EEF 271E
Probability and Statistics

Week #3

INDEPENDENT EVENTS

Two events A and B are **independent** if the occurrence or nonoccurrence of either one does not affect the probability of the occurrence of the other.

A and B are independent if $P(B|A) = P(B)$ and $P(A|B) = P(A)$.

Now, if we substitute $P(B)$ for $P(B|A)$, we get $P(A \cap B) = P(A) \cdot P(B|A)$

$$= P(A) \cdot P(B)$$

DEFINITION Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A) \cdot P(B)$$

EXAMPLE

A coin is tossed three times and the eight possible outcomes, HHH, HHT, HTH, THH, HTT, THT, TTH, and TTT, are assumed to be equally likely. If A is the event that a head occurs on each of the first two tosses, B is the event that a tail occurs on the third toss, and C is the event that exactly two tails occur in the three tosses, show that

(a) events A and B are independent;

(b) events B and C are dependent.

Solution

$$A = \{\text{HHH, HHT}\}$$

$$B = \{\text{HHT, HTT, THT, TTT}\}$$

$$C = \{\text{HTT, THT, TTH}\}$$

yields $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$,

$P(C) = \frac{3}{8}$,

$$A \cap B = \{\text{HHT}\} \qquad P(A \cap B) = \tfrac{1}{8},$$

and

$$B \cap C = \{\text{HTT, THT}\} \qquad P(B \cap C) = \tfrac{1}{4}.$$

(a) Since $P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} = P(A \cap B)$, events A and B are independent.

(b) Since $P(B) \cdot P(C) = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16} \neq P(B \cap C)$, events B and C are not independent.

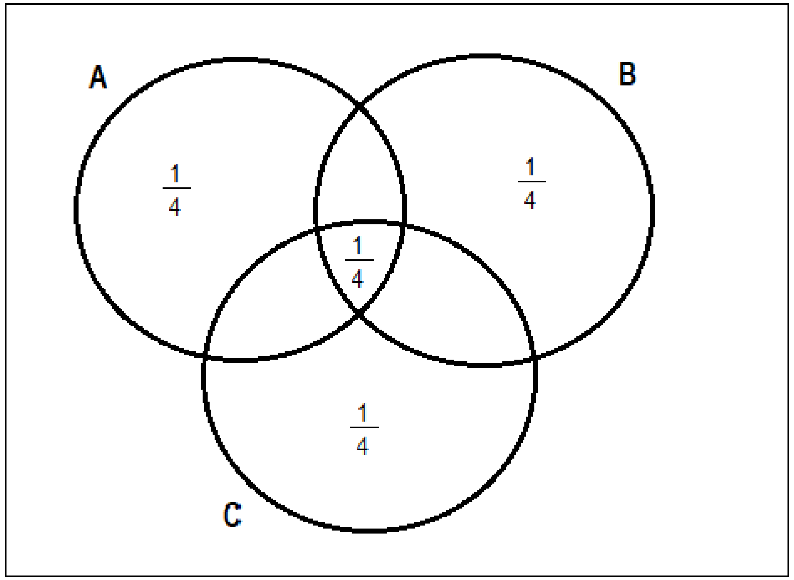
Definition A collection of events $A_1, A_2, A_3, \ldots$ are *independent* if

$$P(A_{i_1} \cap \cdots \cap A_{i_j}) = P(A_{i_1}) \cdots P(A_{i_j})$$

for *any* finite subcollection $A_{i_1}, \ldots, A_{i_j}$ of distinct events.

EXAMPLE

Figure shows a Venn diagram with probabilities assigned to its various regions.



Verify that A and B are independent, A and C are independent, and B and C are independent, but A, B, and C are not independent.

Solution

As can be seen from the diagram,

$P(A) = P(B) = P(C) = \frac{1}{2}$,

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \frac{1}{4}$$

and $P(A \cap B \cap C) = \frac{1}{4}$.

