

Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Fall Semester 2022

EHB 212E

HOMEWORK – 3

Due: December 30, 2022, till 21.00



- You need to UPLOAD THE SOFTCOPY AND bring the hardcopy of the homework to office 7309 in EEB before the deadline (you can throw homework under the door)
- You need to show all the steps during operations. Otherwise, the questions are not graded.
- Do Not forget to write your name!
- The total point is 100 and each question has the same importance.

Q-1) It is known that for an infinite-length coaxial cable with the inner and the outer radii, a , and b , respectively, the capacitance value **per unit length** is $C_0 = \frac{2\pi\epsilon}{\ln(b/a)}$ (F/m) where the region between these conductors' surface is free space ($\epsilon = \epsilon_0$).

- By placing a conductive cylindrical shell with an inner radius d_1 and outer radius d_2 in the region between the cylinders, it is desired to double the capacitance value ($2C_0$) in the first case. Calculate the required ratio d_1/d_2 in terms of a and b to realize this situation.
- In this last case, it is aimed to increase the capacitance value to $8C_0$ by filling $\rho \epsilon(a, d_1)$ and $\rho \epsilon(d_2, b)$ regions with an insulator material with dielectric constant $\epsilon = \epsilon_r \epsilon_0$. Determine the required relative dielectric constant ϵ_r for this case. **Note that, the conductors placed in (a) are still in the system.**

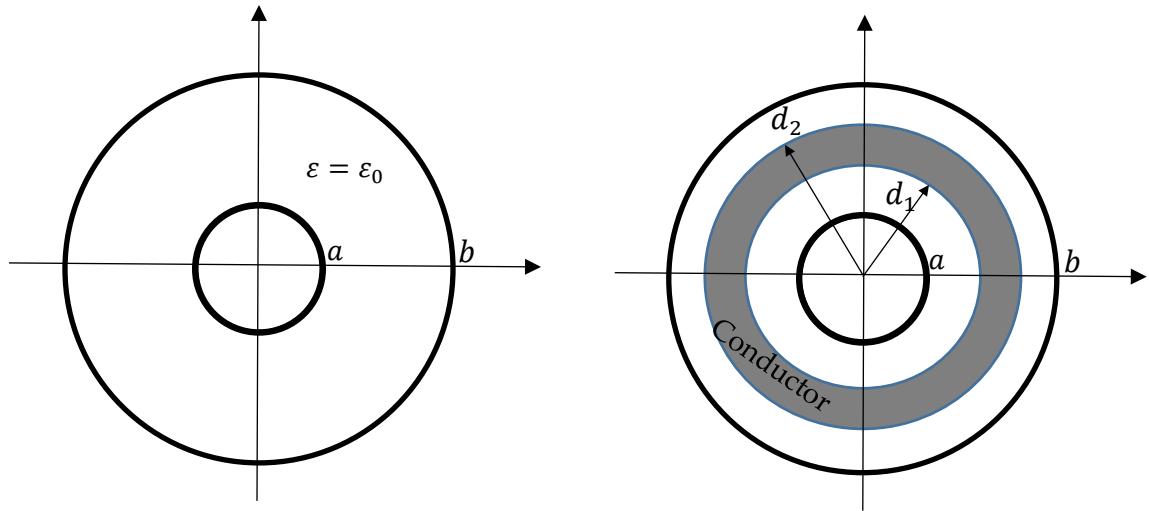


Figure 1. The geometries of the system.

$$1) C = \frac{2\pi\epsilon_0}{\ln(b/a)} [F/m]$$

$$a) C_1 = \frac{2\pi\epsilon}{\ln(d_1/a)}, \quad C_2 = \frac{2\pi\epsilon}{\ln(b/d_2)}$$

$$C_{\text{es}} = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1} \Rightarrow \text{ve } C_{\text{es}} = 2C \Rightarrow$$

$$\frac{4\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a \frac{d_1}{d_2})} \Rightarrow \boxed{\frac{d_1}{d_2} = \sqrt{\frac{a}{b}}}$$

$$b) 8C = \frac{16\pi\epsilon_0}{\ln(b/a)} \Rightarrow$$

$$\frac{\ln(b/a)}{16\pi\epsilon_0} = \frac{\ln(\frac{d_1}{a}) + \ln(\frac{b}{d_2})}{2\pi\epsilon_r\epsilon_0} \Rightarrow$$

$$\epsilon_r \ln(b/a) = 8 \ln\left(\frac{b}{a} \frac{d_1}{d_2}\right) = 8 \ln\sqrt{b/a} \\ = 4 \ln(b/a) \Rightarrow$$

$$\boxed{\underline{\epsilon_r = 4}}$$

Q-2) A constant current I_0 flows uniformly across the cross-section in the positive z-direction through the infinitely long conductor wire of radius a placed along the z-axis as provided in Figure 2.

