A primer on Optimization

Prescriptive Analytics

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Optimization of Functions

1. Single Variable Optimization

For a function $f(x)$, the goal of optimization is to find the point(s) where $f(x)$ reaches a maximum or minimum. The critical points are found by solving:

$$f'(x) = 0$$

Steps:

1. **Differentiate** the function $f(x)$.
2. **Solve** $f'(x) = 0$ for x.
3. **Check** the second derivative $f''(x)$:
   - If $f''(x) > 0$, the critical point is a local minimum.
   - If $f''(x) < 0$, the critical point is a local maximum.

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2. Optimization of Two-Variable Function with Constraints

For a function $f(x, y)$ subject to an equality constraint $g(x, y) = 0$, we use **Lagrange multipliers** to find the extrema. The Lagrange function is:

$$\mathcal{L}(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

Steps:

1. Form the **Lagrange function**.
2. Differentiate with respect to x, y, and λ:

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

3. Solve the system of equations to find the optimal x, y, and λ.

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3. Linear Programming

Linear programming (LP) is used to optimize a linear objective function subject to a set of linear equality and inequality constraints.

General Form:

Maximize:

$$Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n$$

Subject to:

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1$$

$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m$$

Where:

- X_1, X_2, ..., $X_n \geq 0$
- b_1, b_2, ..., $b_m \geq 0$

Steps:

1. **Define** the objective function.
2. **Set up** the constraints.
3. Use algorithms like the **Simplex Method** or **Interior Point Methods** to find the optimal values of X_1, X_2, ..., X_n.

This summarizes the key techniques in single-variable, multi-variable constrained optimization, and linear programming. Most of our efforts will be geared towards translating business problems into linear programming formulation!

Why Not Use Lagrange Multipliers for Linear Programming?

While Lagrange multipliers are useful for constrained optimization, they aren't typically used for **linear programming (LP)**. Here's why:

1. Nature of Functions and Constraints

- **Lagrange Multiplier**: Designed for **nonlinear** problems with differentiable objective functions and equality constraints.
- **Linear Programming**: Involves **linear** objective functions and **inequality** constraints. LP solutions are found at the **vertices** of the feasible region, not where gradients align as in the Lagrange method.

2. Handling Inequality Constraints

- **Lagrange**: Primarily handles **equality constraints**. Extending to inequality constraints (via KKT conditions) is complex.
- **LP**: Methods like **Simplex** and **Interior Point** handle inequalities naturally and efficiently.

3. Efficiency

- **Lagrange**: Solving the system of equations derived from Lagrange multipliers can be computationally expensive for large-scale problems.
- **LP**: **Simplex** and **Interior Point Methods** are highly optimized for solving linear systems quickly, especially for large LP problems.

4. Vertex Solutions vs. Interior Solutions

- **Lagrange**: Finds **interior solutions** where gradients of the objective function and constraints align.
- **LP**: Optimal solutions often lie at the **vertices** of the feasible region, which are better handled by LP-specific methods like **Simplex**.

Conclusion

Lagrange multipliers are suited for nonlinear problems, whereas linear programming methods like **Simplex** are designed to efficiently solve LP problems, especially when inequality constraints and vertex solutions are involved.