by,

$$\begin{aligned} Res[f,2] &= \lim_{z\to 2}\Big[(z-2)\frac{1}{(z-1)^4(z-2)(z-3)}\Big], \\ &= -8, \end{aligned} \tag{7.19a}$$

$$\begin{aligned} Res[f,3] &= \lim_{z\to 2}\Big[(z-3)\frac{1}{(z-1)^4(z-1)(z-3)}\Big] \\ &= \frac{27}{8}, \end{aligned} \tag{7.19b}$$

respectively.

viii) Evaluate the following integral with pole of order $n = 4$

$$\int_0^{\pi}\frac{dz}{(z-i)^4}. \tag{7.20}$$

Answer: The integral has four poles, namely,

$$\begin{aligned} \{z_i\} &= (i)^4 \\ &= e^{\frac{1}{4}(i\pi+2\pi k)} \\ &= \{e^{i\pi/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}\} \quad \because \quad k = 0,1,2,3 \end{aligned} \tag{7.21}$$

Out of them two poles $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$ within range $[0,\pi]$ (i.e., upper half of quadrants), therefore corresponding residues are:

$$\begin{aligned} Res[f, e^{i\pi/4}] &= \lim_{z\to e^{i\pi/4}} \frac{1}{(4-1)!}\frac{d^{4-1}}{dz^{4-1}}\big[(e^{i\pi/4}-i)^4\frac{1}{(z-i)^4}\big] \\ &= -\frac{3}{2}(-1)^{1/4} \\ &= -\frac{3}{2\sqrt{2}}(1+i) \end{aligned} \tag{7.22}$$

$$\begin{aligned} Res[f, e^{3i\pi/4}] &= \lim_{z\to e^{3i\pi/4}} \frac{1}{(4-1)!}\frac{d^4}{dz^4}\big[(z-e^{3i\pi/4})^4\frac{1}{(z-e^{3i\pi/4})^4}\big] \\ &= -\frac{3}{2}(-1)^{3/4} \\ &= \frac{3}{2\sqrt{2}}(1-i) \end{aligned} \tag{7.23}$$

Plucking back Eqs.(7.22) and (7.23) we find,

$$\begin{aligned} \int_0^{\pi} \frac{dz}{(z-i)^4} &= 2\pi i\Big(-\frac{3}{2\sqrt{2}}(1+i) + \frac{3}{2\sqrt{2}}(1-i)\Big) \\ &= 3\sqrt{2}\pi \end{aligned} \tag{7.24}$$

ix) Show that

$$\int_0^{2\pi} \frac{z}{(z-1)(z+1)^2} = \pi i \tag{7.25}$$

Proof: For simple pole at $n = 1$,

$$Res[f, z_0 = 1] = \lim_{z\to 1}(z-1)\big[\frac{z}{(z-1)(z+1)^2}\big] = \frac{1}{4} \tag{7.26}$$

and for pole at $n = -1$ of order $k = 2$,

$$Res[f, z_0 = -1] = \lim_{z\to -1}\frac{1}{(2-1)!}\frac{d}{dz}\big[\frac{z}{(z-1)(z+1)^2}\big] = \frac{1}{4} \tag{7.27}$$

$$\begin{aligned} \therefore \int_0^{2\pi} \frac{z}{(z-1)(z+1)^2} &= 2\pi i(1/4 + 1/4) \\ &= \pi i \quad \text{Q.E.D.} \end{aligned} \tag{7.28}$$

6

6

Consider the evaluation of a general form of the integral involving trigonometric functions:

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)$$

x) Show that:

$$\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \frac{\pi}{3} \tag{7.29}$$

Proof:

To obtain this integral we first consider the following complex integral:

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{1+3\cos^2\theta} &= \oint_C \frac{1}{1+2(2+z+z^{-2})^2}\frac{dz}{iz} \\
&= \oint_C dz \frac{z}{(2z^4+5z^2+2)} \\
&= \oint_C dz \frac{z}{(2z^2+1)(z^2+2)} \\
&= \frac{1}{2}\oint_C dz \frac{z}{(z+i/\sqrt{2})(z-i/\sqrt{2})(z+\sqrt{2}i)(z-\sqrt{2}i)}
\end{aligned} \tag{7.30}$$

