Lecture Notes on Dirac Delta Function

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Instruction to Student/Tutor: If any part of this note appears difficult, a student may avoid that topic in the first-reading. We prefer to keep such advanced topics for the students preparing for competitive exams like JAM, JEST, NET, etc.

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1 Dirac Delta Function

1.1 Introduction

In mathematical physics there are two well known Delta Functions. First, Kronecker Delta, introduced by German mathematician L Kronecker (1823-1891) and second, Dirac Delta Function, introduced by P

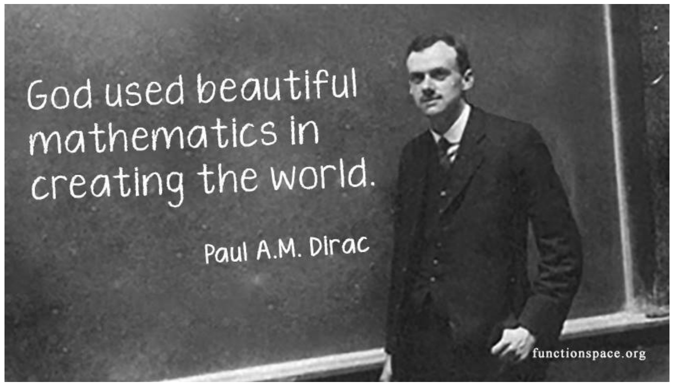

Figure 1: Paul Adrien Maurice Dirac (1902 - 1984)

A M Dirac (1902-1984). The Kronecker Delta function, which is represented by the identity matrix δ_{ij}, applied for a set of discrete variable $(i, j = 1, 2, \ldots..n)$. It is defined as,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.1}$$

It holds following key properties:

$$a) \quad \delta_{ij} = \delta_{ji} \qquad \text{(Symmetric property)}, \tag{1.2a}$$

$$b) \quad \sum_i a_i \delta_{ij} = a_j \qquad \text{(Summation property)}, \tag{1.2b}$$

$$c) \quad \sum_j a_j \delta_{ij} = a_i \qquad \text{(Summation property)} \tag{1.2c}$$

$$d) \quad \sum_k \delta_{ik} \delta_{kj} = \delta_{ij} \qquad \text{(Summation property)} \tag{1.2d}$$

respectively, where the summation is taken over the repeated indices (Called Einstein Sum Convention). A very useful representation of Kronecker delta function is

$$\delta_{nm} = \frac{1}{N} \sum_{k=1}^{\infty} e^{i \frac{2\pi k}{N}(n-m)}, \tag{1.3}$$

which is often referred as Discrete Fourier Transformation.

In contrast, Dirac Delta Function is defined in continuum. It is also not a function in traditional sense, rather '*a unit normalized function*' in a sense that takes its value to infinity at given particular point only. Quantitatively, it is defined as, where

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0; \\ 0 & \text{if } x \neq 0; \end{cases} \tag{1.4}$$

such that

$$\int_{-\infty}^{\infty} dx \delta(x) = 1. \tag{1.5}$$

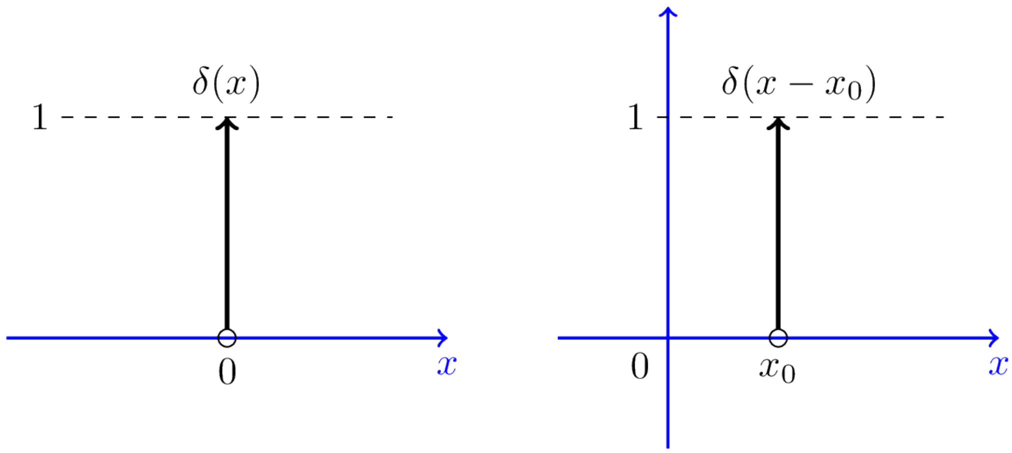

Figure 2: Plot of Dirac delta function with peak $x = 0$ and $x = x_0$

If the spike lies at point $x = x_0$, Eq.(1.4) can be written as,

$$\delta(x - x_0) = \begin{cases} \infty, & \text{if } x = x_0; \\ 0 & \text{if } x \neq x_0; \end{cases} \tag{1.6}$$

