CHAPTER 2

LAMINA STRESS-STRAIN RELATIONSHIPS

2.1 INTRODUCTION

The basic building block of a composite structure is the lamina, which usually consists of one of the fiber/matrix configurations shown in Fig. 1.4. For the purposes of mechanics analysis, however, the "unidirectionally reinforced," or "unidirectional" lamina with an arrangement of parallel, continuous fibers is the most convenient starting point. As shown in subsequent chapters, the stress-strain relationships for the unidirectional lamina form the basis for the analysis of not only the continuous fiber composite laminate [Fig. 1.4(*a*)], but of woven fiber [Fig. 1.4(*b*)] and chopped fiber composites [Figs. 1.4(*c*) and (*d*)] as well.

A composite material is obviously heterogeneous at the constituent material level, with properties possibly changing from point to point. For example, the stress-strain relationships at a point are different for a point in the fiber material from how they are for a point in the matrix material. If we take the composite lamina as the basic building block, however, the "macromechanical" stress-strain relationships of the lamina can be expressed in terms of average stresses and strains and effective properties of an equivalent homogeneous material [2.1]. This chapter is concerned with the development and manipulation of these macromechanical

stress-strain relationships without regard for the constituent ma...als or their interactions. The "micromechanical" relationships between the constituent material properties and the effective lamina properties will be discussed in Chap. 3.

To complicate matters further, the properties of a composite are usually anisotropic. That is, the properties associated with an axis passing through a point in the material generally depend on the orientation of the axis. By comparison, conventional metallic materials are nearly isotropic since their properties are essentially independent of orientation. Fortunately, each type of composite has characteristic material property symmetries that make it possible to simplify the general anisotropic stress-strain relationships. In particular, the symmetry possessed by the unidirectional lamina makes it a so-called orthotropic material. The symmetries associated with various types of composite laminae and the resulting lamina stress-strain relationships are discussed in this chapter, along with certain mathematical manipulations that make it easier to deal with the directional nature of composite properties.

2.2 EFFECTIVE MODULI IN STRESS-STRAIN RELATIONSHIPS

A general three-dimensional state of stress at a point in a material can be described by nine stress components σ_{ij} (where $i, j = 1, 2, 3$), as shown in Fig. 2.1. According to the conventional subscript notation, when $i = j$,

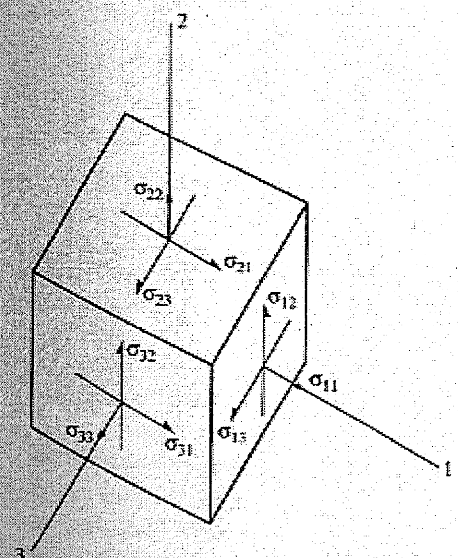

FIGURE 2.1
Three-dimensional state of stress.

the stress component σ_{ij} is a normal stress; and when $i \neq j$, the stress component is a shear stress. The first subscript refers to the direction of the outward normal to the face on which the stress component acts, and the second subscript refers to the direction in which the stress component itself acts.

Corresponding to each of the stress components, there is a strain component ϵ_{ij} describing the deformation at the point. Normal strains ($i=j$) describe the extension or contraction per unit length along the x_i direction, and shear strains ($i \neq j$) describe the distortional deformations associated with lines that were originally parallel to the x_i and x_j axes. It is very important to distinguish between the "tensor" strain ϵ_{ij} and the "engineering" strain γ_{ij}. In the case of normal strain the engineering strain is the same as the tensor strain, but for shear strain $\epsilon_{ij} = \gamma_{ij}/2$. Thus, the engineering shear strain γ_{ij} describes the total distortional change in the angle between lines that were originally parallel to the x_i and x_j axes, but the tensor shear strain ϵ_{ij} describes the amount of rotation of either of the lines.

In the most general stress-strain relationship *at a point* in an elastic material each stress component is related to each of the nine strain components by an equation of the form

$$\sigma_{ij} = f_{ij}(\epsilon_{11}, \epsilon_{12}, \epsilon_{13}, \epsilon_{21}, \epsilon_{22}, \epsilon_{23}, \epsilon_{31}, \epsilon_{32}, \epsilon_{33}) \tag{2.1}$$

where the functions f_{ij} may be nonlinear. For the linear elastic material, which is the primary concern in this book, the most general linear stress-strain relationships *at a point* in the material (excluding effects of environmental conditions) are given by equations of the form

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \\ \sigma_{32} \\ \sigma_{13} \\ \sigma_{21} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} & C_{1132} & C_{1113} & C_{1121} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} & C_{2232} & C_{2213} & C_{2221} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} & C_{3332} & C_{3313} & C_{3321} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{2111} & C_{2122} & C_{2133} & C_{2123} & C_{2131} & C_{2112} & C_{2132} & C_{2113} & C_{2121} \end{bmatrix} \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \\ \epsilon_{32} \\ \epsilon_{13} \\ \epsilon_{21} \end{Bmatrix} \tag{2.2}$$

where $[C]$ is a fully populated 9×9 matrix of stiffnesses or elastic constants (or moduli) having 81 components. Note that the first two subscripts on the elastic constants correspond to those of the stress, whereas the last two subscripts correspond to those of the strain. If no further restrictions are placed on the elastic constants, the material is called anisotropic and Eq. (2.2) is referred to as the generalized Hooke's law for anisotropic materials. In practice, there is no need to deal with this equation and its 81 elastic constants because various symmetry conditions simplify the equations considerably.

As shown in any mechanics of materials book [2.2], both stresses and strains are symmetric (i.e., $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{ij} = \epsilon_{ji}$), so that there are only six independent stress components and six independent strain components. This means that the elastic constants must be symmetric with respect to the first two subscripts and with respect to the last two subscripts (i.e., $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$ where $i, j, k, l = 1, 2, 3$), and that the number of nonzero elastic constants is now reduced to 36. These simplifications lead to a contracted notation that reduces the number of subscripts based on the following changes in notation [2.3–2.6]:

$$\begin{aligned} \sigma_{11} &= \sigma_1 & \epsilon_{11} &= \epsilon_1 \\ \sigma_{22} &= \sigma_2 & \epsilon_{22} &= \epsilon_2 \\ \sigma_{33} &= \sigma_3 & \epsilon_{33} &= \epsilon_3 \\ \sigma_{23} = \sigma_{32} &= \sigma_4 & 2\epsilon_{23} = 2\epsilon_{32} = \gamma_{23} = \gamma_{32} &= \epsilon_4 \\ \sigma_{13} = \sigma_{31} &= \sigma_5 & 2\epsilon_{13} = 2\epsilon_{31} = \gamma_{13} = \gamma_{31} &= \epsilon_5 \\ \sigma_{12} = \sigma_{21} &= \sigma_6 & 2\epsilon_{12} = 2\epsilon_{21} = \gamma_{12} = \gamma_{21} &= \epsilon_6 \end{aligned}$$

