Red Brand Canners Case: Shadow prices and Scenarios

Prescriptive Analytics

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Shadow prices

Definition

Shadow Prices in Linear Programming

In linear programming (LP), **shadow prices** represent the marginal value of relaxing a constraint. More specifically, the shadow price of a constraint is the amount by which the optimal value of the objective function would improve if the right-hand side of that constraint were increased by one unit, assuming all other factors remain constant.

Shadow prices are only valid within certain ranges, known as the **allowable range** for the constraint. If a constraint is not binding in the optimal solution (i.e., it has slack), its shadow price is typically zero, since relaxing it further has no effect on the objective.

Key Points: - Shadow prices indicate the worth of an additional unit of a constrained resource. - They are directly tied to binding constraints in the optimal solution. - Useful for sensitivity analysis and understanding the value of limited resources.

In summary, shadow prices provide insight into the economic trade-offs within an LP model, helping to identify how much the objective can improve by relaxing resource constraints.

Shadow Prices and Lagrange Multipliers in Linear Programming

In linear programming (LP), **shadow prices** are closely related to **Lagrange multipliers**, which arise in constrained optimization problems.

Lagrange Multipliers and LP Consider a standard LP problem:

Maximize (or Minimize) $\quad c^T x$

Subject to: $Ax \leq b$

$x \geq 0$

Where: - x is the vector of decision variables, - c is the coefficient vector in the objective function, - A is the matrix of constraint coefficients, and - b is the vector representing the right-hand side of the constraints.

To solve this using Lagrange multipliers, we introduce a vector λ, where each λ_i is the Lagrange multiplier (or **dual variable**) associated with the i^{th} constraint. These dual variables represent the **rate of change of the objective function** with respect to the right-hand side of the corresponding constraint, effectively measuring how sensitive the objective function is to small changes in the resource availability.

Shadow Prices as Lagrange Multipliers The shadow price of a constraint in LP corresponds to the value of the associated **Lagrange multiplier** at the optimal solution. Mathematically, for each binding constraint $A_i x = b_i$, the Lagrange multiplier λ_i provides the shadow price, indicating how much the objective function will improve per unit increase in b_i.

Thus, in the optimal solution: - $\lambda_i > 0$ for binding constraints (i.e., constraints that are tight and fully utilized), - $\lambda_i = 0$ for non-binding constraints (i.e., constraints with slack).

This means: - For a binding constraint, an increase in the available resource (e.g., increasing b_i will result in an improvement in the objective function, with the rate of improvement given by λ_i, the shadow price. - For a non-binding constraint, increasing the resource does not improve the objective function, hence the shadow price is zero.

Mathematical Interpretation The optimal solution can be found by solving the **Lagrangian function**:

$\mathcal{L}(x, \lambda) = c^T x + \lambda^T (b - Ax)$

The Lagrange multipliers λ are determined as part of the optimization process, and these multipliers are precisely the **shadow prices**.

In Summary: - Shadow prices in LP represent Lagrange multipliers in the dual problem. - These multipliers reflect the marginal value of increasing the right-hand side of a constraint. - Shadow prices are non-zero for binding constraints and zero for non-binding ones, indicating whether a resource is fully utilized or has slack.

By linking shadow prices to Lagrange multipliers, we can better understand how small changes in constraints impact the optimal solution in both linear and nonlinear optimization contexts.

Application to the case

Execution

An inequality constraint is binding if the solution makes it an equality. Otherwise it is nonbinding, and the positive difference between the two sides of the constraint is called slack.

RED BRAND CANNERS					
MIX DECISION	**Whole**	**Juice**	**Paste**	**Total Required**	**Available**
Grade A	525	75	0	600	≤ 600
Grade B	175	225	2,000	2,400	2,400
Total Production	700	300	2,000		3,000
Demand	≤ 14,400	1,000	2,000		
QUALITY	**Whole**	**Juice**	**Paste**	**Quality**	
Grade A	4,725	675	0	9	
Grade B	875	1,125	10,000	5	
Total Quality	5,600	1,800	10,000		
Required Total Quality	≥ 5,600	1,800	10,000		
Average Quality	8.0	6.0	5.0		
Required Average Quality	8.0	6.0	5.0		
PROFIT	**Whole**	**Juice**	**Paste**	**Total Contribution**	**Profit**
Contribution Margin	$247	$198	$222	max $676,300	$136,300

Figure 1: Binding and nonbinding constraints

In order to run sensitivity analysis we need to make sure we are using a linear solver, so please select "COIN-OR CBC" as solver.

