Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Fall Semester 2022

EHB 212E

HOMEWORK – 2

Each student is viewed as a responsible professional in engineering, and thus highest ethical standards are presumed.

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Due: November 2, 2022, till 16.59

- **You need to UPLOAD THE SOFTCOPY AND bring the hardcopy of the homework to office 7309 in EEB before the deadline (you can throw homework under the door)**
- **You need to show all the steps during operations. Otherwise, the questions are not graded.**
- **Do Not forget to write your name!**
- **The total point is 100 and each question has the same importance.**

Q-1)

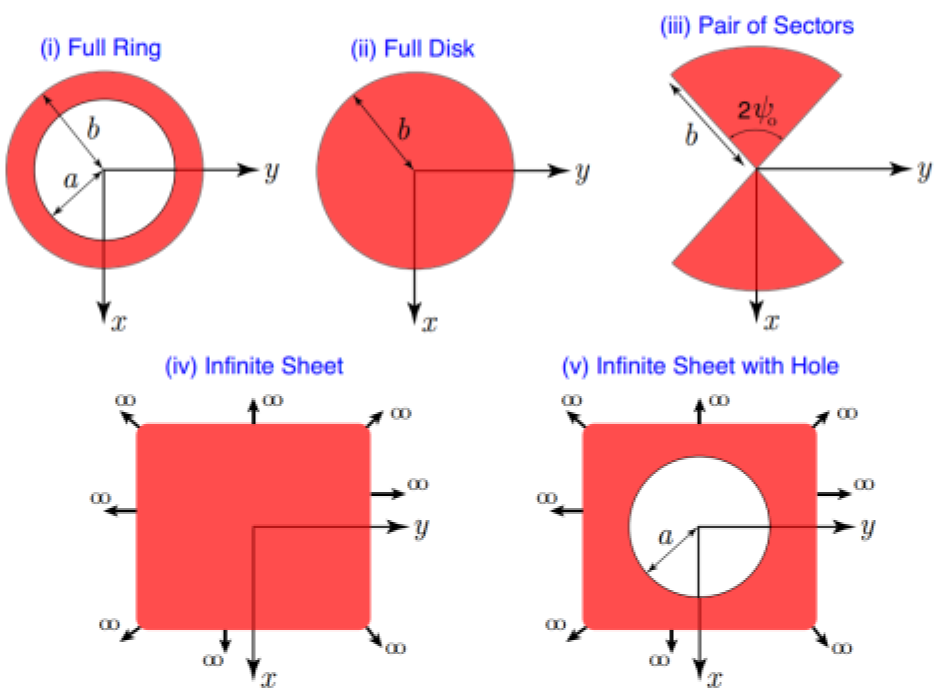


Figure 1. The Geometries of Problem 1.

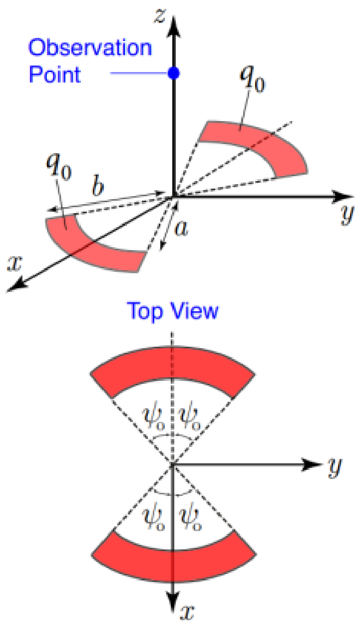


Figure 2. The figure for Problem 1.

(a) Please, notice that the static surface electric charge distributions with different geometries is provided above [(i)–(v)]. In each case, the surface electric charge density is constant [ρ_0 (C/m^2)]. Please, try to obtain the electric field intensity at an arbitrary position on the z-axis (0, 0, z).

Hint: Instead of solving each geometry one by one, try to obtain the electric field intensity for the electric charge distribution on the left and then consider appropriate limits to reach the expressions for the corresponding geometries.

(b) Follow the same procedure for the electric scalar potential. First, try to obtain the electric scalar potential due to the charge distribution on the left at an arbitrary position on the z-axis, and then employ appropriate limits to obtain the electric scalar potential due each distributions above.