$$P(A) \cdot P(B) = \frac{1}{4} = P(A \cap B)$$

$$P(A) \cdot P(C) = \frac{1}{4} = P(A \cap C)$$

$$P(B) \cdot P(C) = \frac{1}{4} = P(B \cap C)$$

but

$$P(A) \cdot P(B) \cdot P(C) = \frac{1}{8}$$

$$\neq P(A \cap B \cap C)$$

Random Variables and probability Distributions

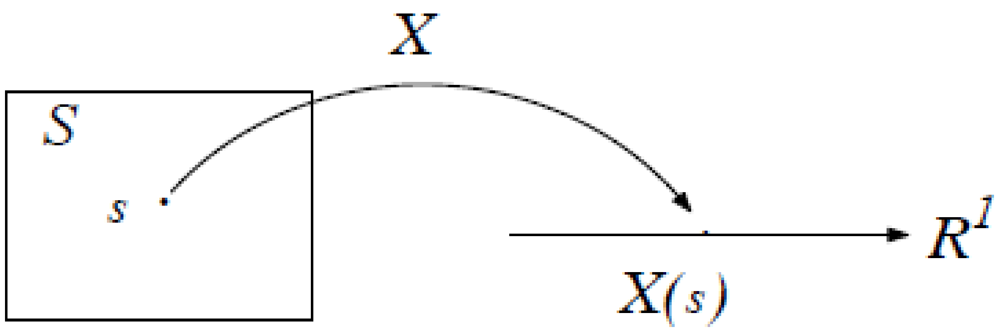
This chapter is concerned with the definitions of random variables, distribution functions, probability functions, density functions, and the development of the concepts necessary for carrying out calculations for a probability model using these entities.

Random Variables

Intuitively, a random variable assigns a numerical value to each possible outcome in the sample space.

For example, if the sample space is {rain, snow, clear}, then we might define a random variable X such that $X = 3$ if it rains, $X = 6$ if it snows, and $X = -2.7$ if it is clear.

Definition A *random variable* is a function from the sample space S to the set R^1 of all real numbers.



A random variable X as a function on the sample space S and taking values in R^1.

The point is, we can define random variables any way we like, as long as they are functions from the sample space to R^1.

we shall always denote random variables by capital letters and their values by the corresponding lowercase letters;

for instance, we shall write x to denote a *value* of the random variable X.

Instead of "random variable," the terms "chance variable," "stochastic variable," and "variate" are also used in some books.

EXAMPLE

A balanced coin is tossed four times. List the elements of the sample space that are presumed to be equally likely, and the corresponding values x of the random varibale X, the total number of heads.

Element of sample space	*Probability*	x
HHHH	$\frac{1}{16}$	4
HHHT	$\frac{1}{16}$	3
HHTH	$\frac{1}{16}$	3
HTHH	$\frac{1}{16}$	3
THHH	$\frac{1}{16}$	3
HHTT	$\frac{1}{16}$	2
HTHT	$\frac{1}{16}$	2

Element of sample space	*Probability*	x
HTTH	$\frac{1}{16}$	2
THHT	$\frac{1}{16}$	2
THTH	$\frac{1}{16}$	2
TTHH	$\frac{1}{16}$	2
HTTT	$\frac{1}{16}$	1
THTT	$\frac{1}{16}$	1
TTHT	$\frac{1}{16}$	1
TTTH	$\frac{1}{16}$	1
TTTT	$\frac{1}{16}$	0

Element of sample space	Probability	x
HHHH	$\frac{1}{16}$	4
HHHT	$\frac{1}{16}$	3
HHTH	$\frac{1}{16}$	3
HTHH	$\frac{1}{16}$	3
THHH	$\frac{1}{16}$	3
HHTT	$\frac{1}{16}$	2
HTHT	$\frac{1}{16}$	2

