Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Spring Semester 2022-2023

EEF 212E

HOMEWORK – 3

Each student is viewed as a responsible professional in engineering, and thus highest ethical standards are presumed.

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Due: May 10, 2023, till 23.30

- **You need to upload HW to Ninova. Other options are not accepted!**
- **You need to show all the steps during operations. Otherwise, the questions are not graded.**
- **Do Not forget to write your name!**
- **The total point is 100 and each question has the same importance.**

Q-1) There exists a conducting ABC triangle wire given in Figure 1. Corners of the triangle have the following coordinates

$$\vec{r}_A = a\hat{a}_x, \vec{r}_B = a\hat{a}_y, \vec{r}_C = a\hat{a}_z$$

A DC current of I_0 Amperes are flowing through in the flowing direction ($C \to A \to B \to C$) and the medium permeability is $|\mu_0$. Find the magnetic flux density vector $\vec{B}$ at the origin $O(0, 0,0)$

Hints: You may one or two expressions below while solving:

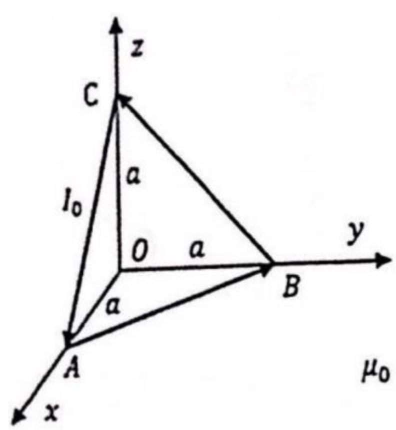


Figure 1. The geometry of Q-1.

Hints:

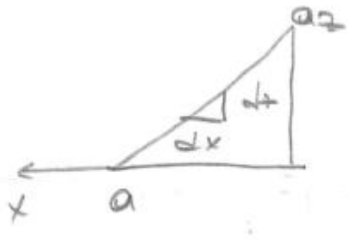
$$\int \frac{1}{\sqrt{Ax^2+Bx+C}}dx = \frac{\ln\left(\frac{B+2Ax}{2\sqrt{A}}+\sqrt{Ax^2+Bx+C}\right)}{\sqrt{A}}$$

$$\int \frac{1}{Ax^2+Bx+C}dx = \frac{2\arctan\left(\frac{B+2Ax}{\sqrt{4AC-B^2}}\right)}{\sqrt{4AC-B^2}}$$

$$\int \frac{1}{(Ax^2+Bx+C)^{\frac{3}{2}}}dx = -\frac{(2B+4Ax)}{(B^2-4AC)\sqrt{Ax^2+Bx+C}}$$

$$\vec{B}(r) = \frac{\mu_0 I}{4\pi} \int_{C'} \frac{dl' \times \vec{e_r}}{(r-r')^2} = \frac{\mu_0 I}{4\pi} \int_{C'} \frac{dl' \times (\vec{r}-\vec{r}')}{|r-r'|^3}$$

$$\begin{cases} r' = x\vec{e_x} + (a-x)\vec{e_z} \\ r = 0e_x + 0\vec{e_z} \end{cases}$$



$$dr' = [\vec{e_x} - \vec{e_z}]\,dx$$

$$r - r' = 0 - r' = -x\vec{e_x} - (a-x)\vec{e_z} \qquad |r-r'| = \sqrt{x^2+(a-x)^2}$$

$$\vec{B}_{CA} = \frac{\mu_0 I_0}{4\pi} \int_{x=0}^{a} \frac{(\vec{e_x}-\vec{e_z}) \times [-x\vec{e_x} - (a-x)e_z]}{[x^2 + (a-x)^2]^{3/2}}\,dx =$$

$$= \frac{\mu_0 I_0}{4\pi} \int_0^a \frac{\begin{vmatrix} e_x & e_y & e_z \\ 1 & 0 & -1 \\ -x & 0 & -(a-x) \end{vmatrix}}{[x^2+(a-x)^2]^{3/2}}\,dx$$