With this contracted notation the generalized Hooke's law can now be written as

$$\sigma_i = C_{ij}\epsilon_j, \qquad i, j = 1, 2, \ldots, 6 \tag{2.3}$$

and the repeated subscript j implies summation on that subscript. Alternatively, in matrix form

$$\{\sigma\} = [C]\{\epsilon\} \tag{2.4}$$

where the elastic constant matrix or stiffness matrix $[C]$ is now 6×6 with 36 components and the stresses $\{\sigma\}$ and strains $\{\epsilon\}$ are column vectors, each having six elements. Alternatively, the generalized Hooke's law relating strains to stresses can be written as

$$\epsilon_i = S_{ij}\sigma_j; \qquad i, j = 1, 2, \ldots, 6 \tag{2.5}$$

or in matrix form as

$$\{\epsilon\} = [S]\{\sigma\} \tag{2.6}$$

where $[S]$ is the compliance matrix, which is the inverse of the stiffness matrix ($[S] = [C]^{-1}$). As shown later, due to the existence of the strain energy density, the stiffness and compliance matrices are symmetric. Note that nothing has been said thus far about any symmetry that the material

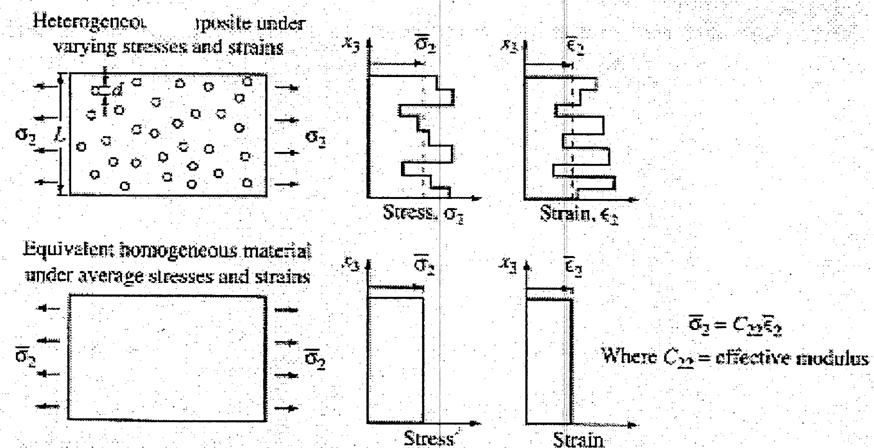

FIGURE 2.2
Concept of an effective modulus of an equivalent homogeneous material.

itself may have. All real materials have some form of symmetry, however, and no known material is completely anisotropic.

Before discussing the various simplifications of the stress-strain relationships, it is appropriate to deal with the problem of heterogeneity in the composite material. Recall that the stress-strain relationships presented up to now are only valid *at a point* in the material, and that the stresses, strains, and elastic moduli will change as we move from point to point in a composite (i.e., the elastic moduli for the matrix material are different from those of the fiber). In order to analyze the macromechanical behavior of the composite, it is more convenient to deal with *averaged* stresses and strains which are related by "effective moduli" of an equivalent homogeneous material.

As shown in Fig. 2.2, if the scale of the inhomogeneity in a material can be characterized by some length dimension, d, then the length dimension, L, over which the macromechanical averaging is to take place, must be much larger than d if the average stresses and strains are to be related by effective moduli of an equivalent homogeneous material. We now define the average stresses, $\bar{\sigma}_i$, and the average strains, $\bar{\epsilon}_i$, $(i = 1, 2, \ldots, 6)$ to be averaged over a volume V, which is characterized by the dimension L, so that [2.1]

$$\bar{\sigma}_i = \int_V \sigma_i \, dv/V \tag{2.7}$$

$$\bar{\epsilon}_i = \int_V \epsilon_i \, dv/V \tag{2.8}$$

where $i = 1, 2, \ldots, 6$ and the σ_i and the ϵ_i are the position-dependent stresses and strains at a point, respectively. If these averaged stresses and strains are used in place of the stresses and strains at a point, the generalized Hooke's law [i.e., Eq. (2.3)] becomes

$$\bar{\sigma}_i = C_{ij}\bar{\epsilon}_j \tag{2.9}$$

and the elastic moduli C_{ij} then become the "effective moduli" of the equivalent homogeneous material in volume V. Similarly, the "effective compliances" S_{ij} may be defined by

$$\bar{\epsilon}_i = S_{ij}\bar{\sigma}_j \tag{2.10}$$

For example, in Fig. 2.2 the scale of the inhomogeneity is assumed to be the diameter of the fiber, d, and the averaging dimension, L, is assumed to be a characteristic lamina dimension such that $L \gg d$. The effective modulus C_{22} of the lamina is thus defined. *In the remainder of this book, lamina properties are assumed to be effective properties as described above.*

2.3 SYMMETRY IN STRESS-STRAIN RELATIONSHIPS

In this section the generalized anisotropic Hooke's law will be simplified and specialized using various symmetry conditions. The first symmetry condition, which has nothing to do with material symmetry, is strictly a result of the existence of a strain energy density function [2.3, 2.6]. The strain energy density function, W, is such that the stresses can be derived according to the equation

$$\sigma_i = \frac{\partial W}{\partial \epsilon_i} = C_{ij}\epsilon_j \tag{2.11}$$

where

$$W = \frac{1}{2} C_{ij}\epsilon_i\epsilon_j \tag{2.12}$$

By taking a second derivative of W, we find that

$$\frac{\partial^2 W}{\partial \epsilon_i \, \partial \epsilon_j} = C_{ij} \tag{2.13}$$

and by reversing the order of differentiation, we find that

$$\frac{\partial^2 W}{\partial \epsilon_j \, \partial \epsilon_i} = C_{ji} \tag{2.14}$$

Since the result must be the same regardless of the order of the differentiation, $C_{ij} = C_{ji}$ and the stiffness matrix is symmetric. Similarly, W can be expressed in terms of compliances and stresses, and by taking two derivatives with respect to stresses, it can be shown that $S_{ij} = S_{ji}$. Thus, the compliance matrix is also symmetric. Due to these mathematical manipulations, only 21 of the 36 anisotropic elastic moduli or compliances are independent, and we still have not said anything about any inherent symmetry of the material itself.

According to the above developments, the stiffness matrix for the linear elastic anisotropic material without any material property symmetry is of the form

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & \text{SYM} & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \tag{2.15}$$

Further simplifications of the stiffness matrix are possible only if the material properties have some form of symmetry. For example, a *monoclinic* material has one plane of material property symmetry. It can be shown that [2.3, 2.7] since the C_{ij} for such a material must be invariant under a transformation of coordinates corresponding to reflection in the plane of symmetry, the number of independent elastic constants for the monoclinic material is reduced to 13. Such a symmetry condition is not of practical interest in composite material analysis, however.

As shown in Fig. 2.3, a unidirectional composite lamina has three mutually orthogonal planes of material property symmetry (i.e., the 12, 23, and 13 planes) and is called an *orthotropic* material. The term "orthotropic" alone is not sufficient to describe the form of the stiffness matrix, however. Unlike the anisotropic stiffness matrix [(Eq. (2.15)], which has the same form (but different terms) for different coordinate systems, the form of the stiffness matrix for the orthotropic material depends on the coordinate system used. The 123 coordinate axes in Fig. 2.3 are referred to as the *principal material coordinates* since they are associated with the reinforcement directions. Invariance of the C_{ij} under transformations of coordinates corresponding to reflections in two orthogonal planes [2.3, 2.7] may be used to show that the stiffness matrix for a so-called *specially orthotropic* material associated with the principal material coordinates is of the form

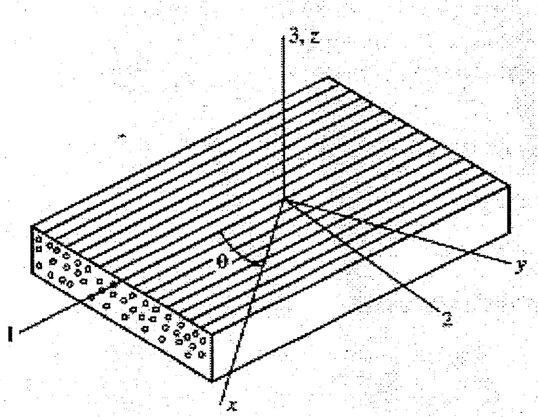

FIGURE 2.3
Orthotropic lamina with principal and nonprincipal coordinate systems.