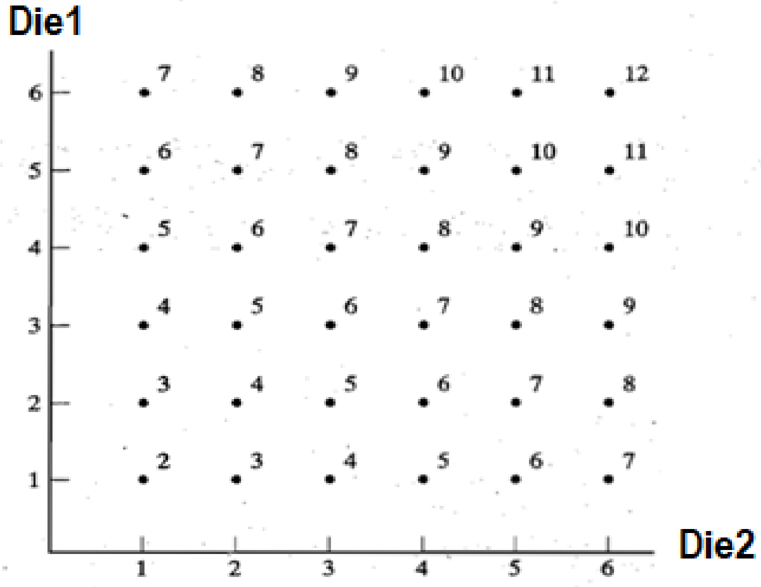
Element of sample space	Probability	x
HTTH	$\frac{1}{16}$	2
THHT	$\frac{1}{16}$	2
THTH	$\frac{1}{16}$	2
TTHH	$\frac{1}{16}$	2
HTTT	$\frac{1}{16}$	1
THTT	$\frac{1}{16}$	1
TTHT	$\frac{1}{16}$	1
TTTH	$\frac{1}{16}$	1
TTTT	$\frac{1}{16}$	0

Thus, for example, for the probability of the event that the random variable X will take on the value 3 we can write $P(X = 3) = \frac{4}{16}$

In all of the examples we have limited our discussion to discrete sample spaces, and hence to **discrete random variables**, namely, random variables whose range is finite or countably infinite.

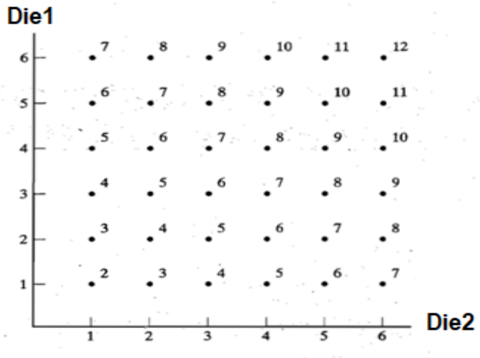
PROBABILITY DISTRIBUTIONS

Definition If X is a discrete random variable, the function given by $f(x) = P(X = x)$ for each x within the range of X is called the **probability distribution** of X.



The total number of points rolled with a pair of dice.

For instance, having assigned the probability $\frac{1}{36}$ to each element of the sample we immediately find that the random variable X, the total rolled with the pair of dice, takes on the value 9 with probability $\frac{4}{36}$;



The total number of points rolled with a pair of dice.

The probabilities associated with all possible values of X are shown in the following table:

x	$P(X = x)$	x	$P(X = x)$
2	$\frac{1}{36}$	7	$\frac{6}{36}$
3	$\frac{2}{36}$	8	$\frac{5}{36}$
4	$\frac{3}{36}$	9	$\frac{4}{36}$
5	$\frac{4}{36}$	10	$\frac{3}{36}$
6	$\frac{5}{36}$	11	$\frac{2}{36}$
		12	$\frac{1}{36}$

Instead of displaying the probabilities associated with the values of a random variable in a table, it is usually preferable to give a formula, that is, to express the probabilities by means of a function such that its values, $f(x)$, equal $P(X = x)$ for each x within the range of the random variable X.

Theorem A function can serve as the probability distribution of a discrete random variable X if and only if its values, $f(x)$, satisfy the conditions

1. $f(x) \geqq 0$ for each value within its domain;

2. $\sum\limits_{x} f(x) = 1$, where the summation extends over all the values within its domain.

EXAMPLE

Check whether the function given by

$$f(x) = \frac{x+2}{25} \quad \text{for } x = 1, 2, 3, 4, 5$$

can serve as the probability distribution of a discrete random variable.

Solution Substituting the different values of x, we get $f(1) = \frac{3}{25}$, $f(2) = \frac{4}{25}$, $f(3) = \frac{5}{25}$, $f(4) = \frac{6}{25}$, and $f(5) = \frac{7}{25}$.

Since these values are all nonnegative, the first condition of the Theorem is satisfied,

and since

$$\begin{aligned} f(1) + f(2) + f(3) + f(4) + f(5) &= \frac{3}{25} + \frac{4}{25} + \frac{5}{25} + \frac{6}{25} + \frac{7}{25} \\ &= 1 \end{aligned}$$

the second condition is satisfied.

Thus, the given function can serve as the probability distribution of a random variable having the range $\{1, 2, 3, 4, 5\}$.