- Find the value $\oint_{C_b} \vec{H} \cdot d\vec{\ell}$ along C_b (closed circle with radius b), with $b < a$.
- Find the value $\oint_{C_d} \vec{H} \cdot d\vec{\ell}$ along C_d (closed circle with radius d) with $d > a$.
- Find the value $\oint_{ABCD} \vec{H} \cdot d\vec{\ell}$ along the square ABCD with side length h , $h > 2a$.
- Find the result of the line integral $\int_A^B \vec{H} \cdot d\vec{\ell}$ along the AB section of the square.

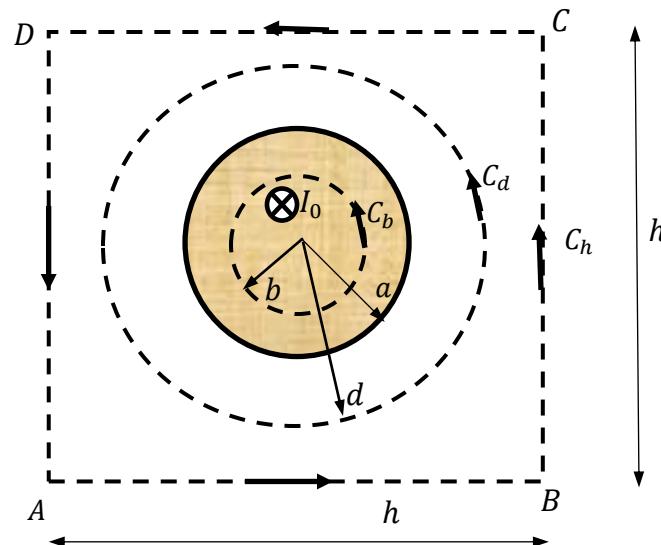


Figure 2. The geometry of Q-2.

a) $\oint \vec{H} \cdot d\vec{l} = \sum I = -I_0 \frac{b^2}{a^2}$

b) $\oint \vec{H} \cdot d\vec{l} = \sum I = -I_0$

c) $\oint \vec{H} \cdot d\vec{l} = \sum I = -I_0$

d) $\int_{AB} \vec{H} \cdot d\vec{l} = \int_A^B \vec{H} \cdot d\vec{l} = \frac{1}{4} \oint \vec{H} \cdot d\vec{l} = \frac{-I_0}{4}$

$\xrightarrow{\text{Simetria}}$

Q-3) Below, the geometry of the problem is given. As it is noticed, steady current I flows through a line. Current originates at infinity and then rotates around the origin on a quarter circle with a radius of a . Then, it goes to infinity. Find B field at origin. The systems in free space.

