

Diferansiyel Denklemler, bilinmeyen bir fonksiyonun tozlu türlerini içeren denklemlerdir.

$$\left. \begin{array}{c} \text{Q} \\ | \\ | \\ | \end{array} \right\} n \quad -mg = \rho h \cdot \frac{d^2 h(t)}{dt^2}$$

$$\frac{d^2 h}{dt^2} = -g \rightarrow \text{dgs denklem}$$

$$\int \frac{dh}{dt^2} dt = \int -g dt$$

$$\int \frac{dh}{dt} dt = \int (-g t + c_1) dt$$

$$h = h(t) = -g \cdot \frac{t^2}{2} + c_1 t + c_2 \rightarrow \text{çözüm. fonksiyonu}$$

Birinci basamak, lineer dgs. denk - integral formu

$$\text{Görünüm} = y'(x) + P(x)y(x) = Q(x)$$

$$(y(x)I(x))' = y'(x) \overset{3}{\cancel{I(x)}} + I'(x)y(x)$$

$$I(x)y'(x) + I(x)P(x)y(x) = I(x)Q(x)$$

$$I'(x) = I(x)P(x) \quad \int \frac{I'(x)}{I(x)} dx = \int P(x) dx$$

$$\ln I(x) = \int P(x) dx \rightarrow I(x) = e^{\int P(x) dx}$$

$I(x) \rightarrow$  integral cəsəbi

$$\int (y(x) \cdot I(x))' dx = \int I(x) Q(x) dx$$

$$y(x) I(x) = \int I(x) Q(x) dx + c$$

1- integral cəsəbini bul.

2- Dərəkləminin her iki tərəfindən  $I(x)$  ləğib cəsəb

3- İki tərəfin integralini al

$$4- I(x) y(x) = \int I(x) Q(x) dx + c$$

5-  $y(x)$  ləğib bul.

Örnək

$y'(x) + x y(x) = x^3$  diff. deq. genel həcmiñiñ bulunus.

$$y'(x) + P(x) y(x) = Q(x) \quad P(x) = x$$

$$I(x) = e^{\int P(x) dx} \quad I(x) = e^{\frac{x^2}{2}}$$

$$e^{\frac{x^2}{2}} y'(x) + e^{\frac{x^2}{2}} x y(x) = e^{\frac{x^2}{2}} x^3 \quad \frac{x^2}{2} = u$$

$$\int (y e^{\frac{x^2}{2}})' dx = \int e^{\frac{x^2}{2}} x^3 dx \quad \frac{du}{dx} = x$$

$$y e^{\frac{x^2}{2}} = \int e^u \cdot 2u du \rightarrow 2(u e^u - e^u)$$

$$\int u du = u^2 - \int v du$$

Hətirlatma

$$\int x \cdot e^x dx \rightarrow x = u \quad e^x \cdot dx = dv \quad du = dx$$

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x \cdot du = x \cdot e^x - e^x + c$$

$$2 \left( \frac{x^2}{2} e^{\frac{x^2}{2}} - e^{\frac{x^2}{2}} \right) + c$$

$$y = e^{\frac{x^2}{2}} = x^2 e^{\frac{x^2}{2}} - 2e^{\frac{x^2}{2}} + c$$

$$y = y(x) = x^2 - 2 + c e^{-\frac{x^2}{2}}$$

Örnək

$$(\cos x) y' + (\sin x) y = \cos^5 x \sin x \quad y(0) = 2$$

$$y' + \tan x y = \cos^4 x \sin x$$

$$I(x) = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x = \frac{1}{\cos x}$$

$$y' \sec x + \tan x \sec x y = \cos^3 x \sin x$$

$$\int (y \cdot \sec x)' dx = \int \cos^3 x \sin x dx \quad \cos x = u \quad -\sin x dx = du$$

$$y \cdot \sec x = \int -u^3 du = -\frac{u^4}{4} + c$$

$$y(x) = -\frac{\cos^5 x}{4} + c \cos x \quad \text{Özel Çözüm}$$

$$2 = -\frac{1}{4} + c \Rightarrow c = \frac{9}{4}$$

$$y_{\text{özel}}(x) = -\frac{\cos^5 x}{4} + \frac{9}{4} \cos x \quad \text{Özel Çözüm}$$