$$\Rightarrow \frac{\vec{e_y}\, \mu_0 I_0 a}{4\pi} \int_{x=0}^{a} \frac{1 \quad dx}{[x^2 + (a-x)^2]^{3/2}} = \vec{e_y}\, \frac{\mu_0 I a}{4\pi} \int_{x=0}^{a} \frac{dx}{[2x^2 - 2ax + a^2]^{3/2}}$$

$$\int \frac{1}{\left[Ax^2+Bx+C\right]^{3/2}} = \frac{-2B+4Ax}{(B^2-4AC)\sqrt{Ax^2+Bx+C}}$$

in our case,

$A=2$

$B=-2A$

$C=a^2$

$$\vec{B}_{CA} = \vec{a}_y \frac{\mu_0 I_0 a}{4\pi}\left[-\frac{a-2x}{a^2\sqrt{2x^2-2ax+a^2}}\right]\Bigg|_{x=0}^{a}$$

$$= \vec{e}_y \frac{\mu_0 I_0 a}{4\pi}\left[\frac{2}{a^2}\right] = \vec{e}_y \frac{\mu_0 I_0}{2\pi a}$$

$\vec{B}_{AB}, \vec{B}_{BC}$ due to cyclic nature of the sides,

$$\vec{B}_{AB} = \vec{e}_z \frac{\mu_0 I_0}{2\pi a} \qquad \vec{B}_{BC} = \vec{e}_x \frac{\mu_0 I_0}{2\pi a}$$

$$\vec{B}(0,0,0) = \frac{\mu_0 I_0}{2\pi a}\left(\vec{e}_x+\vec{e}_y+\vec{e}_z\right) \quad \langle \text{Tesla} \rangle$$

$\uparrow$

total

Q-2) A non-uniform surface current density flows as:

$$\bar{J}_s = \begin{cases} \hat{a}_\phi J_o r' \sin\left(\frac{\pi}{a} r'\right) & \text{over } S': \{a \le r' \le 2a, \quad 0 \le \phi' < 2\pi, z' = 0\} \\ 0 & \text{elsewhere} \end{cases} \text{(Amp/m)}$$

in vacuum where J_0 is a known constant.

a) Find the magnetic vector potential $\boldsymbol{A}$ at the origin O. (***Hint***: $\vec{A} = \frac{\mu_0}{4\pi}\int_{S'} \frac{\vec{J_s}(r')}{|r-r'|} ds'$)

b) Find the magnetic flux density vector $\boldsymbol{B}$ at the origin O.

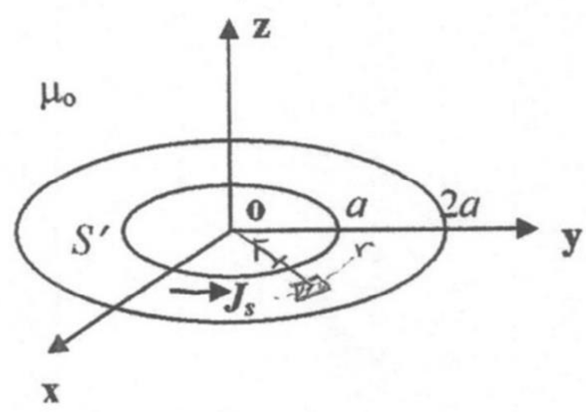
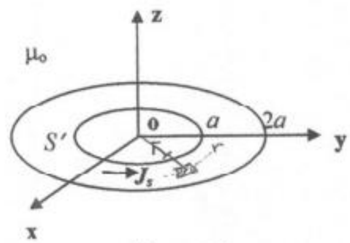


Figure 2. The geometry of Q-2.



Solution:

(a) $\bar{A}(\bar{r}) = \frac{\mu_o}{4\pi}\int_{S'} \frac{\bar{J}_s(\bar{r}')}{R}\, ds'$

where $R = |\bar{R}| = |\bar{r} - \bar{r}'|$ in general.