EXAMPLE

Find a formula for the probability distribution of the total number of heads obtained in four tosses of a balanced coin.

Element of sample space	*Probability*	x	*Element of sample space*	*Probability*	x
HHHH	$\frac{1}{16}$	4	THHT	$\frac{1}{16}$	2
HHHT	$\frac{1}{16}$	3	THTH	$\frac{1}{16}$	2
HHTH	$\frac{1}{16}$	3	TTHH	$\frac{1}{16}$	2
HTHH	$\frac{1}{16}$	3	HTTT	$\frac{1}{16}$	1
THHH	$\frac{1}{16}$	3	THTT	$\frac{1}{16}$	1
HHTT	$\frac{1}{16}$	2	TTHT	$\frac{1}{16}$	1
HTHT	$\frac{1}{16}$	2	TTTH	$\frac{1}{16}$	1
HTTH	$\frac{1}{16}$	2	TTTT	$\frac{1}{16}$	0

Solution Based on the probabilities in the table we find that $P(X=0) = \frac{1}{16}$, $P(X = 1) = \frac{4}{16}$, $P(X = 2) = \frac{6}{16}$, $P(X = 3) = \frac{4}{16}$, and $P(X=4) = \frac{1}{16}$.

Observing that the numerators of these five fractions, 1, 4, 6, 4, and 1, are the binomial coefficients

$$\binom{4}{0}, \binom{4}{1}, \binom{4}{2}, \binom{4}{3}, \text{ and } \binom{4}{4},$$

we find that the formula for the probability distribution can be written as

$$f(x) = \frac{\binom{4}{x}}{16} \quad \text{for } x = 0, 1, 2, 3, 4 .$$

In some problems it is desirable to present probability distributions graphically, and two kinds of graphical presentations used for this purpose .

The one shown in the Figure below is called **probability histogram** and represents the probability distribution of the Example given above.

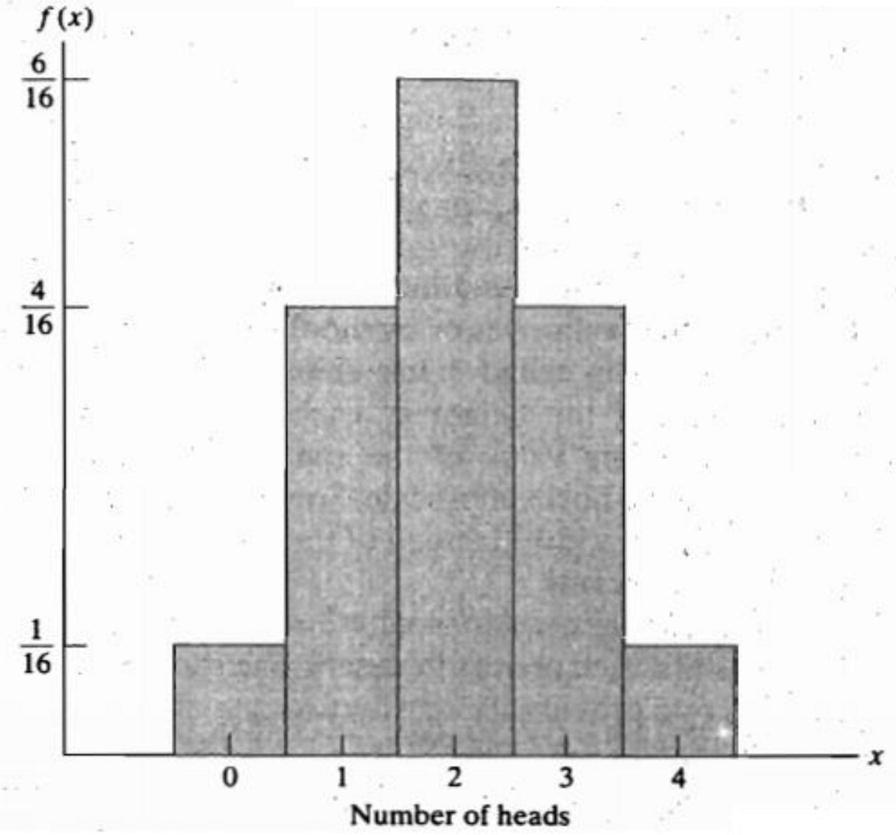


FIGURE : Probability histogram.

