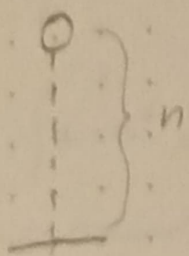


Diferansiyel Denklemler, bilinmeyen bir fonksiyonun bazı türevlerini içeren denklemlerdir.



$$-mg = m \cdot \frac{d^2 h(t)}{dt^2}$$

$$\frac{d^2 h}{dt^2} = -g \rightarrow \text{dif. denklem}$$

$$\int \frac{d^2 h}{dt^2} dt = \int -g dt$$

$$\int \frac{dh}{dt} dt = \int (-g t + C_1) dt$$

$$h = h(t) = -g \frac{t^2}{2} + C_1 t + C_2 \rightarrow \text{çözüm fonksiyonu}$$

Birinci basamaklı lineer dif. denk. - integral çarpanı

$$\text{Görünüm} = y'(x) + P(x)y'(x) = Q(x)$$

$$(y(x) \cdot I(x))' = y'(x) \cdot I(x) + I'(x) y(x)$$

$$I(x)y'(x) + I(x)P(x)y(x) = I(x)Q(x)$$

$$I'(x) = I(x)P(x) \quad \int \frac{I'(x)}{I(x)} dx = \int P(x) dx$$

$$\ln I(x) = \int P(x) dx \rightarrow I(x) = e^{\int P(x) dx}$$

$I(x) \rightarrow$ integral çarpıcı

$$\int (y(x) \cdot I(x))' dx = \int I(x) Q(x) dx$$

$$y(x) I(x) = \int I(x) Q(x) dx + C$$

1- integral çarpıcı bul.

2- Denklemi her iki tarafını $I(x)$ 'le çarp

3- İki tarafın integralini al

$$4- I(x)y(x) = \int I(x) Q(x) dx + C$$

5- $y(x)$ 'i bul.

Örnek

$y'(x) + x y(x) = x^3$ dif. den. genel çözümünü bulunuz.

$$y'(x) + P(x) y(x) = Q(x) \quad P(x) = x$$

$$I(x) = e^{\int P(x) dx} \quad I(x) = e^{\frac{x^2}{2}}$$

$$e^{\frac{x^2}{2}} y'(x) + e^{\frac{x^2}{2}} x y(x) = e^{\frac{x^2}{2}} x^3$$

$$\int (y e^{\frac{x^2}{2}})' dx = \int e^{\frac{x^2}{2}} x^3 dx \quad \frac{d}{dx} \frac{x^2}{2} = x$$

$$y e^{\frac{x^2}{2}} = \int e^u \cdot 2u du \rightarrow 2(u e^u - e^u)$$

$$\int u dv = uv - \int v du$$

Hatırlatma

$$\int x \cdot e^x \cdot dx \rightarrow \begin{matrix} x = u \\ dx = du \end{matrix} \quad \begin{matrix} e^x \cdot dx = du \\ e^x = u \end{matrix}$$

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x du = x \cdot e^x - e^x + C$$

$$2 \left(\frac{x^2}{2} e^{\frac{x^2}{2}} - e^{\frac{x^2}{2}} \right) + C$$

$$y e^{\frac{x^2}{2}} = x^2 e^{\frac{x^2}{2}} - 2 e^{\frac{x^2}{2}} + C$$

$$y = y(x) = x^2 - 2 + C e^{-\frac{x^2}{2}}$$

Örnek

$$(\cos x) y' + (\sin x) y = \cos^5 x \sin x \quad y(0) = 2$$

$$y' + \tan x y = \cos^4 x \sin x$$

$$I(x) = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x = \frac{1}{\cos x}$$

$$y' \sec x + \tan x \sec x y = \cos^3 x \sin x$$

$$\int (y \cdot \sec x)' dx = \int \cos^3 x \sin x dx \quad \begin{matrix} \cos x = u \\ -\sin x dx = du \end{matrix}$$

$$y \cdot \sec x = \int -u^3 du = -\frac{u^4}{4} + C$$

$$y(x) = -\frac{\cos^5 x}{4} + c \cos x \quad \text{Genel Çözüm}$$

$$2 = -\frac{1}{4} + c \Rightarrow c = \frac{9}{4}$$

$$y_{\text{özel}}(x) = -\frac{\cos^5 x}{4} + \frac{9}{4} \cos x \quad \text{Özel Çözüm}$$

