Istanbul Technical University

Faculty of Electrical and Electronics Engineering

Spring Semester 2022-2023

EEF 212E

HOMEWORK – 1

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Each student is viewed as a responsible professional in engineering, and thus highest ethical standards are presumed.

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Due: March 19, 2023, till 23.30

- **You need to upload HW to Ninova. Other options are not accepted!**
- **You need to show all the steps during operations. Otherwise, the questions are not graded.**
- **Do Not forget to write your name!**
- **The total point is 100 and each question has the same importance.**

Q-1) Given the vector field $\vec{A} = x\hat{a}_x + y\hat{a}_y$ and the volume specified by

$$\boldsymbol{v}: a \le R \le b, \quad 0 \le \theta < \pi, \quad 0 \le \phi < 2\pi$$

where, R, θ, ϕ are the usual spherical coordinate parameters, **verify** the Divergence Theorem through this defined volume above.

Hint: Verification of the Divergence Theorem is to show the following equality

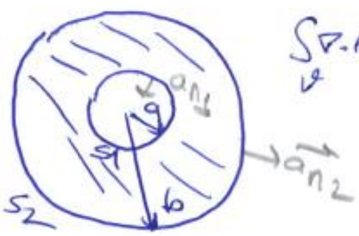
$$\int_v (\vec{\nabla}.\vec{A}).dv = \oint_S \vec{A}.d\vec{S}$$

You can use any tools to evaluate the integrals

One suggestion: https://www.wolframalpha.com/

Question

Verify Div. Theorem



$$\int_v \nabla\cdot A\, dv \overset{?}{=} \oint_S \bar{A}\cdot d\bar{s}$$

$$\bar{\nabla}\cdot\bar{A} = \left[\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}a_y + \frac{\partial}{\partial z}a_z\right]\cdot\left[x\bar{a}_x + y\bar{a}_y\right] = 1+1 = \underline{\underline{2}}$$

$$\int_v (\nabla\cdot A)\,dv = \underbrace{\frac{4}{3}\pi\left[b^3-a^3\right]}_{v}\,2 = \boxed{\frac{8}{3}\pi\left[b^3-a^3\right]}$$

RHS → we have two surfaces S_1 & S_2

$$\oint_S = \int_{S_1} + \int_{S_2} \qquad \boxed{\begin{aligned} d\vec{S}_1 &= -\vec{a}_R\, dS_1 \\ d\vec{S}_2 &= \vec{a}_R\, dS_2 \end{aligned}}$$

$$\int_{S_1} \vec{A}\cdot dS_1 =$$

$\vec{A} = x\bar{a}_x + y\bar{a}_y$

$d\bar{S}_1 = -\bar{a}_R\, a^2 \sin\theta\, d\theta\, d\phi$

$\vec{A}\cdot d\bar{S}_1 =$

$a_R\cdot a_x = \sin\theta\cos\phi$

$x = a\sin\theta\cos\phi$

$a_R\cdot a_y = \sin\theta\sin\phi$

$y = a\sin\theta\sin\phi$

$$\int_{S_1} \vec{A}\cdot dS_1 = -\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overbrace{\left[[a\sin\theta\cos\phi]a_x + [a\sin\theta\sin\phi]a_y\right]}^{\vec{A}}\cdot\left[\vec{a}_R\, a^2\sin\theta\, d\theta\, d\phi\right]$$

$$= -\int_0^{2\pi}\int_0^{\pi}\left[a^3(\sin\theta\cos\phi)(\sin\theta\cos\phi) + (a^3\sin\theta\sin\phi)(\sin\theta\sin\phi)\right]d\theta\, d\phi$$

$$= -a^3\int_0^{2\pi}\int_0^{\pi}(\sin^2\theta\cos\phi^2 + \sin^2\theta\sin^2\phi)\,d\theta\, d\phi$$

$$-a^3\Big[\underbrace{\int_0^{2\pi}\cos^2\phi\,d\phi}_{\pi}\underbrace{\int_0^{\pi}\sin^3\theta\,d\theta}_{\frac{4}{3}}+\underbrace{\int_0^{2\pi}\sin^2\phi\,d\phi}_{\pi}\underbrace{\int_0^{\pi}\sin^3\theta\,d\theta}_{4/3}\Big]$$

$$=-a^3\frac{4\pi}{3}\cdot 2=\boxed{-\frac{8\pi a^3}{3}}$$

~~Second~~ $\int_{S_2}\vec{A}\cdot d\vec{s}_2$, similar to above integral
except $R=b$ $\quad d\vec{s}=ds\,\bar{a}_R$

$$\int_{S_2}\vec{A}\cdot d\vec{s}_2=$$

$\bar{a}_R\cdot\bar{a}_x=\sin\theta\cos\phi$
~~adistant~~ $=$
$\bar{a}_R\cdot\bar{a}_y=\sin\theta\sin\phi$

