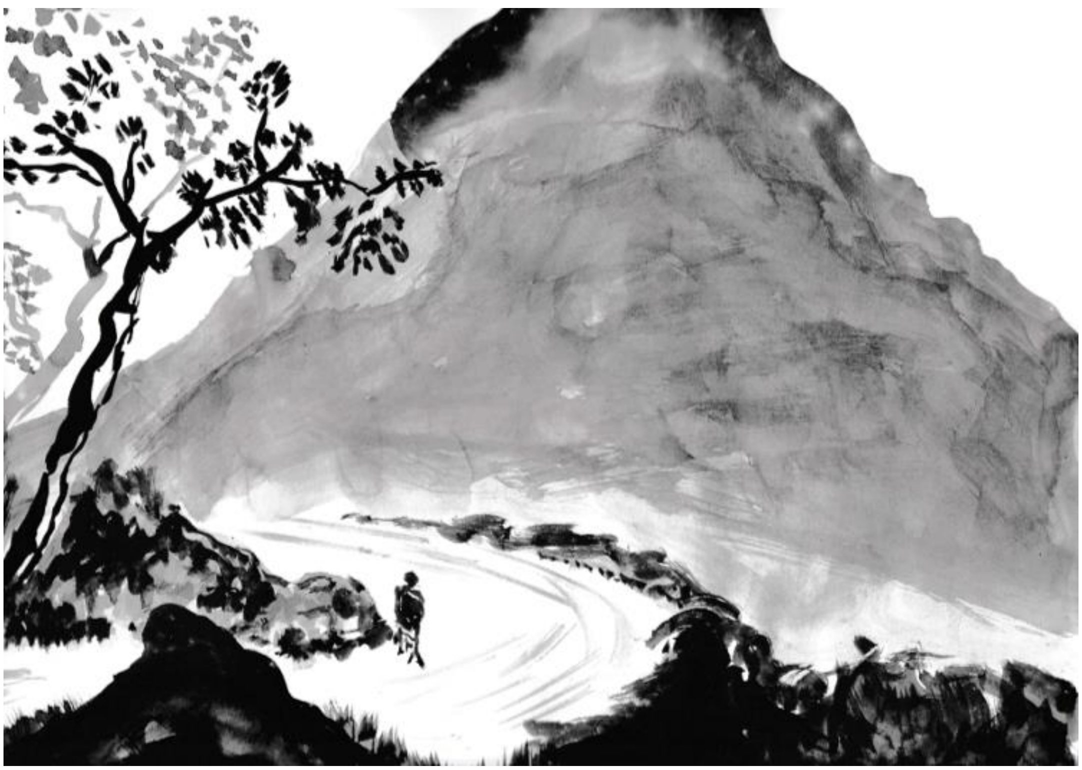
Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Fall Semester 2022

EHB 212E

HOMEWORK – 1



Each student is viewed as a responsible professional in engineering, and thus highest ethical standards are presumed.

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Due: October 17, 2022, till 16.59

- **You need to bring the hardcopy of homework to office 7309 in EEB before the deadline (you can throw homework under the door)**
- **You need to show all the steps during operations. Otherwise, the questions are not graded.**
- **Do Not forget to write your name!**
- **The total point is 100 and each question has the same importance.**

Q-1) Given that $\vec{F} = \rho z \hat{e}_\rho + \frac{cos\phi}{\rho}\hat{e}_\phi + K\rho^2\hat{e}_z$ where K is an unknown constant to be determined. (Figure 1)

a) Find the value of K to make the vector field $\vec{F}$ irrotational (i.e. $\nabla x\, \vec{F} = 0$)

b) For the value of K found above, compute the line integral $I = \int_C \vec{F}.d\vec{l}$ where the contour C is defined on the $z = 1$ plane from point A to B as shown in Figure 1.

