Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Spring Semester 2022-2023

EEF 212E

HOMEWORK – 1

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Each student is viewed as a responsible professional in engineering, and thus highest ethical standards are presumed.

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EEF 212E

HOMEWORK – 2

Due: April 17, 2023, till 23.30

- **You need to upload HW to Ninova. Other options are not accepted!**
- **You need to show all the steps during operations. Otherwise, the questions are not graded.**
- **Do Not forget to write your name!**
- **The total point is 100 and each question has the same importance.**

Q-1) There is a hemispherical surface defined for $R = a, 0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$ is charged surface charge density $\rho_s(R, \theta, \phi) = \theta$

a) Find electrostatic potential V at the center of the sphere (at the origin)

b) Find the electric field $\vec{\boldsymbol{E}}$ at the center of the sphere using Coulomb's Law Approach sphere (at the origin)

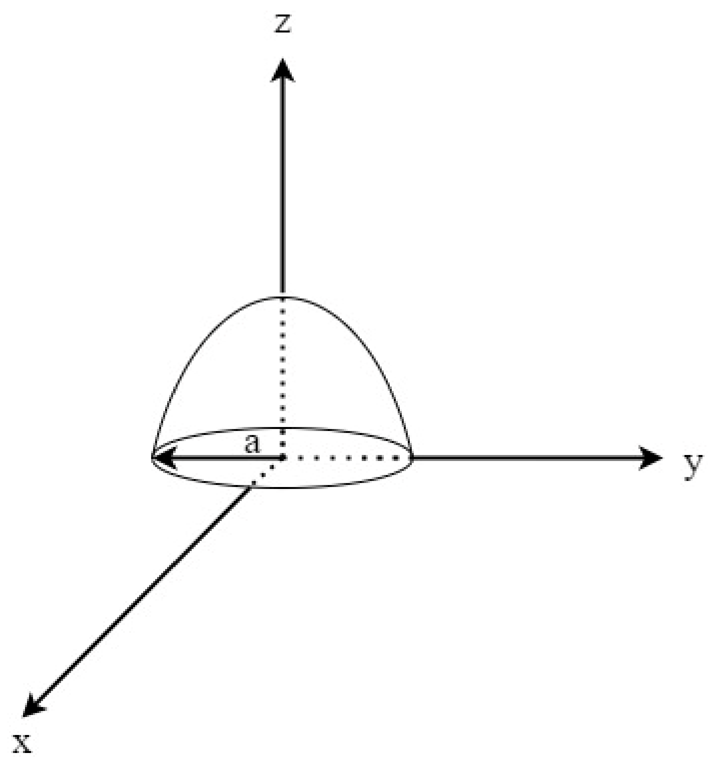


Figure 1. The geometry of Q-1.

ⓐ $dV = \frac{dq}{4\pi\varepsilon_0 R} \Rightarrow \frac{\rho_s dS}{4\pi\varepsilon_0 a}$ as $R=a$

$$dV = \frac{\theta a^2 \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0 a}$$

$$V = \int dV = \frac{a}{4\pi\varepsilon_0} \int_{\theta=0}^{\pi/2} d\theta (\theta \sin\theta) \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \quad *$$

$\theta = u \qquad \sin\theta \, d\theta = dv$

$d\theta = du \qquad -\cos\theta = v \quad \Rightarrow *$

$$\Rightarrow V = \frac{a}{2\varepsilon_0}\left[uv - \int v \, du\right]$$

$$V = \frac{a}{2\varepsilon_0}\left[\cancel{\theta[-\cos\theta]\Big|_0^{\pi/2}} - \int_0^{\pi/2} -\cos\theta \, d\theta\right]$$

$$V(0,0,0) = \frac{a}{2\varepsilon_0}\left(\sin\theta \Big|_0^{\pi/2}\right) = + \frac{a}{2\varepsilon_0} \text{ volt}$$

$$V(0,0,0) = \frac{a}{2\varepsilon_0} \text{ volt}$$

ⓑ $d\bar{E} = \frac{dq}{4\pi\varepsilon_0 R^2} \vec{a}_R$, $\vec{E} = \int d\bar{E}$ where

