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Decidability for type-related problems of 2nd-order λ-calculi and negative translations

Ken-etsu Fujita
Gunma University

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Church-style vs. Curry-style

- Types ($\to$-fragment of intuitionistic prop. logic)

$$A ::= X \mid (A \to A)$$

- λ-terms in Church style

$$M ::= x \mid \lambda x : A.M \mid MM$$

- λ-terms in Curry style

$$M ::= x \mid \lambda x.M \mid MM$$

Type-related problems (TCP, TP, TIP)

- Type checking problem (TCP(s): $\Gamma \vdash_s M : A?$)
  Given a context Γ, a λ-term M in s-style, and a type A,
  determine whether $\Gamma \vdash_s M : A$.
- Type inference problem (TIP(s): $\Gamma \vdash_s M :?$)
  Given Γ and M in s-style,
  determine whether $\Gamma \vdash_s M : A$ for some type A.
- Typability problem (TP(s): $? \vdash_s M :?$)
  Given M in s-style,
  determine whether $\Gamma \vdash_s M : A$ for some context Γ and type A.

$\lambda 2$: 2nd-order lambda-calculus and styles

- $\lambda 2$-types (2nd-order intuitionistic prop. logic with $\to, \forall$)
  $A ::= X \mid (A \to A) \mid \forall X.A$
- $\lambda 2$-terms in Church style
  $M ::= x \mid \lambda x{:}A.M \mid M\,M \mid \Lambda X.M \mid M[A]$
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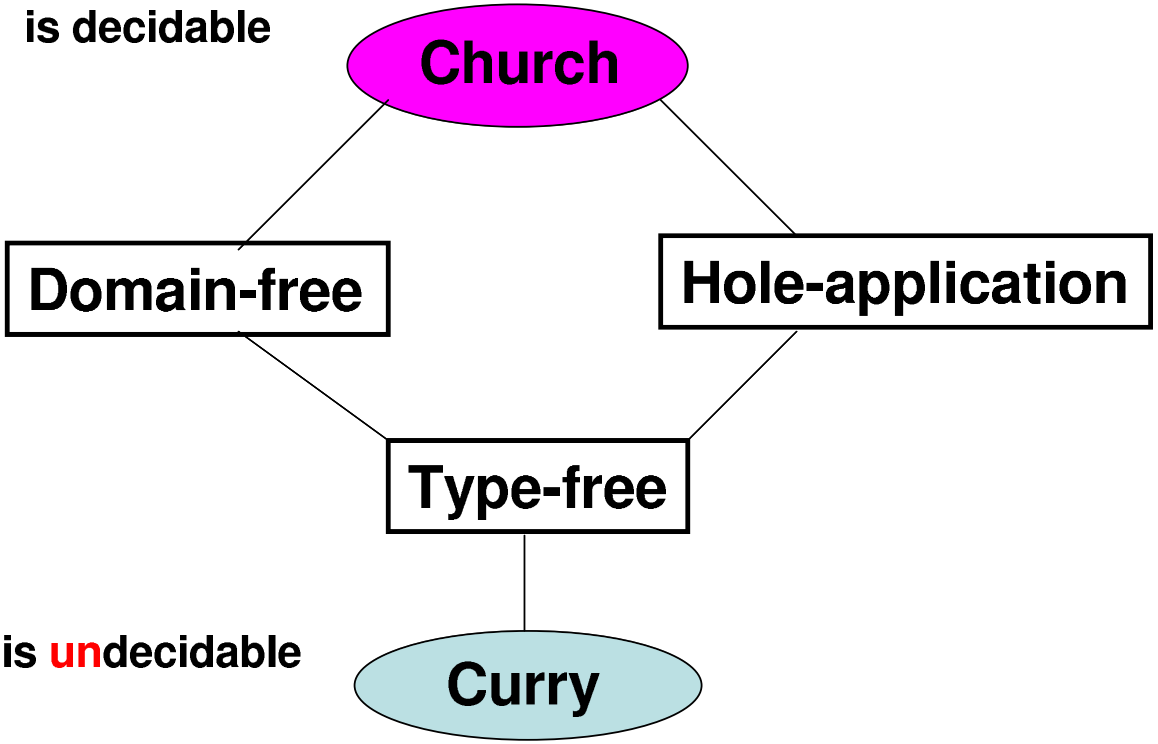
- $\lambda 2$-terms in Curry style
  $M ::= x \mid \lambda x.M \mid M\,M$

Intermediate structures between Church and Curry

TIP(Church) is decidable



TIP(Curry) is undecidable

What is a critical property for the (un)decidability?