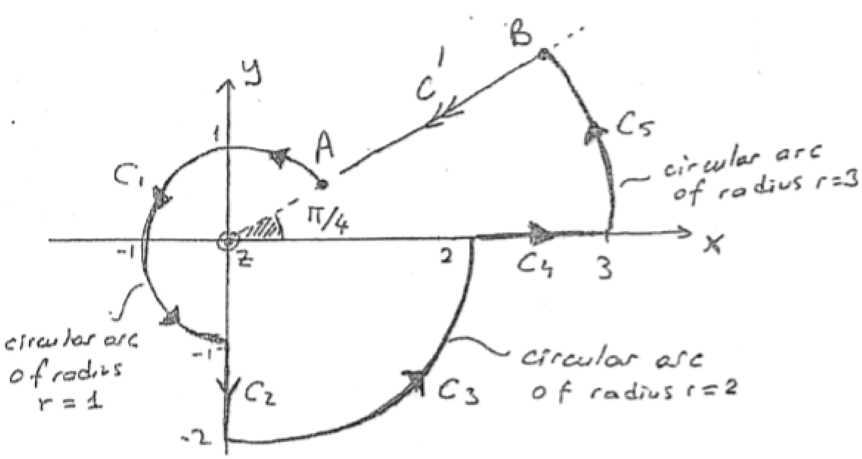


Figure 1. The geometry of Problem 1.

Question 1 CNW #1

Kamil Karagüle

①

$\boxed{C = C_1 + C_2 + C_3 + C_4 + C_5}$

Info

rot $A = 0 \Rightarrow$ A is an irrotational field !

rot$F \Rightarrow 0 \rightarrow$ we should find a "k" value such that rot$F = 0$ ✓.

cylindrical coordinate system is chosen !

$$\mathrm{rot}\vec{F} = \vec{e_r}\left(\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) + \vec{e_\phi}\left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)$$

$$+ \vec{e_z}\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\phi) - \frac{\partial F_r}{\partial \phi}\right)$$

put them into equation

Note that "ρ" sometimes can be used as "r". No problem at all.

$\vec{F} = F_\rho \vec{e_\rho} + F_\phi \vec{e_\phi} + F_z \vec{e_z}$ (decompose $\vec{F}$ vector in terms of three orthogonal vector)

$$\boxed{F_\rho = rz \quad F_\phi = \frac{\cos\phi}{r} \quad F_z = r^2}$$

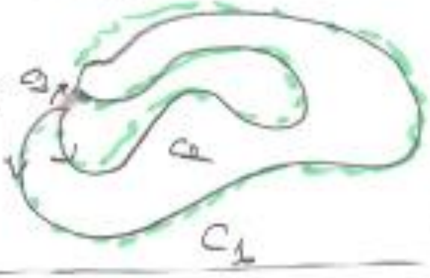
$= \vec{e_\phi}(r - 2kr) = 0 \leftarrow$ we force it to be "zero"

$\boxed{k = 1/2}$

(b) → $\vec{F}$ is irrotational when $\boxed{k=1/2}$.

→ In (B) $\vec{F}$ is irrotational therefore $\boxed{\text{path independent?}}$

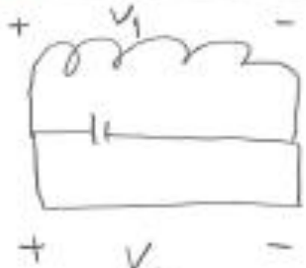
→ c_1



$$\boxed{\int_{C_1} F = \int_{C_2} F = \int_{C_3} F} \xRightarrow{\text{if}} F \text{ is irrotational}$$

