High-Dimensional Hypothesis Testing with the Lasso

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- We rarely have more than 10k samples in the study...
  - We have more variables, p, than samples, n, i.e., high-dimensional data.
  - Ordinary least squares (OLS) do not fit.
- How do we examine the conditional association then?

High-Dimensional Estimation

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  when we have more variables than samples?
- One method to estimate $\boldsymbol{\beta}^*$ is to use regularization, e.g., the lasso:

$$\hat{\boldsymbol{\beta}}_\lambda = \underset{\boldsymbol{b}\in\mathbb{R}^p}{\arg\min}\left\{\|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{b}\|_2^2 + \lambda\|\boldsymbol{b}\|_1\right\}.$$

  - $\lambda > 0$ is the regularization tuning parameter.

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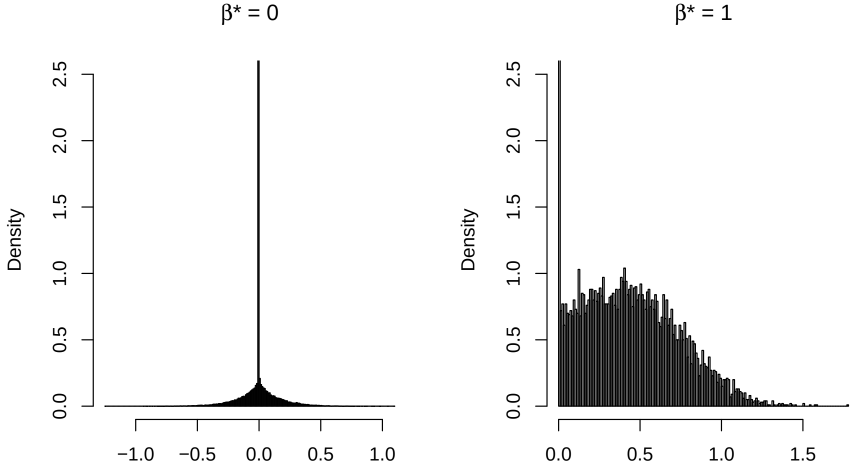
- Can we compute p-values and confidence intervals directly from $\hat{\boldsymbol{\beta}}_\lambda$, just as what we do with OLS estimates?
- Very hard...
  - $\hat{\boldsymbol{\beta}}_\lambda$ is biased, and the bias is unknown.
  - The distribution of $\hat{\boldsymbol{\beta}}_\lambda$ is also unknown – likely not Gaussian.

Distribution of Lasso Estimates



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- References: Javanmard and Montanari (2013, 2014a,b); Zhang and Zhang (2014); van de Geer et al. (2014); Dezeure et al. (2015); Zhao and Shojaie (2016); Ning and Liu (2016).

- **Solution 2**: only use a small subset of variables $\mathcal{M}$ to draw inference:

$$\boldsymbol{y} = \boldsymbol{X}_{\mathcal{M}}\boldsymbol{\beta}^{\mathcal{M}} + \boldsymbol{\epsilon}.$$

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- How about choosing $\mathcal{M}$ using lasso? – <u>*Lasso post-selection inference*</u>.
  - With the lasso (and some mild conditions), $|\mathcal{M}| < n$.

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   - Argument for the usefulness of post-selection inference: in practice, we almost never get a complete list of relevant variables in their correct forms. So we essentially never answer our question of interest.
   - "All models are wrong, but some are useful." – George E. P. Box

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2. We peek the data twice!
   - Once in lasso variable selection and once in OLS sub-model fit.

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- Once in lasso variable selection and once in OLS sub-model fit.
- Estimates from the OLS sub-model fit are no longer Gaussian[1].
- Solutions:
  - Sample-splitting: e.g., Cox (1975); Wasserman and Roeder (2009).
  - Exact inference: e.g., Lee et al. (2016); Tibshirani et al. (2016).

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- Why???

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2. Based on the theoretical result, propose the *lasso score test*, which draws inference on $\boldsymbol{\beta}^*$.
   - Recall that lasso post-selection inference examines hypotheses related to the sub-model.