Here, the observation point is the origin $0 \Rightarrow \bar{r} = 0 \Rightarrow \bar{R} = -\bar{r}' = -r'\hat{a}_r$

$$\bar{A}\Big|_{at\,0} = \frac{\mu_o}{4\pi}\int_{r'=a}^{2a}\int_{\phi'=0}^{2\pi} \hat{a}_\phi J_o \cancel{r'} \sin\left(\frac{\pi}{a}r'\right)\frac{1}{\cancel{r'}}\, \underbrace{r'dr'd\phi'}_{ds'}$$

where $\hat{a}_\phi = -\hat{a}_x \sin\phi' + \hat{a}_y \cos\phi'$

$$\Rightarrow \bar{A}\Big|_{at\,0} = \frac{\mu_o J_o}{4\pi}\left[-\hat{a}_x \int_{r'=a}^{2a} dr' \sin\left(\frac{\pi r'}{a}\right) r' \underbrace{\int_{\phi'=0}^{2\pi} \sin\phi' d\phi'}_{0} + \hat{a}_y \int_{r'=a}^{2a} dr' \sin\left(\frac{\pi r'}{a}\right) r' \underbrace{\int_{\phi'=0}^{2\pi} \cos\phi' d\phi'}_{0}\right]$$

$$\Rightarrow \boxed{\bar{A}\Big|_{at\,0} = 0}$$

(b) As $\bar{A}$ is known only at a single point (at the origin), we can not use the result of part (a) to find $\bar{B}$ from $\bar{B} = \bar{\nabla}\times\bar{A}$. Instead, we need to use the Bio-Savart Law to compute $\bar{B}$ at the origin.

$$\bar{B}\Big|_{at\ 0} = \frac{\mu_0}{4\pi} \int_{S'} \frac{\bar{J}_s \times \hat{a}_R}{R^2} ds' \quad \text{where} \quad \bar{R} = -r'\hat{a}_r \quad , ds' = r'dr'd\phi'$$

$$\Rightarrow \begin{cases} R = |\bar{R}| = r \\ \hat{a}_R = -\hat{a}_r \end{cases} \quad \text{in cylindrical coordinates.}$$

$$= \frac{\mu_0}{4\pi} \int_{r'=a}^{2a} \int_{\phi'=0}^{2\pi} \frac{J_0 \cancel{r'} \sin(\frac{\pi}{a}r') \overbrace{\hat{a}_\phi \times (-\hat{a}_r)}^{\hat{a}_z}}{(r')^{\cancel{2}}} (\cancel{r'} dr' d\phi')$$

$$= \hat{a}_z \frac{\mu_0 J_0}{4\pi} \int_{r'=a}^{2a} dr' \sin(\tfrac{\pi}{a}r') \underbrace{\int_{\phi'=0}^{2\pi} d\phi'}_{2\pi} = \hat{a}_z \frac{\mu_0 J_0}{2} \int_{r'=a}^{2a} \sin(\tfrac{\pi}{a}r') dr'$$

let $u = \frac{\pi}{a} r'$

$\Rightarrow du = \frac{\pi}{a} dr'$

$$\bar{B} = \hat{a}_z \frac{\mu_0 J_0}{2} \int_{u=\pi}^{2\pi} \sin(u) \frac{a}{\pi} du = \hat{a}_z \frac{\mu_0 J_0}{2} \frac{a}{\pi} \underbrace{\left(-\cos u \Big|_{\pi}^{2\pi}\right)}_{-\underbrace{\cos 2\pi}_{1} + \underbrace{\cos\pi}_{-1}}$$

$$\boxed{\bar{B}\Big|_{at\ 0} = -\hat{a}_z \frac{\mu_0 J_0 a}{\pi} \quad (\text{Weber}/m^2)}$$

Q-3) A DC electric current of unknown density $\vec{J}(\frac{Amps}{m^2}\ and\ \vec{J} = J\hat{a}_z)$ flows inside the infinitely long cylinder for $r < a$ as figure below. There is no current outside of the cylinder. The space is a vacuum. The B field vector for $r < a$ is given as $\vec{B} = \hat{a}_\phi \mu_0 G r^3$. Here, G has a known constant value.