---

ex



$\boxed{\text{Voltage is the same}}$

$$V_1 = -\int_{C_1} E \cdot dl = -\int_{C_2} E \cdot dl$$

$E \Rightarrow$ is irrotational

$\nabla \times E = 0$ (for electrostatic)

$$\boxed{\tilde{C} = \boxed{C_1 + C_2 + C_3 + C_4 + C_5} + C'}$$

$$\tilde{C} = C + C'$$

$$\int_{\tilde{C}} F = 0 = \int_{C} F + \int_{C'} F \Rightarrow \boxed{\int_{C} F = -\int_{C'} F}$$

why? → because path independent and $\tilde{C}$ is closed contour therefore

$$\boxed{\int_{A}^{A} F = 0}$$

end and beginning points are the same points

$$I = \int \vec{F} \cdot d\vec{l} = -\int_B^A \vec{F} \cdot d\vec{l} = -\int_{r=3}^{r=1} r\,dr = -\frac{(1-9)}{2} = \boxed{4}$$

3

$$\boxed{d\vec{l} = \vec{e}_r\,dr}$$

direction is given by boundaries !

$$\vec{F} \cdot d\vec{l} = rz$$

$$\boxed{z=1}$$

$$\boxed{I = 4}$$

c) $\boxed{\text{if } k = 1/2}$

$\nabla \times F = 0 \rightarrow$ then F can be electrostatic field !

the end

Q-2) Given the vector field $\vec{A} = x\hat{e}_x$ and the volume specified by

$$\boldsymbol{\mathcal{U}} : a \le R \le b, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le \pi$$

where, R, θ, ϕ are the usual spherical coordinate parameters, **verify** the Divergence Theorem through this defined volume above.

Hint: Verification of Divergence Theorem is to show the following equality

$$\int_v (\vec{\nabla}.\vec{A}).\,dv = \oint_S \vec{A}.\,d\vec{S}$$

<u>Given</u> $\vec{A} = x\vec{e}_x$

(1)

$V: a \leq R \leq b$

$0 \leq \theta \leq \pi$] That is given !!

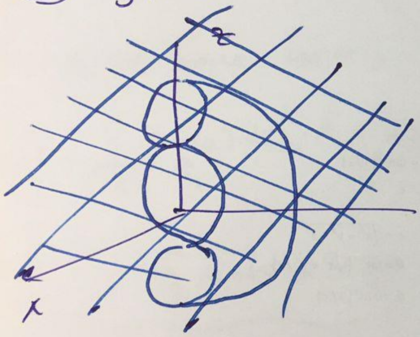
$* \quad 0 \leq \phi < \pi \; *$

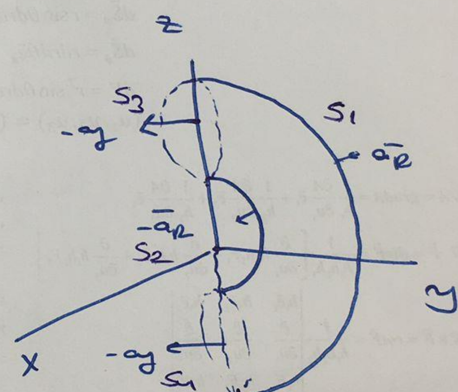
$\nabla \cdot A = \frac{\partial}{\partial x} ax(x e_x) = 1$

$\int \nabla \cdot A \, dv =$

$\int 1 \cdot dv = \int_a^b \int_0^\pi \int_0^\pi R^2 \sin\theta \, d\theta \, d\phi \, dR = \frac{R^3}{3} 2\pi \Big|_a^b = \boxed{\frac{(b^3 - a^3)\, 2\pi}{3}}$