Normality of Post-Selection OLS

- Let $\hat{\mathcal{A}}_\lambda$ be the set of selected variables by the lasso.
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Theorem

With appropriate assumptions (described later), for any $j \in \hat{\mathcal{A}}_\lambda$,

$$\frac{\tilde{\beta}_{\lambda,j} - \beta_j^{\hat{\mathcal{A}}_\lambda}}{\sigma_\epsilon \sqrt{\left[(\boldsymbol{X}_{\hat{\mathcal{A}}_\lambda}^\top \boldsymbol{X}_{\hat{\mathcal{A}}_\lambda})^{-1} \right]_{(j,j)}}} \to_d \mathcal{N}(0,1),$$

where σ_ϵ is the error standard deviation; $\boldsymbol{\beta}^{\hat{\mathcal{A}}_\lambda}$ is the sub-model coefficients.

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- The same distribution as if $\hat{\mathcal{A}}_\lambda$ is selected without looking at the data.

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- We can use naïve confidence intervals to achieve asymptotically correct coverage on $\boldsymbol{\beta}^{\hat{\mathcal{A}}_\lambda}$,

$$\boldsymbol{y} = \boldsymbol{X}_{\hat{\mathcal{A}}_\lambda} \boldsymbol{\beta}^{\hat{\mathcal{A}}_\lambda} + \boldsymbol{\epsilon}.$$

Idea Behind the Result

- Lasso selected active set is fixed with high probability.
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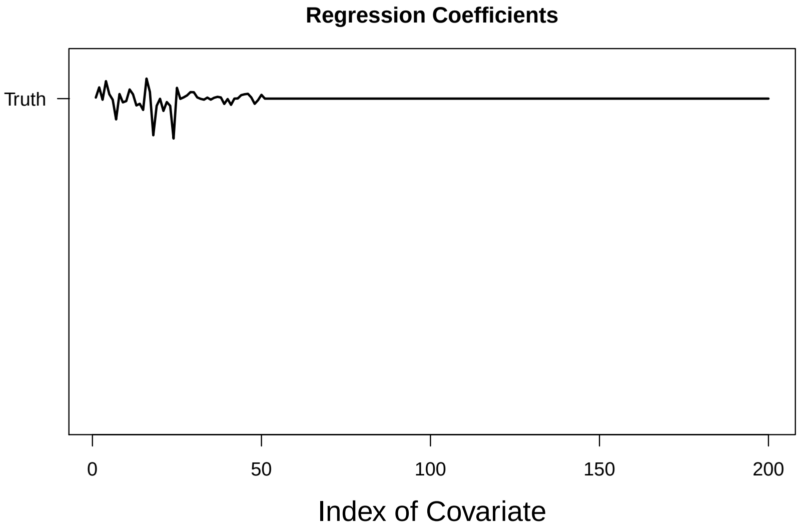
- Lasso selected active set is fixed with high probability.
  - No additional randomness introduced by lasso variable selection.
  - We effectively only look at the data once!

Illustration



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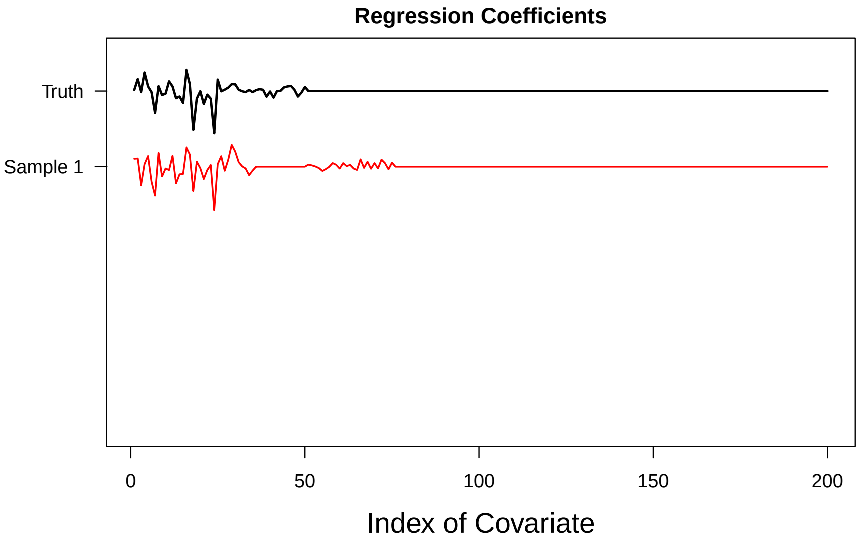

