EEF 271E
Probability and Statistics

Week #6

MATHEMATICAL EXPECTATION

The expected value of a random variable generalization of the weighted average, and is intuitively the arithmetic mean of a large number of independent realizations of X.

The expected value is also known as the expectation, mathematical expectation, mean, average, or first moment.

Expected value is a key concept in economics, finance, and many other subjects.

THE EXPECTED VALUE OF A RANDOM VARIABLE

DEFINITION

If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the **expected value** of X is

$$E(X) = \sum_x x \cdot f(x)$$

Correspondingly, if X is a continuous random variable and $f(x)$ is the value of its probability density at x, the **expected value** of X is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

EXAMPLE

Suppose that $P(X = 0) = P(X = 10) = 1/2$.

Then $E(X) = (0)(1/2) + (10)(1/2) = 5$,

EXAMPLE

Suppose, that $P(Y = 6) = 1/3$, and $P(Y = 15) = 2/3$.

Then

$$E(Y) = (6)(1/3) + (15)(2/3) = 2 + 10 = 12,$$

EXAMPLE

12 television sets includes 2 white cords. If three of the sets are chosen at random for shipment to a hotel, how many sets with white cords can the shipper expect to send to the hotel?

Then all the possible ways we choose $3-x$ of the 10 other sets $\binom{10}{3-x}$

Solution

Assume we choose x sets with white cords

Then all the possible ways we choose sets with white cords is $\binom{2}{x}$

If we choose x sets with white cords this means that we choose $3-x$ of the 10 other sets

possible ways we choose x of the two sets with white cords and $3-x$ of the 10 other sets

$$= \binom{2}{x}\binom{10}{3-x}$$

probability distribution of X,

$$f(x) = \frac{\binom{2}{x}\binom{10}{3-x}}{\binom{12}{3}} \quad \text{for } x = 0, 1, 2$$

x	0	1	2
$f(x)$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

$$\begin{aligned} E(X) &= 0 \cdot \frac{6}{11} + 1 \cdot \frac{9}{22} \\ &\quad + 2 \cdot \frac{1}{22} \\ &= \frac{1}{2} \end{aligned}$$

EXAMPLE

Consider a probability density

$$f(x) = \begin{cases} \dfrac{4}{\pi(1+x^2)} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find the expected value of this random variable.

Solution we have

$$E(X) = \int_0^1 x \cdot \frac{4}{\pi(1+x^2)}\, dx$$

$$= \frac{4}{\pi} \int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \frac{\ln 4}{\pi} = 0.4413$$

NOTE:

There are many problems in which we are interested not only in the expected value of a random variable X, but also in the expected values of random variables related to X.

Thus, we might be interested in the random variable Y, whose values are related to those of X by means of the equation $y = g(x)$;

THEOREM If X is a discrete random variable and $f(x)$ is the value of its probability distribution at x, the expected value of $g(X)$ is given by

$$E[g(X)] = \sum_{x} g(x) \cdot f(x)$$

Correspondingly, if X is a continuous random variable and $f(x)$ is the value of its probability density at x, the expected value of $g(X)$ is given by

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx$$

EXAMPLE

If X has the probability density

$$f(x) = \begin{cases} e^{x} & \text{for } x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

find the expected value of $g(X) = e^{3X/4}$.

Solution we have

$$E[e^{3X/4}] = \int_0^\infty e^{3x/4} \cdot e^{-x}\, dx = \int_0^\infty e^{-x/4}\, dx = 4$$

EXAMPLE

If X is the number of points rolled with a balanced die, find the expected value of $g(X) = 2X^2 + 1$.

Solution Since each possible outcome has the probability $\frac{1}{6}$, we get

$$E[g(X)] = \sum_x g(x) \cdot f(x)$$

$$E[g(X)] = \sum_{x=1}^{6} (2x^2 + 1) \cdot \frac{1}{6}$$

$$= (2 \cdot 1^2 + 1) \cdot \frac{1}{6} + \cdots + (2 \cdot 6^2 + 1) \cdot \frac{1}{6}$$

$$= \frac{94}{3}$$

Theorem If a and b are constants, then

$$E(aX+b) = aE(X)+b$$

Proof. Using $g(X) = aX+b$, we get

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b)\cdot f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\cdot f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= aE(X)+b$$

If we set $b=0$ or $a=0$, it follows

Corollary 1. If a is a constant, then

$$E(aX) = aE(X)$$

Corollary 2. If b is a constant, then

$$E(b) = b$$

THEOREM If $c_1, c_2, \ldots,$ and c_n are constants, then

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{i=1}^{n} c_i E[g_i(X)]$$

Proof. According to $g(X) = \sum_{i=1}^{n} c_i g_i(X)$, we get

$$E\left[\sum_{i=1}^{n} c_i g_i(X)\right] = \sum_{x}\left[\sum_{i=1}^{n} c_i g_i(x)\right] f(x)$$

$$= \sum_{i=1}^{n}\sum_{x} c_i g_i(x) f(x) = \sum_{i=1}^{n} c_i \sum_{x} g_i(x) f(x)$$

$$= \sum_{i=1}^{n} c_i E[g_i(X)]$$

EXAMPLE

Making use of the fact that

$$E(X^2) = (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) \cdot \frac{1}{6} = \frac{91}{6}$$

for the random variable of example $g(X) = 2X^2 + 1$.