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & \text{SYM} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \tag{2.16}$$

A stiffness matrix of this form in terms of engineering constants will be obtained in the next section using observations from simple experiments. Note that there are only 12 nonzero elastic constants and 9 independent elastic constants for the specially orthotropic material.

Table 2.1 summarizes similar results for the different combinations of materials and coordinate systems used in this book. It will also be shown later that if the stress-strain relationshiips for the same orthotropic material are developed for a nonprincipal coordinate system xyz as shown in Fig. 2.3, the stiffness matrix is of the same form as that of the anisotropic material in Eq. (2.15). In such a nonprincipal, or off-axis coordinate system, the material is called *generally orthotropic* (see Table 2.1).

There are two other types of material symmetry that are important in the study of composites. The details will be developed in the next section, but the general forms of the stiffness matrices are given here for completeness. In most composites the fiber-packing arrangement is statistically random in nature, so that the properties are nearly the same in any direction perpendicular to the fibers (i.e., the properties along the 2 direction are the same as those along the 3 direction), and the material is *transversely isotropic*. For such a material we would expect that $C_{22} = C_{33}$, $C_{12} = C_{13}$, $C_{55} = C_{66}$, and that C_{44} would not be independent from the other stiffnesses. It can be shown [2.1] that the complete

TABLE 2.1
Elastic coefficients in the stress-strain relationships for different materials and coordinate systems

Material and coordinate system	Number of nonzero coefficients	Number of independent coefficients
Three-dimensional case		
Anisotropic	36	21
Generally Orthotropic (nonprincipal coordinates)	36	9
Specially Orthotropic (principal coordinates)	12	9
Specially Orthotropic, transversely isotropic	12	5
Isotropic	12	2
Two-dimensional case (lamina)		
Anisotropic	9	6
Generally Orthotropic (nonprincipal coordinates)	9	4
Specially Orthotropic (principal coordinates)	5	4
Balanced orthotropic, or square symmetric (principal coordinates)	5	3
Isotropic	5	2

stiffness matrix for a specially orthotropic, transversely isotropic material is of the form

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{22} & 0 & 0 & 0 \\ & \text{SYM} & & (C_{22}-C_{23})/2 & 0 & 0 \\ & & & & C_{66} & 0 \\ & & & & & C_{66} \end{bmatrix} \tag{2.17}$$

where the 23 plane and all parallel planes are assumed to be planes of isotropy. In the next section a stiffness matrix of the same form will be derived, except that the so-called *engineering constants* will be used instead of the C_{ij}. Note that now there are still 12 nonzero elastic moduli but that only 5 are independent (see Table 2.1).

The simplest form of the stress-strain relationship occurs when the material is *isotropic* and every coordinate axis is an axis of symmetry. Now we would expect that $C_{11} = C_{22} = C_{33}$, $C_{12} = C_{13} = C_{23}$, that $C_{44} = C_{55} = C_{66}$, and that C_{44} again would not be independent from the other C_{ij}. The isotropic stiffness matrix is of the form [2.1]

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & \text{SYM} & & (C_{11}-C_{12})/2 & 0 & 0 \\ & & & & (C_{11}-C_{12})/2 & 0 \\ & & & & & (C_{11}-C_{12})/2 \end{bmatrix} \tag{2.18}$$

Now there are still 12 nonzero elastic constants, but only 2 are independent (see Table 2.1). Similar equations based on the engineering constants will be derived in the next section. Equations of this form can be found in any mechanics of materials book, and the design of metallic components is usually based on such formulations.

2.4 ORTHOTROPIC AND ISOTROPIC ENGINEERING CONSTANTS

In the previous section symmetry conditions were shown to reduce the number of elastic constants (the C_{ij} or S_{ij}) in the stress-strain relationships for several important classes of materials and the general forms of the relationships were presented. When a material is characterized experimentally, however, the so-called "engineering constants" such as Young's modulus (or modulus of elasticity), shear modulus, and Poisson's ratio are usually measured instead of the C_{ij} or the S_{ij}. The engineering constants are also widely used in analysis and design because they are easily defined and interpreted in terms of simple states of stress and strain. In this section, several simple tests and their resulting states of stress and strain will be used to develop the three-dimensional and two-dimensional stress-strain relationships for orthotropic and isotropic materials.

Consider a simple uniaxial tensile test consisting of an applied longitudinal normal stress, σ_1, along the reinforcement direction (i.e., the

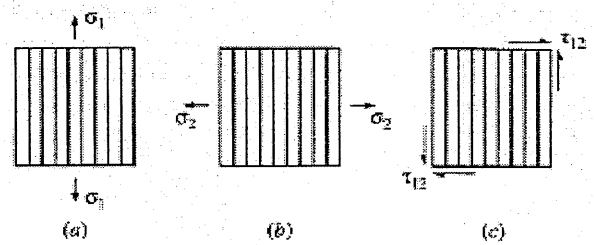

FIGURE 2.4
Simple states of stress used to define lamina engineering constants.

1 direction) of a specimen from an orthotropic material, as shown in Fig. 2.4(a). It is assumed that all other stresses are equal to zero. Within the linear range the experimental observation is that the resulting strains associated with the 123 axes can be expressed empirically in terms of "engineering constants" as

$$\begin{aligned} \epsilon_1 &= \sigma_1/E_1 \\ \epsilon_2 &= -\nu_{12}\epsilon_1 = -\nu_{12}\sigma_1/E_1 \\ \epsilon_3 &= -\nu_{13}\epsilon_1 = -\nu_{13}\sigma_1/E_1 \\ \gamma_{12} &= \gamma_{23} = \gamma_{13} = 0 \end{aligned} \tag{2.19}$$

where E_1 = longitudinal modulus of elasticity associated with the 1 direction

$\nu_{ij} = -\epsilon_j/\epsilon_i$ is the Poisson's ratio, the ratio of the strain in the j direction to the strain in the perpendicular i direction when the applied stress is in the i direction

Recall from mechanics of materials [2.2] that for isotropic materials no subscripts are needed on properties such as the modulus of elasticity and the Poisson's ratio because the properties are the same in all directions. This is not the case with orthotropic materials, however, and subscripts are needed on these properties because of their directional nature. For example, $E_1 \neq E_2$ and $\nu_{12} \neq \nu_{21}$. Note that, as with isotropic materials, a negative sign must be used in the definition of Poisson's ratio. A property like ν_{12} is usually called a *major Poisson's ratio*, whereas a property like ν_{21} is called a *minor Poisson's ratio*. As with isotropic materials, a normal stress induces only normal strains, and all shear strains are equal to zero. This lack of shear/normal interaction is observed only for the principal material coordinate system, however. For any other set of coordinates the so-called "shear-coupling" effect is present. This effect will be discussed in more detail later.

Now consider a similar experiment where a transverse normal stress, σ_2, is applied to the same material as shown in Fig. 2.4(b), with all other stresses being equal to zero. Now the experimental observation is that the resulting strains can be expressed as

$$\begin{aligned} \epsilon_2 &= \sigma_2/E_2 \\ \epsilon_1 &= -\nu_{21}\epsilon_2 = -\nu_{21}\sigma_2/E_2 \\ \epsilon_3 &= -\nu_{23}\epsilon_2 = -\nu_{23}\sigma_2/E_2 \\ \gamma_{12} &= \gamma_{23} = \gamma_{13} = 0 \end{aligned} \tag{2.20}$$

where E_2 is the transverse modulus of elasticity associated with the 2 direction. A similar result for an applied transverse normal stress, σ_3, can be obtained by changing the appropriate subscripts in Eqs. (2.20).