Birinci basamak, ayrılabilir dif. denklemler

- Bu denklemler hem lineer, hem ayrılabilir.
- İntegral sabitini eklemeyi unutma

Örnek

$$y'(x) = \frac{1}{2} y(x) \quad \text{genel çözümünü bulunuz.}$$

$$\frac{dy}{dx} = \frac{1}{2} y \rightarrow 2dy = y dx \rightarrow \boxed{\frac{2}{y} dy = dx} \quad \text{istenilen form}$$

$$\int \frac{2}{y} dy = \int dx \rightarrow 2 \ln|y| = x + c$$

$$y = \underbrace{e^{\frac{x}{2}}}_{\text{bu da bir sabittir.}} \rightarrow \boxed{y = A e^{\frac{x}{2}}}$$

- x ve y'leri bir tarafa toplayıp integral al.

Örnek

$$y'(x) + xy = x^3 \quad \text{genel çözümünü bulunuz.}$$

$$\frac{dy}{dx} + xy = x^3 \rightarrow \frac{dy}{dx} = \underbrace{x^3 - xy}_{\text{ayrılmaz}} \quad \leftarrow \text{integral çarpımıyla}$$

$$p(x) = x \rightarrow P(x) = \int x dx = \frac{x^2}{2}$$

$$e^{\frac{x^2}{2}} y' + e^{\frac{x^2}{2}} xy = e^{\frac{x^2}{2}} x^3$$

$$y \cdot e^{\frac{x^2}{2}} = \int e^{\frac{x^2}{2}} x^3 dx \quad \frac{x^2}{2} = u$$

$$\frac{du}{dx} = x$$

$$\int e^u \cdot 2u \cdot du = 2(u e^u - e^u) + c$$

$$y = x^2 - 2 + c e^{-\frac{x^2}{2}}$$

Örnek

$$3x - 6y \sqrt{x^2 + 1} y'(x) = 0 \quad y(0) = 4 \quad \text{başlangıç değer problemini çözünüz}$$

$$6y \sqrt{x^2 + 1} dy = 3x dx \rightarrow \frac{2}{\sqrt{x^2 + 1}} dy = \frac{3x}{\sqrt{x^2 + 1}} dx \quad \begin{matrix} x^2 + 1 = u \\ \frac{du}{dx} = 2x \end{matrix}$$

$$\frac{2}{\sqrt{x^2 + 1}} dy = \frac{3x}{\sqrt{x^2 + 1}} dx \rightarrow y^2 = u^{\frac{3}{2}} + c$$

$$y^2 = \sqrt{x^2 + 1} + c$$

$$16 = 1 + C \rightarrow C = 15$$

$$y^2 = \sqrt{x^2 + 1} + 15 \rightarrow y = \sqrt{x^2 + 1} + 15 \text{ özel çözüm}$$

Tam Diferansiyel Denklemler

$$M(x,y) + N(x,y)y'(x) = 0 \rightarrow \text{Genel Form}$$

Lineer değil.

$$F(x,y) = C$$

$$\frac{dF(x,y)}{dx} = 0 \quad \frac{\partial F(x,y)}{\partial x} + \frac{\partial F(x,y)}{\partial y} \frac{dy}{dx} = 0$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$F_{xy} = F_{yx}$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad (\text{şart}). \text{ adım 1}$$

$$\int \frac{\partial F(x,y)}{\partial x} dx = \int M(x,y) dx \quad \text{adım 2}$$

$$F(x,y) = \int M(x,y) dx + \underbrace{h(y)}_{\text{integral sabiti}} \quad \text{adım 3}$$

$h(y)$ 'yi yerine yerleştirince herşeyi de çıkarıp eşitleyerek genel çözümünü bulabiliriz.

Örnek

$$2xy - 3x^2 + (2y + x^2 + 1)y'(x) = 0 \text{ genel çözümünü bul.}$$

$M(x,y) \quad N(x,y)$

$$M_y = N_x \quad 2x = 2x \quad \checkmark$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \rightarrow \int \frac{\partial F(x,y)}{\partial x} dx = \int (2xy - 3x^2) dx$$

$$F(x,y) = x^2y - 3x^3 + h(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \rightarrow x^2 + h'(y) = 2y + x^2 + 1$$

$$\int h'(y) dy = \int (2y + 1) dy$$

$$h(y) = y^2 + y + C_1$$

$$F(x,y) = x^2y - 3x^3 + y^2 + y = C$$

Homojen Denklemler

$$\frac{dy}{dx} = f(x,y) \rightarrow \frac{y}{x} \Rightarrow \text{değişim: uygulayınca ayrılabilir dif denkleme oluyor}$$

Örnek
 $(xy + y^2 + x^2) - x^2 y' = 0$ genel çözüm?