Solution

$$E(2X^2 + 1) = 2E(X^2) + 1$$
$$= 2 \cdot \frac{91}{6} + 1 = \frac{94}{3}$$

EXAMPLE

Show that

$$E[(aX + b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

Solution

Since $(ax + b)^n = \sum_{i=0}^{n} \binom{n}{i} (ax)^{n-i} b^i$

$$E[(aX + b)^n] = E\left[\sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i X^{n-i}\right]$$
$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$$

The concept of a mathematical expectation can easily be extended to situations involving more than one random variable.

For instance, if Z is the random variable whose values are related to those of the two random variables X and Y by means of the equation $z = g(x, y)$,

it can be shown that

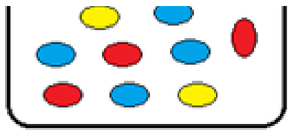
THEOREM If X and Y are discrete random variables and $f(x, y)$ is the value of their joint probability distribution at (x, y), the expected value of $g(X, Y)$ is

$$E[g(X, Y)] = \sum_{x} \sum_{y} g(x, y) \cdot f(x, y)$$

Correspondingly, if X and Y are continuous random variables and $f(x, y)$ is the value of their joint probability density at (x, y), the expected value of $g(X, Y)$ is

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy$$

Two balls are selected at random from a box containing 3 Red 2 Yellow and 4 Blue balls. If X and Y are respectively, the numbers Red and Yellow balls,



EXAMPLE

With reference to Example about Red, Yellow, Blue balls find the expected value of $g(X, Y) = X + Y$.

$f(x,y)$	$x = 0$	$x = 1$	$x = 2$
$y = 0$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$
$y = 1$	$\frac{2}{9}$	$\frac{1}{6}$	
$y = 2$	$\frac{1}{36}$		

Solution

$$E(X+Y) = \sum_{x=0}^{2}\sum_{y=0}^{2}(x+y)\cdot f(x,y)$$

$$= (0+0)\cdot\frac{1}{6} + (0+1)\cdot\frac{2}{9} + (0+2)\cdot\frac{1}{36}$$

$$+ (1+0)\cdot\frac{1}{3} + (1+1)\cdot\frac{1}{6} + (2+0)\cdot\frac{1}{12}$$

$$= \frac{10}{9}$$

MOMENTS

In statistics, the mathematical expectations defined here are called the **moments** of a random variable,

Definition The rth **moment about the origin** of a random variable X, denoted by μ'_r, is the expected value of X^r; symbolically,

$$\mu'_r = E(X^r) = \sum_x x^r \cdot f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete,

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$$

when X is continuous.

Definition μ'_1 is called the **mean** of the distribution of X, or simply the **mean** of X, and it is denoted by μ.

DEFINITION The rth **moment about the mean** of a random variable X, denoted by μ_r, is the expected value of $(X-\mu)^r$; symbolically,

$$\mu_r = E[(X-\mu)^r] = \sum_x (x-\mu)^r \cdot f(x)$$

for $r = 0, 1, 2, \ldots$ when X is discrete,

$$\mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x)\,dx$$

when X is continuous.

The second moment about the mean is of special importance in statistics because it is indicative of the spread or dispersion of the distribution of a random variable; thus, it is given a special symbol and a special name.

DEFINITION μ_2 is called the **variance** of the distribution of X, or simply the **variance** of X, and it is denoted by σ^2, $\mathrm{var}(X)$, or $V(X)$; σ, the positive square root of the variance, is called the **standard deviation.**

In other words, the expectation of the squared deviation of a random variable from its mean is called variance.

$$\sigma^2 = E[(X - \mu)^2]$$

THEOREM $\sigma^2 = \mu_2' - \mu^2$

Proof. $\sigma^2 = E[(X-\mu)^2]$

$$= E(X^2 - 2\mu X + \mu^2)$$

$$= E(X^2) - 2\mu E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu \cdot \mu + \mu^2$$

$$= \mu_2' - \mu^2$$

Or using a different notation:

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{E}\left[(X - \operatorname{E}[X])^2\right] \\ &= \operatorname{E}\left[X^2 - 2X\operatorname{E}[X] + \operatorname{E}[X]^2\right] \\ &= \operatorname{E}\left[X^2\right] - 2\operatorname{E}[X]\operatorname{E}[X] + \operatorname{E}[X]^2 \\ &= \operatorname{E}\left[X^2\right] - \operatorname{E}[X]^2\end{aligned}$$

EXAMPLE

Calculate the variance of X, representing the number of points rolled with a balanced die.

