Lecture Notes on Ordinary Differential Equation

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Instruction to Student/Tutor: If any part of this note appears difficult, a student may avoid that topic in the first-reading. We prefer to keep such advanced topics for the students preparing for competitive exams like JAM, JEST, NET, etc.

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1 Ordinary Differential equation

1.1 Introduction

Nature is dynamical. For a particle or a system of particles, the dynamics is generally described by the time evolution of a coordinate or a system of coordinates. In this respect, the differential equation is probably the most successful tool to describe the time evaluation of a number of phenomena we witness everyday. Broadly speaking, the Differential Equation can be classified into three distinct categories: i) Ordinary Differential equation (ODE) (*One independent variable, one dependent variable*), ii) Partial Differential Differential (PDE) (*Multiple independent variable, one dependent variable*) and iii) Non-linear Differential Equation (NDE) (terms with higher order dependent variable), i.e.,

Ordinary Differential Equation

Ordinary Linear DE | Partial DE | Non-linear DE

In this lecture note we give a cursory overview of different class of ODE which has been recently included in the UG physics course. For a more detail introduction, we refer the readers to visit the following: i) Wiki pages on ODE, ii) MIT Lecture on "Overview ODE", iii) Michel van Biezen and iv) others, v) others.

1.2 General Setup of ODE

A Ordinary Differential Equation (ODE) of n-th order is a function of a unknown variable $y(x)$ and its derivatives with respect to one independent variable x,

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{(n-1)}y}{dx^{(n-1)}}, \ldots, \frac{dy}{dx}, y(x)\right) = Q(x, y(x)), \tag{1.1}$$

where $Q(x, y(x))$ may be a function of dependent and independent variables. If the function Q is only a function of x, i.e., $Q = Q(x)$, the differential equation is said to be Linear ODE (LDE). (1.1) can be also written as,

$$P_n(x)\frac{d^n y}{dx^n} + P_{n-1}(x)\frac{d^{(n-1)}y}{dx^{(n-1)}} + \cdots + P_1(x)\frac{dy}{dx} + P_0(x)y(x) = Q(x, y(x)), \tag{1.2}$$

where $P_n(x)$ be the polynomial of independent variable x. Depending upon the value of n, the equation is referred as first order ($n = 1$), second order ($n = 1$) or third order ($n = 3$) equation etc. It is customary to expand the right hand side as,

$$Q(y, x) = h_0 + yh_1(x) + y^2h_2(x) + \cdots + y^m h_m(x), \tag{1.3}$$

which for $m \leq 1$ is known as Linear DE (LDE) and for $m \geq 2$ called Nonlinear DE (NDE). In general, the ODE exists in two variety: i) *Homogeneous Equation* ($Q(x, y) = 0$), and ii) *Non-homogeneous Equation* ($Q(x, y) \neq 0$), i.e., Ordinary Differential Equation (ODE)

Homogeneous ODE Non-homogeneous ODE

Mathematically, a equation is said to be ***homogeneous*** of degree p if it satisfies,

$$Q(tx, ty) = t^p Q(x, y), \tag{1.4}$$

and on the other hand, if it cannot be written in above form called ***non-homogeneous*** equation. Below we have given some examples of first order differential equations:

- Linear first order homogeneous equation ($P_0(x) = p_0, P_1(x) = 1$, and $Q(x, y) = 0$):

$$\frac{dy(x)}{dx} + p_0 y(x) = 0, \tag{1.5a}$$

  which can be identified with the Radioactive Decay Equation with $y(x) = N(t), x = t$ with $p_0 = \lambda$,

$$\frac{dN(t)}{dt} + \lambda N(t) = 0, \tag{1.5b}$$

- Second order Linear homogeneous equation ($P_2(x) = 1$, $P_0(x) = p_0$ $Q(x, y) = 0$ for all $m > 0$):

$$\frac{d^2 y(x)}{dx^2} + p_0 y(x) = 0, \tag{1.5c}$$

  and identifying $x = t$ and $P_0 = \omega^2$, we readily obtain the Equation of Motion (EOM) of Simple Harmonic Oscillator (SHM):

$$\frac{d^2 y(t)}{dt^2} + \omega^2 y(x) = 0, \tag{1.5d}$$

- Bernoulli's equation $P_1(x) = 1$, $Q_m(x, y) = y^n h_n(x)$:

$$\frac{dy(x)}{dx} + P_0(x) y(x) = y^n h_n(x), \tag{1.5e}$$

  where we can have different degree of nonlinearity depending upon the value of n.