a) Find $\vec{J}$

b) Find $\vec{B}$ for $r > a$

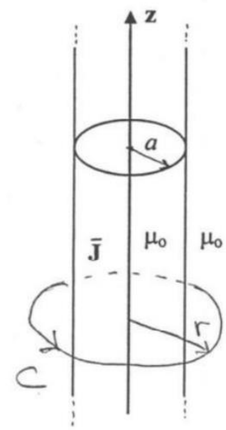
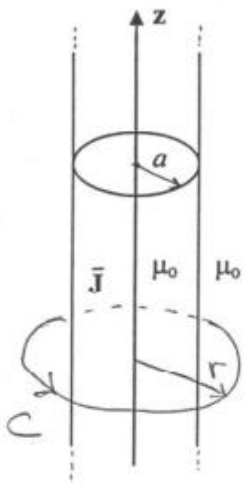


Figure 3. The geometry of the Q-3.

a) Find $\bar{J}$.

b) Find $\bar{B}$ for $r > a$.



a) $\nabla \times \bar{B} = \mu_0 \bar{J}$

$$\nabla \times \bar{B} = \hat{a}_z \frac{1}{r}\frac{\partial}{\partial r}(r B_\phi)$$

$$= \hat{a}_z \frac{1}{r} \underbrace{\frac{\partial}{\partial r}(\mu_0 G r^4)}_{4\mu_0 K r^3}$$

$$= \hat{a}_z \, 4\mu_0 G r^2 = \mu_0 \bar{J}$$

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$$\therefore \boxed{\bar{J} = 4 G r^2 \hat{a}_z} \quad r < a \quad (A/m^2)$$

b) $\oint_C \bar{B} \cdot d\bar{l} = \mu_0 I_{encl}$, $\bar{B} = B_\phi(r)\hat{a}_\phi$

$$B_\phi 2\pi r = \mu_0 \int_0^{2\pi}\int_0^a 4 G r^2 \hat{a}_z \cdot \overbrace{r dr d\phi \, \hat{a}_z}^{d\bar{s}}$$

$$= \mu_0 4 G (2\pi) \int_0^a r^3 dr$$

$$B_\phi \cancel{2\pi} r = \mu_0 \cancel{4} G (\cancel{2\pi}) \frac{a^4}{\cancel{4}}$$

$$\boxed{B_\phi = \frac{\mu_0 G a^4}{r}} \quad r > a \quad (T)$$

(field of a line current)

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Q-4) There exists an infinitely long coaxial structure shown in the figure below. The inner conductor has a radius a and the outer conducting shell has a radius b. The thickness of the outer conductor is ignored as it is very small. Between two conductors, there is a magnetic material with permeability

$$\mu(r) = \mu_0 \ln r$$

Assume that the current I is distributed uniformly over the cross-section of the inner conductor whereas it flows on the surface of the outer conductor. Note that, also in the outer conductor I current is flowing.

a) Find the magnetic field intensity vector $\vec{H}$ and the magnetic flux density vector $\vec{B}$ everywhere ($r < a, a < r < b, r > b$).

b) Find the self-inductance **per unit length** of the coaxial structure.

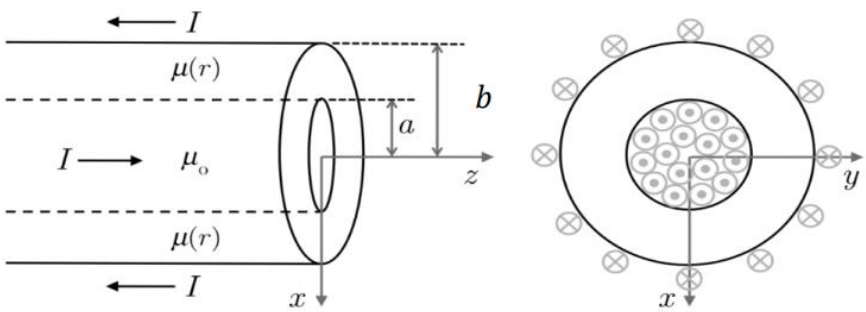
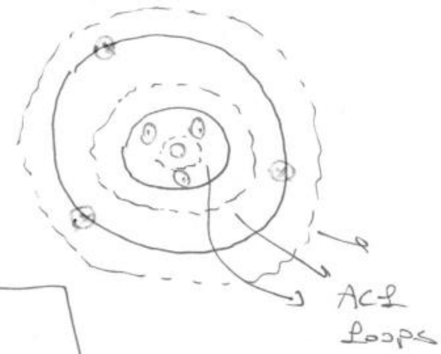


Figure 4. The geometry of the Q-4.