such that point of normalization is shifted to $x = x_0$,

$$\int_{-\infty}^{\infty} dx \delta(x - x_0) = 1. \tag{1.7}$$

For any arbitrary function $f(x)$, Eq.(1.7) can be generalized as

$$\int_a^b dx f(x)\delta(x - x_0) = \begin{cases} f(x_0), & \text{if } a < x < b; \\ 0, & \text{if } elsewhere, \end{cases} \tag{1.8}$$

while within whole range $[-\infty, \infty]$, it is given by

$$\int_{-\infty}^{\infty} dx f(x)\delta(x - x_0) = f(x_0). \tag{1.9}$$

Pictorially the Dirac delta function at $x = 0$ and $x = x_0$ are given below,
The Dirac delta function, its properties and application are discussed in the videos available online: (Overview of Dirac Deleta Function: Link 1) (Link 2), (Link 3), (Link 4), (Link 5), (Problems: Link 6), (Problems from Griffith: Link 7),

1.2 Heaviside (Step) Function and its relation with Dirac Delta Function

The Heaviside function (also called Step Function) is defined as

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0; \\ 0 & \text{if } x < 0; \end{cases} \tag{1.10}$$

[1] Definition of Open and Closed Interval:

$$\begin{aligned} y = f(t) \quad &\text{for} \quad a < x < y, \quad \text{f is defined as open interval (a,b)}, \\ y = f(t) \quad &\text{for} \quad a \leq x \leq y, \quad \text{f is defined as closed interval [a,b]}. \end{aligned}$$

,

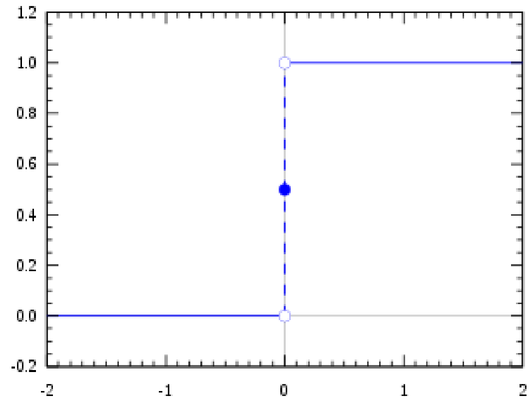

Figure 3: Plot of Heaviside (Step) function

To understand the relation of the Heaviside function with delta function we consider the unit rectangular function of width (Δ_W) and height (Δ_H) such that the area under curve is unity, i.e.,

$$\begin{aligned} Area &= \Delta_W \times \Delta_H \\ &= 1 \end{aligned} \tag{1.11}$$

Figure given below shows that to keep the area of rectangle constant (here unity), we must have $\Delta_W \to 0$ and $\Delta_H \to \infty$ or vice-versa. To describe it quantitatively, we define rectangular function in terms of Heaviside function,

$$rect(x) = \frac{1}{2\epsilon}\big[\Theta\big(x + \frac{1}{\epsilon}\big) - \Theta\big(x - \frac{1}{\epsilon}\big)\big] \tag{1.12}$$

where $\Delta_H = \frac{1}{2\epsilon}$ be the height of the rectangle. Then delta function is defined as,

$$\delta(x) = \lim_{\epsilon \to 0} ract(x). \tag{1.13}$$

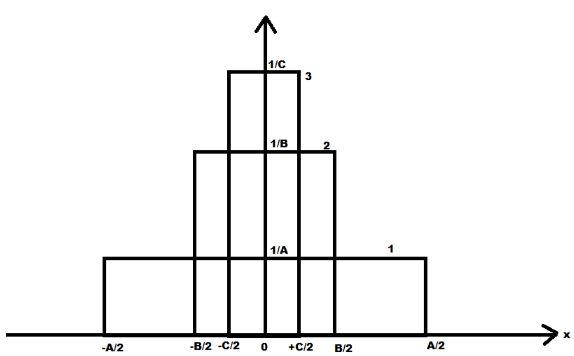

Figure 4: Plot of Rectangular function of different width

From Eq.(1.12) and (1.13) it can be possible to prove the exact relationship given by,

$$\frac{d}{dx}\Theta(x) = \delta(x). \tag{1.14}$$

Proof:
We note that,

$$\frac{d}{dx}[\Theta(x)f(x)] = \Theta'(x)f(x) + \Theta(x)f'(x) \tag{1.15}$$

Integrating both sides,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{d}{dx}\big[\Theta(x)f(x)\big] &= \int_{-\infty}^{\infty} dx\Theta'(x)f(x) + \int_{-\infty}^{\infty} dx\Theta(x)f'(x)\\ \Rightarrow \int_{-\infty}^{\infty} dx\Theta(x)f'(x) &= -\int_{-\infty}^{\infty} dx\Theta'(x)f(x). \end{aligned}\tag{1.16}$$