$R = |\bar{r} - \bar{r}'|$

$\bar{r}' = R\vec{a}_R$

$\bar{r} = 0$

$\vec{a}_R = \frac{\bar{r} - \bar{r}'}{|\bar{r} - \bar{r}'|}$

$\vec{a}_R = -\vec{a}_R$ in (the spherical coordinate system)

$$\Rightarrow d\bar{E} = \frac{\theta a^2 \sin\theta \, d\theta \, d\phi}{4\pi\varepsilon_0 a^2} (-\vec{a}_R)$$

Since $\boxed{dq = \rho \, dS}$

where $\vec{a}_R = \sin\theta\cos\phi \, a_x + \sin\theta\sin\phi \, a_y + \cos\theta \, a_z$

periodic

$$\bar{E} = -\frac{1}{4\pi\varepsilon_0}\left[\vec{a}_x \int_0^{\pi/2} \theta \sin^2\theta \, d\theta \underbrace{\cancel{\int_0^{2\pi} \cos\phi \, d\phi}}_{0} + \vec{a}_y \int_0^{\pi/2} \theta \sin^2\theta \, d\theta \underbrace{\cancel{\int_0^{2\pi} \sin\phi \, d\phi}}_{0} + \vec{a}_z \underbrace{\int_0^{\pi/2} \theta \sin\theta \cos\theta \, d\theta}_{\pi/8} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\right]$$

$$E(0,0,0) = -\frac{\pi}{16\varepsilon_0} \hat{a}_z \text{ V/m}$$

Q-2) There exist infinitely long concentric cylindrical structures as given in Figure right. The region $\rho < a$ is filled with an electric charge of density $\rho_v = \rho_0 \left(1 - \frac{\rho}{a}\right)$ Coul/m³. Here ρ_0 is a constant. Besides, a concentric cylindrical conducting shell is placed outside of the cylinder. Except for the conductor, everywhere is in a vacuum (ϵ_0)

a) Find the electric field intensity everywhere
b) Find the electrostatic potential difference between
   i) $\rho = 0$ and $\rho = a$
   ii) $\rho = a$ and $\rho = b$
   iii) $\rho = b$ and $\rho = c$

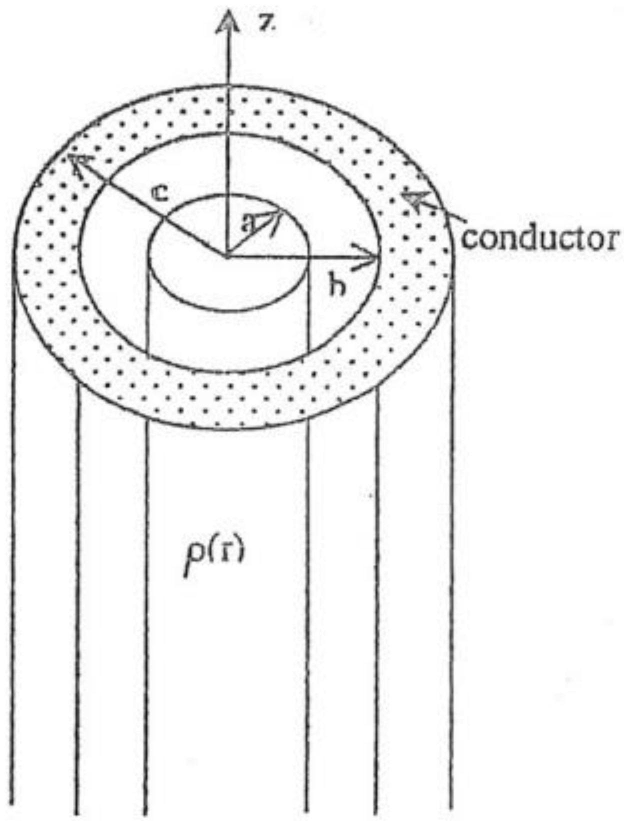


Figure 2. The geometry of Q-2.