The height of each rectangle equals the probability that X takes on the value that corresponds to the midpoint of its base.

The graph of Figure is called a **bar chart**, but it is also referred to as a histogram. As in the Figure the height of each rectangle, or bar, equals the probability of the corresponding value of the random variable.

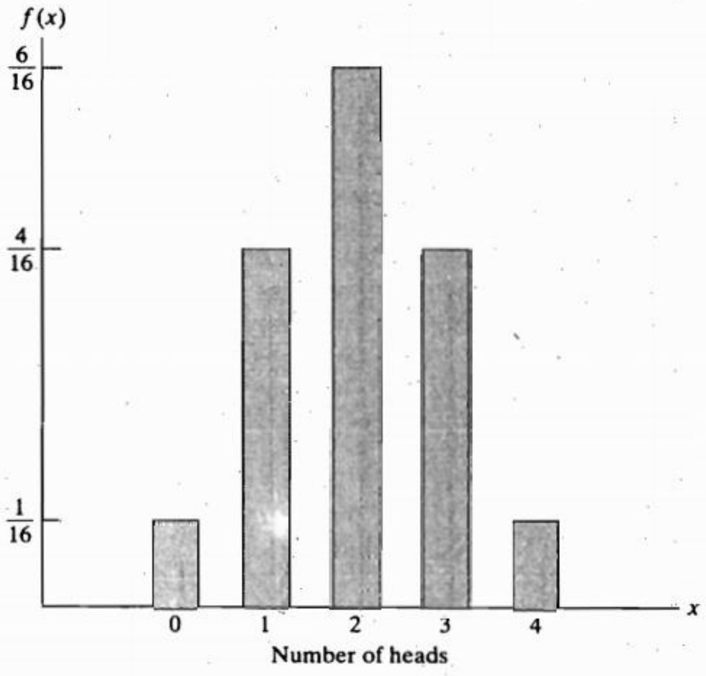
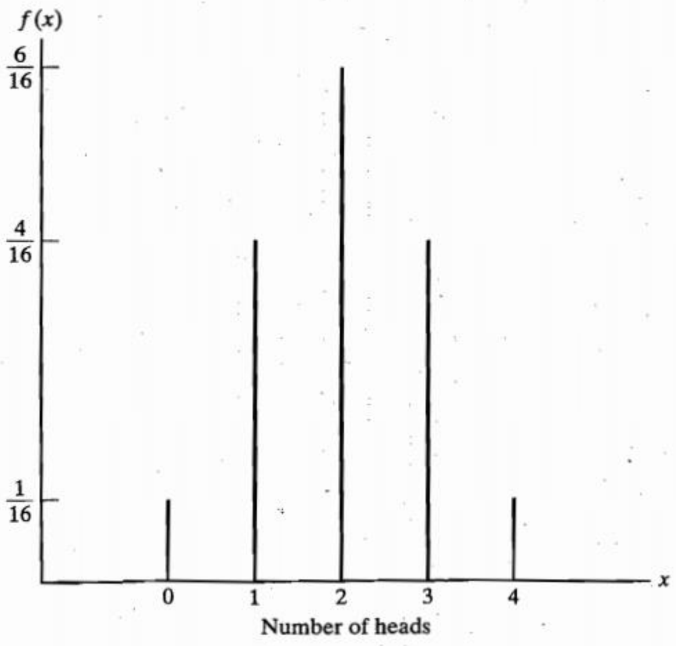


FIGURE : Bar chart.

Sometimes, as shown in the right panel below,
we use lines (rectangles with no width) instead of the rectangles,
but we still refer to the graphs as probability histograms.



Probability histogram.

Definition If X is a discrete random variable,

the function given by

$$F(x) = P(X \leqq x) = \sum_{t \leqq x} f(t)$$

for $-\infty < x < \infty$

is called the **distribution function**, or the **cumulative distribution**, of X.

where $f(t)$ is the value of the probability distribution of X at t.

EXAMPLE

Find the distribution function of the total number of heads obtained in four tosses of a balanced coin.

