



Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Fall Semester 2022

EEB 212E

HOMEWORK – 3

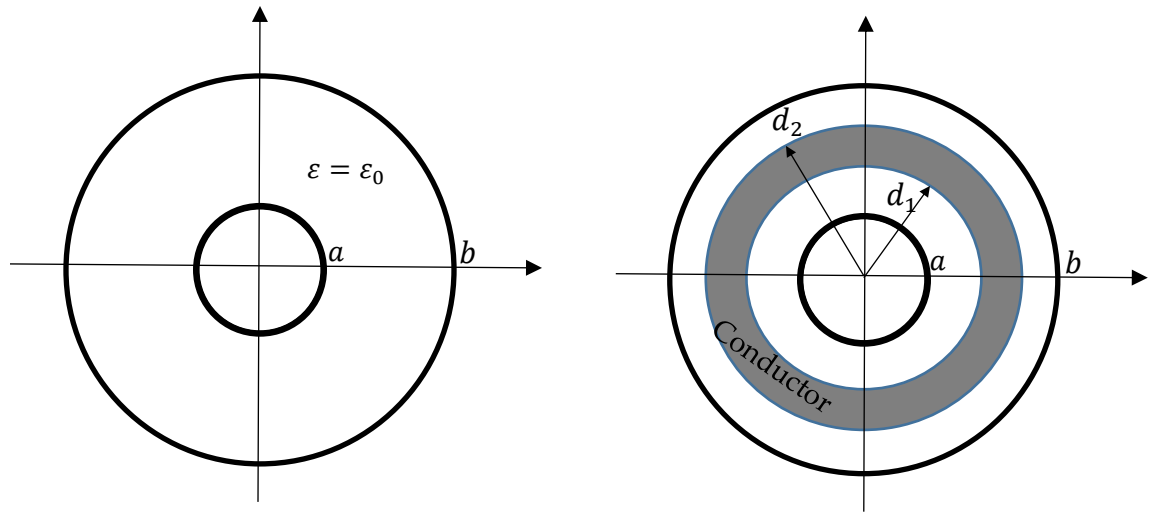
Due: December 30, 2022, till 21.00



- **You need to UPLOAD THE SOFTCOPY AND bring the hardcopy of the homework to office 7309 in EEB before the deadline (you can throw homework under the door)**
- You need to show all the steps during operations. Otherwise, the questions are not graded.
- Do Not forget to write your name!
- The total point is 100 and each question has the same importance.

**Q-1)** It is known that for an infinite-length coaxial cable with the inner and the outer radii,  $a$ , and  $b$ , respectively, the capacitance value **per unit length** is  $C_0 = \frac{2\pi\epsilon}{\ln(b/a)}$  (F/m) where the region between these conductors' surface is free space ( $\epsilon = \epsilon_0$ ).

- By placing a conductive cylindrical shell with an inner radius  $d_1$  and outer radius  $d_2$  in the region between the cylinders, it is desired to double the capacitance value ( $2C_0$ ) in the first case. Calculate the required ratio  $d_1/d_2$  in terms of  $a$  and  $b$  to realize this situation.
- In this last case, it is aimed to increase the capacitance value to  $8C_0$  by filling  $\rho \in (a, d_1)$  and  $\rho \in (d_2, b)$  regions with an insulator material with dielectric constant  $\epsilon = \epsilon_r \epsilon_0$ . Determine the required relative dielectric constant  $\epsilon_r$  for this case. **Note that, the conductors placed in (a) are still in the system.**



**Figure 1.** The geometries of the system.

$$1) C = \frac{2\pi\epsilon_0}{\ln(b/a)} [F/m]$$

$$a) C_1 = \frac{2\pi\epsilon}{\ln(d_1/a)}, C_2 = \frac{2\pi\epsilon}{\ln(b/d_2)}$$

$$C_{eq} = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1} \Rightarrow \text{ve } C_{eq} = 2C \Rightarrow$$

$$\frac{4\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a \cdot d_1/d_2)} \Rightarrow \boxed{\frac{d_1}{d_2} = \sqrt{\frac{a}{b}}}$$