(a) $\oint \vec{H} \cdot d\ell = I_{enclosed} \quad (*)$



for $b > r > a$ $\quad I_{enclosed} = I$

" $\quad r > b$ $\quad$ " $= 0$

for $r < a$ $\quad I_{enclosed} = I\left(\frac{r}{a}\right)^2$

$$= \frac{I\pi r^2}{\pi a^2}$$

Due to symmetry: $\vec{H} = H_\phi \vec{a}_\phi$ is expected

from $(*)$

$$H_\phi \, 2\pi r = I_{enclosed}$$

$$H_\phi = \frac{I_{enclosed}}{2\pi r}$$

$$\longrightarrow \begin{cases} \vec{H} = \dfrac{Ir}{2\pi a^2} \vec{a}_\phi, & r < a \\ \vec{H} = \dfrac{I}{2\pi r} \vec{a}_\phi, & a < r < b \\ \vec{H} = 0, & r > b \end{cases}$$

One ca find easily:

$\vec{B} = \mu \vec{H}$ $\quad$ ($\mu = \mu_0$ or $\mu = \mu(r)$ depending on the region)

$$\vec{B} = \begin{cases} \dfrac{Ir\mu_0}{2\pi a^2}, & r < a \\ \dfrac{I\mu_0 \ell n r}{2\pi r}, & a < r < b \\ 0, & r > b \end{cases}$$

(b)

Let us find $w_m = \frac{1}{2}\int_{v} \vec{B}\cdot\vec{H}\,dv$

$$w_m = w_{m,in} + w_{m,out}$$

$$w_{m,in} = \frac{1}{2}\int_{v,in} \vec{B}\cdot\vec{H}\,dv = \frac{1}{2}\int_0^1\int_0^{2\pi}\int_0^a \frac{\mu_0 I^2 r^2}{(2\pi a^2)^2}\underbrace{r\,d\phi\,dr\,dz}_{dv}$$

$$= \frac{\mu_0 I^2 2\pi}{2(2\pi a^2)}\frac{r^4}{4}\Big|_0^a \Rightarrow \frac{\mu_0 I^2}{16\pi}$$

$$w_{m,in} = \frac{1}{2}L_{in} I^2 \Rightarrow L_{in} = \frac{\mu_0}{8\pi}$$

for $r>b$ $w_m=0$ since B & H are zero.

for $a<r<b$

$$w_{m,out} = \frac{1}{2}\int_0^1\int_0^{2\pi}\int_a^b \frac{I}{2\pi r}\,\frac{I\mu_0 \ln r}{2\pi \cancel{r}}\,\cancel{r}\,dr\,d\phi\,dz$$

$$= \frac{1}{\cancel{2}}\,\frac{I^2\mu_0\,\cancel{2\pi}}{4\pi^{\cancel{2}}}\int_a^b \frac{\ln r}{r}\,dr$$

$\boxed{r>0}$

$\ln r = u$

$\frac{1}{r}dr = du$

$\int u\,du \Rightarrow \frac{u^2}{2} = \frac{\ln^2 r}{2}$

$$= \frac{I^2\mu_0}{4\pi}\left[\frac{\ln^2 b - \ln^2 a}{2}\right]$$

$$= \frac{I^2\mu_0}{8\pi}\left[\ln^2 b - \ln^2 a\right]$$

$$w_{m,out} = \frac{1}{2}\,L_{out}\,\cancel{I}^2 = \frac{\cancel{I}^2\mu_0\left[\ln^2 b - \ln^2 a\right]}{8\pi}$$

$$L_{out} = \frac{\mu_0}{4\pi}\left[\ln^2 b - \ln^2 a\right]$$

$$L = L_{in} + L_{out} \Rightarrow L = \frac{\mu_0}{8\pi} + \frac{\mu_0}{4\pi}\left[\ln^2 b - \ln^2 a\right] \quad [H]$$

(self-inductance)

Q-5) In Figure 5, the magnetic circuit is given. What should be the value of d in meters to have $B = 0.5\ T$ at the air spacing d?