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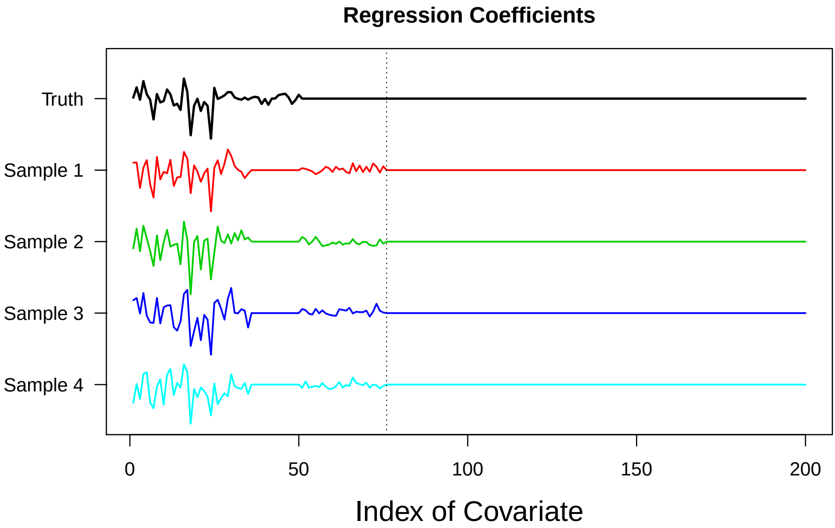

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Let $\hat{\mathcal{A}}_\lambda \equiv \mathrm{supp}(\hat{\boldsymbol{\beta}}_\lambda)$ *and* $\mathcal{A}_\lambda \equiv \mathrm{supp}(\boldsymbol{\beta}_\lambda)$, *where*

$$\text{(lasso):} \qquad \hat{\boldsymbol{\beta}}_\lambda \equiv \operatorname*{arg\,min}_{\boldsymbol{b}\in\mathbb{R}^p} \left\{ \frac{1}{2n} \left\| \boldsymbol{y} - \boldsymbol{X}\boldsymbol{b} \right\|_2^2 + \lambda \left\| \boldsymbol{b} \right\|_1 \right\},$$

$$\text{(noiseless lasso):} \qquad \boldsymbol{\beta}_\lambda \equiv \operatorname*{arg\,min}_{\boldsymbol{b}\in\mathbb{R}^p} \left\{ \frac{1}{2n} \mathrm{E} \left[\left\| \boldsymbol{y} - \boldsymbol{X}\boldsymbol{b} \right\|_2^2 \right] + \lambda \left\| \boldsymbol{b} \right\|_1 \right\}.$$

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- This is not the same as $\lim_{n\to\infty} \Pr[\hat{\mathcal{A}}_\lambda = \mathrm{supp}(\boldsymbol{\beta}^*)] = 1$!
- To recover the true support requires stronger conditions (Meinshausen and Bühlmann, 2006; Zhao and Yu, 2006; Wainwright, 2009).

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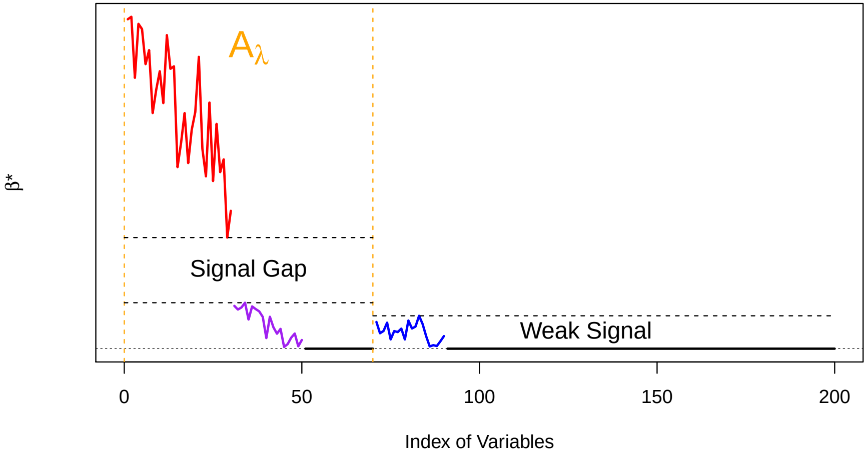
- A technical assumption on the subgradient of the noiseless lasso.