$$b) 8C = \frac{16\pi\epsilon_0}{\ln(b/a)} \Rightarrow$$

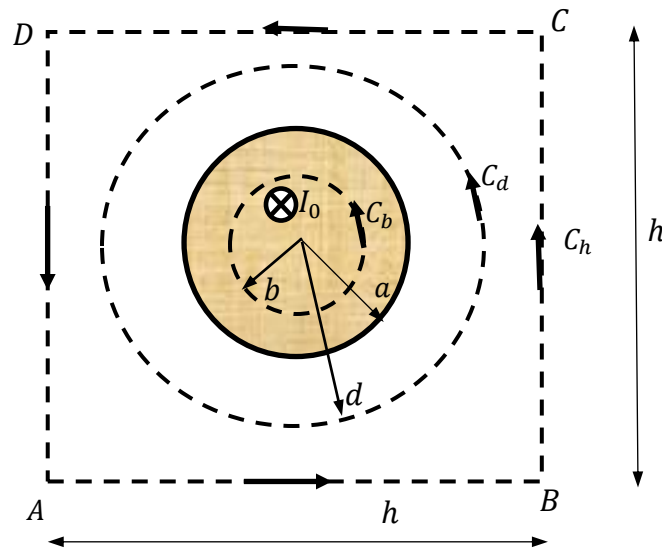
$$\frac{\ln(b/a)}{16\pi\epsilon_0} = \frac{\ln(d_1/a) + \ln(b/d_2)}{2\pi\epsilon_r\epsilon_0} \Rightarrow$$

$$\begin{aligned} \epsilon_r \ln(b/a) &= 8 \ln\left(\frac{b}{a} \frac{d_1}{d_2}\right) = 8 \ln \sqrt{b/a} \\ &= 4 \ln(b/a) \Rightarrow \end{aligned}$$

$$\boxed{\epsilon_r = 4}$$

**Q-2)** A constant current  $I_0$  flows uniformly across the cross-section in the positive  $z$ -direction through the infinitely long conductor wire of radius  $a$  placed along the  $z$ -axis as provided in Figure 2.

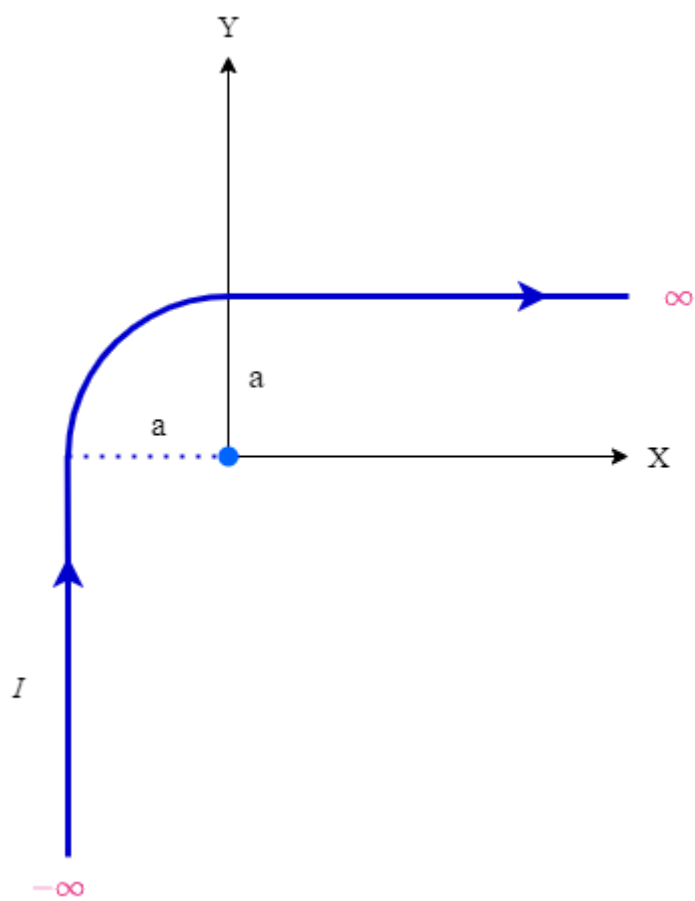
- Find the value  $\oint_{C_b} \vec{H} \cdot d\vec{\ell}$  along  $C_b$  (closed circle with radius  $b$ ), with  $b < a$ .
- Find the value  $\oint_{C_d} \vec{H} \cdot d\vec{\ell}$  along  $C_d$  (closed circle with radius  $d$ ) with  $d > a$ .
- Find the value  $\oint_{ABCD} \vec{H} \cdot d\vec{\ell}$  along the square ABCD with side length  $h$ ,  $h > 2a$ .
- Find the result of the line integral  $\int_A^B \vec{H} \cdot d\vec{\ell}$  along the AB section of the square.