$x=b\sin\theta\cos\phi$ on surface
$y=b\sin\theta\sin\phi$

$$=\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\big[[b\sin\theta\cos\phi]a_x+[b\sin\theta\sin\phi]a_y\big]\cdot(\bar{a}_R\,b^2\sin\theta\,d\theta\,d\phi)$$

$$=\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\big[b^3(\sin^2\theta\cos\phi)(\sin\theta\cos\phi)+(b^3\sin^2\theta\sin\phi)(\sin\theta\sin\phi)\big]d\theta\,d\phi$$

$$=b^3\int_0^{2\pi}\int_0^{\pi}\big[\sin^3\theta\cos^2\phi+\sin^3\theta\sin^2\phi\big]d\theta\,d\phi$$

$$=b^3\Big[\int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin^3\theta\,d\theta+\int_0^{2\pi}\sin^2\phi\,d\phi\int_0^{\pi}\sin^3\theta\,d\theta$$

$$=b^3\frac{4\pi}{3}\cdot 2=\frac{8\pi b^3}{3}$$

$$\oint=\int_{S_1}+\int_{S_2}\Rightarrow\frac{8\pi}{3}[b^3-a^3]\quad \underline{\text{verified}}\quad LHS=RHS$$

Q-2) Verifty the Divergence Theorem for the vector field $\vec{A} = 3R\hat{a}_R$ given in spherical coordinates, and for the conical region (of height $h = 2$ and apex angle $\theta_0 = \frac{\pi}{4}$) shown in the figure below.

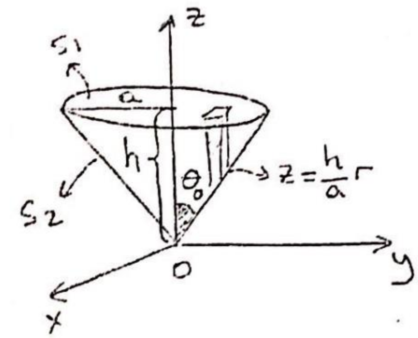


Figure 1. The geometry of Q-2.

Question

$\theta_0 = \pi/4 = 45^0$

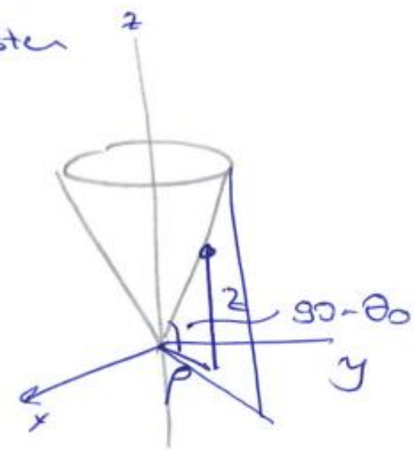
$\int \bar{\nabla}\cdot\bar{A}\, dv = \oint \bar{A}\cdot d\bar{s}$

LHS RHS

$\bar{\nabla}\cdot\bar{A} = \nabla\cdot(A_R \bar{a}_R) = \frac{1}{R^2}\frac{\partial}{\partial R}\left[R^2 A_R\right] = \frac{1}{R^2}\frac{\partial}{\partial R}\left[3R^3\right] = 9$

choose cylindrical coordinate system

$\tan(90-\theta_0) = \frac{z}{\rho}$



Let us write z in term of
ρ and given parameter

$\underbrace{\tan(90-\theta_0) = \frac{h}{a}}_{\text{constant}}$

$z = \tan(90-\theta_0)\,\rho$

$\boxed{z = \frac{h\rho}{a}}$ → z can be obtained in term of ρ

$LHS = \int_V \nabla\cdot A\, dv = \int_{\rho=0}^{a}\int_{z=\frac{h}{a}\rho}^{h}\int_{\phi=0}^{2\pi} 9\rho\, d\phi\, d\rho\, dz = 2\pi 9 \int_{\rho=0}^{a} d\rho\, \rho \times$

$\times\left(h - \frac{h}{a}\rho\right)\Big|_0^a$

$\rho\, d\phi\, d\rho\, dz$

$LHS = 9.2\pi\left(\frac{ha^2}{2} - \frac{ha^2}{3}\right) = \boxed{\frac{9}{3}}\, h\pi a^2$

volume of the cone

$a, h = 2$
$a^2 = 4$
$h = 2$

$\Rightarrow \boxed{24\pi}$ LHS

RHS

$$\oint \bar{A}\cdot d\bar{S} = \int_{S_1}\left[3R\overbrace{\bar{a}_R\cdot\bar{a}_z}^{\cos\theta}\, dS_z\right] + \int_{S_2}\cancel{(3R\bar{a}_R)\cdot\bar{a}_\theta\, dS_\theta}$$

$a_R\cdot a_\theta = 0$

$$= \int_{S_1} 3\underbrace{R\cos\theta}_{h}\, dS_z = 6\int_{S_2} dS_z = \boxed{24\pi} \quad \text{RHS}$$