(2) $\oint \vec{A} \cdot ds$





$\oint \vec{A} \cdot ds \Rightarrow$

$\int_{S_1} + \int_{S_2} + \int_{S_3} + \int_{4} \vec{A} \cdot d\vec{s}$

$d\vec{s}_3 = ds_3(-\vec{a}_y) \qquad \vec{A} \cdot d\vec{s}_3 = 0$ ✓

$ds_4 = ds_4(-\vec{a}_y) \qquad \vec{A} \cdot ds_4 = 0$ ✓

$$\int_{S_1} \vec{A} \cdot ds = \vec{A} \, d\vec{s} = x a_x \cdot \vec{a}_R \, ds$$

$$\Rightarrow R \sin\theta \cos\phi \quad R^2 \sin\theta \, d\theta \, d\phi$$

$$\vec{A} \cdot ds = R^3 \sin^3\theta \cos^2\phi$$

$$\int_{S_1} \vec{A} \cdot ds = \int_{(R=b)} R^3 \sin\theta^3 \cos^2\phi$$

$$= b^3 \int_0^{\pi} \cos^2\phi \, d\phi \int_0^{\pi} \sin^3\theta \, d\theta$$

$$\left(\frac{\pi}{2}\right) \quad \left(\frac{4}{3}\right)$$

$$\int_{S_1} \vec{A} \cdot d\vec{s} = b^3 \frac{\pi}{2} \, \frac{4}{3} = \frac{2}{3} \pi b^3 \; //$$

in the same manner,

$$\int_{S_2} (-a_R) \, ds_2 \cdot \vec{A} \Rightarrow \left(-\frac{2\pi}{3} a^3\right)$$

$$\oint \vec{A} \cdot d\vec{s} = \frac{2\pi}{3} \left(b^3 - a^3\right)$$

Div. Theorem has verified !)

Q-3) If $\vec{A} = \rho\cos\phi\hat{a}_\rho + \sin\phi\hat{a}_\phi$, then, evaluate $\oint \vec{A}.\overrightarrow{dl}$ around the path shown below. Verify Stokes' Theorem.

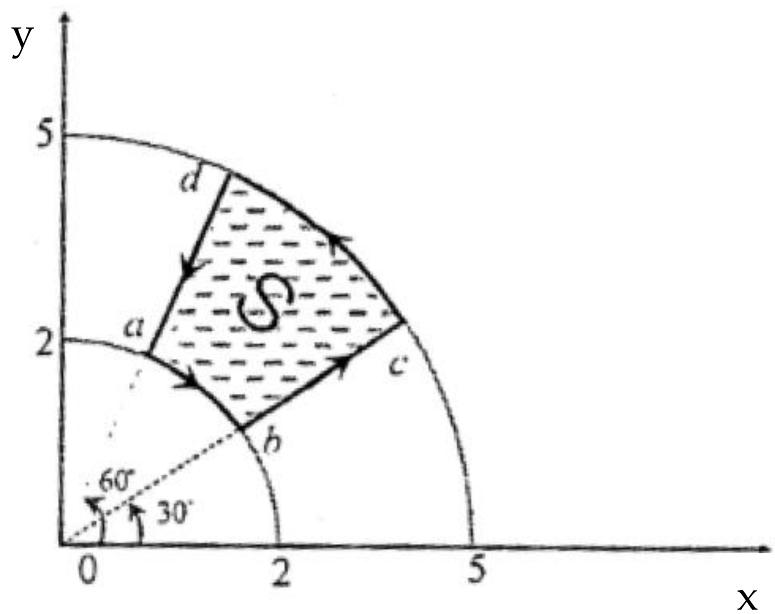
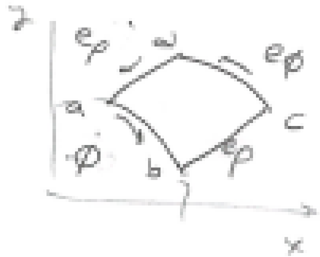


Figure 2. The geometry of Problem 3.

$$\oint_{\ell} \vec{A}\cdot \vec{d\ell} = \int \vec{\nabla}\times\vec{A}\, \vec{ds}$$



$$\vec{A} = \rho\cos\phi\, \vec{e_\rho} + \sin\phi\, \vec{e_\phi}$$

along ab $\quad \rho=2 \quad \vec{d\ell} = \rho\, d\phi\, \vec{e_\phi}$

$$\vec{d\ell} = 2\, d\phi\, \vec{e_\phi}$$

Bandaria $\quad a = \frac{\pi}{3}$

$$b = \frac{\pi}{6}$$

$$\int_a^b \vec{A}\cdot d\ell = \int_{\pi/3}^{\pi/6} 2\sin\phi\, d\phi = -2\cos\phi \Big|_{\pi/3}^{\pi/6} = -(\sqrt{3}-1)$$