$\lambda 2$: 2nd-order lambda-calculus and styles

- $\lambda 2$-types (2nd-order intuitionistic prop. logic with $\rightarrow, \forall$)
  $A ::= X \mid (A \rightarrow A) \mid \forall X.A$
- $\lambda 2$-terms in Church style
  $M ::= x \mid \lambda x\!:\!A.M \mid M\,M \mid \Lambda X.M \mid M[A]$
- $\lambda 2$-terms in hole-application style
  $M ::= x \mid \lambda x\!:\!A.M \mid M\,M \mid \Lambda X.M \mid M[]$
- $\lambda 2$-terms in domain-free style
  $M ::= x \mid \lambda x.M \mid M\,M \mid \Lambda X.M \mid M[A]$
- $\lambda 2$-terms in type-free style
  $M ::= x \mid \lambda x.M \mid M\,M \mid \Lambda.M \mid M[]$
- $\lambda 2$-terms in Curry style
  $M ::= x \mid \lambda x.M \mid M\,M$

* Putting type annotation helps to find type of terms.

* How much of the erased information can be recovered from type inference?

* What annotation determines essentially decidability or undecidability of the problems?

Inference rules for Church $\lambda 2$

$$\frac{x \in \mathrm{dom}(\Gamma)}{\Gamma \vdash_{\mathrm{Ch}} x : \Gamma(x)} \ (\mathrm{var})$$

$$\frac{\Gamma, x : A_1 \vdash_{\mathrm{Ch}} M : A_2}{\Gamma \vdash_{\mathrm{Ch}} \lambda x : A_1 . M : A_1 \to A_2} \ (\to I)$$

$$\frac{\Gamma \vdash_{\mathrm{Ch}} M_1 : A_1 \to A_2 \quad \Gamma \vdash_{\mathrm{Ch}} M_2 : A_1}{\Gamma \vdash_{\mathrm{Ch}} M_1 M_2 : A_2} \ (\to E)$$

$$\frac{\Gamma \vdash_{\mathrm{Ch}} M : A}{\Gamma \vdash_{\mathrm{Ch}} \Lambda X . M : \forall X . A} \ (\forall I)^* \qquad \frac{\Gamma \vdash_{\mathrm{Ch}} M : \forall X . A}{\Gamma \vdash_{\mathrm{Ch}} M[A_1] : A[X := A_1]} \ (\forall E)$$

$*$ Eigenvariable condition: $X \notin \mathrm{FV}(\Gamma)$

Inference rules for $s \in \{\mathrm{Hole}, \mathrm{DF}, \mathrm{TF}, \mathrm{Cu}\}$ are defined similarly.

Summary of decidability

Styles	TCP	TIP	TP
Church	yes	yes	no [Schubert98]
Hole-application	Yes	Yes	No [FS2013]
Domain-free	No	No	No [FS2000]
Type-free	No	No	No [FS2010]
Curry	no	no	no [Wells99]

Figure 1: Decidability of TCP, TIP, and TP for $\lambda 2$

- TCP([]), TIP([]): $Decidable$
  Hole-application style says how to apply $(\to I)$,
  but leaves instance information out of $(\forall E)$.
- TCP(Df), TIP(Df): $Undecidable$
  Domain-free style says how to apply $(\forall E)$,
  but misses polymorphic domains* out of $(\to I)$.
- TP is so hard, and undecidable for any style. Put more annotations!

$\lambda 2$: 2nd-order lambda-calculus and styles

- original Church-style
  $M ::= x^A \mid \lambda x^A.M \mid MM \mid \Lambda X.M \mid M[A]$
- fully annotated Church-style (too much redundant annotations)
  $M ::= x \mid \lambda x{:}A.M^A \mid M^A M^A \mid \Lambda X.M^A \mid M^A[A]$
- partially annotated Church-style
  $M ::= x \mid \lambda x{:}A.M \mid MM \mid \Lambda X.M \mid M^A[A]$
- Church style
  $M ::= x \mid \lambda x{:}A.M \mid MM \mid \Lambda X.M \mid M[A]$
- hole-application style
  $M ::= x \mid \lambda x{:}A.M \mid MM \mid \Lambda X.M \mid M[]$
- domain-free style
  $M ::= x \mid \lambda x.M \mid MM \mid \Lambda X.M \mid M[A]$
- type-free style
  $M ::= x \mid \lambda x.M \mid MM \mid \Lambda.M \mid M[]$
- Curry style
  $M ::= x \mid \lambda x.M \mid MM$

* A forgetful map $|\ |$ is defined from s-style to t-style $(s > t)$.

