How to prove it: Phase transitions in logic

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Overview:

- Introduction
- 2 Lower bounds
- Opper bounds
- 4 Sharpening



Phase transitions for incompleteness are results of the following form:

- \bullet $T \nvdash \varphi_{f_n}$, but
- $2 T \vdash \varphi_f,$

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In this talk we will treat the heuristics for these results. We will use the easy example of miniaturised Dickson's lemma to illustrate.

Definition

Given $a, b \in \mathbb{N}^c$:

- **1** |.| denotes the sup-norm: $|a| = \max_{i < c}(a)_i$,
- **②** ≤ denotes coordinatewise ordering:

$$a \leq b \Leftrightarrow (a)_0 \leq (b)_0 \wedge \cdots \wedge (a)_{c-1} \leq (b)_{c-1}$$
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Definition (MDL_f)

For every l, c there exists $D =: D_f(c, l)$ such that for every sequence m_0, \ldots, m_D of c-tuples, with $|m_i| \leq l + f(i)$, there exist $i < j \leq D$ with $m_i \leq m_j$.

Fact

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Define $A_0(x) = x + 1$ and $A_{n+1}(x) = A_n^{(x)}(x)$. For every primitive recursive function f there exists n such that:

$$f \leq A_n$$

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Exercise

For every I, n there exists a bad sequence

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Corollary

 $RCA_0 \nvdash MDL_{id}$.

Notice that, by the finite pigeonhole principle:

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Our goal is to classify some $f: \mathbb{N} \to \mathbb{N}$ according to the provability of MDL_f .

Examine the following function:

$$a \mapsto D_{x \mapsto a}(c, I)$$
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Exercise

$$D_{x\mapsto a}(c,I)=(I+a+1)^c.$$

Observe: Take $I_n(a) = a^n$, for every n there exist c, I such that:

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Take $u(a) = 2^a$, for every c, l:

$$a \mapsto D_{x \mapsto a}(c, l) \leq u$$
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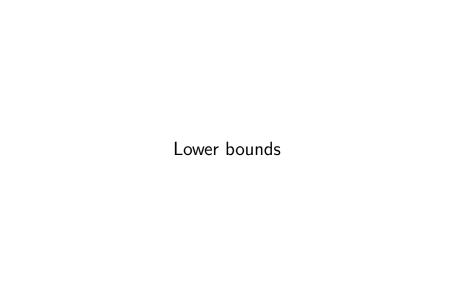
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Experience suggests that using I_n^{-1} as a parameter value will result in *independence*, whilst using u^{-1} will result in *provability*.

Theorem

- $\bullet \ \operatorname{RCA}_0 \nvdash \operatorname{MDL}_{\mathfrak{Y}}, \ \textit{but}$
- **2** $RCA_0 \vdash MDL_{log}$.



Lower bounds

Exercise

For *I* >?:

$$D_{\mathrm{id}}(c, l) \leq D_{\mathcal{N}}(c + n + 1, l).$$

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Corollary

 $RCA_0 \nvdash MDL_{\cancel{0}}$.



Upper bounds

Exercise

If $i \le 2^l$, then $\log i \le l$, so for l > ?:

$$D_{\log}(c, l) \leq D_l(c, l) \leq 2^l$$
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Upper bounds

Exercise

If $i \le 2^l$, then $\log i \le l$, so for l > ?:

$$D_{log}(c, I) \leq D_I(c, I) \leq 2^I$$
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Corollary

 $RCA_0 \vdash MDL_{log}$.



Lemma

$$RCA_0 \nvdash MDL_f$$
 for $f(x) = A^{-1}(x)/x$.

Proof: We examine D_f . Assume, for a contradiction, that:

$$D_f(2I+2,I+1) \leq A(I),$$

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By earlier proofs, we already know:

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Contradiction!

Lemma

 $\mathrm{RCA}_0 \vdash \mathrm{MDL}_{f_n}$ for $f_n(x) = A_n^{-1}(x)/x$.

Proof: Assume, without loss of generality, that $A_n^{-1}(i^{A_n(i)}) = i$,

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Fix $c, l > 2^c$ and k with the above property.

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$$(1+\sqrt[k]{k^{A_n(k)}})^c \le (2\sqrt[k]{k^{A_n(k)}})^c \le k^{A_n(k)}.$$



Thank you for listening.



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