$\phi = \pi/6 \quad \vec{d\ell} = d\rho\, \vec{e_\rho}$

$$\int_b^c \vec{A}\cdot d\ell = \int_{\rho=2}^{5} \rho\cos\phi\, d\rho = \cos\frac{\pi}{6}\left(\frac{\rho^2}{2}\right]_2^5 = \frac{21\sqrt{3}}{4}$$

$d \rightarrow \rho = 5 \quad d\ell = \rho\, d\phi\, \vec{e_\phi}$

$$\int_C \vec{A}\cdot\vec{d\ell} = \int_{\pi/6}^{\pi/3} \rho\sin\phi\, d\phi = -5\cos\phi \Big|_{\pi/6}^{\pi/3} = \frac{5}{2}(\sqrt{3}-1)$$

$a \longrightarrow \quad \phi = 60 \quad \vec{d\ell} = d\rho\, \vec{e_\rho}$

$$\int_d \vec{A}\cdot\vec{d\ell} = \int_{\rho=5}^{2} \rho\cos\phi\, d\rho = \cos 60 \left[\frac{\rho^2}{2}\right]_5^2 = -\frac{21}{4}$$

$$\oint \vec{A}\, d\ell = -(\sqrt{3}-1) + \frac{21\sqrt{3}}{4} + \frac{5}{2}(\sqrt{3}-1) - \frac{21}{4} \cong 4.941$$

$$\oint_C \vec{A}\cdot d\ell = \int_S \mathrm{rot}\,\vec{A}\cdot\vec{ds}$$

$\text{rot} A \Rightarrow \frac{1}{\rho}(1+\rho)\sin\phi\,\vec{e_z}$

close cylindrical coordinates

$\vec{ds} = \rho\, d\phi\, d\rho\, \vec{e_z}$

$$\int \text{rot}\vec{A}\cdot\vec{ds} = \int_{\phi=\pi/6}^{\pi/3}\int_{\rho=2}^{5} \sin\phi\,\frac{(1+\rho)}{\cancel{\rho}}\,d\rho\, d\phi\, \cancel{\rho}\, \underbrace{\vec{e_z}\cdot\vec{e_z}}_{1}$$

$$= -\cos\phi\,\Big|_{\pi/6}^{\pi/3}\left(\rho + \frac{\rho^2}{2}\right)\Big|_2^5$$

$$\cong 4{,}941$$

verified

Q-4) Find $\vec{\nabla}\phi$ when $\phi = 3x^2y - y^3z^2$. Find the directional derivative in the direction of $3\hat{a}_x + 4\hat{a}_y + 12\hat{a}_z$ at (2, -1, 0).

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\vec{e_x} + \frac{\partial\phi}{\partial y}\vec{e_y} + \frac{\partial\phi}{\partial z}\vec{e_z}$$

$$\phi = 3x^2y - y^3z^2$$

$$\nabla\phi = 6xy\,\vec{e_x} + (3x^2 - 3y^2z^2)\vec{e_y} + (-2y^3z)\vec{e_z}$$

at $(2,-1,0) \longrightarrow (-12)\vec{e_x} + 12\vec{e_y}$

Directional derivative $\rightarrow \nabla\phi \cdot \vec{a} = *$

unit vector in the given direction

$$* = (-12\vec{e_x} + 12\vec{e_y}) \cdot \frac{(3\vec{e_x} + 4\vec{e_y} + 12\vec{e_z})}{\sqrt{9+16+144}}$$

$$\cong 0.96$$

=0.92307