The cross-section of the materials $S_1 = S_2 = 10cm^2$

The B-H relation is given in Figure 6.

μ_{1r} stands for the silicon sheet for the material at the left (l_1)

μ_{2r} stands for the cast iron for the material at the right (l_2)

$$l_1 = 0.5 \text{ meters}, l_2 = 1 \text{ meter}, w_1 = 500, w_2 = 200, I_1 = 10\text{A}, I_2 = 7\text{A}, \mu_0 = 12{,}5\cdot 10^{-7}\,\frac{\text{H}}{\text{m}}$$

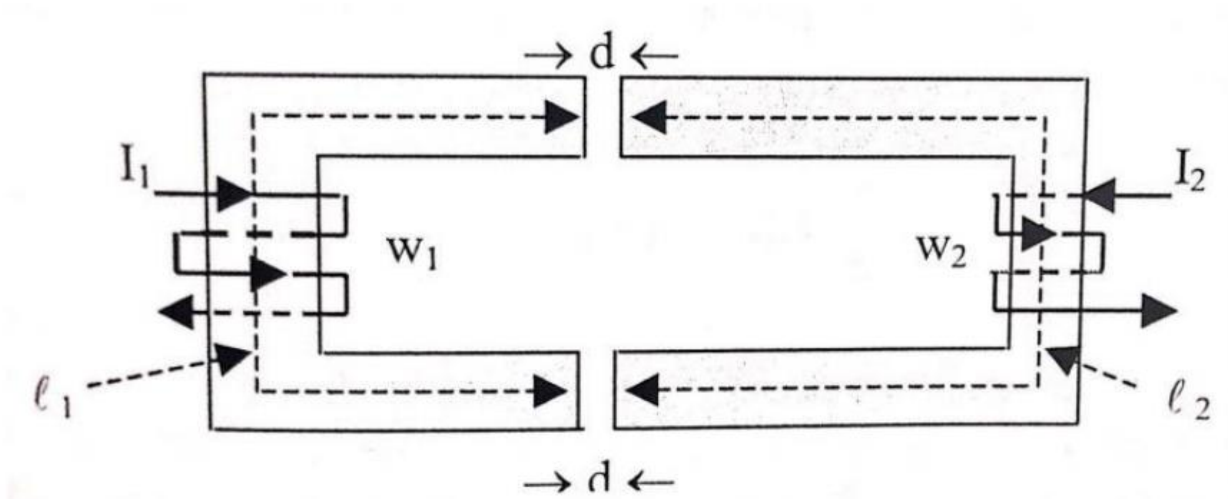


Figure 5. The Figure for Q-5.