Birinci basamak, ayrılanabilir dif. denklemler

- Birinci dereceden denklemler hem linear, hem ayrılanabilirdir.
- Integral sabitini etkileyenin inatına

Örnek

$$y''(x) = \frac{1}{2} y(x) \quad \text{genel çözüm bulunuz.}$$

$$\frac{dy}{dx} = \frac{1}{2} y \rightarrow 2dy = ydx \rightarrow \frac{2}{y} dy = dx$$

isirilen form

$$\int \frac{2}{y} dy = \int dx \rightarrow 2 \ln|y| = x + C$$

$$y = \underbrace{e^{x/2}}_{\text{bu da bir sabitdir.}} \cdot e^{x/2} \rightarrow y = A e^{x/2}$$

- x ve y'leri bir terimde toplayıp integral al.

$$y'(x) + xy = x^3 \quad \text{genel çözüm bulunuz.}$$

integral yöntemle

$$\frac{dy}{dx} + xy = x^3 \rightarrow \frac{dy}{dx} = x^3 - xy$$

ayrılanıyor

$$P(x) = x \rightarrow I(x) = e^{\int x dx} = e^{x^2/2}$$

$$e^{x^2/2} y' + e^{x^2/2} \cdot xy = e^{x^2/2} x^3$$

$$y \cdot e^{x^2/2} = \int e^{x^2/2} x^3 dx \quad \frac{x^2}{2} = u$$

$$\frac{du}{dx} = x$$

$$\int e^u \cdot 2u du = 2(u e^u - e^u) + C$$

$$y = x^2 - 2 + C e^{-x^2/2}$$

Örnek

$$3x - 6y \sqrt{x^2 + 1} y'(x) = 0 \quad y(0) = 4 \quad \text{başlangıç değer problemini çözünüz}$$

$$6y \sqrt{x^2 + 1} dy = 3x dx \rightarrow \frac{2}{3} y dy = \frac{3x}{\sqrt{x^2 + 1}} dx \quad x^2 + 1 = u$$

$$\frac{du}{dx} = 2x$$

$$\frac{2y^2}{3} = \int \frac{1}{2\sqrt{u}} \rightarrow y^2 = u^2 + C$$

$$y^2 = \sqrt{x^2 + 1} + C$$

$$16 = 1 + c \rightarrow c = 15$$

$$y^2 = \sqrt{x^2 + 1} + 15 \rightarrow y = \sqrt{x^2 + 1 + 15} \text{ özel çözüm}$$

### Tan: Diferansiyel Denklemler

$$M(x,y) + N(x,y)y'(x) = 0 \rightarrow \text{Genel Form}$$

linear. depl.

$$f(x,y) = c$$

$$\frac{\partial f(x,y)}{\partial x} = 0 \quad \frac{\partial F(x,y)}{\partial x} + \frac{\partial F(x,y)}{\partial y} \frac{dy}{dx} = 0$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$F_{xy} = F_{yx}$$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad (\text{sart}). \text{ adm!}$$

$$\int \frac{\partial F(x,y)}{\partial x} dx = \int M(x,y) dx \quad \text{adm. 2}$$

$$F(x,y) = \int M(x,y) dx + h(y) \quad \text{adm. 3}$$

$h(y)$  yi yerine yerlestirince harsimizde cihaz esitlik  
genel cozulmusdir.