Question 1 a) $\bar{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \rho_s \frac{(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3}\, ds'$ $\left[\begin{array}{l}\bar{r} = z\hat{a}_z \\ \bar{r}' = x'\hat{a}_x + y'\hat{a}_y\end{array}\right]$

$$= \frac{q_0}{4\pi\epsilon_0}\int_{S'} \frac{[-x'\hat{a}_x - y'\hat{a}_y + z\hat{a}_z]}{[(x')^2+(y')^2+z^2]^{3/2}}\, r'dr'd\phi' \quad \left[\begin{array}{l} x' = r'\cos\phi' \\ y' = r'\sin\phi'\end{array}\right]$$

$$= \frac{q_0}{4\pi\epsilon_0}\int_{S'} \frac{-r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + z\hat{a}_z}{[(r')^2+z^2]^{3/2}}\, r'dr'd\phi'$$

$$\bar{E} = \frac{q_0}{4\pi\epsilon_0}\left[\int_{\phi'=-\psi_0}^{\psi_0}\int_{r'=a}^{b} \frac{-r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + z\hat{a}_z}{[r'^2+z^2]^{3/2}}\, r'dr'd\phi' \right.$$

$$\left. + \int_{\phi'=\pi-\psi_0}^{\pi+\psi_0}\int_{r'=a}^{b} \frac{-r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + z\hat{a}_z}{[r'^2+z^2]^{3/2}}\, r'dr'd\phi'\right]$$

$$\left[\int_{-\psi_0}^{\psi_0}\cos\phi'd\phi' + \int_{\pi-\psi_0}^{\pi+\psi_0}\cos\phi' = 0, \quad \int_{-\psi_0}^{\psi_0}\sin\phi'd\phi' + \int_{\pi-\psi_0}^{\pi+\psi_0}\sin\phi'd\phi' = 0\right]$$

$$\bar{E} = \frac{q_0}{4\pi\epsilon_0}\left[2\int_{\phi'=-\psi_0}^{\psi_0}\int_{r'=a}^{b} \frac{z\hat{a}_z r'}{[r'^2+z^2]^{3/2}}\, dr'd\phi'\right]$$

$$= \hat{a}_z \frac{q_0}{\pi\epsilon_0}\psi_0 z\left[\left.\frac{-1}{[r'^2+z^2]^{1/2}}\right|_{r'=a}^{b}\right]$$

$$\bar{E} = \hat{a}_z \frac{q_0\psi_0}{\pi\epsilon_0} z\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]$$

i) $$\bar{E} = \frac{q_0}{4\pi\epsilon_0}\left[\int_{\phi'=0}^{2\pi}\int_{r'=a}^{b} \frac{-r'\cos\phi'\hat{a}_x - r'\sin\phi'\hat{a}_y + z\hat{a}_z}{[r'^2+z^2]^{3/2}}\, r'dr'd\phi'\right]$$

$$\left[\int_{\phi'=0}^{2\pi}\cos\phi'd\phi' = 0, \int_{\phi'=0}^{2\pi}\sin\phi'd\phi' = 0\right]$$

$$\bar{E} = \hat{a}_z \frac{q_0}{2\epsilon_0} z\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}}\right]$$

ii) $\bar{E} = \frac{q_o}{4\pi\epsilon_o}\left[\int_{\phi'=0}^{2\pi}\int_{r'=0}^{b} \frac{r' z \hat{a}_z}{[r'^2+z^2]^{3/2}}\, dr' d\phi'\right]$

$$\bar{E} = \hat{a}_z \frac{q_o}{2\epsilon_o}\left[1 - \frac{z}{\sqrt{b^2+z^2}}\right]$$

iii) $\bar{E} = \hat{a}_z \frac{q_o \Psi_o}{\pi\epsilon_o}\left[1 - \frac{z}{\sqrt{b^2+z^2}}\right]$

iv) $\bar{E} = \frac{q_o}{4\pi\epsilon_o}\left[\int_{\phi'=0}^{2\pi}\int_{r'=0}^{\infty} \frac{r' z \hat{a}_z}{[r'^2+z^2]^{3/2}}\, dr' d\phi'\right]$

$$\bar{E} = \hat{a}_z \frac{q_o}{2\epsilon_o}$$

v) $\bar{E} = \frac{q_o}{4\pi\epsilon_o}\left[\int_{\phi'=0}^{2\pi}\int_{r'=a}^{\infty} \frac{r' z \hat{a}_z}{[r'^2+z^2]^{3/2}}\, dr' d\phi'\right]$

$$\bar{E} = \hat{a}_z \frac{q_o}{2\epsilon_o} \frac{z}{\sqrt{a^2+z^2}}$$

b) $V = \frac{1}{4\pi\epsilon_o}\int_{s'} \frac{\rho_s}{|\bar{r}-\bar{r}'|}\, ds'$ $\quad \left[\begin{array}{l}\bar{r} = z\hat{a}_z \\ \bar{r}' = -x'\hat{a}_x - y\hat{a}_y\end{array}\right]$