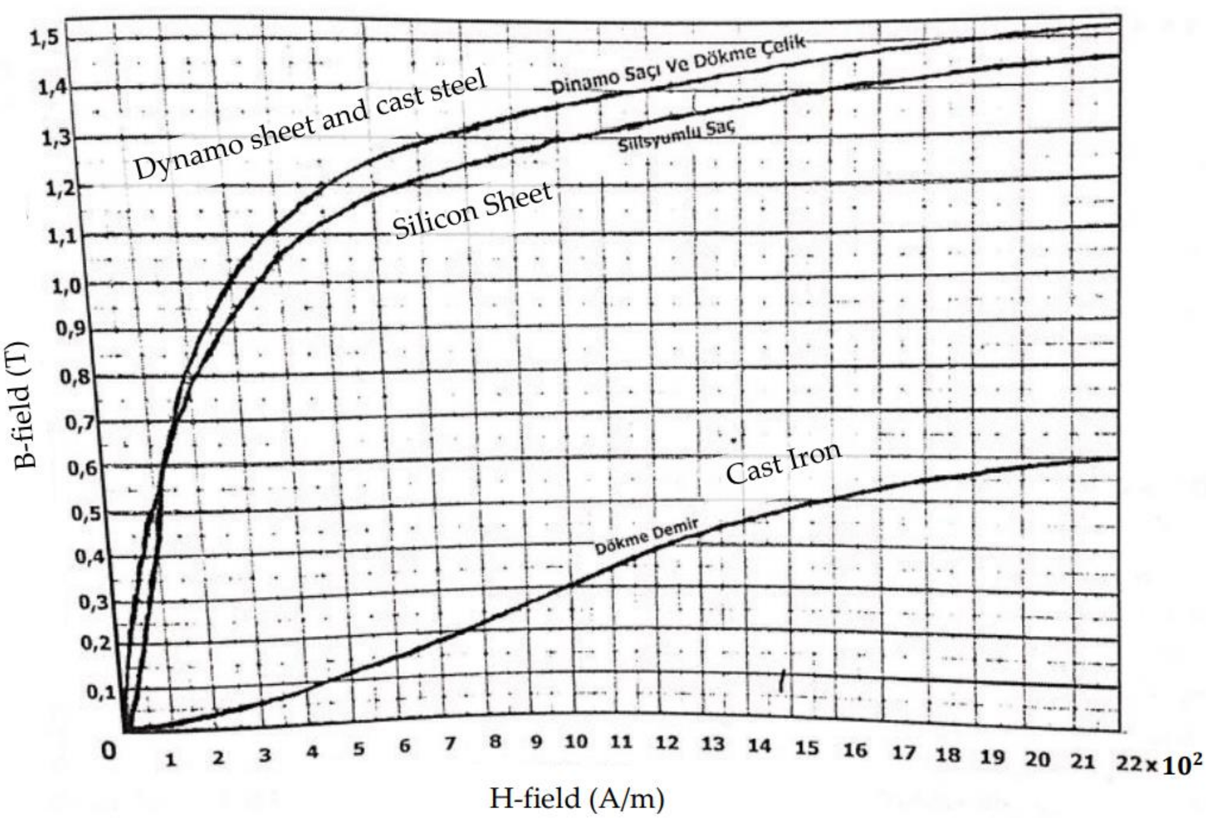


Figure 6. The chart for Q-5.

$$\Phi_p = B_p S \qquad p = 1, 2, d$$

→ flux is constant.

→ Due to having the same cross section,

$$\phi_p = B_p S \Rightarrow \boxed{B_1 = B_2 = B_d} = \frac{\phi_p}{S} = 0.5 \text{ T}$$

↑ given

→ look at the graph and

$H_1 = 100$ A/m

$H_2 = 1600$ A/m

and $\quad H_d = \dfrac{B_d}{\mu_0} \cong 4.10^5$ A/m

Ampere's Law $\Rightarrow$

$$\int H \cdot dl = I_{enclosed} \Rightarrow$$

$$\int_{\ell_1} \vec{H_1} \cdot d\vec{\ell_1} + 2\int_{d} \vec{H_d} \cdot d\vec{\ell_d} + \int_{\ell_2} \vec{H_2} \cdot d\vec{\ell_2} =$$

$$= H_1 \ell_1 + 2H_d d + H_2 \ell_2 = \overset{*}{w_1 I_1 - w_2 I_2}$$

→ * (−) sign due to opposite direction of flux created by different coil!

FOR THIS QUESTION, WHILE GRADING, DIFFERENCES IN NUMERICAL RESULTS SHOULD BE TOLERATED. THE CORRECT APPROACHES, AND UNDERSTANDING THE CONCEPT SHOULD BE REWARDED

$$d = \frac{w_1 I_1 + w_2 I_2 - H_1 e_1 - H_2 e_2}{2H_d}$$

$$\Rightarrow \underline{2{,}44 \text{ MM}}$$