Assumption on β^*



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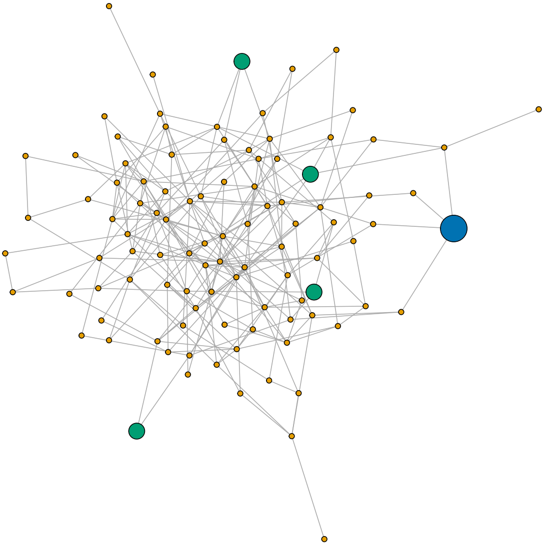
- Let the graph adjacency matrix be the partial correlation matrix of $\boldsymbol{X}$, with partial correlation $\rho = 0.2$.
- Simulation $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}^* + \boldsymbol{\epsilon}$, with

$$\beta_j^* = \begin{cases} 1 & j = 10 \\ 0.1 & j \in \{20, 30, 40, 50\} \\ 0 & \text{otherwise} \end{cases},$$

and signal to noise ratio $SNR \in \{0.1, 0.3, 0.5\}$

Simulations 1 - Graph Structure



Simulations 1 - Coverage Probability

n	300			400			500		
SNR	0.1	0.3	0.5	0.1	0.3	0.5	0.1	0.3	0.5
exact	0.952	0.949	0.950	0.953	0.951	0.951	0.947	0.946	0.946
naïve	0.914	0.932	0.930	0.944	0.932	0.924	0.941	0.940	0.932

Simulations 2 - Setting

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- Simulation $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}^* + \boldsymbol{\epsilon}$, with

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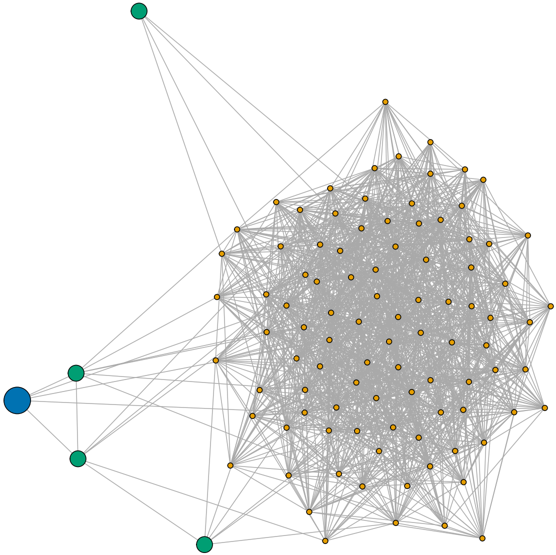
Simulations 2 - Graph Structure



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Inference on the Full Model Coefficients $\boldsymbol{\beta}^*$

- Recall that post-selection inference procedures make inference on $\boldsymbol{\beta}^{\mathcal{M}}$.

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- $\boldsymbol{\beta}^{\mathcal{M}}$ in general is not the same as the fully adjusted regression coefficients, unless lasso does not miss any (strong) confounding variables.
    - It is OK to miss some very weak confounding variables, as long as asymptotically their contributed bias is ignorable.