Ornek

$$2xy - 3x^2 + (2y + x^2 + 1)y'(x) = 0 \quad \text{genel cozumminin bul.}$$

$$M(x,y) \quad N(x,y)$$

$$M_y = N_x \quad 2x = 2x \quad \checkmark$$

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \rightarrow \int \frac{\partial F(x,y)}{\partial x} dx = \int 2xy - 3x^2 dx$$

$$F(x,y) = x^2y - 3x^3 + h(y)$$

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) \rightarrow x^2 + h'(y) = 2y + x^2 + 1$$

$$\int h'(y) dy = \int 2y + 1 dy$$

$$h(y) = y^2 + y + c_1$$

$$F(x,y) = x^2y - 3x^3 + y^2 + y = c$$

### Homogen Denklemler

$$\frac{dy}{dx} = f(x,y) \rightarrow \frac{y}{x} dx \text{ denismi uygulayince}\newline \text{saytabilir dij denk. oluyor}$$

Ornek  
 $(xy + y^2 + x^2) - x^2 y' = 0$  genel çözüm?

$$\frac{dy}{dx} = \frac{xy + y^2 + x^2}{x^2} = \frac{1}{x} + \left(\frac{y}{x}\right)^2 + 1 = \frac{1}{x} + v^2 + 1$$

$$\frac{y}{x} = v \text{ diye} \quad y = vx \quad \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$x + x \frac{dv}{dx} = v + v^2 + 1 \rightarrow \text{ayrılabilir dif. hali} \text{değiş.}$$

$$\int \frac{1}{v^2 + 1} dv = \int \frac{1}{x} dx \rightarrow \arctan v = \ln|x| + C$$

$$v = \tan(\ln|x| + C)$$

$$y = x \cdot \tan(\ln|x| + C)$$

Bernoulli Denklemi:

$$\text{Genel form} \rightarrow y' + P(x)y = Q(x)y^n \rightarrow y' + P(x)y = Q(x)$$

$$V = y^{1-n} \text{ diye} \rightarrow y' = V' \cdot V^{1-n}$$

Sonra integral çarpanı metodıyla çözümlenir.

Ornek  
 $y' - 5y = -\frac{5}{2}xy^3$  genel çözüm?

$$V = y^{-2} \rightarrow \frac{dV}{dx} = -2y^{-3} \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{1}{2}y^3 \frac{dV}{dx}$$

$$\rightarrow y^{-3}y' - 5y^{-2} = -\frac{5}{2}x$$

$$y^{-3} \left[ -\frac{1}{2}y^3 \frac{dV}{dx} \right] - 5y^{-2} = -\frac{5}{2}x$$

$$-\frac{1}{2}V' - 5V = -\frac{5}{2}x \rightarrow V' + 10V = 5x$$

1. basamak  
lin. dif. denk.

$$\int P(x)dx \quad \int 10dx \quad 10x$$

$$I(x) = C \rightarrow C = e^{10x}$$

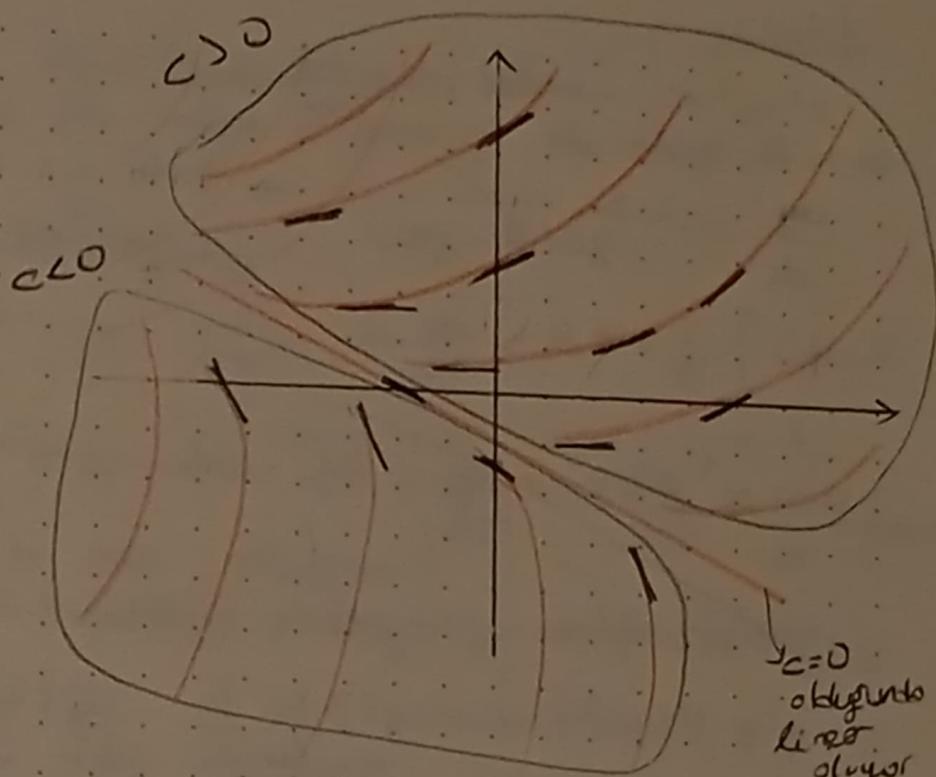
$$\int e^{10x} dx = \int 5x e^{10x} dx$$