ⓐ $\rho < a$ from Gauss law:

$\vec{E} = E\rho\vec{\varphi}$

$E\rho$ is the magnitude

$$E_\rho 2\pi\rho h = \frac{h}{\varepsilon_0}\int_0^{2\pi}\int_0^{\rho} \rho_0\left(1-\frac{\rho}{a}\right)\rho\, d\rho\, d\phi$$

$$\vec{E} = \vec{a_\varphi}\ \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{2}-\frac{r^2}{3a}\right]\quad [V/m]$$

$a < \rho < b$

$\vec{E} = E\rho\vec{\varphi}$

↳ From Gauss law for h meter cylinder

$$\Rightarrow\ E\rho 2\pi\rho h = \frac{h2\pi}{\varepsilon_0}\int_0^a \rho_0\left(1-\frac{\rho}{a}\right)\rho\, d\rho$$

$$\Rightarrow E\rho = \frac{1}{\rho}\left(\frac{\rho_0}{\varepsilon_0}\left(\frac{a^2}{2}-\frac{a^3}{3a}\right)\right)$$

$$\Rightarrow\ \vec{E} = \vec{\varphi}\ \frac{\rho_0 a^2}{\varepsilon_0 6\rho}\quad [V/m]$$

~~b<r<c~~

$b < r < c$ $\quad\vec{E}$

$$\vec{E} = E\rho\vec{\varphi} \Rightarrow \quad 2\pi\rho E\rho = \frac{Q_{enc}}{\varepsilon_0} \Rightarrow \vec{E} = \vec{\varphi}\ \frac{\rho_0 a^2}{6\varepsilon_0\rho}$$

ⓑ ⓘ

$$= \ V(\rho=a) - V(\rho=0) = -\int_{\rho=0}^{\rho=a}\vec{E}\cdot\vec{dl}$$

$$\Rightarrow V(a)-V(0) = -\int_{\rho=0}^{\rho=a}\vec{a_\varphi}\ \frac{\rho_0}{\varepsilon_0}\left(\frac{\rho}{2}-\frac{\rho^2}{3a}\right)\cdot\vec{a_\varphi}\, d\rho$$

$$= -\frac{\rho_0}{\varepsilon_0}\left[\frac{\rho^2}{4}-\frac{\rho^3}{9a}\right]\Bigg|_{\rho=0}^{\rho=a}$$

$$= -\frac{\rho_0}{\varepsilon_0}\left[\frac{a^2}{4}-\frac{a^3}{9a}\right]$$

$$= -\frac{\rho_0}{\varepsilon_0}\left[\frac{a^2}{4}-\frac{a^2}{9}\right]\ V$$

(± errors can be ignored)

(ii) r=a ve r=b

$$V(a)-V(b) = -\int_{\varphi=b}^{\varphi=a} E \cdot dl \Rightarrow$$

$$V(a)-V(b) = -\int_{\rho=b}^{\rho=a} \vec{E} \quad \frac{\rho_0 a^2}{\varepsilon_0 6\rho} \cdot a_\rho \, d\rho$$

$$= -\int_{\rho=b}^{\rho=a} \frac{\rho_0 a^2}{6\varepsilon_0} \frac{1}{\rho} d\rho$$

$$= + \frac{\rho_0 a^2}{6\varepsilon_0} \ln\rho \Big|_a^b$$

$$= \frac{\rho_0 a^2}{6\varepsilon_0} \ln(b/a) \, [V]$$

(iii) $E=0$

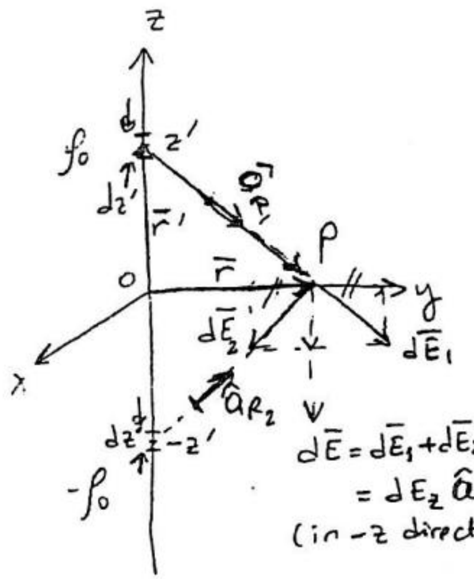
$V=C \Rightarrow V(b)-V(c)=0$

Q-3) Assume the infinitely long line charge distribution defined along the z-axis as

$$\rho(z') = \begin{cases} \rho_0 & for \;\; z' > 0 \\ 0 & for \;\; z' = 0 \\ -\rho_0 & for \;\; z' < 0 \end{cases}$$

where ρ_0 is a constant value

Find the electric field intensity vector at an arbitrary point P on the $z = 0$ plane



$$d\bar{E} = d\bar{E}_1 + d\bar{E}_2 = dE_z\,\hat{a}_z$$
(in $-z$ direction)

Due to symmetry, $\bar{E}$ field is independent of ϕ. Therefore, we can take an observation point P on the $z=0$ plane at an arbitrary ϕ. Let's take $\phi = \frac{\pi}{2}$, for instance (i.e, P is on the y-axis.)

