Subsystems of arithmetic as type theories with inductive definitions

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Introduction of project

- Korea-France joint project
- Started with a discussion with Hugo Herbelin, the team leader of the development of the proof assistant Coq
- Doing mathematical logic using Coq
- Cross-fertilization of computer science and reverse mathematics

Doing mathematics using a proof assistant

- MSR-INRIA joint center (Georges Gonthier: Four Color Theorem)
- Freek Wiedijk: Formalizing 100 Problems (`http://www.cs.ru.nl/~freek/100/`)

Doing reverse mathematics using Coq?

- The current version of Coq is too strong.
- Strength of CIC (Calculus of Inductive Constructions) is known to be stronger than ZFC.

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The Big Five

- RCA_0 = Σ^0_1-IND + CA on Δ^0_1 formulas
- WKL_0 = RCA_0 + Weak König's Lemma
- ACA_0 = IND + CA on Σ^0_1 formulas
- ATR_0 = ACA_0 + Arithmetic Transfinite Recursion
- Π^1_1-CA_0 = ACA_0 + CA on Π^1_1 formulas

Induction axiom versus Σ^0_1-IND

- RCA_0 and WKL_0 have induction over Σ^0_1 formulas:

$$(\Sigma^0_1\text{-IND}) \quad A(0) \to \forall n(A(n) \to A(n+1)) \to \forall n A(n)$$

- The other systems include induction axiom:

$$(\text{IND}) \quad 0 \in X \to \forall n(n \in X \to n+1 \in X) \to \forall n(n \in X)$$

- All 5 systems could be uniformly based on the Σ^0_1-IND: because the strength of Σ^0_1-IND is governed by which sets are definable by the comprehension axioms:

$$(\text{CA}) \quad \exists X \forall n(n \in X \leftrightarrow A(n))$$

- (We also believe that) RCA_0 and WKL_0 could be based on induction axiom too, if all prim. rec. functions are taken primitive.

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Comprehension vs. Separation

(S-CA):

$$\exists X \, \forall n (n \in X \leftrightarrow A(n))$$

for any $A \in S$

(S-SEP)

$$\forall n \, \neg(A_1(n) \wedge A_2(n)) \rightarrow \exists X \, \forall n((A_1(n) \rightarrow n \in X) \wedge (A_2(n) \rightarrow n \notin X))$$

for any $A_1(n), A_2(n) \in S$.

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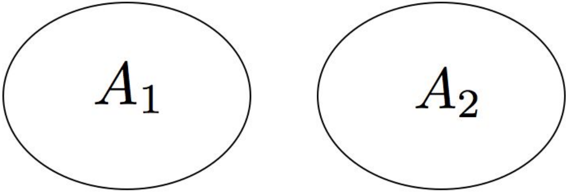
Separation Scheme



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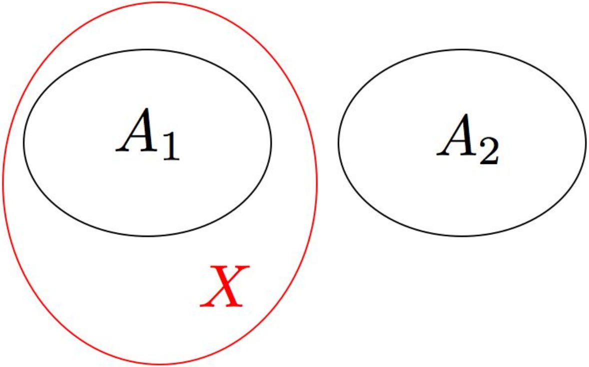


Summary (Simpson '99)

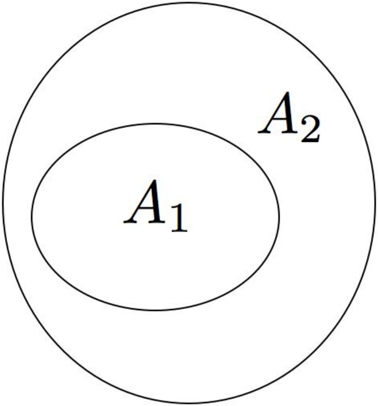
system	characterization
RCA_0	Δ^0_1-CA
WKL_0	Σ^0_1-SEP (or WKL)
ACA_0	Σ^0_1-CA (or Π^0_1-CA)
ATR_0	Σ^1_1-SEP (or ATR)
Π^1_1-CA_0	Σ^1_1-CA (or Π^1_1-CA)

S_1-S_2-Interpolation

$$\forall n\,[\,A_1(n) \to A_2(n)\,] \to \exists X\,\forall n\,[\,(A_1(n) \to n \in X) \wedge (n \in X \to A_2(n))\,]$$

for $A_i(n) \in S_i$.

Interpolation Scheme



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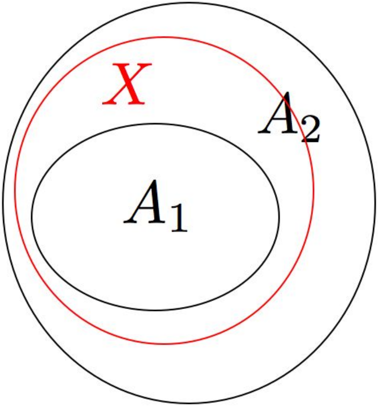

S_1-S_2-Interpolation

$$\forall n\,[\,A_1(n) \to A_2(n)\,] \to \exists X\,\forall n\,[\,(A_1(n) \to n \in X) \wedge (n \in X \to A_2(n))\,]$$

- S-CA iff S-S-INTERPOL.
- S-SEP iff S-$\neg S$-INTERPOL, where $\neg S := \{\neg A \mid A \in S\}$.