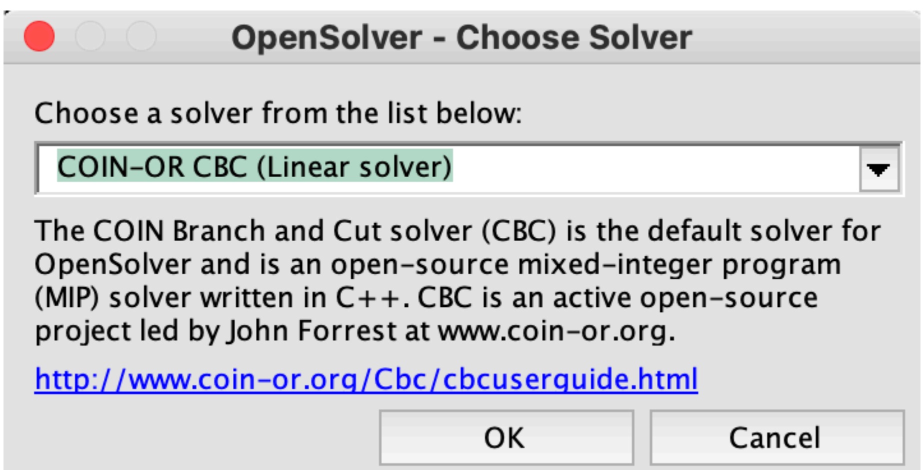

Figure 2: COIN-OR CBC

Note that we will need to "ignore the integer constraints", because sensitivity analysis requires continuous variables. This is not a problem in this problem

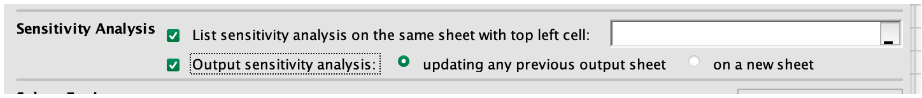

Figure 3: Sensitivity analysis options

because cans are almost continuous (one extra can does not move the needle).

Insights

Mr. Gordon stated that this was not a real limitation. Recently solicited to purchase 80,000 pounds of Grade “A” tomatoes at 25½ cents per pound, he had turned down the offer. He thought, however, that the tomatoes were still available.

Figure 4: Premise

- The constraint Total Grade A tomatoes required is equal to 271.5 with an allowable increase equal to 600.
- How much should RBC be willing to pay for one additional A tomato? Should they buy the additional A tomatoes at 25.5 cents?
- They should be willing to pay up to 27.15 cents.
- Yes they should buy all 80k additional A tomatoes at 25.5 cents since the price is lower than the shadow price (27.15 cents) and 80 is less than the allowable increase (600).
- Bear in mind that we working in 1000s!

OpenSolver Sensitivity Report - CBC
Worksheet: [RBC Lecture.xlsx] Model Sensitivity
Report Created: 23/9/24 22:35:41

Decision Variables

Cells	Name	Final Value	Reduced Costs	Objective Value	Allowable Increase	Allowable Decrease
B4	Grade A Whole	525	0	247	462.6666669	65.3333334
C4	Grade A Juice	75	0	198	65.33333342	462.6666669
D4	Grade A Paste	0	-98	222	98	1E+100
B5	Grade B Whole	175	0	247	1388.000001	65.33333342
C5	Grade B Juice	225	0	198	43.1111112	154.2222223
D5	Grade B Paste	2000	0	222	1E+100	48.5000001

Constraints

Cells	Name	Final Value	Shadow Price	RHS Value	Allowable Increase	Allowable Decrease
B12>=B13	Total Quality Whole	0	-24.5	0	466.6666667	600
C12>=C13	Total Quality Juice	0	-24.5	0	1400	200
D12>=D13	Total Quality Paste	0	0	0	0	1E+100
B6<=B7	Total Production Whole	700	0	14400	1E+100	13700
C6<=C7	Total Production Juice	300	0	1000	1E+100	700
D6<=D7	Total Production Paste	2000	48.5	2000	200	466.6666667
E4<=F4	Grade A Total Required	600	271.5	600	600	466.6666667
E5<=F5	Grade B Total Required	2400	173.5	2400	466.6666667	200

Figure 5: Sensitivity report

Scenario analysis

Premise

Scenario analysis is a strategic planning tool used to evaluate the potential outcomes of different future events by analyzing various possible scenarios. It involves developing and examining multiple, distinct "what-if" situations to understand the impact of uncertainties on decision-making.