Consider two symmetrical sections of length dz' centered at $z=z'$ and $z=-z'$. Due to the form of given $\rho(z')$, the resultant $d\bar{E}$ field, due to charges $(\rho_0 dz')$ at $z=z'$ and $(-\rho_0 dz')$ at $z=-z'$, is in the $(-z)$ direction (radial components of dE_1 and dE_2 cancel out).

$$\bar{r} = r\hat{a}_r$$
$$\bar{r}' = z'\hat{a}_z \quad (\text{at } z=z' \text{ location})$$
$$\bar{R} = \bar{r} - \bar{r}' = r\hat{a}_r - z'\hat{a}_z$$
$$R = |\bar{R}| = \sqrt{r^2 + z'^2}$$
$$\hat{a}_{R_1} = \frac{\bar{R}}{R} = \frac{r\hat{a}_r - z'\hat{a}_z}{\sqrt{r^2 + z'^2}}$$

due to $dq_1 = \rho_0 dz' \longrightarrow d\bar{E}_1 = \dfrac{\rho_0 dz'}{4\pi\epsilon_0 R^2}\hat{a}_{R_1} = \dfrac{\rho_0 dz'(r\hat{a}_r - z'\hat{a}_z)}{4\pi\epsilon_0 (r^2+z'^2)^{3/2}}$

similarly,
due to $dq_2 = -\rho_0 dz' \longrightarrow d\bar{E}_2 = \dfrac{-\rho_0 dz'}{4\pi\epsilon_0 R^2}\hat{a}_{R_2} = \dfrac{-\rho_0 dz'(r\hat{a}_r + z'\hat{a}_z)}{4\pi\epsilon_0 (r^2+z'^2)^{3/2}}$

$$d\bar{E} = d\bar{E}_1 + d\bar{E}_2 = \frac{\rho_0 dz'(-2z')\hat{a}_z}{4\pi\epsilon_0 (r^2+z'^2)}$$
where $z': 0 \longrightarrow \infty$

Q-4) There exist two perfectly conducting spherical shells with radii a and b which form a capacitor and are designed by *ITU Honeycomb*. There is a potential difference V between the shells as shown in the figure. For $0 \le \theta \le \theta_0$, the region between the conductors is filled with an inhomogeneous dielectric of the permittivity $\epsilon(\theta) = \epsilon_0(1 + cos\theta)$ and the rest of the region for which $\theta_0 \le \theta \le \pi$ is filled with permittivity ϵ_0.

a) Find E-field and D-field vectors, everywhere
b) Find the free surface charge density everywhere
c) Find the capacitance of the structure

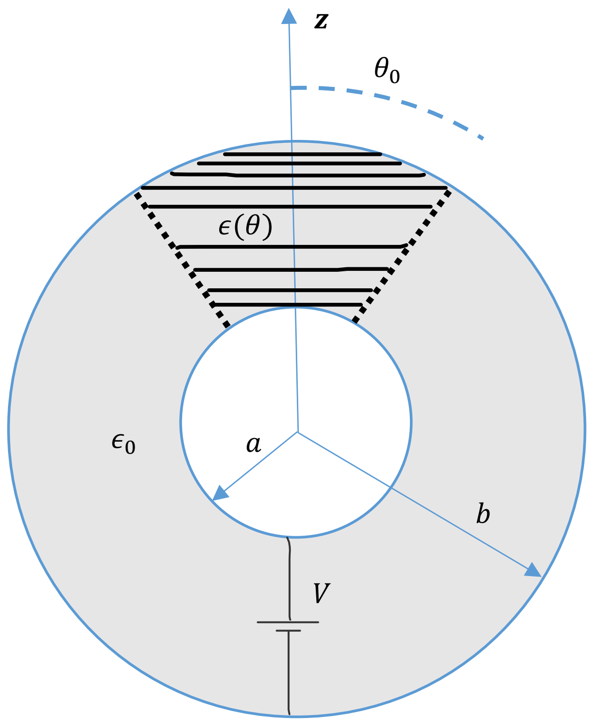
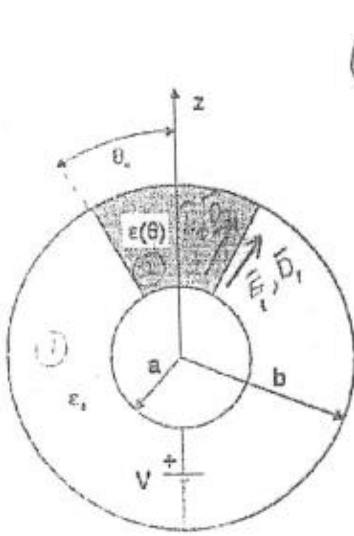