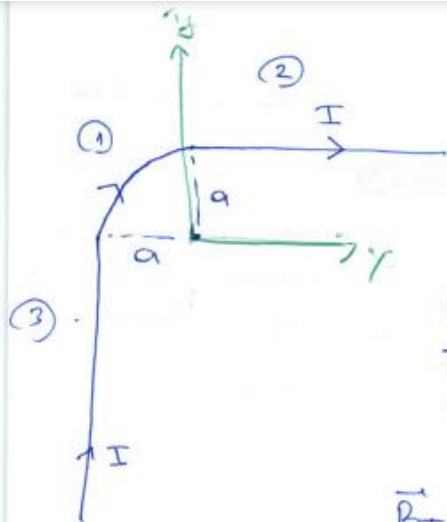
**Figure 2.** The geometry of Q-2.

$$\begin{aligned}
 \text{a) } \oint_{C_b^+} \vec{H} \cdot d\vec{\ell} &= \sum I = -I_0 \frac{b^2}{a^2} \\
 \text{b) } \oint_{C_d^+} \vec{H} \cdot d\vec{\ell} &= \sum I = -I_0 \\
 \text{c) } \oint_{ABCD} \vec{H} \cdot d\vec{\ell} &= \sum I = -I_0 \\
 \text{d) } \int_{AB} \vec{H} \cdot d\vec{\ell} &= \int_A^B \vec{H} \cdot d\vec{\ell} \stackrel{\text{simetri}}{=} \frac{1}{4} \oint_{ABCD} \vec{H} \cdot d\vec{\ell} = \frac{-I_0}{4}
 \end{aligned}$$

**Q-3)** Below, the geometry of the problem is given. As it is noticed, steady current  $I$  flows through a line. Current originates at infinity and then rotates around the origin on a quarter circle with a radius of  $a$ . Then, it goes to infinity. Find  $B$  field at origin. The systems in free space.



**Figure 3.** The Schema for Q-3.



$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{r} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$d\vec{r} = a d\phi \vec{a}_\phi$$

$$\vec{r} = 0 \vec{a}_\rho$$

$$\vec{r}' = a \vec{a}_\rho$$

$$\vec{r} - \vec{r}' = -a \vec{a}_\rho$$

$$|\vec{r} - \vec{r}'| = a$$

$$\vec{B}_I = \frac{\mu_0}{4\pi} \int_{\pi}^{\pi/2} \frac{I a d\phi \vec{a}_\phi \times a \vec{a}_\rho (-\vec{a}_\rho)}{a^3}$$

$$= \frac{\mu_0 I}{4\pi a^2} \int_{\pi}^{\pi/2} \vec{a}_z d\phi$$

$$= \frac{\mu_0 I}{4\pi a} \vec{a}_z \left[ -\frac{\pi}{2} \right] = -\frac{\mu_0 I}{8a} \vec{a}_z$$



$$a_\rho \times a_\phi = a_z$$

$$B_{II} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{(dx' \vec{a}_x) \times (-x' \vec{a}_y - a \vec{a}_y)}{(x'^2 + a^2)^{3/2}}$$

$$dx' \vec{a}_x \times (-x' \vec{a}_x - a \vec{a}_y) = -a dx' \vec{a}_z$$

$$\vec{B}_{II} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{-a dx'}{(x'^2 + a^2)^{3/2}} \vec{a}_z$$

$$= -\frac{\mu_0 I a}{4\pi} \left[ \frac{1}{a^2} \frac{x'}{\sqrt{x'^2 + a^2}} \right]_0^\infty$$

$$= \vec{B}_{II} = -\frac{\mu_0 I}{4\pi a} \vec{a}_z$$

$$r = 0 \vec{a}_\rho$$

$$\vec{r}' = a \vec{a}_y + x' \vec{a}_x$$

$$\vec{r} - \vec{r}' = -a \vec{a}_y - x' \vec{a}_x$$

$$|\vec{r} - \vec{r}'| = \sqrt{a^2 + x'^2}$$

Remember

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} \rightarrow \frac{1}{a^2} \frac{x}{\sqrt{x^2 + a^2}}$$