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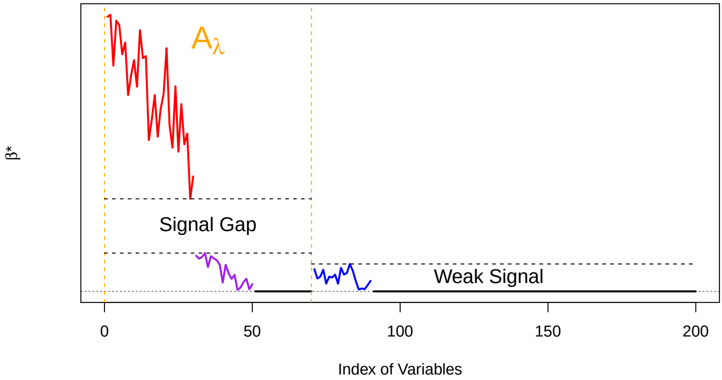
- Recall the set of variable selected by the lasso is the same as the set of variable selected by the noiseless lasso with high probability.

Inference on the Full Model Coefficients β^*

- How to make sure lasso does not miss any strong confounders?
- Recall the set of variable selected by the lasso is the same as the set of variable selected by the noiseless lasso with high probability.
- We can assume the noiseless lasso does not miss any strong confounding variables.

Assumption on $\boldsymbol{\beta}^*$

Inference on $\boldsymbol{\beta}^{\mathcal{A}}$



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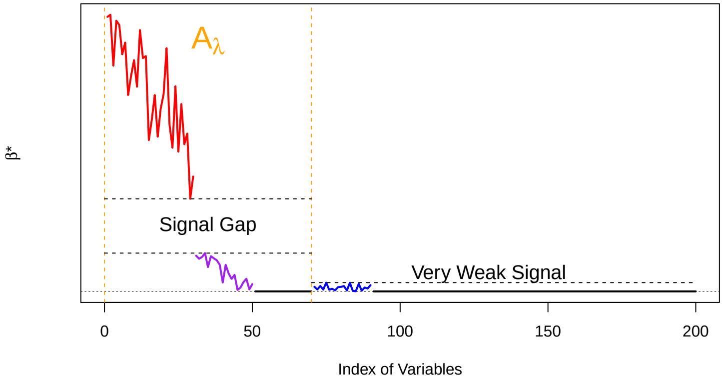

Lasso Score Test

- Let
$$\tilde{S}_j \equiv \boldsymbol{x}_j^\top \left(\boldsymbol{y} - \boldsymbol{X} \tilde{\beta}_\lambda^0 \right),$$
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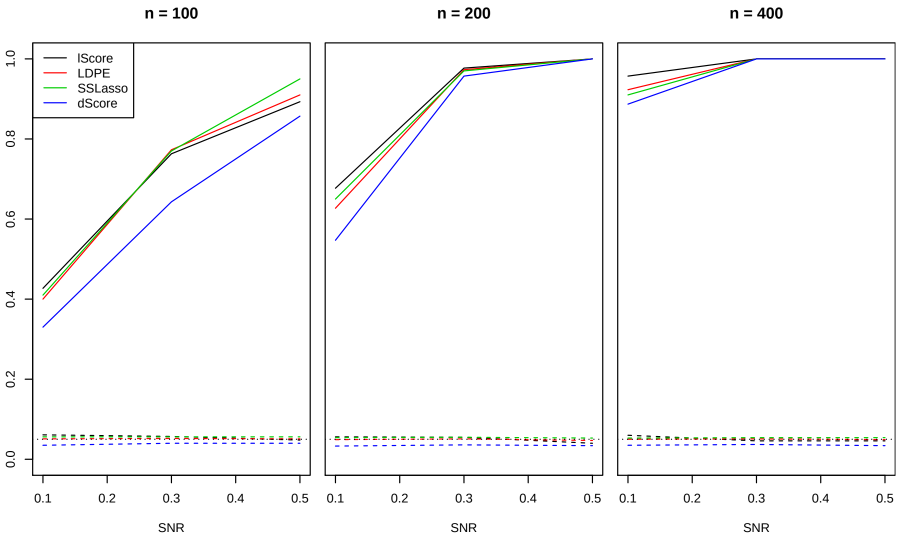
$$\beta_j^* = \begin{cases} 1 & j \in \{30, 60, 90\} \\ 0.1 & j \in \{120, 150, 180, 210, 240, 270, 300\} \\ 0 & \text{otherwise} \end{cases},$$