- Riccati equation ($P_1(x) = 1, P_0(x) = -k$, $h_0 = 0$, $Q_i(x, y) = -h_1(x) + y^2 h_2(x)$:

$$\frac{dy(x)}{dx} - ky(x) = -h_1(x) + y^2 h_2(x), \tag{1.5f}$$

In many situation, $P_i(x)$, instead of a constant, is some simple polynomial of x. Some examples of second order ODE are given below:

- Second order Linear homogeneous equation ($P_2(x) = p_2$, $P_1(x) = p_1$ and $P_0(x) = \omega_0^2$ $Q(x,y) = 0$):

$$p_2 \frac{dy^2(x)}{dx^2} + p_1 \frac{dy(x)}{dx} + \omega_0^2 y(x) = 0, \tag{1.6a}$$

  If we identify, $p_2 = 1$, $p_1 = \gamma$ and $P_0 = 1$, we obtain the equation of motion of Damped Simple Harmonic Motion (D-SHO) :

$$\frac{d^2y(t)}{dt^2} + \gamma \frac{dy(t)}{dt} + \omega^2 y(t) = 0, \tag{1.6b}$$

- Hermite's equation (Second order linear homogeneous equation) ($P_2(x) = 1, P_1(x) = -2x, P_0(x) = 2k$, $Q_i(x) = 0$):

$$\frac{d^2y(x)}{dx^2} - 2x\frac{dy(x)}{dx} + 2ky(x) = 0, \tag{1.6c}$$

- Legendre's equation (Second order linear homogeneous equation) ($P_2(x) = 1-x^2, P_1(x) = -2x, P_0(x) = k(k+1)$, with $Q(x,y) = 0$):

$$(1-x^2)\frac{d^2y(x)}{dx^2} - 2x\frac{dy(x)}{dx} + k(k+1)y(x) = 0, \tag{1.6d}$$

- Bessel's equation ($P_2(x) = x^2, P_1(x) = x, P_0(x) = x^2 - k^2$, with $Q(x) = 0$):

$$x^2\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} + (x^2 - k^2)y(x) = 0, \tag{1.6e}$$

- Gauss's hypergeometric equation (Second order linear homogeneous equation) ($P_2(x) = x(1-x), P_1(x) = -[a-(b+c+1)]x, P_0(x) = -ab$, with $Q_i(x) = 0$ for $i > 0$):

$$x(1-x)\frac{d^2y(x)}{dx^2} - [a-(b+c+1)]\frac{dy(x)}{dx} - aby(x) = 0, \tag{1.6f}$$

  where a, b, c are constants.

- ... and many more.

1.3 A note on Partial Differential Equation (PDE):

If the dependent variable is a function of multiple number of independent variables and its derivatives, i.e.,

$$F\left(x_i, y(x_i), \frac{\partial y}{\partial x_i}, \frac{\partial^2 y}{\partial x_i^2}, \frac{\partial^2 y}{\partial x_i \partial x_j}, \dots\right) = Q(x_i, y(x_i)), \tag{1.7}$$

and the equation is called Partial Differential Equation (PDE). Below we have given some examples of PDE which are frequently used in physics:

- Maxwell Equation:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.8a}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.8b}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.8c}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{1.8d}$$

- Electromagnetic Wave Equation (with $c^2 = \frac{1}{\epsilon\mu}$):

$$\nabla^2 \mathbf{E} + \frac{1}{\epsilon\mu} \frac{\partial^2 \mathbf{E}(\mathbf{r}, t)(t)}{\partial t^2} = -\frac{\rho}{\epsilon_0} \tag{1.8e}$$

$$\nabla^2 \mathbf{B} + \frac{1}{\epsilon\mu} \frac{\partial^2 \mathbf{B}(\mathbf{r}, t)(t)}{\partial t^2} = \mu_0 \mathbf{J} \tag{1.8f}$$

- Heat Equation:

$$\frac{\partial \mathbf{U}(\mathbf{r}, \mathbf{t})(t)}{\partial t} = K \nabla^2 \mathbf{U}(\mathbf{r}, \mathbf{t}) = 0 \tag{1.8g}$$

- Time-dependent Schrödinger Equation

$$\frac{\partial \psi(\mathbf{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t) \tag{1.8h}$$

- Time-independent Schrödinger equation

$$\nabla^2 \Psi(\mathbf{r}) + \frac{2m}{\hbar^2}(E - V(\mathbf{r}))\Psi(\mathbf{r}) = 0 \tag{1.8i}$$

- Euler equation

$$\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + F \quad (Defindforperunitmass) \tag{1.8j}$$

- Navier-Stokes equation

$$\frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla \mathbf{p} + \mathbf{F} + \nu \nabla^2 \mathbf{u} \tag{1.8k}$$

- ... and many more.

Many of these differential equation can be solved either analytically or numerically. Various techniques of solving the PDE will be addressed later.

2 First Order Ordinary Differential Equation

2.1 Introduction

Simplest form of differential equation is called first order ODE,

$$\frac{dy(x)}{dx} = Q(x, y), \tag{2.1}$$

where, the function $Q(x, y)$ determines the nature of its solution. Some well known first order ODE of physics are given below (Symbols have their usual meaning):

- Newton's Law of Cooling:

$$\frac{dT(t)}{dt} + (T(t) - T_0) = 0 \tag{2.2a}$$

- Law of Radioactive Disintegration:

$$\frac{dN(t)}{dt} + \lambda N(t) = 0 \tag{2.2b}$$

- Pressure Altitude Equation:

$$\frac{dp(y)}{dy} = -\rho g \tag{2.2c}$$

- EOM (Equation of Motion) of a body through a viscous liquid:

$$m\frac{dv(y)}{dt} = -m_f g - 6\pi\eta_f Rv \tag{2.2d}$$

- Charging-Discharging of Condenser:

$$R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = V(t) \tag{2.2e}$$

- ...and many more.