$$\int e^{10x} dx = \frac{x}{2} e^{10x} - \frac{1}{20} e^{10x} + C$$

$$\begin{array}{|c|c|} \hline & 5x + e^{10x} \\ \hline 5 & \cancel{\frac{1}{10} e^{10x}} \\ 0 & \cancel{\frac{1}{100} e^{10x}} \\ \hline \end{array}$$

$$y^{-2} = \frac{x}{2} - \frac{1}{20} + C e^{-10x} \quad \text{Genel çözüm}$$

$(x, y)$	$y'$
$(0, 0)$	0
$(1, 0)$	1
$(0, 1)$	2
$(1, 1)$	3
$(-1, 0)$	-1
$(-1, 1)$	1
$(-2, 0)$	-2
$(-2, 1)$	0



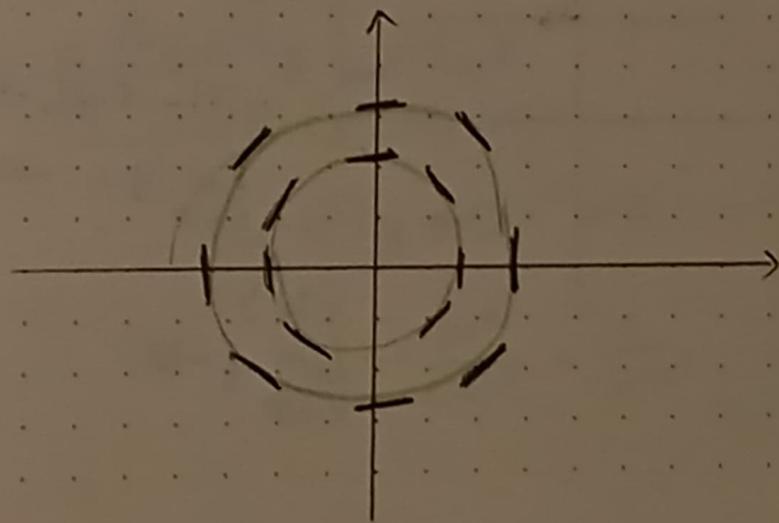
$$\cancel{-\frac{1}{2}x + 2ce^{2x} = x + 2\left(-\frac{1}{2}x - \frac{1}{4} + ce^{2x}\right)}$$

$$0 = 0 \checkmark$$

Onel

- $y' = -\frac{x}{y}$  dif. denk igin
  - ogrultu atari giz
  - başlı güm eğitleri giz
  - $x^2 + y^2 = c$  ışadesinin genel çözümü old. ogrula
- $y(-3) = -4$  bas. deş. ile verilen bas. deş. p. cordim ve temsil etti. güm eğitleri gösterdim.