Figure 3. The geometry of Q-4.



(a) Here, $\nabla^2 V = 0$ (Laplace Eqn.) can be solved within the capacitor.

Because, $\underbrace{\nabla \cdot \bar{D}}_{\epsilon \bar{E}} = \underbrace{\rho}_{0}$ (no free charge within the capacitor)

$$\Rightarrow (\bar{\nabla}\epsilon) \cdot \underbrace{\bar{E}}_{-\nabla V} + \epsilon \bar{\nabla} \cdot \underbrace{\bar{E}}_{-\nabla V} = 0$$

Note that $\underbrace{\bar{\nabla}\epsilon(\theta)}_{\text{in } \hat{a}_\theta \text{ dir.}} \cdot \underbrace{\bar{\nabla} V(R)}_{\text{in } \hat{a}_R \text{ dir.}} = 0$

$$\hat{a}_\theta \cdot \hat{a}_R = 0$$

$$\Rightarrow -\epsilon \nabla^2 V(R) = 0 \Rightarrow \boxed{\nabla^2 V(R) = 0}$$

($V = V(R)$ as there is a fixed potential difference of V between inner and outer electrode, for all possible θ, ϕ values)

In spherical coordinates,

$$\nabla^2 V(R) = \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial V(R)}{\partial R}\right) = \frac{1}{R^2} \frac{d}{dR}\left(R^2 \frac{dV(R)}{dR}\right) = 0$$

$$\Rightarrow \frac{d}{dR}\left(R^2 \frac{dV(R)}{dR}\right) = 0 \Rightarrow R^2 \frac{dV(R)}{dR} = C_1 \Rightarrow \frac{dV(R)}{dR} = \frac{C_1}{R^2}$$

$$\Rightarrow \boxed{V(R) = -\frac{C_1}{R} + C_2}$$

$$V(b) = 0 \Rightarrow -\frac{C_1}{b} + C_2 = 0$$

$$V(a) = 0 \Rightarrow -\frac{C_1}{a} + C_2 = V_0$$

$$\boxed{C_1 = \frac{V_0 ab}{a-b}} \qquad \boxed{C_2 = \frac{V_0 a}{a-b}}$$

$$\boxed{V(R) = \frac{ab}{b-a} V \left[\frac{1}{R} - \frac{1}{b}\right] \text{ volts}}$$

$$\bar{E} = -\bar{\nabla} V = -\hat{a}_R \frac{\partial V(R)}{\partial R}$$

$$\bar{E} = \begin{cases} \frac{ab}{b-a} V \frac{1}{R^2} \hat{a}_R & \text{for } \begin{cases} a < R < b \\ 0 < \theta < \pi \end{cases} \\ 0 & \text{elsewhere.} \end{cases}$$

$$\bar{D} = \epsilon \bar{E}$$

$$\bar{D} = \begin{cases} \epsilon_0 \frac{abV}{b-a} \frac{1}{R^2} \hat{a}_R & \text{(within, in air)} \\ \epsilon_0 (1 + \cos\theta) \frac{abV}{b-a} \frac{1}{R^2} \hat{a}_R & \text{(within, in dielectric)} \\ 0 & \text{elsewhere} \end{cases}$$

(b) at $R=a$ $\rho_s = \bar{D}\cdot\hat{a}_R \Big|_{R=a} = \begin{cases} \epsilon_0 \frac{abV}{b-a}\frac{1}{a^2} & (\text{for } \theta_0<\theta<\pi) \\ \epsilon_0(1+\cos\theta)\frac{abV}{b-a}\frac{1}{a^2} & (\text{for } 0\le\theta<\theta_0) \end{cases}$