2.2 Exact and Non-exact equation

To find the solution of a first order differential equation we consider some distinct cases:

Case-I: When the function Q is factorizable, i.e., $Q(x, y) = -g(x)h(y)$

In such case, two function are easily separable and therefore integrable, i.e.,

$$\int \frac{dy(x)}{h(y)} + \int dxg(x) = c, \tag{2.3}$$

where c be constant which is determined from the boundary condition. This method is also known as *Separation of Variable Method*. The following links give the solution of first order ODE using this method: i) Example: I, ii) Example: II, iii) Example: III, iv) Example: IV

Case-II: When the function Q is a quotient, i.e., $Q(x,y) = -\frac{M(x,y)}{N(x,y)}$

It is worth mentioning here that if the quotient satisfies the relation,

$$\frac{M(tx,ty)}{N(tx,ty)} = \frac{M(x,y)}{N(x,y)}, \tag{2.4}$$

the differential equation is said to be ***homogeneous***, in contrast, if satisfies

$$\frac{M(tx,ty)}{N(tx,ty)} = t^p \frac{M(x,y)}{N(x,y)} \tag{2.5}$$

$(p \neq 0)$ it is said to be ***Non-homogeneous***, respectively. Plucking back the value of $Q(x,y)$ in (2.1) we obtain,

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.6}$$

If the function $M(x,y)$ and $N(x,y)$ satisfy the condition, [1]

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}, \tag{2.7}$$

the differential equation is said to be ***Exact***, otherwise, it is called ***Non-Exact*** differential equation, i.e.,

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}. \tag{2.8}$$

i) Solution of Exact Equation:

For *Exact Equation*, the function $u(x,y)$ is a perfect differential (See, footnote [1]) and it is straight forward to show,

$$u(x,y) = \int dx M(x,y) + c_1(x) \tag{2.9a}$$

$$u(x,y) = \int dy N(x,y) + c_2(y), \tag{2.9b}$$

where $c_1(x)$ and $c_2(x)$ are arbitrary functions which as said earlier, are to be determined from the boundary conition. Some examples of solving the exact equations are discussed here: i) Example: I, ii) Example: II, iii) Example: III, iv) Example: IV

[1] Consider the partial derivative of a function $u = u(x,y)$,

$$du(x,y) = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy$$

Comparing this with (2.7) we find,

$$M(x,y) = \frac{\partial u(x,y)}{\partial x}, \quad N(x,y) = \frac{\partial u(x,y)}{\partial y},$$

If $u(x,y)$ is an exact differential i.e.,

$$du(x,y) = 0 \Rightarrow u(x,y) = constant,$$

then we readily obtain,

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

ii) Solution of Non-exact Equation

For ***Non-exact*** equation, it is necessary to multiply M and N with a suitable polynomial $\mu(x,y)$ which is called ***Integrating factor (I.F.)***. The I.F. reduces (2.7) into following form,

$$\frac{\partial G(x,y)}{\partial y} = \frac{\partial H(x,y)}{\partial x}, \tag{2.10}$$

where, $G(x,y) = \mu(x,y)M(x,y)$ and $H(x,y) = \mu(x,y)N(x,y)$, respectively. There exists some rules to obtain the I.F. of first order ODE:

- Rule-I: When the equation is ***Exact*** and ***Homogeneous***
  In this case the I.F. is given by,
  $$\mu_+(x,y) = \frac{1}{M(x,y)x + N(x,y)y} \tag{2.11a}$$
  where, $M(x,y)x + N(x,y)y \neq 0$.
- Rule-II: When the equation is ***Non-exact*** and ***Non-homogeneous***
  Here the I.F. is given by,
  $$\mu_-(x,y) = \frac{1}{M(x,y)x - N(x,y)y} \tag{2.11b}$$
  such that, $M(x,y)x - N(x,y)y \neq 0$.
- Rule-III: The equation is ***Non-exact*** and ***Non-homogeneous***:
  i) If $M(x,y)$ and $M(x,y)$ satisfy the condition,
  $$\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right) = g(x), \tag{2.11c}$$
  then the differential equation is solved by using the I.F.,
  $$\mu_y = e^{\int dx g(x)}, \tag{2.11d}$$
  ii) On the other hand, if they satisfy,
  $$\frac{1}{M(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right) = h(y), \tag{2.11e}$$
  the equation can be solved using the I.F.,
  $$\mu_x = e^{\int dy h(y)}. \tag{2.11f}$$
- Rule-V: The equation is ***Non-exact*** and ***homogeneous***
  If we have the following condition,
  $$\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right) = K, \tag{2.11g}$$
  $$\frac{1}{M(x,y)}\left(\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right) = K, \tag{2.11h}$$
  then the differential equation is solved by using the I.F.,
  $$\mu = e^{\int dxK} = e^{\int dyK} \equiv e^{Kx} \tag{2.11i}$$

In the following links we have solved some first order ODE using above technique: Example: I, Example: II, Example: III and Example: IV.