* We propose a framework to handle related systems not only $\lambda 2$ but also $\lambda^{\exists}$.

Inference rules for fully annotated Church $\lambda^{\exists}$

$$\frac{x \in \mathrm{dom}(\Gamma)}{\Gamma \vdash x : \Gamma(x)}\ (\mathrm{var})$$

$$\frac{\Gamma, x\!:\!A \vdash M : \bot}{\Gamma \vdash \lambda x^A.M : \neg A}\ (\neg I) \qquad \frac{\Gamma \vdash M_1 : \neg A \quad \Gamma \vdash M_2 : A}{\Gamma \vdash M_1 M_2 : \bot}\ (\neg E)$$

$$\frac{\Gamma \vdash M_1 : A \quad \Gamma \vdash M_2 : B}{\Gamma \vdash \langle M_1, M_2 \rangle : A \wedge B}\ (\wedge I)$$

$$\frac{\Gamma \vdash M_1 : A \wedge B \quad \Gamma, x\!:\!A, y\!:\!B \vdash M_2 : C}{\Gamma \vdash \mathtt{let}\ \langle x^A, y^B \rangle = M_1\ \mathtt{in}\ M_2 : C}\ (\wedge E)$$

$$\frac{\Gamma \vdash M : B[X := A]}{\Gamma \vdash \langle A, M \rangle_{\exists X.B} : \exists X.B}\ (\exists I)$$

$$\frac{\Gamma \vdash M_1 : \exists X.A \quad \Gamma, x\!:\!A \vdash M_2 : B}{\Gamma \vdash \mathtt{let}\ \langle X, x^A \rangle = M_1\ \mathtt{in}\ M_2 : B}\ (\exists E)^*$$

$*$ eigenvariable condition: $X \notin \mathrm{FV}(\Gamma, B)$

TP($\lambda 2$) and TP($\lambda^{\exists}$)

Styles	TCP($\lambda 2$)	TCP($\lambda^{\exists}$)	TP($\lambda 2$)	TP($\lambda^{\exists}$)
full Church	yes	yes	yes	yes
partial Church	yes	yes	yes/no	yes/no [F2013]
Church	yes	yes	no	no [F2011,13]
Hole-application	yes	yes	no	no [F2013]
Domain-free	no	no	no	no [Nakazawa,et.al.08]
Type-free	no	no	no	no [FS2009]
Curry	no	no	no	no [F2011]

Figure 2: Decidability of TCP and TP for $\lambda 2, \lambda^{\exists}$

Framework:

- Undecidability of $\lambda 2$ implies that of a corresponding $\lambda^{\exists}$ via negative trans. (CPS-trans.)
- Decidability of $\lambda^{\exists}$ implies that of a corresponding $\lambda 2$

Framework

$$
\begin{tikzcd}
\lambda 2: & \text{Church} \arrow[r, "|\,|^{\mathrm{ch}}_{\mathrm{df}}"] \arrow[d, shift right, "*^{\mathrm{ch}}"'] & \mathrm{DF} \arrow[r, "|\,|^{\mathrm{df}}_{\mathrm{tf}}"] \arrow[d, shift right, "*^{\mathrm{df}}"'] & \mathrm{TF} \arrow[r, "|\,|^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[d, shift right, "*^{\mathrm{tf}}"'] & \text{Curry} \arrow[d, shift right, "*^{\mathrm{cu}}"'] \\
\lambda^{\exists}: & \text{Church} \arrow[r, "|\,|^{\mathrm{ch}}_{\mathrm{df}}"] \arrow[u, shift right, "\sharp^{\mathrm{ch}}"'] & \mathrm{DF} \arrow[r, "|\,|^{\mathrm{df}}_{\mathrm{tf}}"] \arrow[u, shift right, "\sharp^{\mathrm{df}}"'] & \mathrm{TF} \arrow[r, "|\,|^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[u, shift right, "\sharp^{\mathrm{tf}}"'] & \text{Curry} \arrow[u, shift right, "\sharp^{\mathrm{cu}}"']
\end{tikzcd}
$$