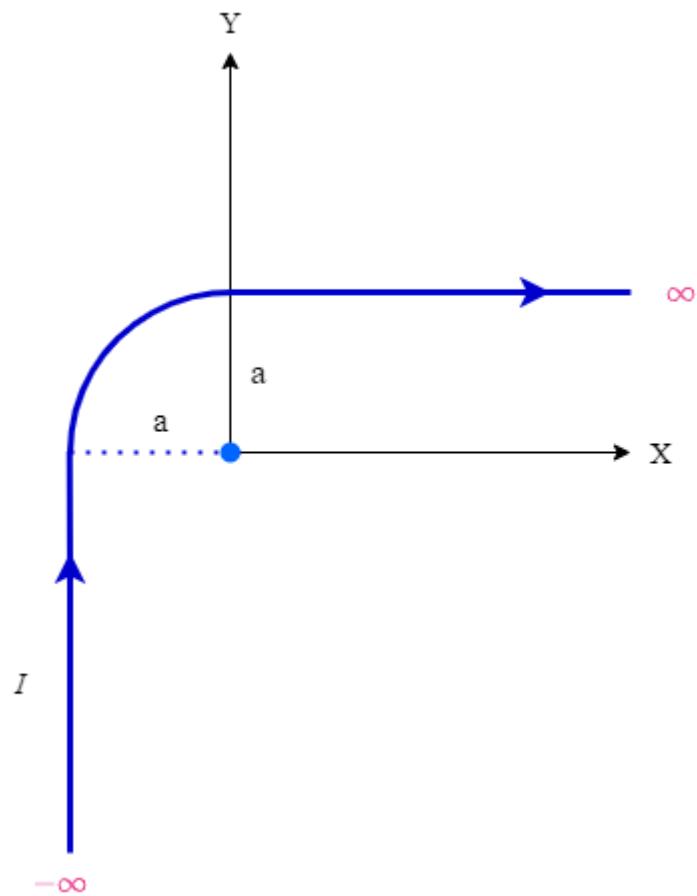
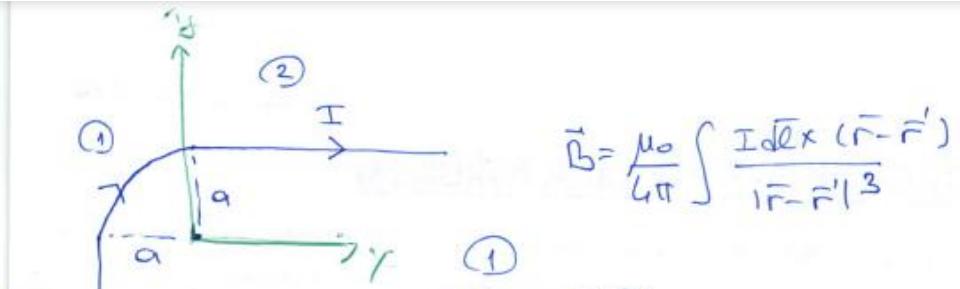


Figure 3. The Schema for Q-3.



$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{I d\vec{x} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

① $d\vec{l} = a d\phi \hat{a}_\phi$

$$\begin{aligned}\vec{r} &= 0 \hat{a}_r \\ \vec{r}' &= a \hat{a}_\phi \\ |\vec{r} - \vec{r}'| &= a\sqrt{1 + \sin^2 \phi} \\ |\vec{r} - \vec{r}'| &= a\end{aligned}$$

$$\vec{B}_I = \frac{\mu_0}{4\pi} \int_{\pi}^{\pi/2} \frac{I a d\phi \hat{a}_\phi \times \hat{a}_r (-a)}{a^3} d\phi$$



$$= \frac{\mu_0 I}{4\pi a^2} \int_{\pi}^{\pi/2} a_2 d\phi$$

$$\hat{a}_\phi \times -\hat{a}_r = a_2$$

$$= \frac{\mu_0 I}{4\pi a} \hat{a}_2 \left[-\frac{\pi}{2} \right] = -\frac{\mu_0 I}{8a} \hat{a}_2$$

$$B_{II} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{(dx' \hat{a}_x) \times (-a \hat{a}_y - x' \hat{a}_y)}{(x'^2 + a^2)^{3/2}}$$

$$\vec{r} = 0 \hat{a}_r$$

$$\vec{r}' = a \hat{a}_y + x' \hat{a}_x$$

$$\vec{r} - \vec{r}' = -a \hat{a}_y - x' \hat{a}_x$$

$$|\vec{r} - \vec{r}'| = \sqrt{a^2 + x'^2}$$

$$dx' \hat{a}_x \times (-a \hat{a}_y - x' \hat{a}_y) = -adx' \hat{a}_z$$

Remember

$$\vec{B}_{II} = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{-adx'}{(x'^2 + a^2)^{3/2}} \hat{a}_z$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} \rightarrow \frac{1}{a^2} \frac{x}{\sqrt{x^2 + a^2}}$$

$$= \vec{B}_{II} = -\frac{\mu_0 I}{4\pi a} \hat{a}_z$$

B_{III}

$$\vec{r} = a\hat{a}_\theta \quad d\ell = dy' \hat{a}_y$$

$$\vec{r}' = a\hat{a}_x - a\cos\theta \hat{a}_x + y' \hat{a}_y$$

$$\vec{r} - \vec{r}' = a\hat{a}_x - y' \hat{a}_y$$

$$|\vec{r} - \vec{r}'| = \sqrt{a^2 + y'^2}$$

$$B = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{r} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$



$$\hat{a}_y \times \hat{a}_x = -\hat{a}_z$$

$$B_{III} = \frac{\mu_0 I}{4\pi} \int_{-\infty}^0 \frac{dy' \hat{a}_y \times [a\hat{a}_x - y' \hat{a}_y]}{(a^2 + y'^2)^{3/2}}$$