3 Second order linear differential equation

3.1 Introduction

Since the discovery of Newton's law of motion, the second order differential equation was always at the centre-stage of understanding the diverse physical phenomena of nature. A list of such equation from different branches of physics is given below:

- Simple Harmonic Motion (SHO):

$$\frac{d^2y(t)}{dt^2} + \omega_0^2 y(t) = 0 \tag{3.1a}$$

- Damped Simple Harmonic Motion (DSHO):

$$\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \omega_0^2 y(t) = 0 \tag{3.1b}$$

- Forced Simple Harmonic Motion (FSHO):

$$\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \omega_0^2 y(t) = F_0 \sin\omega t \tag{3.1c}$$

- The pendulum equation:

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{l}\sin\theta(t) = 0 \tag{3.1d}$$

- The equations that govern Kepler law of motion:

$$\frac{d^2r(t)}{dt^2} - \left(\frac{d\theta(t)}{dt}\right)^2 = -\frac{GM}{r^2} \tag{3.1e}$$

$$\frac{d^2\theta(t)}{dt^2} + 2\frac{dr(t)}{dt}\frac{d\theta(t)}{dt} = 0 \tag{3.1f}$$

- Equation of LCR circuit in series:

$$L\frac{d^2I(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = V(t) \tag{3.1g}$$

- Time-independent Schrödinger equation (One dimension)

$$\frac{d^2\psi(x)}{dx^2} + \frac{2m}{\hbar^2}(E - V(x)\psi(x) = 0 \tag{3.1h}$$

- ... and many more.

3.2 General setup

The generic n-th order LDE is given by

$$\mathbf{L}y(x) = Q(x) \tag{3.2}$$

where $\mathbf{L}$ be the differential operator given by,

$$\mathbf{L} = a_n \frac{d^n}{dx^n} + a_{n-1}\frac{d^{(n-1)}}{dx^{(n-1)}} + \cdots + a_1 \frac{d}{dx} + a_0 \tag{3.3}$$

with $a_i (i = 0, 2, \ldots n)$ be the constant coefficients. The solution a second order LDE can be done by the following methods: i) Undetermined Coefficients Method (Method using Complementary Function and Particular Integral), ii) Linear Operator Method, iii) Series Solution Method, iv) Variation of Parameters Method, v) Change of Dependent Variable Method, vi) Numerical Method etc. In this section we shall give the outline of first two methods with some examples.

3.3 The Method of Undetermined Coefficient

Most general second order differential equation is given by,

$$A(x)\frac{d^2y}{dx^n} + B(x)\frac{dy}{dx} + C(x)y(x) = Q(x). \tag{3.4}$$

The homogenous LDE with $Q(x) = 0$ is called *Auxiliary Equation* and its solution is known as Complementary Function (C.F.). If $y_h^1(x)$ and $y_h^2(x)$ are two *linearly independent* (See next chapter) solutions of the auxiliary equation, then its C.F. is given by

$$y_h(x) = c_1 y_h^1(x) + c_2 y_h^2(x) \tag{3.5}$$

where c_i $(i = 1, 2)$ be the constants which are evaluated from the boundary condition. On the contrary, for non-zero value of $Q(x)$, which makes the equation non-homogeneous, the solution is obtained by introducing a quantity called 'Particular Integral' (P.I.). Thus the general solution is a sum of the C.F. $(y_h(x))$ and P.I. $(y_p(x))$, i.e.,

$$y(x) = y_h(x) + y_p(x). \tag{3.6}$$

Below we have given some links which illustrate the solution of the second order Homogeneous LDE $(Q(x) = 0)$: Example: I, Example: II, Example: III, Example: IV, Example: V, Example: IV (With Boundary Condition), iv) Example: VI.

The evaluation of the P.I. $(y_p(x))$ is, however, a tricky issue which is guided by the form of the function $Q(t)$. Some well known representation of $Q(x)$ are: i) Polynomial (Linear, quadratic etc), ii) Trigonometric, iii) Exponential etc. In the following table we have given different form of $Q(x)$ and corresponding trial functions used to evaluate the P.I.:

Sl no.	$Q(x)$	Trial solution $y_p(x)$
1	$\alpha e^{\pm kx}$	$Ae^{\pm kx}$
2	$\alpha x + \beta$	$Ax + B$
3	$\alpha \cos kx$	A $\sin kx + B\cos kx$
4	$\beta \sin kx$	$A\sin kx + B\cos kx$
5	$P_n(x)^2$	$y_p^n(x)^3$

Below we have illustrated the evaluation C.F. and P.I. of a second order ODE with following examples:

- Example I:

  Choosing the coefficients to be, $A(x) = a, B(x) = b, C(x) = c$ and $Q(x) = Q_0 e^{kx}$, Eq.(3.4) becomes

$$a\frac{d^2y}{dx^n} + b\frac{dy}{dx} + cy(x) = Q_0 e^{kx}. \tag{3.7}$$

  i) Calculation of Complimentary Function (C.F.):