Element of sample space	*Probability*	x	*Element of sample space*	*Probability*	x
HHHH	$\frac{1}{16}$	4	THHT	$\frac{1}{16}$	2
HHHT	$\frac{1}{16}$	3	THTH	$\frac{1}{16}$	2
HHTH	$\frac{1}{16}$	3	TTHH	$\frac{1}{16}$	2
HTHH	$\frac{1}{16}$	3	HTTT	$\frac{1}{16}$	1
THHH	$\frac{1}{16}$	3	THTT	$\frac{1}{16}$	1
HHTT	$\frac{1}{16}$	2	TTHT	$\frac{1}{16}$	1
HTHT	$\frac{1}{16}$	2	TTTH	$\frac{1}{16}$	1
HTTH	$\frac{1}{16}$	2	TTTT	$\frac{1}{16}$	0

Element of sample space	Probability	x	Element of sample space	Probability	x
HHHH	$\frac{1}{16}$	4	THHT	$\frac{1}{16}$	2
HHHT	$\frac{1}{16}$	3	THTH	$\frac{1}{16}$	2
HHTH	$\frac{1}{16}$	3	TTHH	$\frac{1}{16}$	2
HTHH	$\frac{1}{16}$	3	HTTT	$\frac{1}{16}$	1
THHH	$\frac{1}{16}$	3	THTT	$\frac{1}{16}$	1
HHTT	$\frac{1}{16}$	2	TTHT	$\frac{1}{16}$	1
HTHT	$\frac{1}{16}$	2	TTTH	$\frac{1}{16}$	1
HTTH	$\frac{1}{16}$	2	TTTT	$\frac{1}{16}$	0

Solution Given $f(0) = \frac{1}{16}$, $f(1) = \frac{4}{16}$, $f(2) = \frac{6}{16}$, $f(3) = \frac{4}{16}$, and $f(4) = \frac{1}{16}$

$$F(0) = f(0) = \frac{1}{16}$$

$$F(1) = f(0) + f(1) = \frac{5}{16}$$

$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16}$$

$$F(3) = f(0) + f(1) + f(2) + f(3) = \frac{15}{16}$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1$$

Hence, the distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ \frac{1}{16} & \text{for } 0 \leqq x < 1 \\ \frac{5}{16} & \text{for } 1 \leqq x < 2 \\ \frac{11}{16} & \text{for } 2 \leqq x < 3 \\ \frac{15}{16} & \text{for } 3 \leqq x < 4 \\ 1 & \text{for } x \geqq 4 \end{cases}$$

THEOREM The values $F(x)$ of the distribution function of a discrete random variable X satisfy the conditions

1. $F(-\infty) = 0$ and $F(\infty) = 1$;
2. if $a < b$, then $F(a) \leqq F(b)$

   for any real numbers a and b.

EXAMPLE

Two socks are selected at random and removed in succession from a drawer containing five brown socks and three green socks.

List the elements of the sample space, the corresponding probabilities, and the corresponding values w of the random variable W, where W is the number of brown socks selected.

Solution

If B and G stand for brown and green, the probabilities for BB, BG, GB, and GG are, respectively,

$$\frac{5}{8} \cdot \frac{4}{7} = \frac{5}{14}, \quad \frac{5}{8} \cdot \frac{3}{7} = \frac{15}{56},$$

$$\frac{3}{8} \cdot \frac{5}{7} = \frac{15}{56}, \quad \text{and } \frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$$

Element of sample space	*Probability*	w
BB	$\frac{5}{14}$	2
BG	$\frac{15}{56}$	1
GB	$\frac{15}{56}$	1
GG	$\frac{3}{28}$	0

Element of sample space	*Probability*	*w*
BB	$\frac{5}{14}$	2
BG	$\frac{15}{56}$	1
GB	$\frac{15}{56}$	1
GG	$\frac{3}{28}$	0

Find the distribution function of the random variable W and plot its graph.

Solution Based on the probabilities given in the table write $f(0) = \frac{3}{28}$,

$$f(1) = \frac{15}{56} + \frac{15}{56} = \frac{15}{28},$$

and $f(2) = \frac{5}{14}$

$$F(0) = f(0) = \frac{3}{28}$$

$$F(1) = f(0) + f(1) = \frac{9}{14}$$

$$F(2) = f(0) + f(1) + f(2) = 1$$

Hence, the distribution function of W is given by

$$F(w) = \begin{cases} 0 & \text{for } w < 0 \\ \frac{3}{28} & \text{for } 0 \leqq w < 1 \\ \frac{9}{14} & \text{for } 1 \leqq w < 2 \\ 1 & \text{for } w \geqq 2 \end{cases}$$