(coul/m²)

at $R=b$ $\rho_s = \bar{D}\cdot(-\hat{a}_R)\Big|_{R=b} = \begin{cases} -\epsilon_0 \frac{abV}{b-a}\frac{1}{b^2} & (\text{for } \theta_0<\theta<\pi) \\ -\epsilon_0(1+\cos\theta)\frac{abV}{b-a}\frac{1}{b^2} & (\text{for } 0\le\theta<\theta_0) \end{cases}$

Note that $\bar{D}_{air}, \bar{D}_{diel.}$ within the capacitor are tangential to the boundary at $\theta=\theta_0$ (no normal component of $\bar{D}$ to boundary)

$\Rightarrow \rho_s = 0$ at that boundary.

(c) Total charge at $R=a$ conductor surface is

$$Q = Q_1 + Q_2 = \int_{S_1} \rho_{s_1}\, dS + \int_{S_2} \rho_{s_2}\, dS$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=\theta_0}^{\pi} \epsilon_0 \frac{abV}{b-a}\frac{1}{a^2} a^2 \sin\theta\, d\theta\, d\phi + \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\theta_0} \epsilon_0(1+\cos\theta)\frac{abV}{b-a}\frac{1}{a^2}\underbrace{a^2\sin\theta\, d\theta\, d\phi}_{dS}$$

$$= \epsilon_0 \frac{abV}{b-a} 2\pi \underbrace{\left(-\cos\theta\Big|_{\theta_0}^{\pi}\right)}_{1+\cos\theta_0} + \epsilon_0 \frac{abV}{b-a}\frac{1}{a^2}a^2 2\pi \underbrace{\int_{\theta=0}^{\theta_0}(1+\cos\theta)\sin\theta\, d\theta}_{-\cos\theta\Big|_0^{\theta_0} + \frac{1}{2}\underbrace{\int_0^{\theta_0}\sin 2\theta\, d\theta}_{-\frac{1}{2}\cos 2\theta\Big|_0^{\theta_0}}}$$

$$= \epsilon_0 \frac{abV}{b-a} 2\pi(1+\cos\theta_0) + \epsilon_0 \frac{2\pi abV}{b-a}\left[1-\cos\theta_0 - \frac{1}{4}(\cos 2\theta_0 - 1)\right]$$

$$Q = V\left[\epsilon_0 \frac{ab}{b-a} 2\pi\left(1+\cancel{\cos\theta_0}+1-\cancel{\cos\theta_0}+\frac{1}{4}-\frac{1}{4}\cos 2\theta_0\right)\right]$$

$$C \triangleq \frac{Q}{V} \Rightarrow \boxed{C = \epsilon_0 \frac{ab\,2\pi}{b-a}\left(\frac{9}{4}-\frac{1}{4}\cos 2\theta_0\right) \text{ Farad}}$$

Q-5) A cylindrical resistor of length h is designed by ITU BEE by using two different conducting materials as shown below. The region $0 < \rho < a$ has a conductivity σ_1, and the region $a < \rho < b$ has a conductivity σ_2. A voltage V_0 is applied between the perfectly conducting electrodes at $z = 0$ and $z = h$

a) Find $\boldsymbol{E}$ and $\boldsymbol{J}$ vectors in each region
b) Find the resistance of the structure

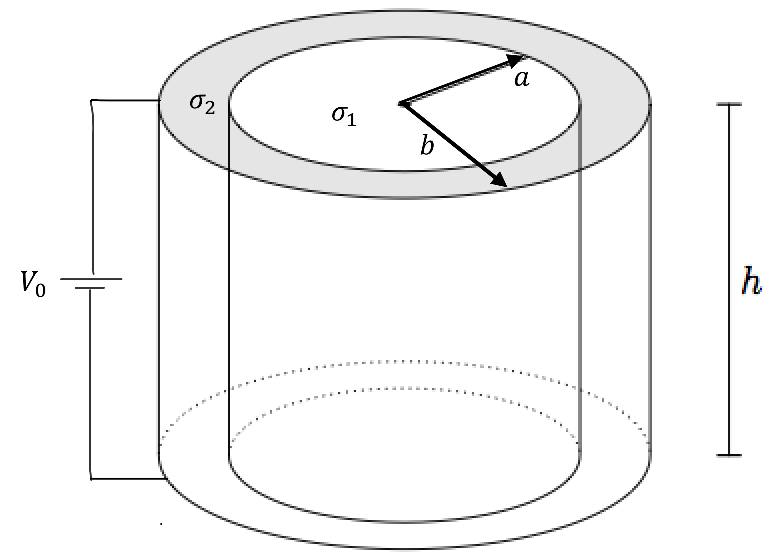
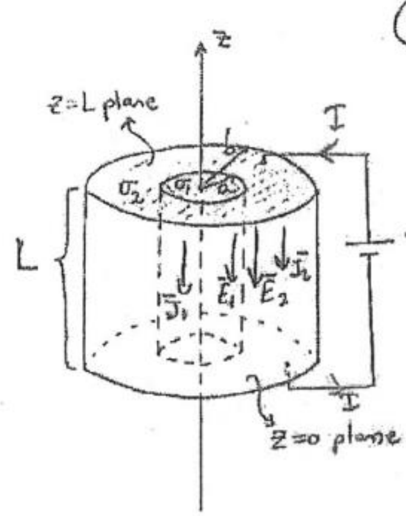