Soundness of t-style $\lambda 2$:

$$
\begin{tikzcd}
\Gamma \vdash_{s\text{-}\lambda 2} N : A \text{ for some } N \text{ s.t. } |N| = M \arrow[d, "\text{Translation } *^{s}"'] & \Gamma \vdash_{t\text{-}\lambda 2} M : A \arrow[l, "\text{lifting } |\,|^{-1}"'] \arrow[d, "M^{*^{t}} = |N|^{*^{t}} = |N^{*^{s}}|"] \\
\neg\Gamma^{*}, a\!:\!A^{*} \vdash_{s\text{-}\lambda^{\exists}} N^{*} : \bot \arrow[r, "\text{erasing } |\,|"] & \neg\Gamma^{*}, a\!:\!A^{*} \vdash_{t\text{-}\lambda^{\exists}} |N^{*}| : \bot
\end{tikzcd}
$$

Completeness of t-style $\lambda 2$:

$$
\begin{tikzcd}
\Gamma \vdash_{s\text{-}\lambda 2} Q^{\sharp} : A \arrow[r, "|\,|"] & \Gamma \vdash_{t\text{-}\lambda 2} |Q^{\sharp}| : A \\
\neg\Gamma^{*}, a\!:\!A^{*} \vdash_{s\text{-}\lambda^{\exists}} Q : \bot \text{ for some } Q \text{ s.t. } |Q| = P \arrow[u, "\text{Inverse } \sharp^{s}"] & \neg\Gamma^{*}, a\!:\!A^{*} \vdash_{t\text{-}\lambda^{\exists}} P : \bot \arrow[l, "|\,|^{-1}"'] \arrow[u, "P^{\sharp^{t}} = |Q|^{\sharp^{t}} = |Q^{\sharp^{s}}|"']
\end{tikzcd}
$$

Negative translation (CPS-translation) from $\lambda 2$ into $\lambda^{\exists}$

1. CPS-translation:
   $X^* = X, (A \to B)^* = \neg A^* \wedge B^*, (\forall X.A)^* = \exists X.A^*$.
2. fully annotated Church $\lambda 2$:
   $M ::= x \mid \lambda x^A.M^B \mid M M^A \mid \Lambda X.M^A \mid M^{\forall X.A} B$
3. fully annotated Church $\lambda^{\exists}$:
   $$\begin{aligned} M ::= {} & x \mid \lambda x : A.M \mid M M \\ & \mid \langle M, M \rangle \mid \texttt{let}\ \langle x : A, x : B \rangle = M\ \texttt{in}\ M \\ & \mid \langle A, M \rangle_{\exists X.B} \mid \texttt{let}\ \langle X, x : A \rangle = M\ \texttt{in}\ M \end{aligned}$$
4. CPS-translation: $(x)^* = xa$,
   $(\lambda x^A.M^B)^* = (\texttt{let}\ \langle x : \neg A^*, a : B^* \rangle = a\ \texttt{in}\ M^*)$,
   $(M N^A)^* = M^*[a := \langle \lambda a : A^*.N^*, a \rangle]$,
   $(\Lambda X.M^A)^* = (\texttt{let}\ \langle X, a : A^* \rangle = a\ \texttt{in}\ M^*)$,
   $(M^{\forall X.A} B)^* = M^*[a := \langle B^*, a \rangle_{\exists X.A^*}]$
5. Theorem (sound and complete):
   $\Gamma \vdash_{\mathrm{fullCh}\lambda 2} M : A$ if and only if $\neg \Gamma^* \vdash_{\mathrm{fullCh}\lambda^{\exists}} \lambda a : A^*.M^* : \neg A^*$.

Commutativity of CPS-translation $* : \lambda 2 \to \lambda^{\exists}$, inverse $\sharp : \lambda^{\exists} \to \lambda 2$ and forgetful mapping $|\ |^s_t$

Case of $(M^{\forall X.A} B)$:

$$\begin{tikzcd}
M^{\forall X.A} B \arrow[r, "|\ |^{\mathrm{fch}}_{\mathrm{ch}}"] \arrow[d, "*^{\mathrm{fch}}"'] & |M|^{\mathrm{fch}}_{\mathrm{ch}} B \arrow[r, "|\ |^{\mathrm{ch}}_{\mathrm{tf}}"] \arrow[d, "*^{\mathrm{ch}}"'] & |M|^{\mathrm{fch}}_{\mathrm{tf}}[] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[d, "*^{\mathrm{tf}}"'] & |M|^{\mathrm{fch}}_{\mathrm{cu}} \arrow[d, "*^{\mathrm{cu}}"'] \\
M^*[a := \langle B^*, a\rangle_{\exists X.A^*}] \arrow[r, "|\ |^{\mathrm{fch}}_{\mathrm{ch}}"] & |M^*|^{\mathrm{fch}}_{\mathrm{ch}}[a := \langle B^*, a\rangle] \arrow[r, "|\ |^{\mathrm{ch}}_{\mathrm{tf}}"] & |M^*|^{\mathrm{fch}}_{\mathrm{tf}}[a := \langle a\rangle] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] & |M^*|^{\mathrm{fch}}_{\mathrm{cu}}
\end{tikzcd}$$