Next, consider a shear test where a pure shear stress, $\sigma_{12} = \tau_{12}$, is applied to the material in the 12 plane, as shown in Fig. 2.4(c). Now the experimental observation is that resulting strains can be written as

$$\begin{aligned} \gamma_{12} &= \tau_{12}/G_{12} \\ \epsilon_1 &= \epsilon_2 = \epsilon_3 = \gamma_{13} = \gamma_{23} = 0 \end{aligned} \tag{2.21}$$

where G_{12} is the shear modulus associated with the 12 plane. Similar results can be obtained for pure shear in the 13 and 23 planes by changing the appropriate subscripts in Eq. (2.21). Again, notice that there is no shear/normal interaction (or shear coupling). As before, however, this is only true for the principal material axes.

Finally, consider a general three-dimensional state of stress consisting of all possible normal and shear stresses associated with the 123 axes as shown in Fig. 2.1. Since we are dealing with linear behavior, it is appropriate to use superposition and add all the resulting strains due to the simple uniaxial and shear tests, as given in Eqs. (2.19), (2.20), (2.21), and similar equations as described above. The resulting set of equations is given below:

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} \tag{2.22}$$

Note that the compliance matrix is of the same form as the stiffness matrix for a specially orthotropic material [Eq. (2.16)] as it should be because $[S] = [C]^{-1}$. Note also that due to symmetry of the compliance matrix, $\nu_{ij}/E_i = \nu_{ji}/E_j$ and only nine of the engineering constants are independent.

If we now consider a simple uniaxial tensile test consisting of an applied normal stress, σ_x, along some arbitrary x axis as shown in Fig. 2.3, we find that the full complement of normal strains and shear strains are developed. The generation of shear strains due to normal stresses and normal strains due to shear stresses is often referred to as the "shear-coupling effect." As a result of shear coupling, all the zeros disappear in the compliance matrix and it becomes fully populated for the general three-dimensional state of stress associated with the arbitrary xyz axes; this is the *generally orthotropic* material. The stiffness or compliance matrices for the generally orthotropic material are of the same form as those for the general anisotropic material [Eq. (2.15)], although the material still has its orthotropic symmetries with respect to the principal material axes. Obviously then, the experimental characterization of such a material is greatly simplified by testing it as a specially orthotropic material along the principal material directions. As shown later, once we have the stiffnesses or compliances associated with the 123 axes, we can obtain those for an arbitrary off-axis coordinate system such as xyz by transformation equations involving the angles between the axes.

If the material being tested is specially orthotropic and transversely isotropic, the subscripts 2 and 3 in Eqs. (2.22) are interchangeable, and we have $G_{13} = G_{12}$, $E_2 = E_3$, $\nu_{21} = \nu_{31}$, and $\nu_{23} = \nu_{32}$. In addition, the familiar relationship among the isotropic engineering constants [2.2] is now valid for the engineering constants associated with the 23 plane, so that

$$G_{23} = \frac{E_2}{2(1+\nu_{32})} \tag{2.23}$$

Now the compliance matrix is of the same form as Eq. (2.17) and only five of the engineering constants are independent.

Finally, for the isotropic material there is no need for subscripts and $G_{13} = G_{23} = G_{12} = G$, $E_1 = E_2 = E_3 = E$, $\nu_{12} = \nu_{23} = \nu_{13} = \nu$, and $G = E/2(1+\nu)$. Now the compliance matrix is of the same form as Eq. (2.18) and only two of the engineering constants are independent.

2.5 THE SPECIALLY ORTHOTROPIC LAMINA

As shown later in the analysis of laminates, the lamina is often assumed to be in a simple two-dimensional state of stress (or plane stress). In this case the specially orthotropic stress-strain relationships in Eqs. (2.22) can be simplified by letting $\sigma_3 = \tau_{23} = \tau_{31} = 0$, so that

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \tag{2.24}$$

where the compliances S_{ij} and the engineering constants are related by the equations

$$\begin{aligned} S_{11} &= \frac{1}{E_1}, \qquad S_{22} = \frac{1}{E_2}, \\ S_{12} = S_{21} &= -\frac{\nu_{21}}{E_2} = -\frac{\nu_{12}}{E_1}, \qquad S_{66} = \frac{1}{G_{12}} \end{aligned} \tag{2.25}$$

Thus, there are five nonzero compliances and only four independent compliances for the specially orthotropic lamina (Table 2.1). The lamina stresses in terms of *tensor* strains are given by

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & 2Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12}/2 \end{Bmatrix} \tag{2.26}$$

where the Q_{ij} are the components of the lamina stiffness matrix, which are related to the compliances and the engineering constants by

$$\begin{aligned} Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} = \frac{E_1}{1-\nu_{12}\nu_{21}} \\ Q_{12} &= -\frac{S_{12}}{S_{11}S_{22} - S_{12}^2} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = Q_{21} \\ Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} = \frac{E_2}{1-\nu_{12}\nu_{21}} \\ Q_{66} &= \frac{1}{S_{66}} = G_{12} \end{aligned} \tag{2.27}$$

Note that the factor of 2 has been introduced in the Q_{66} term of Eq. (2.26) to compensate for the use of tensor shear strain. The reason for this will become apparent in the next section. As shown later, the experimental characterization of the orthotropic lamina involves the measurement of four independent engineering constants such as E_1, E_2, G_{12}, and ν_{12}. Typical values of these properties for several composites are shown in Table 2.2.

The balanced orthotropic lamina shown schematically in Fig. 2.5 often occurs in practice when the fiber reinforcement is woven or cross-plied at 0° and 90°. In this case the number of independent elastic constants in Eqs. (2.24) to (2.27) is reduced to 3 because of the double symmetry of properties with respect to the 1 and 2 axes (Table 2.1).

TABLE 2.2
Typical values of lamina engineering constants for several composites

Material	E_1 Msi (GPa)	E_2 Msi (GPa)	G_{12} Msi (GPa)	ν_{12}	v_f
T300/934 graphite/epoxy	19.0 (131)	1.5 (10.3)	1.0 (6.9)	0.22	0.65
AS/3501 graphite/epoxy	20.0 (138)	1.3 (9.0)	1.0 (6.9)	0.3	0.65
p-100/ERL 1962 pitch graphite/epoxy	68.0 (468.9)	0.9 (6.2)	0.81 (5.58)	0.31	0.62
Kevlar® 49/934 aramid/epoxy	11.0 (75.8)	0.8 (5.5)	0.33 (2.3)	0.34	0.65
Scotchply® 1002 E-glass/epoxy	5.6 (38.6)	1.2 (8.27)	0.6 (4.14)	0.26	0.45
Boron/5505 boron/epoxy	29.6 (204.0)	2.68 (18.5)	0.81 (5.59)	0.23	0.5
Spectra® 900/826 polyethylene/epoxy	4.45 (30.7)	0.51 (3.52)	0.21 (1.45)	0.32	0.65
E-glass/470-36 E-glass/vinylester	3.54 (24.4)	1.0 (6.87)	0.42 (2.89)	0.32	0.30

Kevlar® is a registered trademark of DuPont Company, Wilmington, Delaware; Scotchply® is a registered trademark of 3M Company, St. Paul, Minnesota; and Spectra® is a registered trademark of Allied-Signal Company, Petersburg, Virginia.

Thus, for the balanced orthotropic lamina we have $E_1 = E_2$, $Q_{11} = Q_{22}$, and $S_{11} = S_{22}$.