  If $y_h(x) = Ae^{kx}$ be the solution (called exponential ansatz) of the auxiliary equation, then, Eq.(3.7) becomes,

$$(ak^2 + bk + c)Ae^{kx} = 0. \tag{3.8}$$

  The C.F. is given by,

$$y_h(x) = A_+ e^{k_+} + A_- e^{k_-}, \tag{3.9}$$

  where, $A_\pm$ be the constants and $k_\pm = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})$, respectively. Finally writing $k_\pm = \alpha \pm \beta$, where $\alpha = \frac{-b}{2a}$ and $\beta = \frac{1}{2a}\sqrt{b^2 - 4ac} = |\beta|$, we obtain three possible solutions:

  Case-I: When roots ($|\beta| > 0$) are real and distinct,

$$y_h(x) = A_+ e^{k_+ x} + A_- e^{k_- x}, \tag{3.10a}$$

  Case-II: When roots ($|\beta| = 0$) are real and equal,

$$y_h(x) = (A_+ + A_- x)e^{\alpha x}, \tag{3.10b}$$

  Case-III: When roots ($|\beta| < 0$) are imaginary,

$$y_h(x) = e^{\alpha x}(A_+ \cos|\beta|x + A_- \sin|\beta|x), \tag{3.10c}$$

  where, $A_\pm$ be the complex constants. ii) Calculation of Particular Integral (P.I.):

  To evaluate P.I., following above Table, we work with a trial function $y_p(x) = Ae^{kx}$ and proceeding in the similar way we obtain,

$$A = \frac{Q_0}{ak^2 + bk + c} \equiv \frac{Q_0}{(k - k_+)(k - k_-)}. \tag{3.11}$$

  Thus the P.I. is given by,

$$y_p(x) = \frac{Q_0 e^{kx}}{(k - k_+)(k - k_-)}, \tag{3.12}$$

  such that $k \neq k_\pm$. Thus the complete solution of second order LDE for three distinct cases are given by:

  Case-I: For $|\beta| > 0$, the roots are real and distinct

$$y_p(x) = A_+ e^{k_+ x} + A_- e^{k_- x} + \frac{Q_0 e^{kx}}{(k - \alpha + |\beta|)(k - \alpha - |\beta|)}, \tag{3.13a}$$

[2] $P_n(x) = a_p x^p + a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$

[3] $y_p^n(x) = A_p x^p + A_{p-1}x^{p-1} + \cdots + A_1 x + A_0$

Case-II: For $|\beta| = 0$, the roots are real and equal

$$y(x) = (A_+ + xA_-)e^{kx} + \frac{Q_0}{(k + \frac{b}{2a})^2} \tag{3.13b}$$

Case-III: For $|\beta| < 0$, the roots become imaginary

$$y(x) = e^{\alpha x}(A_+ \cos|\beta|x + A_- \sin|\beta|x) + \frac{Q_0 e^{kx}}{(k - \alpha + i|\beta|)(k - \alpha - i|\beta|)} \tag{3.13c}$$

- Example II: Forced Simple Harmonic Oscillator:

  The Forced simple Harmonic oscillator is a non-homogeneous equation with trigonometric inhomogeneity which described by the Equation of Motion (EOM)

$$\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \omega_0^2 y(t) = F_0 \cos\omega t \tag{3.14}$$

where ω_0 and γ be the systems's natural frequency and decay constant, respectively and $F(t) = F_0 \cos\omega t$ be the external driving for with frequency ω and amplitude F_0, respectively. To find the solution we first consider the auxiliary equation by switching off the forced term,

$$\frac{d^2y(t)}{dt^2} + \gamma\frac{dy(t)}{dt} + \omega_0^2 y(t) = 0 \tag{3.15}$$

Using $y_h(t) = Ae^{i\omega t}$ as ansatz solution, the straight forward evaluation the C.F. gives,

$$y_h(t) = Ae^{i\omega_+ t} + Be^{-i\omega_- t} \tag{3.16}$$

where,

$$\omega_\pm = \frac{1}{2}\left(1 \pm \sqrt{\gamma^2 - 4\omega_0^2}\right) \tag{3.17}$$

To find the P.I., we consider the trial function from Table-I,

$$y_p(t) = A\sin\omega t + B\cos\omega t \tag{3.18}$$

Further, noting the fact that,

$$y_p'(t) = A\omega\cos\omega t - B\omega\sin\omega t, \tag{3.19a}$$

$$y_p''(t) = -A\omega^2\sin\omega t - B\omega^2\cos\omega t, \tag{3.19b}$$

and plugging back them into (3.14) we obtain,

$$(-A\omega^2\sin\omega t - B\omega^2\cos\omega t) + \gamma(A\omega\cos\omega t - B\omega\sin\omega t) + \omega_0^2(A\sin\omega t + B\cos\omega t) = F_0\sin\omega t \tag{3.20}$$

Comparing the coefficients of $\sin\omega t$ and $\cos\omega t$ we obtain,

$$A = \frac{\gamma\omega F_0}{(\omega - \omega_0)^2 + \gamma^2\omega^2}, \tag{3.21a}$$

$$B = \frac{(\omega^2 - \omega_0^2)F_0}{(\omega_0 - \omega)^2 + \gamma^2\omega^2} \tag{3.21b}$$