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1 = v + v^2 + 1$$

$\frac{y}{x} = v$ diyelim $y = vx$ $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$x + x \frac{dv}{dx} = v + v^2 + 1 \rightarrow$ ayrılabilir dif. haline getirebiliriz

$$\int \frac{1}{v^2+1} dv = \int \frac{1}{x} dx \rightarrow \arctan v = \ln|x| + c$$

$$v = \tan(\ln|x| + c)$$

$$y = x \cdot \tan(\ln|x| + c)$$

Bernoulli Denklemi

Genel Form $y' + P(x)y = Q(x)y^n \rightarrow y' + P(x)y = Q(x)$

$v = y^{1-n}$ dönüşümü uygulayacağız.

Sonra integral çarpma metoduyla çözeceğiz.

Örnek
 $y' - 5y = -\frac{5}{2}xy^3$ genel çözüm?

$$v = y^{-2} \rightarrow \frac{dv}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2}y^3 \frac{dv}{dx}$$

$$\rightarrow y^{-3} y' - 5y^{-2} = -\frac{5}{2}x$$

$$y^{-3} \left[-\frac{1}{2} y^3 \frac{dv}{dx} \right] - 5y^{-2} = -\frac{5}{2}x$$

$$-\frac{1}{2}v' - 5v = -\frac{5}{2}x \rightarrow v' + 10v = 5x$$

1. basamak lin dif. den.