Case of $\langle B^*, C\rangle_{\exists X.A^*}$:

$$\begin{tikzcd}
C^{\sharp}[[\,]^{\forall X.A^{*\sharp}} B^{*\sharp}] \arrow[r, "|\ |^{\mathrm{fch}}_{\mathrm{ch}}"] & |C^{\sharp}|^{\mathrm{fch}}_{\mathrm{ch}}[B^{*\sharp}] \arrow[r, "|\ |^{\mathrm{ch}}_{\mathrm{tf}}"] & |C^{\sharp}|^{\mathrm{fch}}_{\mathrm{tf}}[[]] \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] & |C^{\sharp}|^{\mathrm{fch}}_{\mathrm{cu}} \\
\langle B^*, C\rangle_{\exists X.A^*} \arrow[r, "|\ |^{\mathrm{fch}}_{\mathrm{ch}}"] \arrow[u, "\sharp^{\mathrm{fch}}"'] & \langle B^*, |C|^{\mathrm{fch}}_{\mathrm{ch}}\rangle \arrow[r, "|\ |^{\mathrm{ch}}_{\mathrm{tf}}"] \arrow[u, "\sharp^{\mathrm{ch}}"'] & \langle |C|^{\mathrm{fch}}_{\mathrm{tf}}\rangle \arrow[r, "|\ |^{\mathrm{tf}}_{\mathrm{cu}}"] \arrow[u, "\sharp^{\mathrm{tf}}"'] & |C|^{\mathrm{fch}}_{\mathrm{cu}} \arrow[u, "\sharp^{\mathrm{cu}}"']
\end{tikzcd}$$

Note that for CPS-types $A^*, B^* ::= X \mid \neg A^* \wedge B^* \mid \exists X.A^*$,

we have a unique $\lambda 2$-type A s.t. $A^{*\sharp} = A$.

Undecidability of TP(partial Church $\lambda 2$) implies that of TP(partial Church $\lambda^{\exists}$)

1. (Schubert 1998): 2nd order unification of simple instances is undecidable

$$\begin{aligned} \mathsf{F} A_1 \ldots A_n &\doteq (\mathsf{F} A'_1 \ldots A'_n \to A_0) \\ \mathsf{G} B_1 \ldots B_m &\doteq (\mathsf{G} B'_1 \ldots B'_m \to \mathsf{F} A'_1 \ldots A'_n) \end{aligned}$$

   F, G are functional unification variables, and $A, B ::= X \mid (A \to A)$.
2. partially annotated Church $\lambda 2$:
   $M ::= x \mid \lambda x^A.M^B \mid M\,M \mid \Lambda X.M^A \mid M\,A$
   $M_0 \stackrel{\text{def}}{=} z\,(z_1((x_{\mathsf{F}} A_1 \ldots A_n)(x_{\mathsf{F}} A'_1 \ldots A'_n)))\ \ (\lambda x\!:\!A_0.(z_1 x)^X)$
   $\qquad (z_2((y_{\mathsf{G}} B_1 \ldots B_m)(y_{\mathsf{G}} B'_1 \ldots B'_m)))\ \ (z_2(x_{\mathsf{F}} A'_1 \ldots A'_n))$
3. Theorem: The simple instance is solvable iff M_0 is typable.
   This implies that TP(partial Church $\lambda 2$) is undecidable. Moreover, the following TP(partial Church $\lambda^{\exists}$) is undecidable under the framework.
4. partially annotated Church $\lambda^{\exists}$
   $M ::= x \mid \lambda x.M \mid M\,M \mid \langle M, M\rangle \mid \texttt{let}\ \langle x\!:\!A, x\!:\!B\rangle = M\ \texttt{in}\ M$
   $\qquad \mid \langle A, M\rangle \mid \texttt{let}\ \langle X, x\!:\!A\rangle = M\ \texttt{in}\ M$

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