$(x, y)$	$y'$
$(0, 0)$	$\tan^{-1} \frac{1}{2}$
$(1, 0)$	$\infty$
$(1, 1)$	-1
$(0, 1)$	0
$(-1, 0)$	$\infty$
$(-1, 1)$	1
$(-1, -1)$	-1



$$x^2 + y^2 = c \rightarrow 2x + 2y \frac{dx}{dy} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \rightarrow 3+16 = c = 25$$

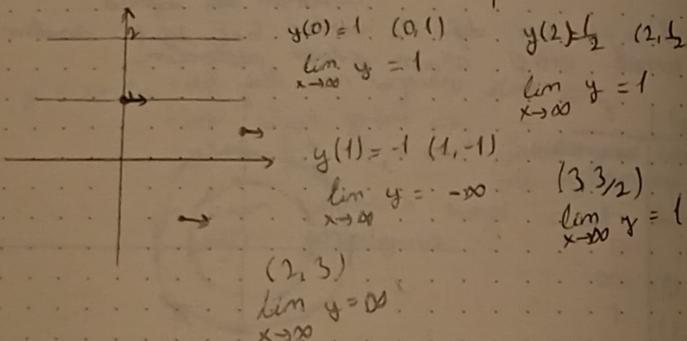
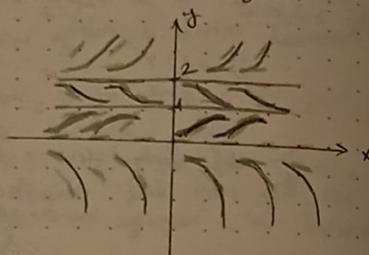
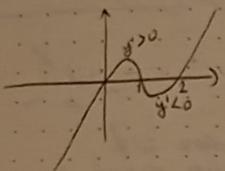
$$x^2 + y^2 = 25 \rightarrow \text{circle centered at the origin}$$

$$y' = y^3 - 3y^2 + 2y \quad \text{dsg: dekh iain}$$

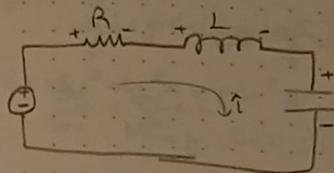
- dognultu dekh ve basit çözümüne  
 $y(0) = 1 \quad y(1) = -1 \quad y(2) = \frac{1}{2}$

$y(3) = \frac{3}{2} \quad y(2) = 3$  nondecreasing  
 baslangicde 0'da x.  $x \rightarrow \infty$  zoom out  
 limit devresim. yorumlayin

$$y=0, y=1, y=2$$



Week 1  
 Consider the  $R-L-C$  circuit given below.  
 Try to express the voltage in terms of the charge  $Q$  on cap.

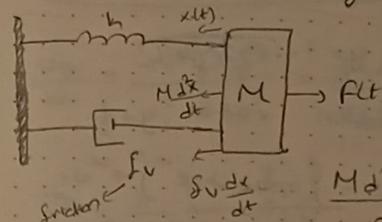


Using KVL

$$L \frac{di}{dt} + R i + \frac{1}{C} \int idt = V$$

$$i = \frac{dq}{dt} \rightarrow L \frac{dq}{dt} + R q + \frac{1}{C} q = V$$

Consider the following mass-spring-damper system



Newton's Law

$$m \frac{d^2x}{dt^2} = \sum F$$

$$m \frac{d^2x(t)}{dt^2} + h \frac{dx(t)}{dt} + kx(t) = F(t)$$

Example

$$(4+t^2) \frac{dy}{dt} + 2ty = 4t$$

$$(4+t^2) \frac{dy}{dt} + 2ty = \underbrace{\frac{d}{dt}[(4+t^2)y]}_{L.H.S.} = L.H.S.$$

$$(4+t^2)y = 2t^2 + C$$

$$y = \frac{2t^2}{4+t^2} + \frac{C}{4+t^2}$$

Week 2

$$\text{Solve } \frac{dy}{dx} = \frac{x^2}{1-y^2}$$

$$\frac{dy}{dx} = \frac{3x^2 + 6x + 2}{2(y-1)}; y(0)=1$$

$$-x^2 + (1-y^2) \frac{dy}{dx} = 0 \quad (2(y-1)dy = (3x^2 + 6x + 2)dx)$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$\int -x^2 dx + \int (1-y^2) dy = C_1$$