2.6 THE GENERALLY ORTHOTROPIC LAMINA

In the analysis of laminates having multiple laminae it is often necessary to know the stress-strain relationships for the *generally orthotropic*

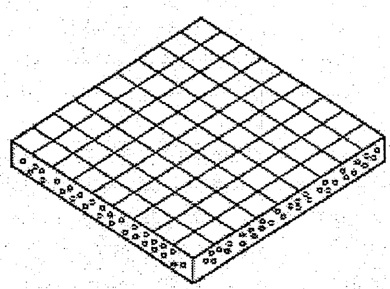

FIGURE 2.5
Balanced orthotropic lamina consisting of fibers oriented at 0° and 90°.

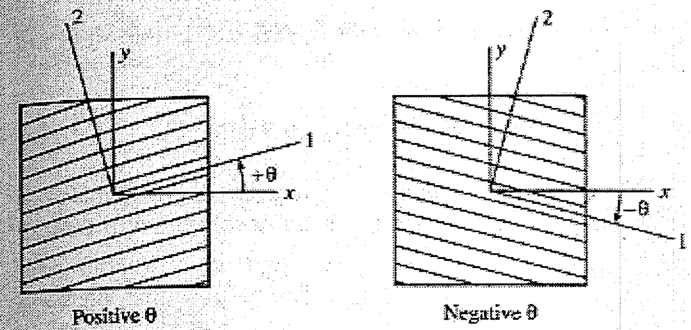

FIGURE 2.6
Sign convention for lamina orientation.

lamina in nonprincipal coordinates (or "off-axis" coordinates) such as x and y in Fig. 2.3. Fortunately, the elastic constants in these so-called "off-axis" stress-strain relationships are related to the four independent elastic constants in the principal material coordinates and the lamina orientation angle. The sign convention for the lamina orientation angle, θ, is given in Fig. 2.6. The relationships are found by combining the equations for transformation of stress and strain components from the 12 axes to the xy axes.

Relationships for transformation of stress components between coordinate axes may be obtained by writing the equations of static equilibrium for the wedge-shaped differential element in Fig. 2.7. For example, the force equilibrium along the x direction is given by

$$\sum F_x = \sigma_x \, dA - \sigma_1 \, dA \cos^2\theta - \sigma_2 \, dA \sin^2\theta + 2\tau_{12} \, dA \sin\theta\cos\theta = 0 \tag{2.28}$$

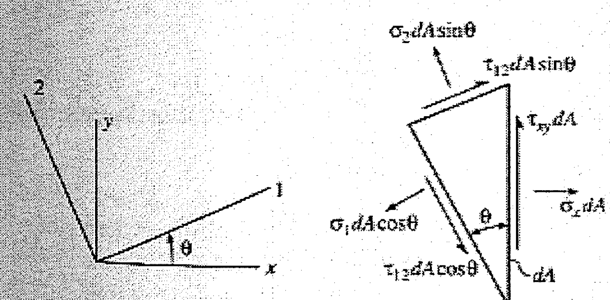

FIGURE 2.7
Differential element under static equilibrium with forces in two coordinate systems.

which, after dividing through by dA, gives an equation relating σ_x to the stresses in the 12 system:

$$\sigma_x = \sigma_1 \cos^2\theta + \sigma_2 \sin^2\theta - 2\tau_{12}\sin\theta\cos\theta \tag{2.29}$$

Using a similar approach, the complete set of transformation equations for the stresses in the xy-coordinate system can be developed and written in matrix form as

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs\\ s^2 & c^2 & 2cs\\ cs & -cs & c^2-s^2\end{bmatrix}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [T]^{-1}\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} \tag{2.30}$$

and the stresses in the 12 system can be written as

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \tau_{12}\end{Bmatrix} = [T]\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} \tag{2.31}$$

where $c = \cos\theta$, $s = \sin\theta$, and the transformation matrix, $[T]$, is defined as

$$[T] = \begin{bmatrix} c^2 & s^2 & 2cs\\ s^2 & c^2 & -2cs\\ -cs & cs & c^2-s^2\end{bmatrix} \tag{2.32}$$

Methods for determining the matrix inverse $[T]^{-1}$ are described in any book dealing with matrices. It can also be shown [2.2, 2.3] that the *tensor strains* transform the same way as the stresses, and that

$$\begin{Bmatrix}\epsilon_1\\ \epsilon_2\\ \gamma_{12}/2\end{Bmatrix} = [T]\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}/2\end{Bmatrix} \tag{2.33}$$

or $\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{bmatrix} = RTR^{-1}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$

$R = \begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{vmatrix}$

Substituting Eqs. (2.33) into Eqs. (2.26), and then substituting the resulting equations into Eqs. (2.30), we find that

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = [T]^{-1}[Q][T]\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}/2\end{Bmatrix} \tag{2.34}$$

where the stiffness matrix $[Q]$ in Eqs. (2.34) is defined in Eqs. (2.26).

Carrying out the indicated matrix multiplications and converting back to engineering strains, we find that

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} \tag{2.35}$$

where the $\bar{Q}_{ij}$ are the components of the transformed lamina stiffness matrix which are defined as follows:

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\theta + Q_{22}\sin^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta\\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\cos^4\theta+\sin^4\theta)\\
\bar{Q}_{22} &= Q_{11}\sin^4\theta + Q_{22}\cos^4\theta + 2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta\\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})\cos^3\theta\sin\theta - (Q_{22}-Q_{12}-2Q_{66})\cos\theta\sin^3\theta\\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})\cos\theta\sin^3\theta - (Q_{22}-Q_{12}-2Q_{66})\cos^3\theta\sin\theta\\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{2.36}$$

Although the transformed lamina stiffness matrix now has the same form as that of an anisotropic material with nine nonzero coefficients, only four of the coefficients are independent because they can all be expressed in terms of the four independent lamina stiffnesses of the specially orthotropic material. That is, the material is still orthotropic, but it is not recognizable as such in the off-axis coordinates. As in the three-dimensional case, it is obviously much easier to characterize the lamina experimentally in the principal material coordinates than in the off-axis coordinates. Recall that the engineering constants, the properties that are normally measured, are related to the lamina stiffnesses by Eqs. (2.27).

Alternatively, the strains can be expressed in terms of the stresses as

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16}\\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26}\\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} \tag{2.37}$$

where the $\bar{S}_{ij}$ are the components of the transformed lamina compliance matrix which are defined by equations similar to, but not exactly the same form as, Eqs. (2.36).

The lamina engineering constants can also be transformed from the principal material axes to the off-axis coordinates. For example, the modulus of elasticity associated with uniaxial loading along the x

direction is defined as

$$E_x = \frac{\sigma_x}{\epsilon_x} = \frac{\sigma_x}{\bar{S}_{11}\sigma_x} = \frac{1}{\bar{S}_{11}} \tag{2.38}$$

where the strain ϵ_x in the denominator has been found by substituting the stress conditions $\sigma_x \neq 0$, $\sigma_y = \tau_{xy} = 0$ in Eqs. (2.37). By replacing $\bar{S}_{11}$ with an equation similar to the first of Eqs. (2.36) and then using Eqs. (2.25), we find that

$$E_x = \frac{1}{\dfrac{1}{E_1}c^4 + \left[-\dfrac{2\nu_{12}}{E_1} + \dfrac{1}{G_{12}}\right]c^2s^2 + \dfrac{1}{E_2}s^4} \tag{2.39}$$

where c and s are as defined in Eqs. (2.30). Similar transformation equations may be found for other off-axis engineering constants such as ν_{xy} and G_{xy} [2.6, 2.8]. The variation of these properties with lamina orientation for a nylon fiber-reinforced elastomer composite is shown graphically in Fig. 2.8 from Ref. [2.5]. As intuitively expected, E_x varies from a maximum at $\theta = 0°$ to a minimum at $\theta = 90°$ for this particular material. It is not necessarily true that the extremum values of such material properties occur along the principal material directions, however [2.6]. What may not be intuitively expected is the sharp drop in modulus as the angle changes slightly from 0° and the fact that over much of the range of lamina orientations the modulus is very low. This is why transverse reinforcement is needed in most composites.