$\pi a^2 = 4\pi$

$R\cos\theta$ on surface

$(h=2)$

LHS = RHS = 24π

Div. Theorem is verified

Q-3) A vector field $\vec{F}$, is defined with the following expression:

$$\vec{F} = \hat{a}_x + \hat{a}_y z^4 + \hat{a}_z(4yz^3 + 5)$$

(a) *Determine the divergence of* $\vec{F}$

(b) *Determine the curl of* $\vec{F}$

(c) *Find* $\int_C \vec{F}.\overrightarrow{dl}$ *where the contour C is defined from the point* $P_1(0,0,3)$ *to point* $P_2(1,0,3)$ *as shown in the figure*

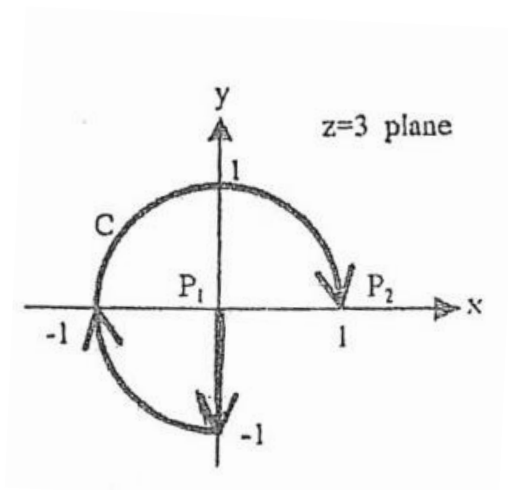


Figure 2. The geometry of Q-3.

question

$\vec{F} = \vec{a_x} + z^4 \vec{a_y} + (4yz^3+5)\vec{a_z}$

$\boxed{z=3}$

(a)

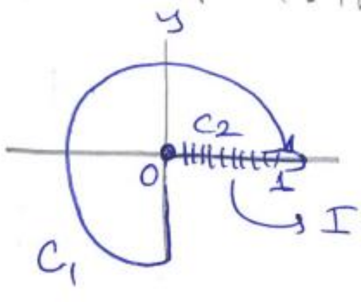
$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}a_x + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}a_z\right)\cdot\left(a_x + z^4 \vec{a_y} + (4yz^3+5)\vec{a_z}\right)$

$$\frac{\partial}{\partial z}(4yz^3+5) = 12yz^2 = \underline{\underline{12yz^2}}$$

(b)

$$\nabla \times F = \begin{vmatrix} a_x & a_y & a_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1 & z^4 & 4yz^3+5 \end{vmatrix} = \underline{\underline{0}}$$

F is irrotational, conservative field.
path independent. Therefore I choose simple contour



$\int_{C_1} \bar{F}\cdot d\bar{l} = \int_{C_2} \bar{F}\cdot d\bar{l} \Rightarrow$ all of them the full credit

I choose this path (since path independent)

$C_2: d\vec{l} = dx\,\vec{a_x}$

$$\int_{C_1} \vec{F}\cdot d\bar{l} = \int_{C_2} \bar{F}\cdot d\bar{l} \Rightarrow \int_0^1 \left(a_x + z^4 a_y + 4yz^3 + 5a_z\right)\cdot\vec{a_x}\,dx$$

$(z=3) \Rightarrow 1$

$\int_C \bar{F}\cdot d\bar{l} = \underline{\underline{1}}$

~~Any path~~ Any path can be chosen.
Since it is path independent

Q-4) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point $(2, -1, 2)$

Hint: Think about the Gradient operator and then Dot Product

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2 + y^2 + z^2 = 9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z = x^2 + y^2 - 3$ or $x^2 + y^2 - z = 3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2 + y^2 - z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2| \cos\theta$, where θ is the required angle. Then

$$(4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})\cdot(4\mathbf{i} - 2\mathbf{j} - \mathbf{k}) = |4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}|\,|4\mathbf{i} - 2\mathbf{j} - \mathbf{k}| \cos\theta$$

$$16 + 4 - 4 = \sqrt{(4)^2 + (-2)^2 + (4)^2}\,\sqrt{(4)^2 + (-2)^2 + (-1)^2} \cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \arccos 0.5819 = 54^\circ 25'$.

Q-5) Prove that $\nabla^2 \left(\frac{1}{r}\right) = 0$

Here $\boldsymbol{r}$ can be taken as $r = \sqrt{x^2 + y^2 + z^2}$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left[-x\,(x^2+y^2+z^2)^{-3/2}\right]$$

$$= 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} \quad \text{and} \quad \frac{\partial^2}{\partial z^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$$

Then by addition, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = 0.$

The equation $\nabla^2\phi = 0$ is called *Laplace's equation*. It follows that $\phi = 1/r$ is a solution of this equation.