$$x=0, y=-1$$

$$1+2 = C_1 = 3$$

$$\frac{x^3}{3} + 2x^2 + \frac{y^3}{3} = C_1$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3$$

$$-x^3 + 3y - y^3 = C$$

$$y = 1 - \sqrt[3]{x^3 + 2x^2 + 2x + 3}$$

$$xy^2 dx + e^y dy = 0 \quad \text{initial condition } x \rightarrow \infty, y \rightarrow \frac{1}{2}$$

$$\int x e^y dx + \int y^2 dy = 0 \quad uv - \int v du = -x e^{-x} \int -e^{-x} dx$$

$$= -x e^{-x} - e^{-x}$$

$$dx/dx = 1$$

$$v = -e^{-x}$$

$$-x e^{-x} - e^{-x} - y^{-1} = C$$

$$-\lim_{x \rightarrow \infty} \frac{1}{e^x} - \lim_{x \rightarrow \infty} \frac{1}{e^x} = -\lim_{y \rightarrow 0} \frac{1}{y} = C$$

$$x e^{-x} + e^{-x} + \frac{1}{y} = 2$$

$$\frac{dy}{dx} = \cos^2 x \cos y$$

$$\{ \ln(\sec y + \tan y) = \frac{51\pi x}{4} + \frac{\pi}{2} \}$$

$$\int \sec y dy = \int \cos^2 x dx = \frac{\cos 2x}{2} dx + \frac{dx}{2}$$

$$\frac{dy}{dx} = \frac{x^2 + xy^2}{x^2} \rightarrow \frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$\text{define: } y = vx$$

$$\frac{y}{x} = v \rightarrow \frac{du}{dx} x + u = 1 + v + v^2$$

$$\frac{dy}{dx} = \frac{du}{dx} x + v \quad \int \frac{du}{1+u^2} = \int \frac{dx}{x}$$

$$\arctan v = \ln|x| + C$$

$$\text{so then } \frac{y}{x} = \ln|x| + C$$

$$\frac{dy}{dx} = \frac{4x^2 y - y^3}{x^3 + 2x y^2} = \frac{y(4x^2 - y^2)}{x(x^2 + 2y^2)} = \frac{y x^2 (4 - \frac{y^2}{x^2})}{x \cdot x^2 (1 + \frac{2y^2}{x^2})} \quad \text{W.L.O.G.}$$

$$\frac{dy}{dx} = \frac{y \left( 4 - \frac{y^2}{x^2} \right)}{\left( 1 + \frac{2y^2}{x^2} \right)} \quad v = \frac{y}{x} \quad \frac{y}{x} = v \quad \text{computation}$$

$$\frac{dy}{dx} = \frac{du}{dx} x + v$$

$$\frac{dv}{dx} x + v = v \left( \frac{4 - v^2}{1 + 2v^2} \right) = \frac{4v - v^3}{1 + 2v^2}$$

$$\begin{cases} (A+B)v^2 + Cv + 3A \\ -2 & 0 & 1 \end{cases}$$

$$\frac{dv}{dx} = \frac{4v - v^3 - v + 2v^3}{1 + 2v^2} = \frac{v^3 + 3v}{1 + 2v^2}$$

$$A = \frac{1}{3}$$

$$\frac{1-2v^2}{\sqrt{v^2+3}} dv = \frac{dx}{x} \rightarrow \frac{A}{v} + \frac{Bv + C}{v^2 + 3} = \frac{1-2v^2}{v(v^2+3)}$$

$$B = -\frac{7}{3}$$

$$C = 0$$

$$\frac{1}{3} \int \frac{dv}{v} - \frac{7}{3} \int \frac{v dv}{v^2 + 3} = \int \frac{dx}{x} \rightarrow \frac{1}{3} \ln|v| - \frac{7}{6} \ln|v^2 + 3| = \ln|x| + C$$