$$I(x) = e^{\int P(x)dx} \Rightarrow e^{\int 10dx} = e^{10x}$$

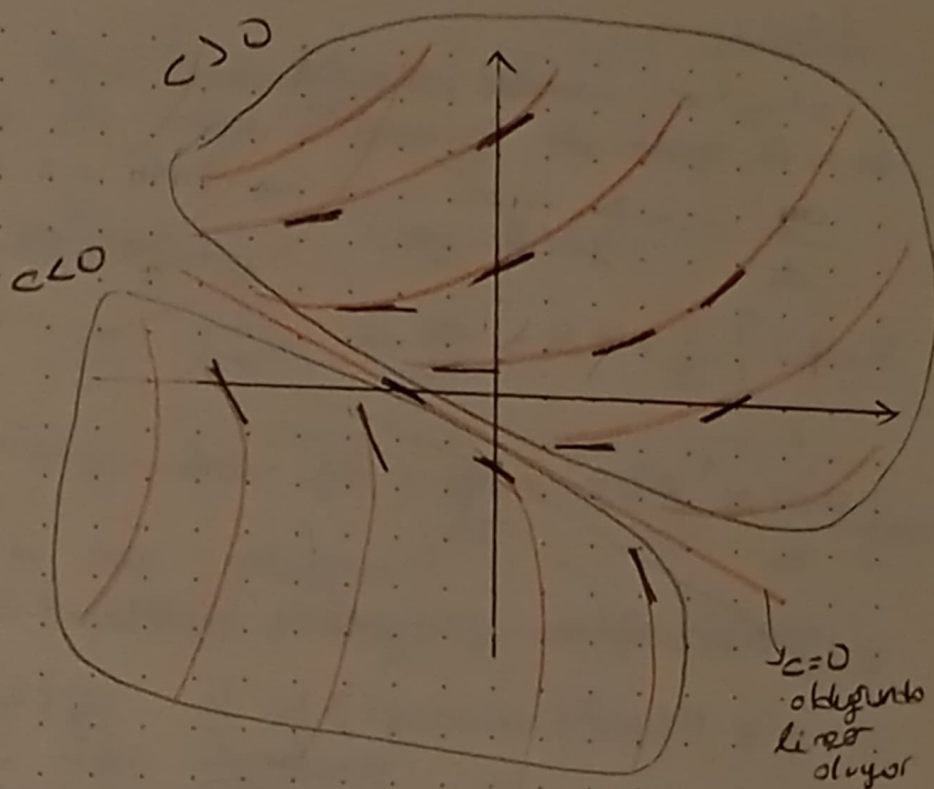
$$v e^{10x} = \int 5x e^{10x} dx$$

$$v e^{10x} = \frac{x}{2} e^{10x} - \frac{1}{20} e^{10x} + c$$

5x	e^{10x}
5	$\frac{1}{10} e^{10x}$
0	$\frac{1}{100} e^{10x}$

$$y^{-2} = \frac{x}{2} - \frac{1}{20} + c e^{-10x} \quad \text{Genel çözüm}$$

(x, y)	y'
$(0, 0)$	0
$(1, 0)$	1
$(0, 1)$	2
$(1, 1)$	3
$(-1, 0)$	-1
$(-1, -1)$	1
$(-2, 0)$	-2
$(-2, -1)$	0



$$-\frac{1}{2} + 2c e^{2x} = x + 2\left(-\frac{1}{2}x - \frac{1}{4} + c e^{2x}\right)$$

$$0 = 0 \checkmark$$

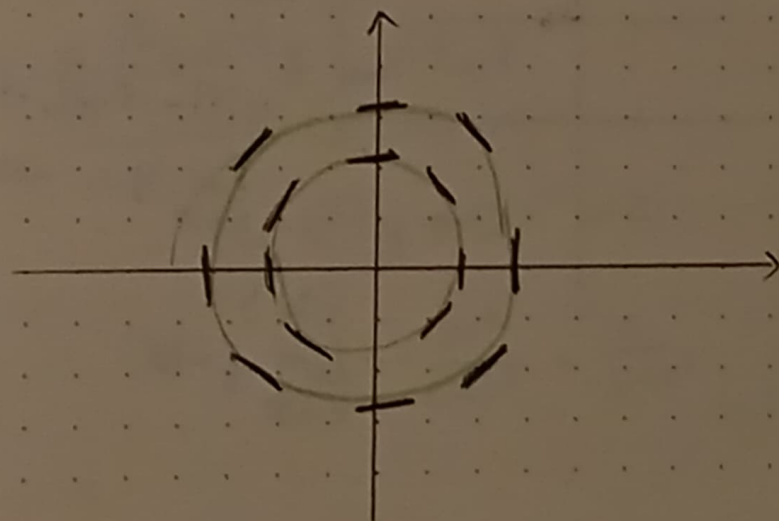
örnek

$$y' = -\frac{x}{y} \text{ dif. denk için}$$

- doğrultu okları çiz
- bazı çözümler çizeriz
- $x^2 + y^2 = c$ ifadesinin genel çözümüne abd. doğrultu

• $y(-3) = -4$ baş. değ. ile verilen baş. değ. pr. çözüm ve tanımlı ettiği çözümleri gösterelim.

(x, y)	y'
$(0, 0)$	tanımsız
$(1, 0)$	∞
$(1, 1)$	-1
$(0, 1)$	0
$(-1, 0)$	∞
$(-1, 1)$	1
$(-1, -1)$	-1



$$x^2 + y^2 = c \rightarrow 2x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \checkmark \quad \rightarrow 9+16 = c = 25$$

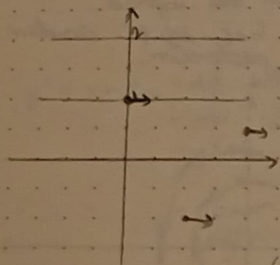
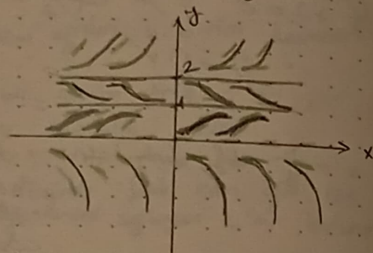
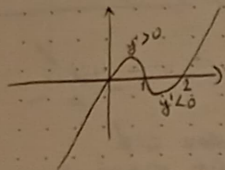
$$x^2 + y^2 = 25 \rightarrow \text{3. ell çizim}$$

$$y' = y^3 - 3y^2 + 2y \quad \text{dis. denk. için}$$

$$y' = y(y^2 - 3y + 2)$$

$$y' = y(y-1)(y-2)$$

$$y=0, y=1, y=2$$



$$y(0) = 1 \quad (0, 1)$$

$$\lim_{x \rightarrow \infty} y = 1$$

$$y(2) = \frac{1}{2} \quad (2, \frac{1}{2})$$

$$\lim_{x \rightarrow \infty} y = 1$$

$$y(1) = -1 \quad (1, -1)$$

$$\lim_{x \rightarrow \infty} y = -\infty$$

$$(3, \frac{3}{2})$$

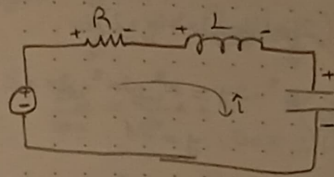
$$\lim_{x \rightarrow \infty} y = 1$$

$$(2, 3)$$

$$\lim_{x \rightarrow \infty} y = \infty$$

Week 1

Consider the RLC circuit given below.
Try to express the voltage in terms of the charge Q on cap.