  and signal to noise ratio $SNR \in \{0.1, 0.3, 0.5\}$

Simulations 1 - Power and Type-I Error Rate



Simulations 2 - Setting

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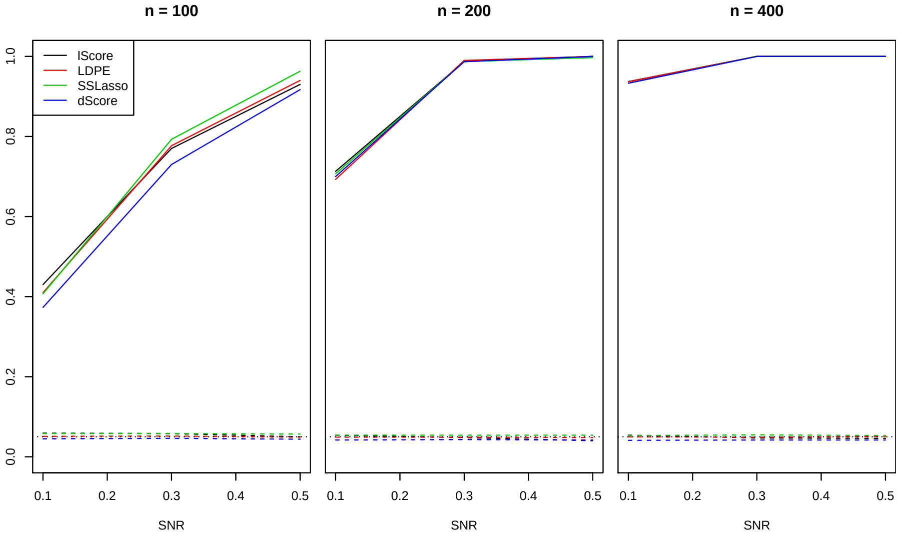

Discussion

- Empirical evidence suggests the largest λ whose MSE is within one standard error of the minimum MSE (Hastie et al., 2009) works well.
  - λ should be larger than the prediction optimal ones.

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- Empirical evidence suggests the largest λ whose MSE is within one standard error of the minimum MSE (Hastie et al., 2009) works well.
  - λ should be larger than the prediction optimal ones.
- Lasso score test is closely related to both post-selection inference and debiased tests.
  - The two classes of inference procedures are in fact very similar.

A Bigger Picture

Classical statistical inference:

1. Formulate the model: what variables to adjust for? in what form, e.g., linear, polynomial, log-transformed, interactions?
2. Collect data.
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Bottomline: post-selection inference is a promising area of research; there are still lots of questions to be solved...

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Thanks!

Questions?

Assumption on the Subgradient

- Denote $\hat{\boldsymbol{\Sigma}} = \boldsymbol{X}^\top \boldsymbol{X}/n$. Define the subgradient of the noiseless lasso

$$\boldsymbol{\tau}_\lambda = \frac{1}{\lambda} \hat{\boldsymbol{\Sigma}} \left(\boldsymbol{\beta}^* - \boldsymbol{\beta}_\lambda \right).$$

Then,

$$\limsup_{n\to\infty} \|\boldsymbol{\tau}_{\lambda,\mathcal{A}_\lambda^c}\|_\infty < 1,$$

$$\frac{1}{\min_{j\in\mathcal{A}_\lambda\setminus\mathcal{S}^*} \left| \left(\hat{\boldsymbol{\Sigma}}_{(\mathcal{A}_\lambda,\mathcal{A}_\lambda)} \right)^{-1} \boldsymbol{\tau}_{\lambda,\mathcal{A}_\lambda} \right|_j} = \mathcal{O}\left(\lambda \sqrt{\frac{n}{\log(p)}} \right),$$

where $\mathcal{S}^*$ is the set of strong signal variables.