$$\frac{v^4 + 3}{v^2 + 3} = u \quad \frac{v dv}{v^2 + 3} = \frac{du}{2u} \rightarrow \frac{1}{2} \ln|u|$$

$$C' = \ln C \quad \text{let's say}$$

$$\ln \left[ \frac{|y|^{1/3}}{|y^2+3x^2|^{1/6}} \right] = \ln|x| + \ln C \quad \frac{|y|^{1/3}}{|y^2+3x^2|^{1/6}} = C|x|$$

$$\frac{|y|^{1/3}}{|y^2+3x^2|^{1/6}} = C|x| \quad \frac{|y|^{1/3} x^2}{(y^2+3x^2)^{1/6}} = C|x|$$

Exact equations and integrating factors.

$2x + y^2 + 2xy y' = 0 \rightarrow$  Neither linear nor separable

$$\Psi(x,y) = x^2 + xy^2 \quad \frac{\partial \Psi}{\partial x} = 2x + y^2 \quad \frac{\partial \Psi}{\partial y} = 2xy$$

$$\frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{dy}{dx} = 0 \quad \frac{\partial \Psi(x,y)}{\partial x} = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial x}$$

Chain Rule

$$\frac{\partial \Psi(x,y)}{\partial x} = 0 \quad \frac{d}{dx}(x^2 + xy^2) = 0$$

Let's say:  $M(x,y) + N(x,y)y' = 0$

$$\text{Suppose } \frac{\partial \Psi}{\partial x}(x,y) = M(x,y) \quad \frac{\partial \Psi}{\partial y}(x,y) = N(x,y)$$

$$(y \cos x + 2x e^y) + \underbrace{(\sin x + x^2 e^y - 1)}_{M} y' = 0$$

$$\underbrace{(y \cos x + 2x e^y)}_{N} + (\sin x + x^2 e^y - 1) y' = 0$$

$$M_y = \cos x + 2x e^y = N_x = \cos x + 2x e^y$$

There is a function  $\Psi(x,y)$  such that:

$$\Psi_x(x,y) = y \cos x + 2x e^y$$

$$\Psi_y(x,y) = \sin x + x^2 e^y - 1 \quad \text{Integrate one of them}$$

$$\Psi(x,y) = y \sin x + x^2 e^y + h(y)$$

$$\text{We must determine } h(y): \Psi_y = N = \sin x + x^2 e^y + h'(y)$$

$$h(y) = -y + c \quad = \sin x + x^2 e^y - 1$$

$$\Psi(x,y) = y \sin x + x^2 e^y - y + c$$

must be constant  $y \sin x + x^2 e^y - y = C \rightarrow$  implicit solution

$$(y e^{2xy} + x) dx + (b x e^{2xy}) dy = 0$$

$$M_y = e^{2xy} + 2xy e^{2xy} \quad N_x = b e^{2xy} + 2y b x e^{2xy}$$

$$\Psi_x = y e^{2xy} + x \quad \Psi_y = x e^{2xy} \quad b = 1$$

$$\Psi_x = \frac{2y e^{2xy}}{2} + h'(x) \leftarrow \Psi = \frac{x e^{2xy}}{2} + h(x)$$

$$h'(x) = x \quad h(x) = \frac{x^2}{2} + C_1 \quad \Psi(x,y) = e^{2xy} + x^2 + C$$

$$\left(\frac{y}{x} + 6x\right)dx + (\ln x + 2)dy = 0$$

$$M_y = \frac{1}{x} \quad N_x = \frac{1}{x}$$

$$\Psi_x = \frac{y}{x} + 6x \quad \Psi_y = \ln x - 2$$

$$\Psi(x,y) = y \ln x + \frac{6x^2}{2} + h(y)$$

$$\Psi_y = \ln x + h'(y) = \ln x - 2 \quad h'(y) = -2$$

$$h(y) = -2y + 4 \quad \Psi = y \ln x + 3x^2 - 2y + C_1$$