⊗

$$= \frac{\mu_0 I (-a)}{4\pi} \int_{-\infty}^0 \frac{dy'}{(a^2 + y'^2)^{3/2}}$$

$$\int \frac{dy}{(a^2 + y^2)^{3/2}} = \frac{1}{a^2} \frac{y}{\sqrt{y^2 + a^2}}$$

$$= -\frac{\mu_0 I a \hat{a}_z}{4\pi} \frac{1}{a^2} \frac{y}{\sqrt{y^2 + a^2}} \Big|_{-\infty}^0 = -\frac{\mu_0 I a \hat{a}_z}{4\pi a} \frac{y}{\sqrt{y^2 + a^2}} \Big|_{-\infty}^0 = -\frac{\mu_0 I}{4\pi a} \hat{a}_z$$

$$\vec{B} = \vec{B}_{I+} + \vec{B}_{II+} + \vec{B}_{III} = -\frac{\mu_0 I}{4a} \left(1 + \frac{2}{\pi}\right) \hat{a}_z \quad [\text{Wb/m}^2]$$

Q-4) Consider a spherical structure of inner and outer radii a and $2a$ meters, respectively. Assuming the inner conductor is kept V_0 Volt while the outer is grounded (i.e. at 0 Volt), **in terms of a, V_0, σ_0 and ϵ_0 :**

- (a) Find the electrostatic potential expression $V(r)$ inside the capacitor, i.e. $a \leq r \leq b$. **Note that the potential varies the same way in Region I and II.**
- (b) First, find the total resistance of the system
- (c) What is the capacitance C of this structure?

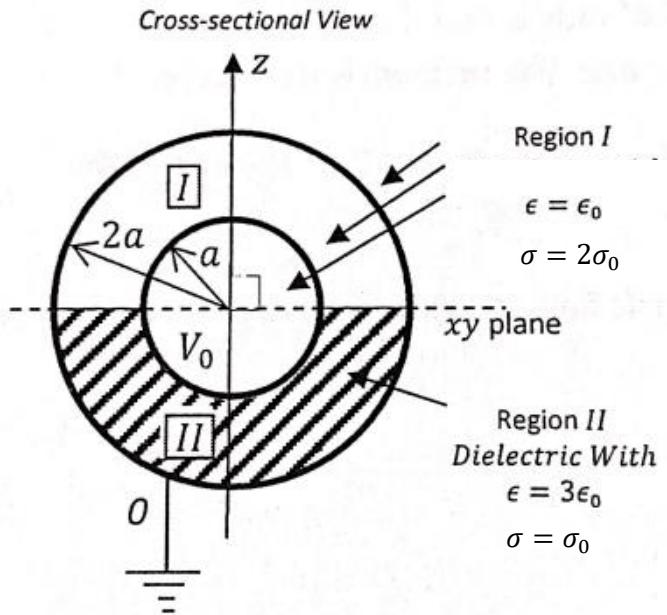


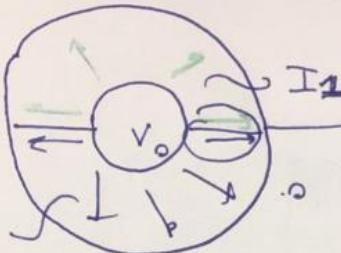
Figure 3. The geometry for Q-4.