Figure 4. The geometry of Q-5.



(a) As both regions are homogeneous, free charge is not accumulated within the resistor volume, i.e., $\rho_v = 0$ everywhere within the resistor. $\Rightarrow$ Laplace Eqn. can be solved. Furthermore, the potential difference between the electrodes is kept fixed (for arbitrary values of r and ϕ)

$\Rightarrow V = V(z)$, solve $\nabla^2 V(z) = 0$

$$\Rightarrow \frac{\partial^2 V(z)}{\partial z^2} = 0 \Rightarrow V(z) = C_1 z + C_2$$

$$V(z=0) = C_2 = 0$$
$$V(z=L) = C_1 L = V_0 \Rightarrow C_1 = \frac{V_0}{L}$$

$$\Rightarrow \boxed{V(z) = \frac{V_0}{L} z \quad (\text{Volts})}$$

$$\bar{E} = -\bar{\nabla} V = -\frac{\partial}{\partial z} V(z)\,\hat{a}_z = -\frac{V_0}{L}\hat{a}_z$$ (Note that $\bar{E}$ is tangential to the boundary at $r = a$)

$$\Rightarrow \boxed{\bar{E}_1 = \bar{E}_2 = -\frac{V_0}{L}\hat{a}_z \ (V/m)} \Rightarrow \bar{J}_1 = \sigma_1 \bar{E}_1, \quad \bar{J}_2 = \sigma_2 \bar{E}_2 \Rightarrow \boxed{\begin{array}{c} \bar{J}_1 = -\frac{\sigma_1 V_0}{L}\hat{a}_z \ (A/m^2) \\ \hline \bar{J}_2 = -\frac{\sigma_2 V_0}{L}\hat{a}_z \ (A/m^2) \end{array}}$$

(b)
$$I_1 = \int_{S_1} \bar{J}_1 \cdot d\bar{S} = \int_{r=0}^{a}\int_{\phi=0}^{2\pi} \left(-\frac{\sigma_1 V_0}{L}\hat{a}_z\right)\cdot(-\hat{a}_z\, r\,dr\,d\phi) = \frac{\sigma_1 V_0}{L} \underbrace{\int_{S_1} dS}_{\pi a^2} = \frac{\sigma_1 V_0}{L}\pi a^2 \ (\text{Amp})$$

$$\boxed{I_1 = \frac{\sigma_1 V_0}{L}\pi a^2 \ (\text{Amp})}$$ and, similarly, $$\boxed{I_2 = \frac{\sigma_2 V_0}{L}(\pi b^2 - \pi a^2) \ (\text{Amp})}$$

$$I_{total} = I_1 + I_2 = \frac{V_0}{L}\pi\left[\sigma_1 a^2 + \sigma_2 b^2 - \sigma_2 a^2\right] \quad (\text{Amp})$$

$$\boxed{R \triangleq \frac{V_0}{I_{total}} = \frac{L}{\pi(\sigma_1 a^2 - \sigma_2 a^2 + \sigma_2 b^2)} \quad (\Omega)}$$

Note that $R = R_1 // R_2$ where $\boxed{R_1 = \frac{L}{\pi\sigma_1 a^2}}$ $\boxed{R_2 = \frac{L}{\pi\sigma_2 (b^2 - a^2)}}$