Finally substituting (3.21) into (3.18) the P.I. reads,

$$y_p(t) = \frac{F_0}{(\omega-\omega_0)^2+\gamma^2\omega^2}\big(\gamma\omega\sin\omega t + (\omega_0^2-\omega^2)\cos\omega t\big) \tag{3.22}$$

Finally collecting (3.16) and (3.22), the complete solution is given by

$$y(t) = Ae^{i\omega_+ t} + Be^{-i\omega_- t} + \frac{F_0}{(\omega-\omega_0)^2+\gamma^2\omega^2}\big(\gamma\omega\sin\omega t + (\omega_0^2-\omega^2)\cos\omega t\big). \tag{3.23}$$

Introducing the phase angle, $\tan\phi = \frac{\gamma\omega}{\omega-\omega_0}$, (3.23) can be written in more convenient form,

$$y(t) = Ae^{i\omega_+ t} + Be^{-i\omega_- t} + \frac{F_0}{(\omega-\omega_0)^2+\gamma^2\omega^2}\cos(\omega t-\phi). \tag{3.24}$$

- Below we have given some examples of simple non-homogeneous equation solved by above method [?]:

$$\text{Set}-\text{I} = \begin{cases} Example : 1 & \text{Q(x) is Exponential} \\ Example : 2 & \text{Q(x) is Polynomial} \\ Example : 3 & \text{Q(x) is Trigonometric} \end{cases} \tag{3.25a}$$

$$\text{Set}-\text{II} = \begin{cases} Example : 1 & \text{Q(x) is Exponential} \times \text{Polynomial} \\ Example : 2 & \text{Q(x) is Exponential} \times \text{Trigonometric} \\ Example : 3 & \text{Q(x) is Polynomial} \times \text{Trigonometric} \end{cases} \tag{3.25b}$$

$$\text{Set}-\text{III} = \begin{cases} Example : 1 & \text{Q(x) is Exponential} + \text{Polynomial} \\ Example : 2 & \text{Q(x) is Polynomial} + \text{Trigonometric} \\ Example : 3 & \text{Q(x) is Trigonometric} + \text{Exponential} \end{cases} \tag{3.25c}$$

$$\text{Set}-\text{IV} = \begin{cases} Example : 1 & \text{Q(x) is Exponential} \times \text{Polynomial} \times \text{Trigonometric} \\ Example : 2 & \text{Q(x) is Exponential} + \text{Polynomial} + \text{Trigonometric} \end{cases} \tag{3.25d}$$

In next section we discuss 'Operator Method' (Also called 'Annihilation Method') to solve the non-homogeneous equation.

3.4 Linear Operator Method (Advanced Part)

To solve LDE using Linear Operator Method it is convenient to write (3.3) in the following form,

$$\begin{aligned} (a^n D_n + a_{n-1}D_{(n-1)} + \cdots + a_1 D_1 + a_0)y(x) &= Q(x), \\ \hat{p}_n(D)y(x) &= Q(x) \end{aligned} \tag{3.29}$$

where $\hat{p}_n(D) = a_n D_n + a_{n-1}D_{(n-1)} + \cdots + a_1 D_1 + a_0$ be the linear differential operator. Similar to previous method, here the C.F. can be derived from the auxiliary equation ($Q(x) = 0$), while the P.I. function is governed by the function $Q(x)$. If $\{r_1, r_2, \dots r_i\}$ be the roots of the equation, then formally the Particular Integral can be written in the following form,

$$y_p(x) = \frac{Q(x)}{(D-r_1)(D-r_2)(D-r_3)\dots(D-r_n)} \tag{3.27a}$$
$$= \frac{1}{\prod_{i=1}^{n}(D-r_i)} Q(x). \tag{3.30}$$

However, the successive operation of the operator on $y(x)$ is quite nontrivial as it depends on the nature of the function $Q(x)$. So before its evaluation, let us recall some useful properties of the D operator:

$$i) \frac{1}{D_x} y(x) = \int dx y(x) \tag{3.28a}$$
$$ii) \frac{1}{D+1} = \sum_{n=0}^{n} (-1)^n D^n = 1 - D + D^2 - D^3 + \dots \tag{3.28b}$$
$$iii) \frac{1}{D-1} = \sum_{n=0}^{n} D^n = 1 + D + D^2 + D^3 + \dots \tag{3.28c}$$
$$iv) e^D = \sum_{n=0}^{n} \frac{D^n}{n!}$$
$$= 1 - \frac{1}{2!} D + \frac{1}{3!} D^2 + \frac{1}{3!} D^3 + \dots + \frac{1}{n!} D^n \tag{3.28d}$$

Below we have given the rules which are commonly used to evaluate the P.I. of a inhomogeneous second order LDE: [4]