The shear-coupling effect has been described previously as the generation of shear strains by off-axis normal stresses and the generation

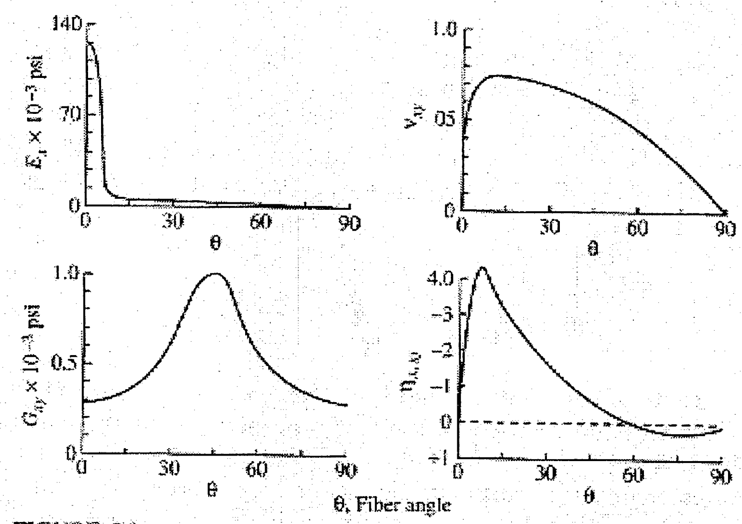

FIGURE 2.8
Variation of lamina engineering constants with lamina orientation for a nylon-reinforced elastomer composite. (*From Halpin [2.5]. Reprinted by permission of Technomic Publishing Co.*)

of normal strains by off-axis shear stresses. One way to quantify the degree of shear coupling is by defining dimensionless shear-coupling ratios [2.4, 2.5] or mutual influence coefficients [2.9] or shear-coupling coefficients [2.10]. For example, when the state of stress is defined as $\sigma_x \neq 0$, $\sigma_y = \tau_{xy} = 0$, the ratio

$$\eta_{x,xy} = \frac{\gamma_{xy}}{\epsilon_x} = \frac{\bar{S}_{16}}{\bar{S}_{11}} \tag{2.40}$$

is a measure of the amount of shear strain generated in the xy plane per unit normal strain along the direction of the applied normal stress, σ_x. Thus, the shear-coupling ratio is analogous to the Poisson's ratio, which is a measure of the coupling between normal strains. As shown in Fig. 2.8, $\eta_{x,xy}$ strongly depends on orientation and has its maximum value at some intermediate angle. Since there is no coupling along principal material directions, $\eta_{x,xy} = 0$ for $\theta = 0°$ and $\theta = 90°$. As the shear-coupling ratio increases, the amount of shear coupling increases. Other shear-coupling ratios can be defined for different states of stress. For example, when the stresses are $\tau_{xy} \neq 0$, $\sigma_x = \sigma_y = 0$, the ratio

$$\eta_{xy,y} = \frac{\epsilon_y}{\gamma_{xy}} = \frac{\bar{S}_{26}}{\bar{S}_{66}} \tag{2.41}$$

characterizes the normal strain response along the y direction due to a shear stress in the xy plane.

The effects of lamina orientation on stiffness are difficult to assess from inspection of stiffness transformation equations such as Eqs. (2.36) and (2.39). In addition, the eventual incorporation of lamina stiffnesses into laminate analysis requires integration of lamina stiffnesses over the laminate thickness, and integration of such complicated equations is also difficult. In view of these difficulties, a more convenient "invariant" form of the lamina stiffness transformation equations has been proposed by Tsai and Pagano [2.11]. By using trigonometric identities to convert from power functions to multiple angle functions and then using additional mathematical manipulations, Tsai and Pagano showed that Eqs. (2.36) could also be written as

$$\begin{aligned}
\bar{Q}_{11} &= U_1 + U_2\cos 2\theta + U_3\cos 4\theta \\
\bar{Q}_{12} &= U_4 - U_3\cos 4\theta \\
\bar{Q}_{22} &= U_1 - U_2\cos 2\theta + U_3\cos 4\theta \\
\bar{Q}_{16} &= \frac{U_2}{2}\sin 2\theta + U_3\sin 4\theta \\
\bar{Q}_{26} &= \frac{U_2}{2}\sin 2\theta - U_3\sin 4\theta \\
\bar{Q}_{66} &= \frac{1}{2}(U_1 - U_4) - U_3\cos 4\theta
\end{aligned} \tag{2.42}$$

where the set of "invariants" is defined as

$$
\begin{aligned}
U_1 &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}) \\
U_2 &= \frac{1}{2}(Q_{11} - Q_{22}) \\
U_3 &= \frac{1}{8}(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}) \\
U_4 &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})
\end{aligned}
\tag{2.43}
$$

As the name implies, the invariants, which are simply linear combinations of the Q_{ij}, are invariant to rotations in the plane of the lamina. Note that there are four independent invariants, just as there are four independent elastic constants. Equations (2.42) are obviously easier to manipulate and interpret than Eqs. (2.36). For example, all the stiffness expressions except those for the coupling stiffnesses consist of one constant term and terms which vary with lamina orientation. Thus, the effects of lamina orientation on stiffness are easier to interpret.

Invariant formulations of lamina compliance transformations are also useful. It can be shown [2.5, 2.10] that the off-axis compliance components in Eqs. (2.37) can be written as

$$
\begin{aligned}
\bar{S}_{11} &= V_1 + V_2 \cos 2\theta + V_3 \cos 4\theta \\
\bar{S}_{12} &= V_4 - V_3 \cos 4\theta \\
\bar{S}_{22} &= V_1 - V_2 \cos 2\theta + V_3 \cos 4\theta \\
\bar{S}_{16} &= V_2 \sin 2\theta + 2V_3 \sin 4\theta \\
\bar{S}_{26} &= V_2 \sin 2\theta - 2V_3 \sin 4\theta \\
\bar{S}_{66} &= 2(V_1 - V_4) - 4V_3 \cos 4\theta
\end{aligned}
\tag{2.44}
$$

where the invariants are

$$
\begin{aligned}
V_1 &= \frac{1}{8}(3S_{11} + 3S_{22} + 2S_{12} + S_{66}) \\
V_2 &= \frac{1}{2}(S_{11} - S_{22}) \\
V_3 &= \frac{1}{8}(S_{11} + S_{22} - 2S_{12} - S_{66}) \\
V_4 &= \frac{1}{8}(S_{11} + S_{22} + 6S_{12} - S_{66})
\end{aligned}
\tag{2.45}
$$

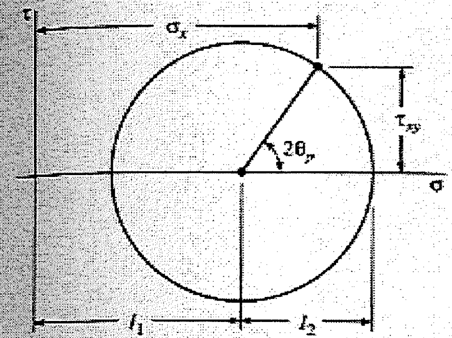

FIGURE 2.9
Mohr's circle for stress transformation.