$$V = \frac{1}{4\pi\epsilon_o}\int_{s'} \frac{q_o}{\sqrt{(x')^2+(y')^2+z^2}}\, r' dr' d\phi' \quad \left[\begin{array}{l}x' = r'\cos\phi' \\ y' = r'\sin\phi'\end{array}\right]$$

$$V = \frac{q_o}{4\pi\epsilon_o}\left[\int_{\phi'=-\Psi_o}^{\Psi_o}\int_{r'=a}^{b} \frac{1}{\sqrt{r'^2+z^2}}\, r' dr' d\phi' + \int_{\phi'=\pi-\Psi_o}^{\pi+\Psi_o}\int_{r'=a}^{b} \frac{1}{\sqrt{r'^2+z^2}}\, r' dr' d\phi'\right]$$

$$V = \frac{q_o \Psi_o}{\pi\epsilon_o}\left[\sqrt{r'^2+z^2}\,\Big|_{r'=a}^{b}\right] = \frac{q_o\Psi_o}{\pi\epsilon_o}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$$

i) $V = \frac{q_o}{2\epsilon_o}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$ $\quad$ ii) $V = \frac{q_o}{2\epsilon_o}\left[\sqrt{b^2+z^2} - |z|\right]$

iii) $V = \frac{q_o \Psi_o}{\pi\epsilon_o}\left[\sqrt{b^2+z^2} - |z|\right]$ $\quad$ iv) $V \to \infty$ $\quad$ v) $V \to \infty$

Q-2) There exist finite static line electric charge densities $\rho_l(\boldsymbol{r}) = q_0$ at $(x = a, z = 0)$ and another finite static line electric charge densities $\rho_l(\boldsymbol{r}) = -q_0$ at $(x = -a, z = 0)$. (Look at the figure below). The length of each line is $2a$. They are parallel to each other and lie along the y-axis. Find the electric field intensity and electric scalar potential at $\boldsymbol{r} = a\hat{a}_{\boldsymbol{z}}$. (*Hint: It is preferred to use Cartesian Coordinate System)*

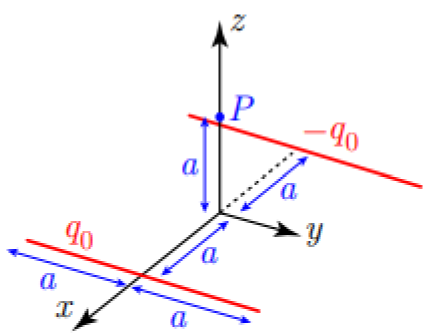


Figure 3. The figure for Problem 2.

For q_0, at $x=a$

$\vec{r} = a.\hat{a}_z$

$\vec{r}' = a.\hat{a}_x + y.\hat{a}_y$

$\vec{r} - \vec{r}' = a.\hat{a}_z - a.\hat{a}_x - y.\hat{a}_y$

$R = |\vec{r} - \vec{r}'| = (2a^2 + y^2)^{1/2}$

$$\Rightarrow E_{q_0} = \frac{q_0}{4\pi\varepsilon_0}\int_{-a}^{a} \frac{a.\hat{a}_z - a.\hat{a}_x - y.\hat{a}_y}{(2a^2+y^2)^{3/2}}\, dy$$

$$= \frac{q_0}{4\pi\varepsilon_0}\left(\int_{-a}^{a} \frac{a.\hat{a}_z - a.\hat{a}_x}{(2a^2+y^2)^{3/2}} - \underbrace{\int_{-a}^{a} \frac{-y\hat{a}_y}{(2a^2+y^2)^{3/2}}\, dy}_{=0 \text{ (odd)}}\right)$$

$$= \frac{q_0}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{3}.a}(\hat{a}_z - \hat{a}_x)$$

$$\Rightarrow \Phi_{q_0} = \frac{q_0}{4\pi\varepsilon_0}\int_{-a}^{a} \frac{1}{\sqrt{2a^2+y^2}}\, dy = \frac{q_0}{4\pi\varepsilon_0} \ln\left(\frac{a+\sqrt{3}a}{-a+\sqrt{3}a}\right) = \frac{q_0}{4\pi\varepsilon_0} \ln\left(\frac{1+\sqrt{3}}{-1+\sqrt{3}}\right)$$