In the last equation we have used the fact that the total integral vanishes at the boundary, i.e., $\int_{-\infty}^{\infty} \frac{d}{dx}\big[\Theta(x)f(x)\big] = 0$. Thus we have, [2]

$$\begin{aligned} L.H.S. &= \int_{-\infty}^{\infty} dx\Theta(x)f'(x)\\ &= \Theta(x)f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\Theta'(x)f(x)\\ &= 0 - \int_{-\infty}^{\infty} dx\Theta'(x)f(x)\\ &= R.H.S \qquad Q.E.D. \end{aligned}\tag{1.17}$$

1.3 Representation of Dirac Delta Function

Some popular representation of delta function are given below:

$$a) \quad \delta_\sigma(x-y) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}} \quad (Gausian form) \tag{1.18a}$$

$$b) \quad \delta(x) = \lim_{\epsilon\to 0} \frac{\sin Nx}{\sin \pi x} \quad (Trigonometric form) \tag{1.18b}$$

$$c) \quad \delta(x) = \frac{1}{\pi} \lim_{\epsilon\to 0} \frac{\epsilon}{x^2+\epsilon^2} \quad (Lorenzian form) \tag{1.18c}$$

$$d) \quad \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx} \quad (Fourier form) \tag{1.18d}$$

$$e) \quad \delta(x) = \lim_{\epsilon\to\infty} \frac{1}{2\epsilon}[\Theta(x+\epsilon) - \Theta(x-\epsilon)] \quad (Heaviside form) \tag{1.18e}$$

$$f) \quad \delta(x) = \frac{1}{2}\frac{d^2}{dx^2}|x| \tag{1.18f}$$

$$g) \quad \delta(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t(t-x)} \tag{1.18g}$$

Below we have discussed Gaussian, Trigonometric, Lorentzian and Fourier representation which are frequently used in physics.

[2] We use the formula,

$$\int u\frac{dv}{dx} = uv - \int dxv\frac{du}{dx}$$

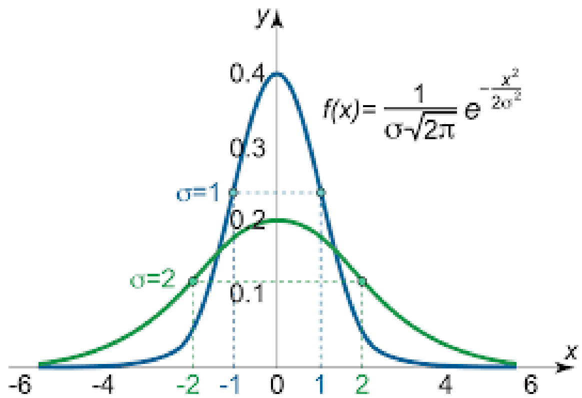

Figure 5: Plot of Gaussian (Bell-shaped) curve

1.4 Some important representations

I) **Gaussian Representation**

Prove that,

$$\delta_\sigma(x-y) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}}, \tag{1.19}$$

such that in the limit $\sigma \to 0$, it is normalized to unity, i.e.,

$$\int_{-\infty}^{\infty} \delta_\sigma(x-y) = 1. \tag{1.20}$$

The Gaussian function is defined as,

$$f_\sigma(x-y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-y)^2}{2\sigma^2}}; \qquad \sigma > 0, \tag{1.21}$$

Proof:

$$\begin{aligned}
L.H.S. &= \int_{-\infty}^{\infty} dx \delta_\sigma x) \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}}, \\
&= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dt e^{-t^2} \quad \text{we set} \quad x = \sqrt{2\sigma^2} t \\
&= 2\sqrt{\frac{1}{\pi}} \int_{0}^{\infty} dt e^{-t^2} \\
&= 2\sqrt{\frac{1}{\pi}} \cdot \frac{\sqrt{\pi}}{2} \\
&= 1 = R.H.S. \qquad Q.E.D.
\end{aligned} \tag{1.22}$$

where in the last equation we have used the well known property of Gausian Integral (See, for example).