Solution First we compute

$$\mu = E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$$

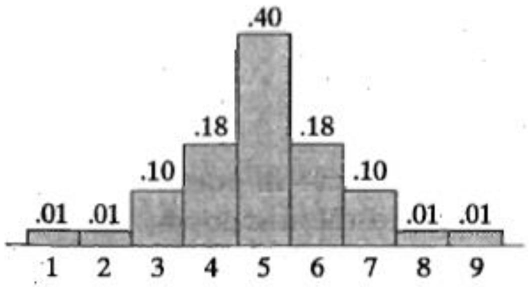
Now,

$$\mu_2' = E(X^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} \cdot + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6}$$

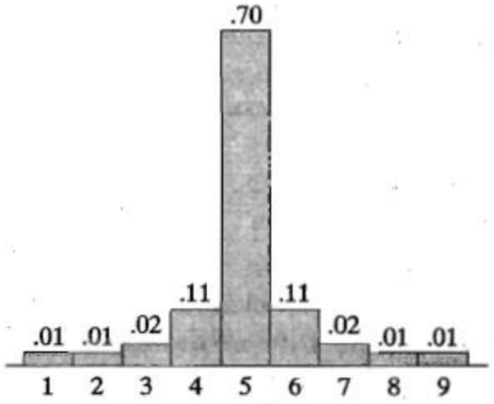
and it follows that

$$\sigma^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

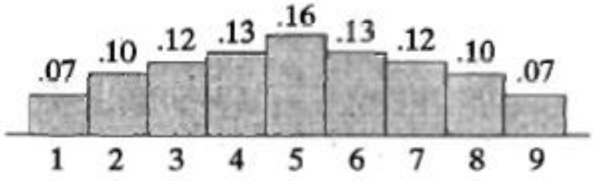
Distributions with different dispersions.



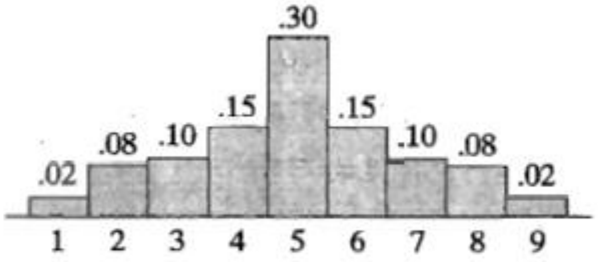
$\mu = 5$ and $\sigma^2 = 1.66$



$\mu = 5$ and $\sigma^2 = 0.88$



$\mu = 5$ and $\sigma^2 = 5.26$



$\mu = 5$ and $\sigma^2 = 3.18$

PRODUCT MOMENTS

let us now present the **product moments** of two random variables.

DEFINITION The rth and sth **product moment about the origin** of the random variables X and Y, when X and Y are discrete, denoted by $\mu'_{r,s}$,

is the expected value of $X^r Y^s$;

$$\mu'_{r,s} = E(X^r Y^s) = \sum_x \sum_y x^r y^s \cdot f(x, y)$$

when X and Y are continuous.

$$\mu'_{r,s} = E(X^r Y^s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^r y^s \cdot f(x, y)\, dx\, dy$$

for $r = 0, 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$

Definition The rth and sth **product moment about the means** of the random variables X and Y, denoted by $\mu_{r,s}$, is the expected value of $(X-\mu_X)^r(Y-\mu_Y)^s$; when X and Y are discrete,

$$\begin{aligned}\mu_{r,s} &= E[(X-\mu_X)^r(Y-\mu_Y)^s] \\ &= \sum_x \sum_y (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\end{aligned}$$

for $r = 0, 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$

when X and Y are continuous

$$\begin{aligned}\mu_{r,s} &= E[(X-\mu_X)^r(Y-\mu_Y)^s] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)^r(y-\mu_Y)^s \cdot f(x,y)\,dx\,dy\end{aligned}$$

In statistics, $\mu_{1,1}$ is of special importance because it is indicative of the relationship, if any, between the values of X and Y; thus, it is given a special symbol and a special name.