This method is commonly used in finance, business strategy, and risk management to:

- Assess how different external factors, such as market shifts or regulatory changes, might influence key outcomes.
- Prepare for extreme cases (best-case, worst-case, and baseline scenarios).
- Identify potential risks and opportunities under various conditions.

Scenario analysis helps organizations make informed decisions by considering a range of future possibilities rather than relying on a single forecast.

Application to the case

Let's work with a small variation on the actual case:

Suppose today is May 1st, so RBC has to determine how much tomato crop to purchase first and then allocates Grade A and B tomatoes across tomato products for next year. Fruit costs per pound are 18 cents

Everything else is the same but the demand can be either high or low

Scenario	Probability	Whole Tomato (lbs)	Tomato Juice (lbs)	Tomato Paste (lbs)
High	80%	14,400K	1,000K	2,000K
Low	20%	600K	500K	1,000K

Let's start with the high demand scenario. We need to make a few adjustments in our model:

- We need to maximize profit, not just total contribution
- The profit now depends on the total production
- The total production has become a decision variable

With that in mind, we can adjust and rerun out model:

We can then proceed with the low demand scenario, which is identical in construction and changes only in the values of the demand constraints.

What happens when we err in our forecast? There are two types of mistakes we can make.

RED BRAND CANNERS					
MIX DECISION	**Whole**	**Juice**	**Paste**	**Total Required**	**Available**
Grade A	477	250	0	727	≤ 727
Grade B	159	750	2,000	2,909	2,909
Total Production	636	1,000	2,000		3,636
Demand	≤ 14,400	1,000	2,000		
QUALITY	**Whole**	**Juice**	**Paste**	**Quality**	
Grade A	4,295	2,250	0	9	
Grade B	795	3,750	10,000	5	
Total Quality	5,091	6,000	10,000		
Required Total Quality	≥ 5,091	6,000	10,000		
Average Quality	8.0	6.0	5.0		
Required Average Quality	8.0	6.0	5.0		
PROFIT	**Whole**	**Juice**	**Paste**	**Total Contribution**	**Profit**
Contribution Margin	$247	$198	$222	$799,182	max $144,636

Figure 6: High demand scenario

RED BRAND CANNERS					
MIX DECISION	**Whole**	**Juice**	**Paste**	**Total Required**	**Available**
Grade A	239	125	0	364	≤ 364
Grade B	80	375	1,000	1,455	1,455
Total Production	318	500	1,000		1,818
Demand	≤ 600	500	1,000		
QUALITY	**Whole**	**Juice**	**Paste**	**Quality**	
Grade A	2,148	1,125	0	9	
Grade B	398	1,875	5,000	5	
Total Quality	2,545	3,000	5,000		
Required Total Quality	≥ 2,545	3,000	5,000		
Average Quality	8.0	6.0	5.0		
Required Average Quality	8.0	6.0	5.0		
PROFIT	**Whole**	**Juice**	**Paste**	**Total Contribution**	**Profit**
Contribution Margin	$247	$198	$222	$399,591	max $72,318

Figure 7: Low demand scenario

First, let's assume that the order for tomato crop is determined assuming the demand state will be high, but the actual demand state turns out to be low, i.e. we overestimate demand.

- The total production becomes "fixed" by the time we can optimize, so it is no longer a decision variable
- The demand constraint needs to be updates
- We can still optimize the allocation of tomatoes into *whole*, *juice*, and *paste*

As we see in the picture, despite still having some room to manoeuver we cannot avoid losses.

RED BRAND CANNERS					
MIX DECISION	**Whole**	**Juice**	**Paste**	**Total Required**	**Available**
Grade A	450	125	0	575	≤ 727
Grade B	150	375	1,000	1,525	2,909
Total Production	600	500	1,000		3,636
Demand	≤ 600	500	1,000		
QUALITY	**Whole**	**Juice**	**Paste**	**Quality**	
Grade A	4,050	1,125	0	9	
Grade B	750	1,875	5,000	5	
Total Quality	4,800	3,000	5,000		
Required Total Quality	≥ 4,800	3,000	5,000		
Average Quality	8.0	6.0	5.0		
Required Average Quality	8.0	6.0	5.0		
PROFIT	**Whole**	**Juice**	**Paste**	**Total Contribution**	**Profit**
Contribution Margin	$247	$198	$222	$469,200	max -$185,345

Figure 8: High demand overestimate