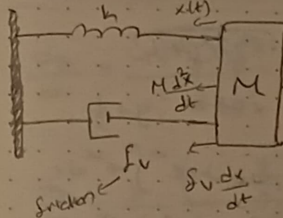


Using KVL

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = V$$

$$i = \frac{dQ}{dt} \rightarrow L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = V$$

Consider the following mass-spring-damper system



Newton's Law

$$m \frac{d^2x}{dt^2} = \sum F$$

$$\frac{M d^2 x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + kx(t) = F(t)$$

Example

Week 2

$$(4+t^2) \frac{dy}{dt} + 2ty = 4t$$

$$(4+t^2) \frac{dy}{dt} + 2ty = \frac{d}{dy} [(4+t^2)y] = 4t$$

$$(4+t^2)y = 2t^2 + C$$

$$y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$$

Solve $\frac{dy}{dx} = \frac{x^2}{1-y^2}$

$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$, $y(0) = -1$

$-x^2 + (1-y^2) \frac{dy}{dx} = 0$

$(2(y-1))dy = (3x^2 + 4x + 2)dx$

$y^2 - 2y = x^3 + 2x^2 + 2x + C$

$\int -x^2 dx + \int (1-y^2) dy = C$

$x=0, y=-1$

$1+2 = C = 3$

$-\frac{x^3}{3} + y - \frac{y^3}{3} = C$

$y^2 - 2y = x^3 + 2x^2 + 2x + 3$

$-x^3 + 3y - y^3 = C$

$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 3}$

$xy^2 dx + e^x dy = 0$ initial condition $x \rightarrow \infty, y \rightarrow \frac{1}{2}$

$\int x e^x dx + \int y^2 dy = 0$ $uv - \int v du = -x e^x - \int -e^{-x} dx$

$dx dx \cdot v = -e^{-x}$

$-x e^{-x} - e^{-x} - y^{-1} = C$

$-\lim_{x \rightarrow \infty} \frac{1}{e^x} - \lim_{x \rightarrow \infty} \frac{1}{e^x} - \lim_{y \rightarrow \frac{1}{2}} \frac{1}{y} = C$

$x e^{-x} + e^{-x} + \frac{1}{y} = 2$

$\frac{dy}{dx} = \cos^2 x \cos y$

$\ln|\sec y + \tan y| = \frac{\sin 2x}{4} \cdot \frac{x}{2}$

$\int \sec y dy = \int \cos^2 x dx = \int \frac{\cos 2x}{2} dx + \int \frac{dx}{2}$

$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \rightarrow \frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$

define: $y = vx$

$\frac{y}{x} = v \rightarrow \frac{dv}{dx} x + v = 1 + v + v^2$

$\frac{dy}{dx} = \frac{dv}{dx} x + v \quad \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$

$\arctan v = \ln|x| + C$

$\arctan \frac{y}{x} = \ln|x| + C$

$\frac{dy}{dx} = \frac{4x^2 y - y^3}{x^3 - 2xy^2} = \frac{y(4x^2 - y^2)}{x(x^2 - 2y^2)} = \frac{y x^2 (4 - \frac{y^2}{x^2})}{x x^2 (1 - 2\frac{y^2}{x^2})}$

$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{(4 - \frac{y^2}{x^2})}{(1 - 2\frac{y^2}{x^2})}$ $v = \frac{y}{x}$ $y = vx$ $\frac{dy}{dx} = \frac{dv}{dx} x + v$

$\frac{dv}{dx} x + v = v \left(\frac{4 - v^2}{1 - 2v^2} \right) = \frac{4v - v^3}{1 - 2v^2}$

$\frac{dv}{dx} x = \frac{4v - v^3 - v + 2v^3}{1 - 2v^2} = \frac{v^3 + 3v}{1 - 2v^2}$