Invariant formulations also lend themselves well to graphical interpretation. As shown in any mechanics of materials book [2.2], stress transformation equations such as Eqs. (2.30) can be combined and manipulated so as to generate the equation of Mohr's circle. As shown in Fig. 2.9, the transformation of a normal stress component σ_x can be described by the invariant formulation

$$\sigma_x = I_1 + I_2 \cos 2\theta_p \tag{2.46}$$

where $I_1 = \dfrac{\sigma_x + \sigma_y}{2} = \text{invariant}$

$I_2 = \sqrt{\left[\dfrac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2} = \text{invariant}$

θ_p = angle between the x axis and the principal stress axis

In this case the invariants are I_1, which defines the position of the center of the circle, and I_2, which is the radius of the circle. Note that, as with Eqs. (2.42), the invariant formulation consists of a constant term and a term which varies with orientation. Similarly, the invariant forms of the stiffness transformations can also be interpreted graphically using Mohr's circles. For example, Tsai and Hahn [2.10] have shown that the stiffness transformation equation

$$\bar{Q}_{11} = U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \tag{2.47}$$

can be represented graphically by using two Mohr's circles, as shown in Fig. 2.10. The distance between points on each of the two circles represents the total stiffness $\bar{Q}_{11}$, whereas the distance between the centers of the two circles is given by U_1. The radius and angle associated with one circle are U_2 and 2θ, respectively, and the radius and angle associated with the other circle are U_3 and 4θ, respectively. Thus, the

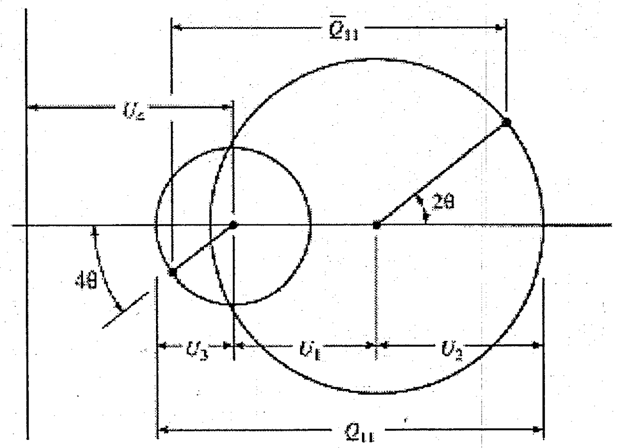

FIGURE 2.10
Mohr's circles for stiffness transformation. (*From Tsai and Hahn* [2.10]. *Reprinted by permission of Technomic Publishing Co.*)

distance between the centers of the circles is a measure of the isotropic component of stiffness, whereas the radii of the circles indicates the strength of the orthotropic component. If $U_2 = U_3 = 0$, the circles shrink to points and the material is isotropic.

Invariants will prove to be very useful later in the analysis of randomly oriented short fiber composites and laminated plates. For additional applications of invariants in composite analysis the reader is referred to books by Halpin [2.5] and Tsai and Hahn [2.10].

Example 2.1. A filament wound cylindrical pressure vessel (Fig. 2.11) of mean diameter $d = 1$ m and wall thickness $t = 20$ mm is subjected to an internal pressure, p. The filament winding angle is $\theta = 53.1°$ from the longitudinal axis of the pressure vessel, and the glass/epoxy material has the following properties: $E_1 = 40$ GPa $= 40(10^3)$ MPa, $E_2 = 10$ GPa, $G_{12} = 3.5$ GPa, and $\nu_{12} = 0.25$. By the use of a strain gage, the normal strain along the fiber direction is determined to be $\epsilon_1 = 0.001$. Determine the internal pressure in the vessel.

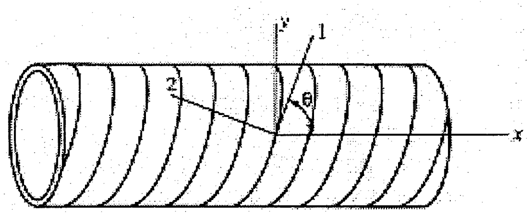

FIGURE 2.11
Filament wound pressure vessel.

Solution. From mechanics of materials, the stresses in a thin-walled cylindrical pressure vessel are given by

$$\sigma_x = \frac{pr}{2t} = \frac{0.5p}{2(0.02)} = 12.5p \qquad \tau_{xy} = 0$$

$$\sigma_y = \frac{pr}{t} = \frac{0.5p}{0.02} = 25p$$

(Note that $r = d/2 = 0.5$ m.)

These equations are based on static equilibrium and geometry only. Thus, they apply to a vessel made of any material. Since the given strain is along the fiber direction, we must transform the above stresses to the 12 axes. Recall that in the "netting analysis" in Problems 1.5 and 1.6 only the fiber longitudinal normal stress was considered. This was because the matrix was ignored, and the fibers alone cannot support transverse or shear stresses. In the current problem, however, the transverse normal stress, σ_2, and the shear stress, τ_{12}, are also considered because the fiber and matrix are now assumed to act as a composite. From Eqs. (2.31), the stresses along the 12 axes are

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$= (12.5p)(0.6)^2 + (25p)(0.8)^2 + 0 = 20.5p\,(\text{MPa})$$

$$\sigma_2 = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$= (12.5p)(0.8)^2 + (25p)(0.6)^2 - 0 = 17.0p\,(\text{MPa})$$

$$\tau_{12} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$= -(12.5p)(0.8)(0.6) + (25p)(0.6)(0.8) + 0 = 6.0p\,(\text{MPa})$$

where the pressure p is in MPa. From the first of Eqs. (2.24), the normal strain ϵ_1 is

$$\epsilon_1 = \frac{\sigma_1}{E_1} - \frac{\nu_{12}\sigma_2}{E_1} = \frac{20.5p}{40(10^3)} - \frac{0.25(17.0p)}{40(10^3)} = 0.001$$

and the resulting pressure is $p = 2.46$ MPa.

Example 2.2. A tensile test specimen is cut out along the x direction of the pressure vessel described in Example 2.1. What effective modulus of elasticity would you expect to get during a test of this specimen?

Solution. The modulus of elasticity, E_x, associated with the x direction is given by Eq. (2.39) with $\theta = 53.1°$.

$$E_x = \frac{1}{\dfrac{1}{E_1}c^4 + \left[-\dfrac{2\nu_{12}}{E_1} + \dfrac{1}{G_{12}}\right]c^2s^2 + \dfrac{1}{E_2}s^4}$$

$$E_x = \frac{1}{\dfrac{1}{40}(0.6)^4 + \left[-\dfrac{2(0.25)}{40} + \dfrac{1}{3.5}\right](0.6)^2(0.8)^2 + \dfrac{1}{10}(0.8)^4} = 9.33 \text{ MPa}$$

Example 2.3. A lamina consisting of continuous fibers randomly oriented in the plane of the lamina is said to be "planar isotropic," and the elastic properties in the plane are isotropic in nature. Find expressions for the lamina stiffnesses for a planar isotropic lamina.

Solution. Since the fibers are assumed to be randomly oriented in the plane, the "planar isotropic stiffnesses" can be found by averaging the transformed lamina stiffnesses as follows:

$$\tilde{Q}_{ij} = \frac{\int_0^\pi \bar{Q}_{ij}\,d\theta}{\int_0^\pi d\theta}$$

It is convenient to use the invariant forms of the transformed lamina stiffnesses because they are easily integrated. For example, if we substitute the first of Eqs. (2.42) in the above equation, we get

$$\tilde{Q}_{11} = \frac{\int_0^\pi \bar{Q}_{11}\,d\theta}{\int_0^\pi d\theta} = \frac{\int_0^\pi [U_1 + U_2\cos 2\theta + U_3\cos 4\theta]\,d\theta}{\pi} = U_1$$

Note that the averaged stiffness equals the isotropic part of the transformed lamina stiffness, and that the orthotropic parts drop out in the averaging process. Similarly, the other averaged stiffnesses can be found in terms of the invariants. The derivations of the remaining expressions are left as an exercise.