$$\begin{aligned}
 \nabla^2 V(r) = 0 &\rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \\
 \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0 &\rightarrow r^2 \frac{dV}{dr} = C_1 \rightarrow dV = \frac{C_1}{r^2} dr \rightarrow V(r) = -\frac{C_1}{r} + C_2 \\
 V_0 = -\frac{1}{a} C_1 + C_2 \\
 0 = -\frac{1}{2a} C_1 + C_2
 \end{aligned}
 \left. \begin{array}{l} \left. \begin{array}{l} \nabla^2 V(r) = 0 \rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \rightarrow \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = 0 \\ \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right) = 0 \rightarrow r^2 \frac{dV}{dr} = C_1 \rightarrow dV = \frac{C_1}{r^2} dr \rightarrow V(r) = -\frac{C_1}{r} + C_2 \end{array} \right. \\ \left. \begin{array}{l} V_0 = -\frac{1}{a} C_1 + C_2 \\ 0 = -\frac{1}{2a} C_1 + C_2 \end{array} \right. \end{array} \right\} \rightarrow C_1 = -2V_0a, \quad C_2 = -V_0 \rightarrow V(r) = \boxed{\frac{2V_0a}{r} - V_0}$$

$$\begin{aligned}
 \mathbf{E}(r) &= -\nabla V(r) = -\hat{\mathbf{a}}_r \frac{\partial}{\partial r} V(r) = \hat{\mathbf{a}}_r \frac{2V_0a}{r^2} \\
 \mathbf{D}_I(r) = \epsilon_0 \mathbf{E}(r) &= \boxed{\hat{\mathbf{a}}_r \epsilon_0 \frac{2V_0a}{r^2}} \quad \mathbf{D}_{II}(r) = 3\epsilon_0 \mathbf{E}(r) = \boxed{\hat{\mathbf{a}}_r \epsilon_0 \frac{6V_0a}{r^2}}
 \end{aligned}$$

$$\begin{aligned}
 Q &= 2\pi a^2 \rho_{sI} + 2\pi a^2 \rho_{sII} = 2\pi a^2 \hat{\mathbf{a}}_r \cdot \mathbf{D}_I(a) + 2\pi a^2 \hat{\mathbf{a}}_r \cdot \mathbf{D}_{II}(a) \\
 Q &= 2\pi a^2 \epsilon_0 \frac{2V_0}{a} (1+3) = \boxed{16\pi a \epsilon_0 V_0}
 \end{aligned}$$

$$C = \frac{Q}{V_0} = \boxed{16\pi a \epsilon_0}$$



$$\vec{E}_1 = \vec{E}_2 \text{ (since tangential components are continuous)}$$

$$I_2 \quad \text{Hemisphere} \quad \frac{4\pi R^2 L}{2}$$

$$S_1 \boxed{\frac{2\pi R^2}{2}}$$

$$S_2 \boxed{\frac{2\pi R^2}{2}}$$

$$\begin{aligned} \vec{J}_2 &= \sigma_2 \vec{E}_2 = \sigma_2 \vec{E}_1 = \vec{J}_1 \\ \vec{J}_1 &= \sigma_0 \vec{E}_1 \\ \vec{E}_1 &= \sigma_0 \frac{2V_0 a}{R^2} \vec{a}_r \\ \vec{J}_1 &= 2\sigma_0 \frac{2V_0 a}{R^2} \vec{a}_r \end{aligned}$$

$$\begin{aligned} \vec{I}_2 &= S_2 \vec{J}_2 = 2\pi R^2 \sigma_0 \frac{2V_0 a}{R^2} \vec{a}_r = 2\pi V_0 \sigma_0 a \vec{a}_r \\ I_2 &= S_2 J_2 \\ &\quad \boxed{2\pi R^2 2\sigma_0 \frac{2V_0 a}{R^2} \vec{a}_r} = 2\pi V_0 2\sigma_0 a \vec{a}_r \end{aligned}$$

$$\vec{I}_2 = 4\pi V_0 \sigma_0 a \vec{a}_r$$

$$\vec{I}_2 = 8\pi V_0 \sigma_0 a \vec{a}_r$$

$$\frac{V}{I} = R \Rightarrow \frac{V_0}{I_2} = R_2 \quad \frac{V_0}{I_2} = R_1$$

$$\Downarrow \quad \frac{1}{4\pi V_0 a} \quad \boxed{R}$$

$$\Downarrow \quad \frac{1}{8\pi V_0 a} \quad \boxed{R}$$

$$R = \frac{R_1 + R_2}{2}$$

$$= \frac{1}{R_{\text{total}}} = \frac{1}{4\pi V_0 a} + \frac{1}{8\pi V_0 a}$$

$$\frac{1}{R_{\text{total}}} = \frac{1}{12\pi V_0 a}$$

$$R_{\text{total}} = \frac{1}{\frac{1}{12\pi V_0 a}} \quad \boxed{R}$$