Summary 2

system	new characterization	old characterization
RCA_0	Π^0_1-Σ^0_1-INTERPOL	(i.e. Δ^0_0-CA)
WKL_0	Σ^0_1-Π^0_1-INTERPOL	(i.e. Σ^0_1-SEP)
ACA_0	Σ^0_1-Σ^0_1-INTERPOL	(i.e.,Σ^0_1-CA)
ATR_0	Σ^1_1-Π^1_1-INTERPOL	(i.e., Σ^1_1-SEP)
Π^1_1-CA_0	Σ^1_1-Σ^1_1-INTERPOL	(i.e.,Σ^1_1-CA)

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Computational content of proofs

- The computational content of proofs can be explained by the Curry-Howard correspondence.
- Curry (1958): Hilbert-style propositional logic = simply-typed combinatory logic
- Howard (1969): Gentzen's natural deduction = some simply-typed λ-calculus
- Martin-Löf's type theory with W-type (around 1980):
  - an intuitionistic logic with the strength Π^1_1-CA_0
  - also a programming language
- Griffin [1990] showed that proofs in classical logic are closely related to the control operators (`callcc/throw`).

Inference rules

$$\frac{\Gamma \vdash (0 \in P) \quad \Gamma, n \in P \vdash n+1 \in P \quad n \text{ fresh}}{\Gamma \vdash t \in P} \ (\text{IND})$$

$$\frac{\Gamma \vdash A_1[t/n] \quad A_1 \in S_1}{\Gamma \vdash t \in \{n \mid A_1 \lhd A_2\}} \ (\text{INTERPOL}_I)$$

$$\frac{\Gamma \vdash t \in \{n \mid A_1 \rhd A_2\} \quad A_2 \in S_2 \quad \Gamma, A_1 \vdash A_2}{\Gamma \vdash A_2[t/n]} \ (\text{INTERPOL}_E)$$

Inference rules

$$\frac{\Gamma \vdash p : (0 \in P) \quad \Gamma, a : (n \in P) \vdash q : (n+1 \in P) \quad n \text{ fresh}}{\Gamma \vdash \texttt{ind}\ t\ \texttt{of}\ [\, p \mid (n,a).q \,] : (t \in P)}\ (\text{IND})$$

$$\frac{\Gamma \vdash p : A_1[t/n] \quad A_1 \in S_1}{\Gamma \vdash \texttt{interpol}\ p : (t \in \{n \mid A_1 \lhd A_2\})}\ (\text{INTERPOL}_I)$$

$$\frac{\Gamma \vdash p : (t \in \{n \mid A_1 \rhd A_2\}) \quad A_2 \in S_2 \quad \Gamma, a : A_1 \vdash q : A_2}{\Gamma \vdash \texttt{compose}\ p\ \texttt{as}\ a\ \texttt{in}\ q : A_2[t/n]}\ (\text{INTERPOL}_E)$$

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RCA_0 with cosepration

- Onc can show that RCA_0 is conservative over $i\mathsf{RCA}_0$ on Σ^0_1 formulae.

- For intuitionistic analysis, Σ^0_1-coseparation behaves better than Π^0_1-separation or Δ^0_1-comprehension.

  (Σ^0_1-coSEP)
  $\forall n(A_1(n) \vee A_2(n)) \to$
  $\exists X\, Y\{\forall n(n \in X \vee n \in Y) \;\wedge\; \forall n[\,(n \in X \to A_1(n)) \wedge (n \in Y \to A_2(n))\,]\}$

- (Σ^0_1-coSEP) is classically equivalent to Π^0_1-separation.

RCA_0 with cosepration

- A variation of Coquand-Hofmann (1999) translation is used:

$$
\begin{aligned}
\bot^X &:= \exists n(n \in X) \\
\top^X &:= \top \\
(t = u)^X &:= t = u \\
(t \in S)^X &:= t \in S \\
(B \vee C)^X &:= B^X \vee C^X \\
(B \wedge C)^X &:= B^X \wedge C^X \\
(B \to C)^X &:= \forall Y \supseteq X(B^Y \to T_Y(C)) \\
(\exists x B)^X &:= \exists x\, B^X \\
(\exists Y B)^X &:= \exists Y\, B^X \\
(\forall x B)^X &:= \forall x\, T_X(B) \\
(\forall Y B)^X &:= \forall Y\, T_X(B)
\end{aligned}
$$

where $T_X(A) := \forall Y \supseteq X[\forall Y' \supseteq Y(A^{Y'} \to \exists n(n \in Y')) \to \exists n(n \in Y)]$.

RCA_0 with cosepration

- $A^X = A$ for any Σ^0_1 formula A.
- $\vdash_{\mathsf{RCA}_0} A$, then $\vdash_{i\mathsf{RCA}_0} T_X(A)$.
- $\vdash_{\mathsf{RCA}_0} A$, then $\vdash_{i\mathsf{RCA}_0} A$ if A is Σ^0_1.

Thank you!