PROBLEMS

2.1. A representative section from a composite lamina is shown in Fig. 2.12, along with the transverse stress and strain distributions across the fiber and matrix materials in the section. Assuming that the dimensions of the section do not change along the longitudinal direction (perpendicular to the page), find the numerical value of the effective transverse modulus for the section.

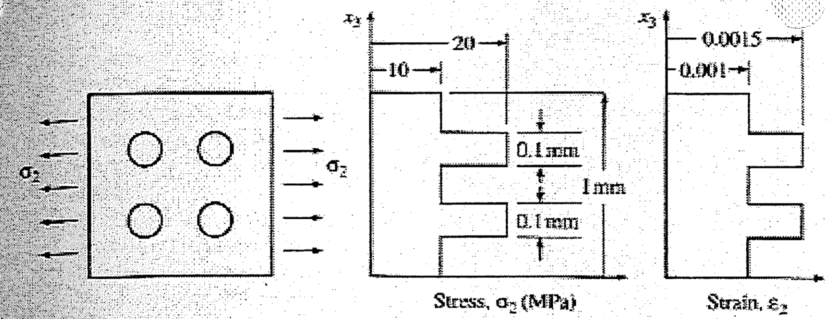

FIGURE 2.12
Transverse stress and strain distribution over a section of lamina.

2.2. Derive Eq. (2.39) for the off-axis modulus, E_x.

2.3. Find an expression for the off-axis shear modulus, G_{xy}, in terms of E_1, E_2, ν_{12}, G_{12}, and θ.

2.4. Using the result from Problem 2.3:
(*a*) Find the value of the angle θ (other than 0° or 90°) where the curve of G_{xy} vs. θ has a possible maximum, minimum, or inflection point.
(*b*) For the value of θ found in part (*a*), find the bounds on G_{12} which must be satisfied if G_{xy} is to have a maximum or minimum.
(*c*) Qualitatively sketch the variation of G_{xy} vs. θ for the different cases and identify each curve by the corresponding bounds on G_{12} which give that curve.
(*d*) Using the bounds on G_{12} from part (*b*), find which conditions apply for E-glass/epoxy composites. The bounds on G_{12} in part (*b*) should be expressed in terms of E_1, E_2, and ν_{12}.

2.5. Describe a series of tensile tests that could be used to measure the four independent engineering constants for an orthotropic lamina without using a pure shear test. Give the necessary equations for the data reduction.

2.6. A balanced orthotropic, or square symmetric lamina, is made up of 0° and 90° fibers woven into a fabric and bonded together, as shown in Fig. 2.5.
(*a*) Describe the stress-strain relationships for such a lamina in terms of the appropriate engineering constants.
(*b*) For a typical glass/epoxy composite lamina of this type, sketch the expected variations of all the engineering constants for the lamina from 0° to 90°. Numerical values are not required.

2.7. An element of a balanced orthotropic graphite/epoxy lamina is under the state of stress shown in Fig. 2.13. If the properties of the woven graphite fabric/epoxy material are $E_1 = 70$ GPa, $\nu_{12} = 0.25$, $G_{12} = 5$ GPa, determine all the strains along the fiber directions.

2.8. Derive Eqs. (2.27).

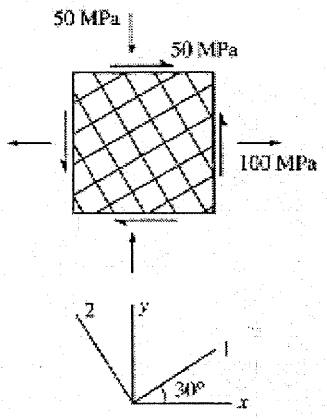

FIGURE 2.13
Stresses acting on an element of balanced orthotropic lamina.

2.9. Express the stress-strain relationships in Eqs. (2.37) in terms of off-axis engineering constants such as the moduli of elasticity, shear modulus, Poisson's ratios, and shear-coupling ratios.

2.10. Derive the first two equations of Eqs. (2.42).

2.11. Find all components of the stiffness and compliance matrices for a specially orthotropic lamina made of AS/3501 graphite/epoxy.

2.12. Using the results of Problems 2.11, determine the invariants U_i and V_i for the AS/3501 lamina, where $i = 1, 2, 3, 4$.

2.13. Using the results of Problems 2.11 or 2.12, compare the transformed lamina stiffnesses for AS/3501 graphite/epoxy plies oriented at +45° and −45°.

2.14. Show how the Mohr's circles in Fig. 2.10 can be used to interpret the transformed lamina stiffness $\bar{Q}_{12}$.

2.15. Using the approach described in Example 2.3, derive the expressions for all the averaged stiffnesses for the planar isotropic lamina in terms of invariants. Use these results to find the corresponding averaged engineering constants (modulus of elasticity, shear modulus, and Poisson's ratio) in terms of invariants.

2.16. For a specially orthotropic, transversely isotropic material the "plane strain bulk modulus," K_{23}, is an engineering constant that is defined by the stress conditions $\sigma_2 = \sigma_3 = \sigma$ and the strain conditions $\epsilon_1 = 0$, $\epsilon_2 = \epsilon_3 = \epsilon$. Show that these conditions lead to the stress-strain relationship $\sigma = 2K_{23}\epsilon$, and find the relationship among K_{23}, E_1, E_2, G_{23}, and ν_{12}.

2.17. Describe the measurements that would have to be taken and the equations that would have to be used to determine G_{23}, ν_{32}, and E_2 for a specially orthotropic, transversely isotropic material from a single tensile test.

2.18. An off-axis tensile specimen of an orthotropic lamina with fiber orientation $\theta = 45°$ is subjected to a known uniaxial stress, σ_x, and the resulting normal strains ϵ_x and ϵ_y are measured with strain gages. Describe how the in-plane shear modulus, G_{12}, can be determined from these data. Assume that no other data are available.

REFERENCES

2.1. Christensen, R. M., *Mechanics of Composite Materials*, John Wiley & Sons, New York (1979).
2.2. Crandall, S. H., Dahl, N. C., and Lardner, T. J., *An Introduction to the Mechanics of Solids*, 2d ed. with SI units, McGraw-Hill, Inc., New York (1978).
2.3. Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, McGraw-Hill, Inc., New York (1956).
2.4. Ashton, J. E., Halpin, J. C., and Petit, P. H., *Primer on Composite Materials: Analysis*, Technomic Publishing Co., Lancaster, PA (1969).
2.5. Halpin, J. C., *Primer on Composite Materials: Analysis*, Rev., Technomic Publishing Co., Lancaster, PA (1984).
2.6. Jones, R. M., *Mechanics of Composite Materials*, Hemisphere Publishing Co., New York (1975).
2.7. Vinson, J. R. and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, Martinus Nijhoff Publishers, Dordrecht, The Netherlands (1986).
2.8. Agarwal, B. D. and Broutman, L. J., *Analysis and Performance of Fiber Composites*, 2d ed., John Wiley & Sons, New York (1990).
2.9. Lekhnitski, S. G., *Theory of Elasticity of an Anisotropic Body*, Mir Publishing Co., Moscow, USSR (1981).
2.10. Tsai, S. W. and Hahn, H. T., *Introduction to Composite Materials*, Technomic Publishing Co., Lancaster, PA (1980).
2.11. Tsai, S. W. and Pagano, N. J., "Invariant Properties of Composite Materials," in S. W. Tsai, J. C. Halpin, and N. J. Pagano (eds.), *Composite Materials Workshop*, pp. 233–253, Technomic Publishing Co., Lancaster, PA (1968).

6