$\frac{1-2v^2}{(v^2+3)} dv = \frac{dx}{x} \rightarrow \frac{A}{v} + \frac{Bv+C}{v^2+3} = \frac{1-2v^2}{v(v^2+3)}$

$\frac{1}{3} \int \frac{dv}{v} - \frac{7}{3} \int \frac{v dv}{v^2+3} = \int \frac{dx}{x} \rightarrow \frac{1}{3} \ln|v| - \frac{7}{6} \ln|v^2+3| = \ln|x| + C$

$v^2+3=u$ $\frac{v dv}{v^2+3} = \frac{du}{2u} \rightarrow \frac{1}{2} \ln|u|$

$C' = \ln C$ let's say

$\frac{(A+B)v^2 + Cv + 3A}{-2 \quad 0 \quad 1}$
 $A = \frac{1}{3}$
 $B = -\frac{7}{3}$
 $C = 0$

$$\ln \left[\frac{|v|^{1/3}}{|v^2+3|^{7/6}} \right] = \ln|x| + \ln C \cdot \frac{|v|^{1/3}}{|v^2+3|^{7/6}} = C(x)$$

$$\frac{\left| \frac{y}{x} \right|^{1/3}}{\left| \frac{y^2+3x^2}{x^2} \right|^{7/6}} = C(x) \quad \frac{|y|^{1/3} x^2}{(y^2+3x^2)^{7/6}} = C(x)$$

Exact equations and integrating factors.

$$2x + y^2 + 2xy \cdot y' = 0 \rightarrow \text{Neither linear nor separable}$$

$$\psi(x,y) = x^2 + xy^2 \quad \frac{\partial \psi}{\partial x} = 2x + y^2 \quad \frac{\partial \psi}{\partial y} = 2xy$$

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0 \quad \frac{\partial \psi(x,y)}{\partial x} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

Chain Rule

$$\frac{\partial \psi(x,y)}{\partial x} = 0 \quad \frac{d}{dx} (x^2 + xy^2) = 0$$

$$\text{Let's say: } M(x,y) + N(x,y) y' = 0$$

$$\text{Suppose } \frac{\partial \psi}{\partial x}(x,y) = M(x,y) \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

$$\underbrace{(y \cos x + 2xe^y)}_M + \underbrace{(\sin x + x^2 e^y - 1)}_N y' = 0$$

$$M_y = \cos x + 2xe^y = N_x = \cos x + 2xe^y$$

There is a function $\psi(x,y)$ such that:

$$\psi_x(x,y) = y \cos x + 2xe^y$$

$$\psi_y(x,y) = \sin x + x^2 e^y - 1$$

Integrate one of them

$$\psi(x,y) = y \sin x + x^2 e^y + h(y)$$

$$\text{We must determine } h(y): \psi_y = N = \sin x + x^2 e^y + h'(y)$$

$$h(y) = -y + C$$

$$= \sin x + x^2 e^y - 1$$

$$\psi(x,y) = y \sin x + x^2 e^y - y + C$$

$$\text{must be constant} \quad y \sin x + x^2 e^y - y = C \rightarrow \text{implicit solution}$$

$$(y e^{2xy} + x) dx + (b x e^{2xy}) dy = 0$$

$$M_y = e^{2xy} + 2xy e^{2xy} \quad N_x = b e^{2xy} + 2y b x e^{2xy}$$

$$\psi_x = y e^{2xy} + x$$

$$\psi_y = x e^{2xy}$$

$$b = 1$$

$$\psi_x = \frac{2y e^{2xy}}{2} + h'(x) \leftarrow \psi = \frac{x e^{2xy}}{2} + h(x)$$

$$h'(x) = x \quad h(x) = \frac{x^2}{2} + C$$

$$\psi(x,y) = e^{2xy} + x^2 + C$$

$$\left(\frac{y}{x} + 6x\right) dx + (\ln x + 2) dy = 0$$

$$M_y = \frac{1}{x}$$

$$N_x = \frac{1}{x}$$

$$\psi_x = \frac{y}{x} + 6x$$

$$\psi_y = \ln x + 2$$

$$\psi(x, y) = y \ln x + \frac{6x^2}{2} + h(y)$$

$$\psi_y = \ln x + h'(y) = \ln x + 2 \quad h'(y) = 2$$

$$h(y) = 2y + C_1 \quad \psi = y \ln x + 3x^2 + 2y + C_1$$