DEFINITION $\mu_{1,1}$ is called the **covariance** of X and Y, and it is denoted by σ_{XY}, $\mathrm{cov}(X, Y)$, or $C(X, Y)$.

THEOREM

$$\sigma_{XY} = \mu'_{1,1} - \mu_X \mu_Y$$

(Covariance of two random variables X,Y is equal to the mean of XY minus the multiplication of their distinct mean values.)

Proof.

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E(XY - X\mu_Y - Y\mu_X + \mu_X\mu_Y)$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y$$

$$= E(XY) - \mu_Y\mu_X - \mu_X\mu_Y + \mu_X\mu_Y$$

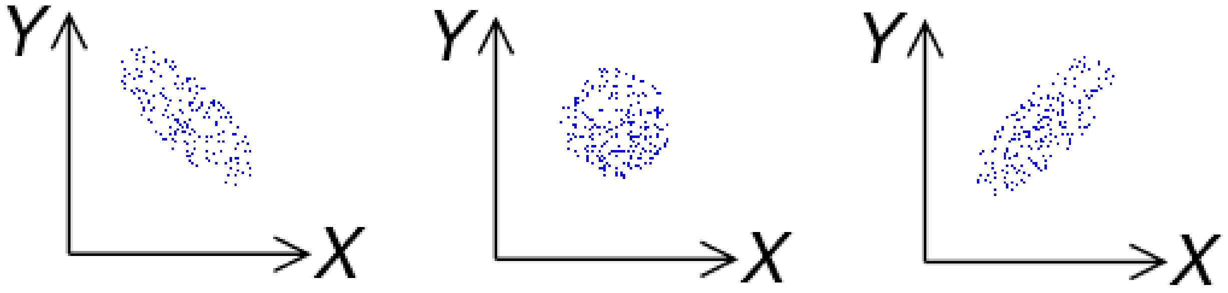
$$= \mu'_{1,1} - \mu_X\mu_Y$$

Special case: X=Y;

$$\sigma_{XY} = E(X^2) - 2\mu \cdot \mu + \mu^2$$

$$\sigma_{XY} = \mu'_2 - \mu^2 = \sigma^2$$

Variance is the covariance of a random variable by itself.



$\mathrm{cov}(X,Y)<0$ $\mathrm{cov}(X,Y)\approx 0$ $\mathrm{cov}(X,Y)>0$

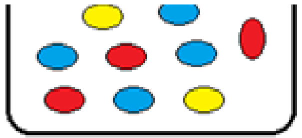
covariance is a measure of the joint variability of two *random variables*

If the greater values of one variable mainly correspond with the greater values of the other variable, the covariance is positive.

In the opposite case, when the greater values of one variable mainly correspond to the lesser values of the other, the covariance is negative.

The sign of the covariance therefore shows the tendency in the linear relationship between the variables.

Two balls are selected at random from a box containing 3 Red 2 Yellow and 4 Blue balls. If X and Y are respectively, the numbers Red and Yellow balls,



EXAMPLE Find the covariance of X and Y.

		$x=0$	$x=1$	$x=2$	
y	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{12}$
	1	$\frac{2}{9}$	$\frac{1}{6}$		$\frac{7}{18}$
	2	$\frac{1}{36}$			$\frac{1}{36}$
		$\frac{5}{12}$	$\frac{1}{2}$	$\frac{1}{12}$	

Solution

$$\sigma_{XY} = \mu'_{1,1} - \mu_X \mu_Y$$

$$\mu'_{r,s} = E(X^r Y^s) = \sum_x \sum_y x^r y^s \cdot f(x, y)$$

$$\mu'_{1,1} = E(XY) = \sum_x \sum_y x\ y \cdot f(x, y)$$

$$\mu'_{1,1} = E(XY) = 0 \cdot 0 \cdot \frac{1}{6} + 0 \cdot 1 \cdot \frac{2}{9} + 0 \cdot 2 \cdot \frac{1}{36} + 1 \cdot 0 \cdot \frac{1}{3} + 1 \cdot 1 \cdot \frac{1}{6} + 2 \cdot 0 \cdot \frac{1}{12} = \frac{1}{6}$$

and using the marginal probabilities, we get

$$\mu_X = E(X) = 0 \cdot \frac{5}{12} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{12} = \frac{2}{3}$$

and

$$\mu_Y = E(Y) = 0 \cdot \frac{7}{12} + 1 \cdot \frac{7}{18} + 2 \cdot \frac{1}{36} = \frac{4}{9}$$

It follows that

$$\sigma_{XY} = \frac{1}{6} - \frac{2}{3} \cdot \frac{4}{9} = -\frac{7}{54}$$

The negative result suggests that the more red balls we get the fewer yellow balls we will get and vice versa.