Out of four poles, only two simple poles at $z_1 = i/\sqrt{2}$ and $z_2 = -i/\sqrt{2}$ are inside the unit radius and therefore the integral by CRT is given by,

$$\begin{aligned}
\int_0^{2\pi} dz \frac{z}{(2z^4+5z^2+2)} &= 2\pi i \Big(\frac{z}{2z^4+5z^2+2}; \frac{i}{\sqrt{2}} + \frac{z}{(2z^4+5z^2+2)}; -\frac{i}{\sqrt{2}} \Big) \\
&= 2\pi i \left(\frac{1}{8z^2+10}\bigg|_{z=i/\sqrt{2}} + \frac{1}{8z^2+10}\bigg|_{z=i/\sqrt{2}} \right) \\
&= \frac{2\pi}{3}
\end{aligned} \tag{7.31}$$

where in the last step the L' Hospital's Rule is used. Q.E.D

xi) Show that (*Integral with Branch Cut*)

$$\int_0^{\infty} dx \frac{logx}{x^2+1} = 0 \tag{7.32}$$

Proof:

$$\begin{aligned}
\oint_C \frac{logz dz}{z^2+1} &= \int_r^R dz \frac{logz}{z^2+1} + \int_{C_R} dz f(z) + \int_{-R}^{-r} dz \frac{(logz+i\pi)}{z^2+1} + \int_{C_r} dz f(z) \\
&= 2\pi i \sum_i Res[f, z_i]
\end{aligned} \tag{7.33}$$

Taking a unit circle $z = e^{i\theta}$ and $d\theta = \frac{dz}{iz}$, it can be written as

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) = \oint_C dz f(z) \quad \because \quad \cos z = \frac{1}{2}(z+z^{-1}), \sin z = \frac{1}{2i}(z-z^{-1})$$

where $f(z) = \frac{1}{2i}F\big(\frac{1}{2}(z+\bar{z}), \frac{1}{2i}(z-\bar{z})\big)$, respectively. Now if the function is analytic except at $z = z_k$ within the contour C, then using residue theorem we get,

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) = 2\pi i \sum_{k=0}^{n} Res[f, z_k]$$

$$Res[f, z_0, a_{-1}^{(k)}] = \lim_{z\to z_0} \frac{k!}{(k-1)!}\frac{d^k}{dz^k}\big[(z-z_0)^k f(x)\big]$$

Now for $z^2 + 1 = 0$ we have two poles at $z = \pm i$. Out of them only $z = i$ is inside the contour and therefore we calculate that residue only,

$$
\begin{aligned}
Res[f, z_0 = i] &= \lim_{z\to i}(z-i)f(z) \\
&= \lim_{z\to i}(z-i)\frac{logz}{(z-i)(z+i)} \\
&= \lim_{z\to i}\frac{logz}{(z+i)} \\
&= \frac{logi}{2i} \\
&= \frac{log|i| + iarg(i)}{2i} \\
&= \frac{i}{2i}\cdot\frac{\pi}{2} \\
&= \frac{\pi}{4}
\end{aligned} \tag{7.34}
$$

Thus we have

$$
\oint_C \frac{logzdz}{z^2+1} = 2\pi i\frac{\pi}{4} = \frac{i\pi^2}{2} \tag{7.35}
$$

Now using the inequality (ML Identity) it is easy to see, [7]

$$
\lim_{R\to\infty}\oint_{C_R} f(z) \to 0 \tag{7.36a}
$$

$$
\lim_{r\to 0}\oint_{C_r} f(z) \to 0 \tag{7.36b}
$$

Plucking back Eq.((7.36)) into Eq.((7.33)), we obtain,

[7] We note that $|\int_C f(z)dz| \le \int_C dz|f(z)|$

$$
\begin{aligned}
|\int_{C_R} f(z)dz| &= \left|\int_{C_R}\frac{\log z}{z^2+1}dz\right| \\
&\le \left|\int_{C_R}\frac{\log z}{z^2+1}\right||dz| \\
&\le \int_{C_R}\frac{|\log z|}{|z^2+1|}|dz| \\
&\le \int_{C_R}\frac{|\log r + i\theta|}{|r^2e^{2i\theta}|-1}|ire^{i\theta}d\theta| \\
&\le \frac{(\log R + \pi)}{R^2-1}\pi R \\
&\le \Big(\frac{\pi R\log R}{R^2-1} + \frac{\pi}{R^2-1}\pi R\Big)
\end{aligned}
$$

Thus we have

$$
\lim_{R\to 0}|\int_{\gamma_R} f(z)dz| \equiv \Big(\frac{\pi r\log r}{R^2-1} + \frac{\pi}{R^2-1}\pi R\Big) \to \infty.
$$