II) **Trigonometric Representation**

We note that,

$$\begin{aligned}\frac{1}{2\pi}\int_{-g}^{g} dk\,\exp(ikx) &= \frac{1}{2\pi}\left[\frac{\exp(ikx)}{ix}\right]_{-g}^{g}\\ &= \frac{1}{\pi x}.\frac{1}{2i}\Bigg(\exp(igx)-\exp(-igx)\Bigg)\\ &= \frac{\sin gx}{\pi x}. \qquad (1.23a)\end{aligned}$$

Now taking the limit, namely,

$$\begin{aligned}\lim_{g\to\infty}\frac{\sin gx}{\pi x} &= \lim_{g\to\infty}\frac{1}{2\pi}\int_{-g}^{g} dk\,\exp(ikx)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\exp(ikx)\\ &= \delta(x) \qquad Q.E.D. \qquad (1.24a)\end{aligned}$$

III) **Lorenzian Representation**

Show that

$$\delta(x) = \frac{1}{\pi}\lim_{\epsilon\to 0}\frac{\epsilon}{x^2+\epsilon^2} \qquad (1.25)$$

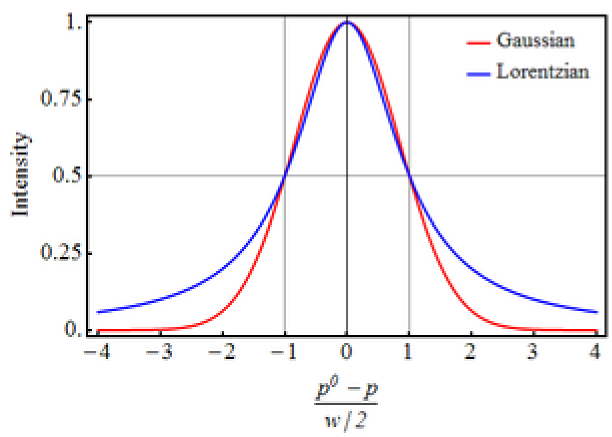

Figure 6: Comparison of Gaussian and Lorentzian plot

$$
\begin{aligned}
\delta(x) &= \int_{-\infty}^{\infty} dk e^{ikx} \\
&= \int_{-\infty}^{0} dk e^{ikx} + \int_{0}^{\infty} dk e^{ikx} \\
&= \int_{0}^{\infty} dk e^{-ikx} + \int_{0}^{\infty} dk e^{ikx} \\
&= \int_{0}^{\infty} dk \Big(e^{ikx} + e^{-ikx} \Big) \quad \text{(We take limiting value of } \epsilon) \\
&= \lim_{\epsilon \to 0^+} \int_{0}^{\infty} dk \Big(e^{ikx} + e^{-ikx} \Big) e^{-k\epsilon} \\
&= \lim_{\epsilon \to 0^+} \int_{0}^{\infty} dk \Big(e^{i(k+i\epsilon)} + e^{-i(k-i\epsilon)} \Big) \\
&= \lim_{\epsilon \to 0^+} \left[\frac{e^{i(k+i\epsilon)}}{i(x-i\epsilon)} + \frac{e^{-i(k-i\epsilon)}}{i(x+i\epsilon)} \right] \\
&= \lim_{\epsilon \to 0^+} \left[0 + 0 - \frac{1}{i(x-i\epsilon)} + \frac{1}{i(x+i\epsilon)} \right] \\
&= \lim_{\epsilon \to 0^+} \left[\frac{i}{(x-i\epsilon)} - \frac{i}{(x+i\epsilon)} \right] \\
&= \lim_{\epsilon \to 0^+} \frac{2\epsilon}{(x^2+\epsilon^2)} \\
&= \begin{cases} \infty, & \text{if } x = 0; \\ 0. & \text{if } x \neq 0; \qquad Q.E.D. \end{cases}
\end{aligned} \tag{1.25}
$$

IV) **Fourier Representation**

The Fourier Transformation (F.T.) of a function $f(x)$ is defined as,

$$
\mathfrak{F}[f(x)] = \int_{-\infty}^{\infty} dk F(k) \exp{[-ikx]} \tag{1.27}
$$

where $F(k)$ is a function which is connected with $f(x)$ by so called *Inverse Fourier Transformation*,

$$
\mathfrak{F}^{-1}[f(x)] \equiv F(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx f(x) \exp{[ikx]} \tag{1.28}
$$

To find the F.T. of delta function, i.e., for $\mathfrak{F}[\delta(x)]$, we have,

$$
\mathfrak{F}[\delta(x)] = \int_{-\infty}^{\infty} dk \delta(k) \exp{[ikx]} \tag{1.29}
$$