For $-q_0$, at $x=-a$

$$E_{-q_0} = \frac{-q_0}{4\pi\varepsilon_0}\frac{1}{\sqrt{3}a}(-\hat{a}_z - \hat{a}_x) \Rightarrow \boxed{E_P = E_{q_0} + E_{-q_0} = \frac{q_0}{2\sqrt{3}\pi\varepsilon_0 a}(-\hat{a}_x)}$$

$$\Phi_{-q_0} = -\Phi_{q_0} \Rightarrow \boxed{\Phi_P = 0}$$

Q-3) Consider a static electric field intensity vector defined as:

$$\boldsymbol{E}(\boldsymbol{r}) = e_0\left(\hat{a}_x axyz + \hat{a}_y x^2 z + \hat{a}_z x^2 y\right)$$

where e_0 and a are constants. Find the electric scalar potential at $(x, y, z) = (2, 2, 2)$ meter **if the potential at the origin is assumed to be zero**.

$$\bar{E}(r) = e_0 \cdot axyz \cdot \hat{a}_x + e_0 x^2 z \hat{a}_y + e_0 \cdot x^2 y \hat{a}_z$$

In orderto be able to calculate electrical potential one must be sure that $\bar{E}$-field is irrotational.

$$\bar{\nabla} \times \bar{E} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{a}_x \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \hat{a}_y \left(\frac{\partial}{\partial z} E_x - \frac{\partial}{\partial x} E_z \right)$$

$$+ \hat{a}_z \left(\frac{\partial}{\partial x} E_y - \frac{\partial}{\partial y} E_x \right) = 0$$

must satisfie

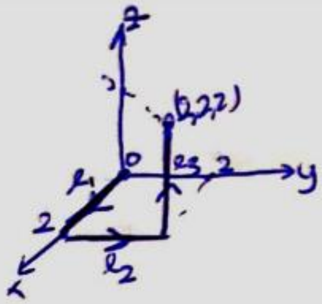
$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0 \Rightarrow x^2 - x^2 = 0 \quad \text{already satisfied.}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0 \Rightarrow axy - 2xy = 0 \rightarrow \boxed{a = 2} \text{ must be selected}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \Rightarrow 2xz - axz = 0 \rightarrow \overset{a=2}{\text{satisfies.}}$$

Therefore, E-field becomes;

$$\bar{E}(\bar{r}) = e_0 \left(2xyz \hat{a}_x + x^2 z \hat{a}_y + x^2 y \hat{a}_z \right)$$



$$V_{(2,2,2)} = V_{(2,2,2)} - V_{(0,0,0)}$$

$$= \underbrace{V_{(2,2,2)} - V_{(2,2,0)}}_{\text{along } l_3} + \underbrace{V_{(2,2,0)} - V_{(2,0,0)}}_{\text{along } l_2} + \underbrace{V_{(2,0,0)} - V_{(0,0,0)}}_{\text{along } l_1}$$

$$V_{(2,0,0)} - V_{(0,0,0)} = -\int_0^2 \bar{E}\cdot dx \quad (\text{when } y=0,\ z=0)$$

$$= -x^2yz\Big|_0^2 = 0 \quad \text{as } y=0,\ z=0$$

$$V_{(2,2,0)} - V_{(2,0,0)} = -\int_0^2 \bar{E}\cdot \bar{dy} \quad (\text{when } x=2,\ z=0)$$

$$= -x^2yz\Big|_0^2 = 0 \quad \text{as } z=0$$

$$V_{(2,2,2)} - V_{(2,2,0)} = -\int_0^2 \bar{E}\cdot \bar{dz} \quad (\text{when } x=2,\ y=2)$$

$$= -x^2yz\Big|_0^2 = -16\ e_0\ V.$$

$$V_{(2,2,2)} = 0 + 0 = 16\,V = \overline{-16 e_0\ eV}$$

Q-4) Let us consider the volume bounded by two spherical surfaces having the same center (concentric spherical surfaces). The surfaces have R=a and R=b radii, respectively. Inside two surfaces, the volume is filled with the electric charge of unknown density $\rho_v(R)$ $[C/m^3]$. In the space, there is no charge except this volume and the dielectric constant of the space is ϵ_0, everywhere. The electric field is given as:

$$\boldsymbol{E}(\boldsymbol{r}) = 10\left(1 - \frac{a^2}{R^2}\right)\hat{a}_R[\text{V/m}],\ R\ \epsilon\ [a, b]$$

a) Find unknown charge density distribution function (R)

b) Find $\boldsymbol{E}(\boldsymbol{r})$ for R<a and R>b

c) Determine the electric potential $V(\mathbf{r})$ in the space (everywhere)