B<sub>III</sub>

$$\vec{r} = 0a\hat{r} \quad dl = dy'a\hat{y}$$

$$\vec{r}' = -a\hat{x} + y'a\hat{y}$$

$$\vec{r} - \vec{r}' = a\hat{x} - y'a\hat{y}$$

$$|\vec{r} - \vec{r}'| = \sqrt{a^2 + y'^2}$$

$$B = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

$$B_{III} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{dy'a\hat{y} \times [a\hat{x} - y'a\hat{y}]}{(a^2 + y'^2)^{3/2}}$$

$$= \frac{\mu_0 I}{4\pi} (-a\hat{z}) a \int_{-\infty}^{\infty} \frac{dy'}{(a^2 + y'^2)^{3/2}}$$

$$\hat{y} \times \hat{x} = -\hat{z}$$

$$\int \frac{dy}{(a^2 + y^2)^{3/2}} = \frac{1}{a^2} \frac{y}{\sqrt{y^2 + a^2}}$$

$$= -\frac{\mu_0 I}{4\pi} a\hat{z} \frac{1}{a^2} \frac{y}{\sqrt{y^2 + a^2}} \Big|_{-\infty}^{\infty} = -\frac{\mu_0 I a\hat{z}}{4\pi a} \frac{y}{\sqrt{y^2 + a^2}} \Big|_{-\infty}^{\infty} = -\frac{\mu_0 I}{4\pi a} a\hat{z}$$

$$\vec{B} = \vec{B}_I + \vec{B}_{II} + \vec{B}_{III} = -\frac{\mu_0 I}{4a} \left(1 + \frac{2}{\pi}\right) a\hat{z} \quad [W/m^2]$$

**Q-4)** Consider a spherical structure of inner and outer radii  $a$  and  $2a$  meters, respectively. Assuming the inner conductor is kept  $V_0$  Volt while the outer is grounded (i.e. at 0 Volt), **in terms of  $a, V_0, \sigma_0$  and  $\epsilon_0$ :**

- Find the electrostatic potential expression  $V(r)$  inside the capacitor, i.e.  $a \leq r \leq b$ . **Note that the potential varies the same way in Region I and II.**
- First, find the total resistance of the system
- What is the capacitance  $C$  of this structure?

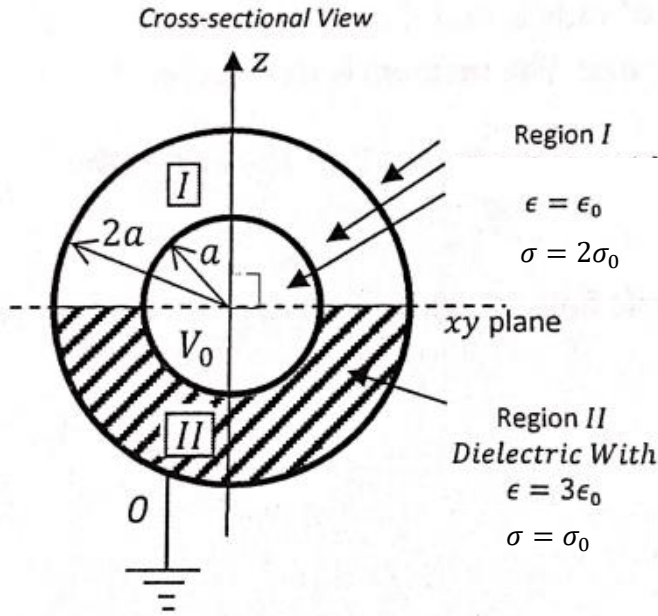


Figure 3. The geometry for Q-4.