Proceeding similar way, for the little circle γ_r of radius $r < 1$ we can show

$$
\lim_{r\to 0}|\int_{C_r} f(z)dz| \equiv \Big(\frac{\pi r\log r}{1-r^2} + \frac{\pi}{1-r^2}\pi r\Big) \to 0
$$

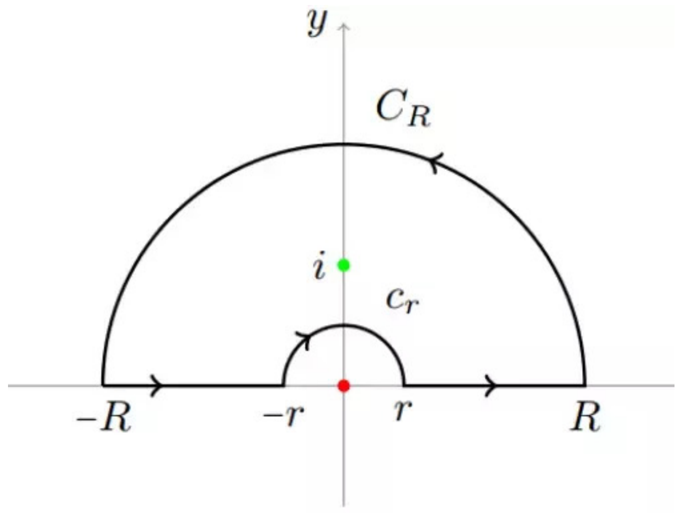

Figure 19: Circular contour bypassing the origin at $z_0 = 0$.

$$
\begin{aligned}
\frac{i\pi^2}{2} &= \int_r^R dz \frac{logz}{z^2+1} + 0 + \int_{-R}^{-r} dz \frac{(logz + i\pi)}{z^2+1} + 0 \\
&= \int_0^\infty dx \frac{logx}{x^2+1} + \int_{-\infty}^0 dx \frac{log|x|}{x^2+1} + i\pi \int_{-\infty}^0 dx \frac{1}{x^2+1} \\
&= 2\int_0^\infty dx \frac{logx}{x^2+1} + i\pi \arctan x\Big|_0^\infty \\
&= 2\int_0^\infty dx \frac{logx}{x^2+1} + \frac{i\pi^2}{2},
\end{aligned} \tag{7.37}
$$

Which readily gives,

$$
\int_0^\infty dx \frac{logx}{x^2+1} = 0 \quad \text{Q.E.D.} \tag{7.38}
$$

xii) Show that

$$
\frac{1}{2\pi^2} \int_{-\infty}^\infty dx \frac{y}{x^2+y^2} e^{-iky} = \frac{e^{|k|y}}{2\pi} \tag{7.39}
$$

Proof: To start with let us first evaluate the following integral

$$
\begin{aligned}
&\frac{1}{2\pi^2} \int_{-\infty}^\infty dz \frac{y}{z^2+y^2} e^{-iky} \\
&= \frac{1}{2\pi^2} \frac{y}{(z+iy)(z-iy)} e^{-iky} \\
&= 2\pi i \sum_i Res[f, z_i]
\end{aligned} \tag{7.40}
$$

Out two poles at $z = -iy$ and $z = iy$, the first one is lying in the upper half of the semicircle, thus the residue reads,

$$
\begin{aligned}
Res[f,i] &= \lim_{z\to iy}(z-iy)\frac{ye^{ikz}}{2\pi^2(z+iy)(z-iy)} \\
&= \frac{e^{ky}}{4\pi^2 i} && (7.41a) \\
Res[f,i] &= \lim_{z\to -iy}(z+iy)\frac{ye^{ikz}}{2\pi^2(z+iy)(z-iy)} \\
&= -\frac{e^{-ky}}{4\pi^2 i} && (7.41b)
\end{aligned}
$$

Plucking back Eq.((7.41a)) into Eq.((7.40)) we obtain,

$$
\begin{aligned}
&\frac{1}{2\pi^2}\int_{-\infty}^{\infty} dx \frac{y}{x^2+y^2}e^{-iky} \\
&= 2\pi i(\frac{e^{ky}}{4\pi^2 i} + \frac{e^{-ky}}{4\pi i^2}) \\
&= \frac{e^{ky}}{2\pi} + \frac{e^{-ky}}{2\pi} \\
&= \frac{e^{-|k|y}}{2\pi} \quad \text{Q.E.D.} && (7.42)
\end{aligned}
$$

xiii) Prove the following integral ($a > 0$)

$$\int_0^{\infty} dx \frac{\cos ax}{x^2} = -\pi a \tag{7.43}$$

Proof:

We start by evaluating the contour integral shown in Fig.(20) bypassing the origin at $z_0 = 0$,

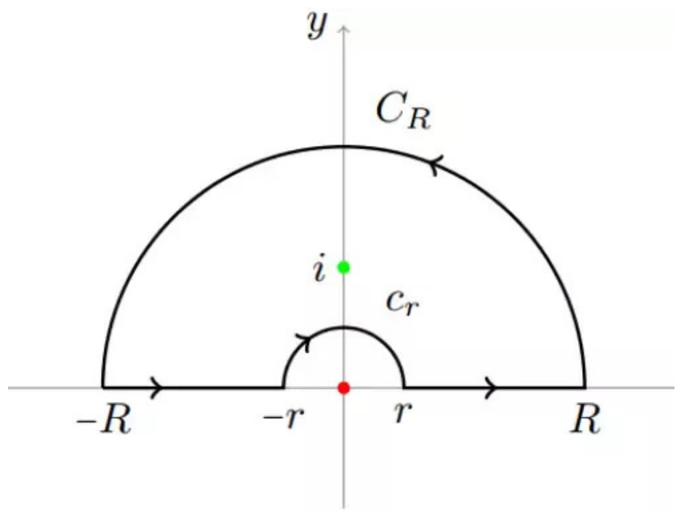

Figure 20: Circular contour bypassing the origin at $z_0 = 0$.

$$\oint dz\frac{e^{iaz}}{z^2} = \int_{-R}^{-r} dz\frac{e^{iaz}}{z^2} + \int_{-r}^{r} dz\frac{e^{iaz}}{z^2} + \int_{r}^{R} dz\frac{e^{iaz}}{z^2} + \int_{R}^{-R} dz\frac{e^{iaz}}{z^2}$$
$$= \int_{-R}^{-r} dz\frac{e^{iaz}}{z^2} + \int_{C_r} dz\frac{e^{iaz}}{z^2} + \int_{r}^{R} dz\frac{e^{iaz}}{z^2} + \int_{C_R} dz\frac{e^{iaz}}{z^2} \tag{7.44}$$

In the above integral, from first and third terms give,

$$\begin{aligned}&\int_{r}^{R} dz\frac{e^{iaz}}{z^2} + \int_{-R}^{-r} dz\frac{e^{iaz}}{z^2}\\ &= \int_{r}^{R} dz\frac{e^{iaz}}{z^2} + \int_{r}^{R} dz\frac{e^{-iaz}}{z^2}\\ &= 2\int_{r}^{R} \frac{dz}{z^2}(\frac{e^{iaz}+e^{-iaz}}{2})\\ &= 2\int_{r}^{R} dz\frac{\cos az}{z^2}\end{aligned} \tag{7.45}$$

[8] and

$$\oint dz\frac{e^{iaz}}{z^2} = 0 \quad \text{(By Cauchy Residue Theorem)} \tag{7.46a}$$

$$\int_{C_R} dz\frac{e^{az}}{z^2} = 0 \quad \text{(By Jordan's lemma)} \tag{7.46b}$$

$$\begin{aligned}\int_{C_r} dz\frac{e^{iaz}}{z^2} &= -2i\pi Res[f; z_0 = 0] \quad \text{(By Cauchy's Residue Theorem moving counter clockwise path } C_r \text{)}\\ &= -2i\pi(ia)\\ &= 2\pi a\end{aligned} \tag{7.46c}$$

Finally plucking back Eq.(7.45) and Eq.(7.46a) into Eq.(7.44), we obtain

$$0 = 2\int_{r}^{R} dz\frac{\cos az}{z^2} + 2\pi a + 0 \tag{7.47}$$

and then taking the limit $R \to \infty$ and $r \to 0$ we obtain,

$$\int_{0}^{\infty} dx\frac{\cos ax}{x^2} = -\pi a \quad \text{Q.E.D.} \tag{7.48}$$

References

[1] Complex Analysis Demytified: David McMahon, The McGraw-Hill Companies Inc, 2008

[2] An useful Internet resource including advanced topics

[3] A first course in complex analysis with applications: D.G.Zill ,P.D.Shanahan, Jones and Bartlett Publishers Inc, 2003

[4] Complex Variable: Schaum's Outlines Series: M Spiegel, S Lipschutz, J Schiller, D Spellman, The McGraw-Hill Companies Inc, 2009

[8] Here we have used $Res[f; z_0 = 0] = \frac{1}{(2-1)!}\frac{d}{dz}\left[z^2\frac{e^{ikz}}{z^2}\right]\Big|_{z_0=0} = ia$