- Rule I: $Q(x)$ is an exponential function, $Q(x) = e^{\alpha x}$
  If $p_n(\alpha) \neq 0$, the solution is given by,

$$y_p(x) = \frac{1}{p(D)} e^{\alpha x}$$
$$= e^{\alpha x} \frac{1}{p_n(\alpha)} \tag{3.32}$$

  while for $p_n(\alpha) = 0$, we use the identity $p_n(D) y(x) = (D-\alpha)^k p_{n-k}(D) y(x)$ $(1 < k < n)$ and get,

$$y(x) = \frac{A x^k e^{\alpha x}}{k! p_{n-\alpha}(\alpha)} \tag{3.30}$$

  For example, see, Example: I, Example: II, Example: III, Example: IV

- Rule II: $Q(x)$ is an trigonometric function, $Q(x) = A\cos\beta x + B\cos\beta x$ The trigonometric function can be only solved provided, $p_n(D) = \mathcal{O}(p_n(D^2))$ and we need to use the following identity,

$$p_n(D^2)\sin\beta x = \sin\beta x p_n(-\beta^2) \tag{3.31a}$$
$$p_n(D^2)\cos\beta x = \cos\beta x p_n(-\beta^2) \tag{3.31b}$$

  For example, see, i) Example: I, ii) Example: II

[4]W.Chen, Differential Operator Method of Finding A Particular Solution to An Ordinary Nonhomogeneous Linear Differential Equation with Constant Coefficients The readers may avoid this section in first reading.

- Rule III: $Q(x)$ is a n-th order polynomial function, i.e., $Q(x) = Q_n(x)$

  The solution in this case is given by,

$$y_p(x) = \frac{1}{p_n(D)} Q_n(x). \tag{3.32}$$

  We shall solve it for two distinct cases:
  Case I: For $a_0 = a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k \neq 0$

$$\begin{aligned} y_p(x) =& p_n(D)^{-1} Q_n(x) \\ =& \frac{1}{a^n D_n + a_{n-1} D_{(n-1)} + \cdots + a_1 D_1)} D^{-k} Q_n(x) \end{aligned} \tag{3.36a}$$

  Case II: For $a_0 \neq 0$,

$$\begin{aligned} y_p(x) =& p_n(D)^{-1} Q_n(x) \\ =& \sum_{p=1}^{\infty} (-1)^p \frac{1}{a_0} \big(\frac{a_n}{a_0} D^n + \frac{a_{n-1}}{a_0} D^{n-1} + \cdots + \frac{a_1}{a_0} D^1 \big)^p Q_n(x) \end{aligned} \tag{3.36b}$$

  For example, see, Example: I, Example: II

- Rule IV: $Q(x)$ is a product of polynomial and exponential function, i.e., $Q(x) = e^{ax} h_n(x)$

  The solution in this case is given by,

$$\begin{aligned} y_p(x) =& \frac{1}{p(D)} e^{\alpha x} h_n(x) \\ =& e^{ax} \frac{1}{p_n(D + \alpha)} h_n(x) \end{aligned} \tag{3.38}$$

  For example, see, Example: I, Example: II, Example: III

- Rule V: $Q(x)$ is a polynomial and trigonometric function,$Q(x) = h_n(x)\{A \cos \beta x + B \cos \beta x)$ The solution in this case is given by,

$$\begin{aligned} y_p(x) =& \frac{1}{p(D)} h_n(x) \{A \cos \beta x + A \cos \beta x) \\ =& \frac{A + iB}{2} e^{\beta x} \frac{1}{p_n(D + i\beta)} h_n(x) + \frac{A - iB}{2} e^{-\beta x} \frac{1}{p_n(D - i\beta)} h_n(x). \end{aligned} \tag{3.39}$$

  For example, see, Example: I, Example: II, Example: III

- Rule VI: $Q(x)$ is a product of exponential and trigonometric function and $p_n(\alpha + i\beta) \neq 0$, respectively, i.e., $Q(x) = e^{\alpha x}\{A \cos \beta x + A \cos \beta x)$ The solution in this case is given by,

$$\begin{aligned} y_p(x) =& \frac{1}{p(D)} e^{\alpha x} \{A \cos \beta x + A \cos \beta x) \\ =& \frac{A + iB}{2p_n(\alpha - i\beta)} e^{(\alpha - i\beta)x} + \frac{A - iB}{2p_n(\alpha + i\beta)} e^{(\alpha - i\beta)x} \end{aligned} \tag{3.40}$$

  Example: I, Example: II

- Rule VII: $Q(x)$ is a product of polynomial, exponential and trigonometric function, respectively, i.e., $Q(x) = h_m(x)e^{\alpha x}\{A\cos\beta x + A\cos\beta x)$ The solution in this case is given by,

$$\begin{aligned} y_p(x) =& \frac{1}{p(D)} h_m(x)e^{\alpha x}\{A\cos\beta x + A\cos\beta x) \\ =& \frac{A+iB}{2}\frac{1}{p_n(D+\alpha-i\beta)}h_m(x) + \frac{A-iB}{2}\frac{1}{p_n(D+\alpha+i\beta)}h_m(x) \end{aligned} \tag{3.41}$$

For example, see, Example: I, Example: II, Example: III, Example: IV

Some more examples of solving non-homogeneous equation with Boundary Condition are given here:

4 Properties of solution of ODE

4.1 Introduction

In this section we shall consider three properties of the solution of ODE: i) Existence and Uniqueness Theorem for Initial Value Problem (IVP) (or Boundary Condition) , ii) Linear independence of solutions of ODE, iii) Wronskian of a solution of ODE and its relation with linear independence.