$$\nabla^2 V(r) = 0 \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) = 0 \rightarrow \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) = 0$$

$$\frac{d}{dr} \left( r^2 \frac{dV}{dr} \right) = 0 \rightarrow r^2 \frac{dV}{dr} = C_1 \rightarrow dV = \frac{C_1}{r^2} dr \rightarrow V(r) = -\frac{C_1}{r} + C_2$$

$$\left. \begin{aligned} V_0 &= -\frac{1}{a} C_1 + C_2 \\ 0 &= -\frac{1}{2a} C_1 + C_2 \end{aligned} \right\} \rightarrow C_1 = -2V_0 a, \quad C_2 = -V_0 \rightarrow \boxed{V(r) = \frac{2V_0 a}{r} - V_0}$$

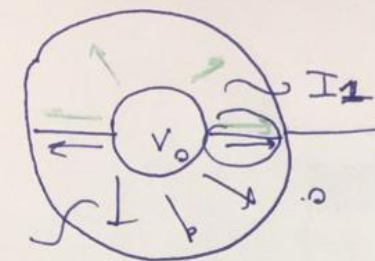
$$\mathbf{E}(r) = -\nabla V(r) = -\hat{\mathbf{a}}_r \frac{\partial}{\partial r} V(r) = \hat{\mathbf{a}}_r \frac{2V_0 a}{r^2}$$

$$\mathbf{D}_I(r) = \epsilon_0 \mathbf{E}(r) = \boxed{\hat{\mathbf{a}}_r \epsilon_0 \frac{2V_0 a}{r^2}} \quad \mathbf{D}_{II}(r) = 3\epsilon_0 \mathbf{E}(r) = \boxed{\hat{\mathbf{a}}_r \epsilon_0 \frac{6V_0 a}{r^2}}$$

$$Q = 2\pi a^2 \rho_{sl} + 2\pi a^2 \rho_{sII} = 2\pi a^2 \hat{\mathbf{a}}_r \cdot \mathbf{D}_I(a) + 2\pi a^2 \hat{\mathbf{a}}_r \cdot \mathbf{D}_{II}(a)$$

$$Q = 2\pi a^2 \epsilon_0 \frac{2V_0}{a} (1 + 3) = \boxed{16\pi a \epsilon_0 V_0}$$

$$C = \frac{Q}{V_0} = \boxed{16\pi a \epsilon_0}$$



$\vec{E}_1 = \vec{E}_2$  (since tangential components are continuous)

$I_2$  Hemisphere  $\frac{4\pi R^2}{2}$

$S_1$   $2\pi R^2$

$S_2$   $2\pi R^2$

$$\vec{J}_2 = \sigma_2 \vec{E}_2 = \sigma_2 \vec{E}_1 = \vec{J}_1$$

$$\vec{J}_2 = \sigma_2 \vec{E}_2 = \sigma_2 \vec{E}_1 = \vec{J}_1$$

$$\vec{J}_2 = \sigma_0 \frac{2V_0 a}{R^2} \vec{a}_R$$

$$\vec{J}_2 = 2\sigma_0 \frac{2V_0 a}{R^2} \vec{a}_R$$

$$\vec{I}_2 = S_2 \vec{J}_2 = 2\pi R^2 \sigma_0 \frac{2V_0 a}{R^2} \vec{a}_R = 2\pi 2V_0 \sigma_0 a \vec{a}_R$$

$$I_2 = S_2 J_2$$

$$\underline{\quad} 2\pi R^2 2\sigma_0 \frac{2V_0 a}{R^2} \vec{a}_R = 2\pi 2V_0 2\sigma_0 a \vec{a}_R$$

$$\vec{I}_2 = 4\pi V_0 \sigma_0 a \vec{a}_R$$

$$\vec{I}_2 = 8\pi V_0 \sigma_0 a \vec{a}_R$$

$$\frac{V}{I} = R \Rightarrow \frac{V_0}{I_2} = R_2 \quad \frac{V_0}{I_2} = R_2$$

$$\Downarrow \quad \Downarrow$$

$$\frac{1}{4\pi \sigma_0 a} \Omega \quad \frac{1}{8\pi \sigma_0 a} \Omega$$

$$R = R_1 \parallel R_2$$

$$= \frac{1}{R_{\text{total}}} = 4\pi \sigma_0 a + 8\pi \sigma_0 a$$

$$\frac{1}{R_{\text{total}}} = 12\pi \sigma_0 a$$

$$R_{\text{total}} = \frac{1}{12\pi \sigma_0 a} \Omega$$