Plucking back Eq.(1.26) in Eq.(1.25) we obtain,

$$
\begin{aligned}
\mathfrak{F}[\delta(x)] &= \int_{-\infty}^{\infty} dk\delta(k)\exp\left[-ikx\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx'\delta(x')\exp\left[-i(x-x')k\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\delta(x') \int_{-\infty}^{\infty} dk\exp\left[-i(x-x')k\right] \\
&= \int_{-\infty}^{\infty} dx'\delta(x')\delta(x-x') \quad \Big(\because \delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\delta(k)\exp\left[ik(x-x')\right]\Big) \\
&= \delta(x) \quad Q.E.D.
\end{aligned} \tag{1.30}
$$

Thus the Fourier transformed of delta function has constant value only at $x = 0$.

1.5 Some properties of delta function

(I) **General properties**:

$$
\begin{aligned}
&a) \quad \delta(x) = \delta(-x) \quad \text{(Symmetry property)} && (1.31a)\\
&b) \quad \delta(x-a) = \delta(a-x) \quad \text{(Symmetry property)} && (1.31b)\\
&c) \quad x\delta(x) = 0 && (1.31c)\\
&d) \quad \delta(ax) = \frac{1}{|a|}\delta(x) \quad \text{(Scaling property)} && (1.31d)\\
&e) \quad \delta(x^2-a^2) = \frac{1}{|2a|}[\delta(x-a)+\delta(x-b)] && (1.31e)\\
&f) \quad \delta\big((x-a)\delta(x-b)\big) = \frac{1}{(a-b)}[\delta(x-a)+\delta(x-b)] && (1.31f)\\
&g) \quad \delta(f(x)) = \sum_i \frac{1}{f'(x_i)}\delta(x-x_i) \quad \text{iff} \quad f(x_i)=0 \quad \text{and} \quad f'(x_i)\neq 0 && (1.31g)
\end{aligned}
$$

(II) **Derivative & integral of delta function**:

$$a) \quad \int_{-\infty}^{\infty} dx\delta(x) = 1 \tag{1.32a}$$

$$b) \quad \int_{-\infty}^{\infty} dx f(x)\delta(x - x_0) = f(x_0) \tag{1.32b}$$

$$c) \quad \frac{d}{dx}\delta(x) = -\frac{d}{dx}\delta(-x) \tag{1.32c}$$

$$d) \quad x\frac{d}{dx}\delta(x) = -\delta(x) \tag{1.32d}$$

$$e) \quad \int_{-\infty}^{\infty} dx f(x)\delta'(x - x_0) = -f'(x_0) \tag{1.32e}$$

$$f) \quad \int_{-\infty}^{\infty} dx f(x)\frac{d^n}{dx^n}[\delta(x - x_0)] = (-1)^n \frac{d^n}{dx^n} f(x)\Big|_{x=x_0} \tag{1.32f}$$

$$g) \quad \int_{-\infty}^{\infty} \delta(x - a)\delta(x - b) = \delta(a - b) \tag{1.32g}$$

Proof of Property-I:

The Fourier representation of delta function is given by,

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ikx}. \tag{1.33}$$

from which it follows,

$$\delta(-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-ikx} \tag{1.34}$$

Now substituting $k = -p$ we have,

$$\begin{aligned}\delta(-x) &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{ipx} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{-ipx} \\ &= \delta(x) \qquad Q.E.D.\end{aligned} \tag{1.35}$$

Alternative proof,

$$\begin{aligned}
f(0) &= \int_{-\infty}^{\infty} dx f(x)\delta(x) \\
&= \int_{-\infty}^{\infty} d(-x) f(-x)\delta(-x) \\
&= -\int_{-\infty}^{\infty} dx f(-x)\delta(-x) \\
&= \int_{\infty}^{-\infty} dx f(-x)\delta(-x) \qquad \text{we change the variable to,} \quad z = -x \\
&= -\int_{\infty}^{-\infty} dz f(z)\delta(z) \\
&= \int_{-\infty}^{\infty} dz f(z)\delta(z) \\
&= f(0)
\end{aligned} \tag{1.36}$$

which implies $\delta(x) = \delta(-x)$ since $z = -x$.

2 Some Applications of Dirac Delta Function

2.1 Electromagnetic Theory

2.2 Heat

2.3 Quantum Mechanics

2.4 Fourier Transformation, Delta function and Parseval theorem

2.5 Laplace Transformation of Delta function

References

[1] (Overview of Dirac Deleta Function)