ⓐ $\nabla \cdot E = \rho / \varepsilon_0$ $\quad$ $E \to$ has only a_R direction

$$\nabla \cdot E = \frac{1}{R^2}\left\{\frac{\partial}{\partial R}\left[R^2 E_R\right]\right\} = \frac{1}{R^2}\underbrace{\frac{\partial}{\partial R}\left[10(R^2-a^2)\right]}_{20R} = \frac{20}{R} = \rho/\varepsilon_0$$

$$\boxed{\rho(R) = \frac{20\varepsilon_0}{R}} \quad [\text{Coulomb}/m^3]$$

$$\frac{Q_{total}}{\varepsilon_0} = \oint \vec{E} \cdot \vec{ds}$$

$$Q_{total} = \int_v \rho \cdot dv \Rightarrow \int_a^b \int_0^{2\pi} \int_0^{\pi} \frac{20\varepsilon_0}{\cancel{R}} R^{\cancel{2}} \sin\theta \, d\theta \, d\phi$$

$$= 20\varepsilon_0 \left[R^2/2\right]\Big|_a^b (4\pi)$$

$$= 40\pi\varepsilon_0 (b^2 - a^2) \to Q \text{ (coulomb)}$$

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = 4\pi$$

ⓑ $\underline{\underline{R > b}}$

$$\oint \varepsilon_0 E \cdot ds = Q_{enclosed}$$

we found

$\vec{E} = E \vec{a_R}$

$\vec{ds} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{a_R}$

$E \cdot ds = E R^2 \sin\theta \, d\theta \, d\phi$

$$\to E \int_0^{2\pi}\int_0^{\pi} R^2 \sin\theta \, d\theta \, d\phi = Q_{enclosed}$$

$$E \, 4\pi R^2 = 40\pi\varepsilon_0 (b^2 - a^2)$$

$$E = \frac{40\pi\varepsilon_0 (b^2-a^2)}{\varepsilon_0 4\pi R^2} = 10\frac{b^2-a^2}{R^2}\vec{a_R}$$

$\boxed{R<a}$ $E = 0$ (since $Q_{enclosed} = 0$ ②

ⓒ $a \leq R \leq b$

ⓒ for $R \geq b$ $V(R) = \dfrac{Q_{total}}{4\pi\varepsilon_0 R}$

$$= \frac{4\pi\varepsilon_0 (b^2 - a^2)}{4\pi\varepsilon_0 R} = 10\,\frac{b^2 - a^2}{R}$$

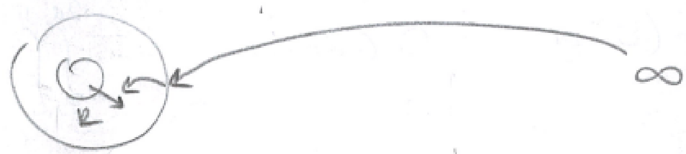
or

$$E = -\nabla\Phi \Rightarrow -\int \frac{10(b^2-a^2)}{R^2}\,dR = \frac{10(b^2-a^2)}{R}$$

$R \to \infty$ $V \to 0$

for $a \leq R \leq b$

$V(R) - V(b)$ → why



from infinity to the surface first, bring then
Bring from surface to where you search for

(inside) $\vec{E} = 10\left(1 - \dfrac{a^2}{R^2}\right)\vec{a_R}$ (given)

$$V(R) - V(b) = -\int_b^R 10\left(1 - \frac{a^2}{R^2}\right)dR$$

$$= -10\left[R + \frac{a^2}{R}\right]\Big|_b^R = 10\left(b - R + \frac{a^2}{b} - \frac{a^2}{R}\right)$$

$$V(R) - V(b) = 10\left(b - R + \frac{a^2}{b} - \frac{a^2}{R}\right)$$

$$V(R) = \underbrace{V(b)}_{\downarrow} + 10\left(b - R + \frac{a^2}{b} - \frac{a^2}{R}\right)$$

for $R \geq b \rightarrow V = \dfrac{10(b^2 - a^2)}{R}$ (found previously)

$$\Rightarrow V = \frac{10(b^2 - a^2)}{b}$$

$$V(R) = 10\frac{(b^2 - a^2)}{b} + 10\left(b - R + \frac{a^2}{b} - \frac{a^2}{R}\right)$$

$$= 10\left(2b - R - \frac{a^2}{R}\right)$$

for $R \leq a$ $\quad V = \text{constant}$ because E field is zero

$V(R) = V(a)$ $\qquad V(a) = 10(2b - a - a) = 20(b - a)$

constant [volt]