4.2 Existence and Uniqueness Theorem

To prove these theorems for first order ODE, let us suppose $f(x, y)$ is a continuous function in (x, y) plane which follows the Initial Value Problem (IVP):

$$\frac{\partial y(x,y)}{\partial x} = f(x, y(x)), \qquad y(x_0) = y_0. \tag{4.1}$$

For a nonhomogeneous second order ODE,

$$\frac{d^2y(x)}{dx^2} + p(x)\frac{dy(x)}{dx} + q(x)y(x) = r(x), \tag{4.2}$$

where $p(x)$, $q(x)$ and $r(x)$ be the continuous but function of x, respectively, the Initial Value Problem (IVP) is defined with a boundary condition,

$$y(x_0) = y_0 \qquad y'(x_0) = y'_0, \tag{4.3}$$

respectively, where, x_0, y_0 and y'_0 be the constants. Now we need to answer the following question:

Q I: With aforesaid boundary condition, does there exist any solution of ODE?

Q II: What will be the condition for getting *unique* solution within that range?

Then *Existence and Uniqueness Theorem* precisely gives answer to these questions. Below we give the definitions:

- i) *Existence and Uniqueness Theorem for First order ODE:*

Theorem: Given the Initial Value Problem (IVP)

$$\frac{\partial y(x,y)}{\partial x} = f(x, y(x)), \qquad y(x_0) = y_0, \tag{4.4}$$

If $f(x)$ and $\frac{\partial f(x)}{\partial x}$ are continuous within a region $\mathbf{R}$ such that,

$$\mathbf{R} = \{(x, y)|a < x < b, c < y < d\}, \tag{4.5}$$

then there exists a point (x_0, y_0) which is indeed a unique solution of the IVP within that range.

- ii) *Existence and Uniqueness Theorem for Second order ODE:*

Theorem: Consider the following IVP with the second order nonhomogeneous ODE,

$$\frac{d^2y(x)}{dx^2} + P(x)\frac{dy(x)}{dx} + Q(x)y(x) = R(x), \tag{4.6}$$

where, $P(x)$, $Q(x)$, and $R(x)$ be continuous functions of x on a closed interval $[a, b]$ (i.e., $\mathbf{R}$: $|x - x_0| \leq a, \quad |y - y_0| \leq b$. If x_0 is any point in $[a, b]$, and if y_0 and y'_0 are any numbers whatever, then Eq.(4.x) has one and only one (i.e., *unique*) solution $y(x)$ on the entire interval such that $y(x_0) = y_0$ and $y'(x_0) = y'_0$.

4.3 Linear independence

To start with let us consider a linear homogeneous ODE (LDE) of n-th order,

$$y^{(n)} + a_{n-1}y^{(n-1)} + a_{n-2}y^{(n-2)} + \cdots + a_2y^{(2)}(x) + a_1y^{(1)}(x) + a_0y(x) = 0. \tag{4.7}$$

where $y^{(n)}$ represents the n-th order derivative. Let $\{y_1(x), y_2(x), y_3(x), \ldots y_n(x)\}$ be a set of solutions which are called fundamental set of solution. These solutions are said to be *linearly independent* if their linear combination satisfies the condition,

$$c_1y_1(x) + c_2y_2(x) + c_3y_3(x), \cdots + c_ny_n(x) = 0, \tag{4.8}$$

with c_i be the non-zero constants. On the contrary, if it satisfies,

$$c_1y_1(x) + c_2y_2(x) + c_3y_3(x), \cdots + c_ny_n(x) = \phi(x), \tag{4.9}$$

they are said to be *linearly dependent*. The dependence is always associated with a range within which the solution is well defined with a given boundary condition.

4.4 Wronskian and Linear Independence

It is customary to define a determinant, called *Wronskian*, involving the multiple solutions of ODE,

$$W[y_1, y_2, \ldots y_n] = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & y_3^{(1)} & \cdots & y_n^{(1)} \\ y_1^{(2)} & y_2^{(2)} & y_3^{(2)} & \cdots & y_n^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)}, \end{vmatrix} \tag{4.10}$$

where $y^{(n)}(x)$ be the n-th order derivative with respect to n. Here we have two possible choices:

- If $W[y_1, y_2, \ldots y_n] = 0$, then "The solution are linearly dependent",

- If $W[y_1, y_2, \dots y_n] \neq 0$, then "The solutions are linearly independent".

In a word, the *Wronskian* helps to check the linear independence of the solutions of a homogeneous LDE of any arbitrary order. Some examples of calculating Wronskian are given below:

Example: I, Example: II, Example: III, Example: IV.

References

[1] W. E. Boyce, R. C. Di Prima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley & Sons, Inc., New Jersey, 2009.

[2] G. F. Simmons, *Differential Equation with Application and Historical Note*, CRC Press, Boca Raton, FL, 2017.

[3] E. Kreyszig, *Advanced Engineering Mechanics*, John Wiley & Sons, Inc., New Jersey, 2011.