Chapter 8

Maximum Likelihood for Location-Scale Based Distributions

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Chapter 8
Maximum Likelihood for Location-Scale Based Distributions
Objectives

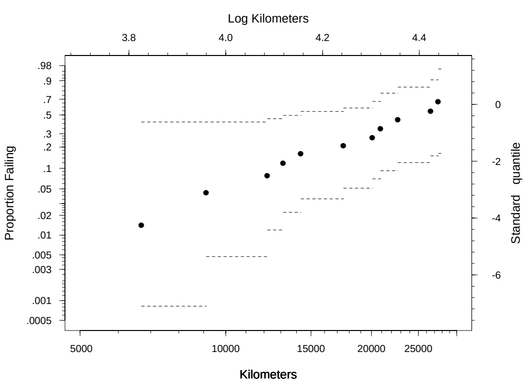
- Illustrate likelihood-based methods for parametric models based on log-location-scale distributions (especially Weibull and Lognormal).

- Construct and interpret likelihood-ratio-based confidence intervals/regions for model parameters and for **functions** of model parameters.

- Construct and interpret normal-approximation confidence intervals/regions.

- Describe the advantages and pitfalls of assuming that the log-location-scale distribution shape parameter is known.

Weibull Probability Plot of the Shock Absorber Data



Weibull Distribution Model Likelihood for Right Censored Data

- The Weibull distribution model is

$$\Pr(T \le t) = F(t;\mu,\sigma) = \Phi_{\text{sev}}\left\{[\log(t) - \mu]/\sigma\right\}.$$

- The likelihood has the form

$$\begin{aligned} L(\mu,\sigma) &= \prod_{i=1}^{n} L_i(\mu,\sigma;\text{data}_i) \\ &= \prod_{i=1}^{n} \left[f(t_i;\mu,\sigma)\right]^{\delta_i}\left[1 - F(t_i;\mu,\sigma)\right]^{1-\delta_i} \\ &= \prod_{i=1}^{n} \left[\frac{1}{\sigma t_i}\phi_{\text{sev}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{\delta_i} \times \left[1 - \Phi_{\text{sev}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{1-\delta_i} \end{aligned}$$

$$\delta_i = \begin{cases} 1 & \text{if } t_i \text{ is an exact observation} \\ 0 & \text{if } t_i \text{ is a right censored observation} \end{cases}$$

$\phi_{\text{sev}}(z)$ is the standardized smallest extreme value density.

Lognormal Distribution Model Likelihood for Right Censored Data

- The lognormal distribution model is

$$\Pr(T \le t) = F(t;\mu,\sigma) = \Phi_{\text{nor}}\left\{[\log(t) - \mu]/\sigma\right\}.$$

- The likelihood has the form

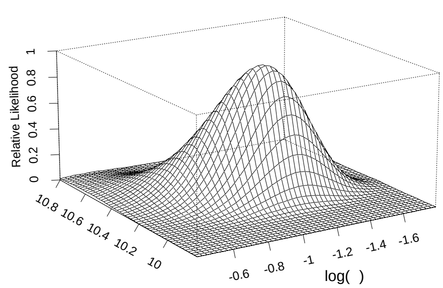
$$\begin{aligned} L(\mu,\sigma) &= \prod_{i=1}^{n} L_i(\mu,\sigma;\text{data}_i) \\ &= \prod_{i=1}^{n} \left[f(t_i;\mu,\sigma)\right]^{\delta_i}\left[1 - F(t_i;\mu,\sigma)\right]^{1-\delta_i} \\ &= \prod_{i=1}^{n} \left[\frac{1}{\sigma t_i}\phi_{\text{nor}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{\delta_i} \times \left[1 - \Phi_{\text{nor}}\left(\frac{\log(t_i)-\mu}{\sigma}\right)\right]^{1-\delta_i} \end{aligned}$$

$$\delta_i = \begin{cases} 1 & \text{if } t_i \text{ is an exact observation} \\ 0 & \text{if } t_i \text{ is a right censored observation} \end{cases}$$

$\phi_{\text{nor}}(z)$ is the standardized normal density.

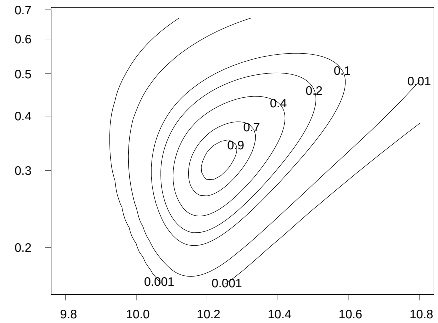
Weibull Relative Likelihood for the Shock Absorber Data
ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = .3164$

$$R(\mu, \log(\sigma)) = L(\mu, \log(\sigma))/L(\widehat{\mu}, \log(\widehat{\sigma}))$$



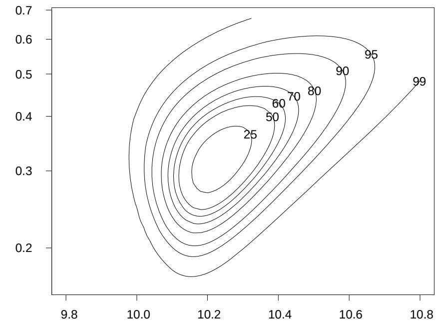
Weibull Relative Likelihood for the Shock Absorber Data ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = .3164$

$$R(\mu, \sigma) = L(\mu, \sigma)/L(\widehat{\mu}, \widehat{\sigma})$$

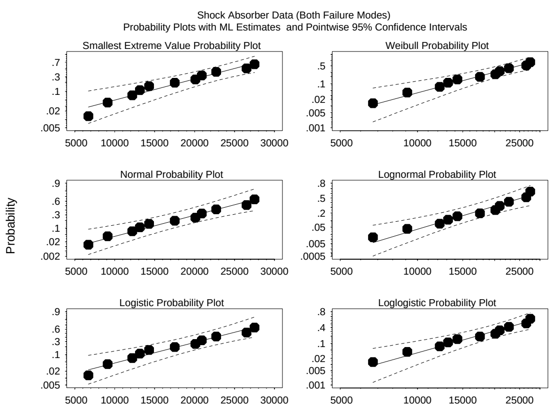


Weibull Likelihood-Based Joint Confidence Regions for μ and σ for the Shock Absorber Data ML Estimates: $\widehat{\mu} = 10.23$ and $\widehat{\sigma} = .3164$

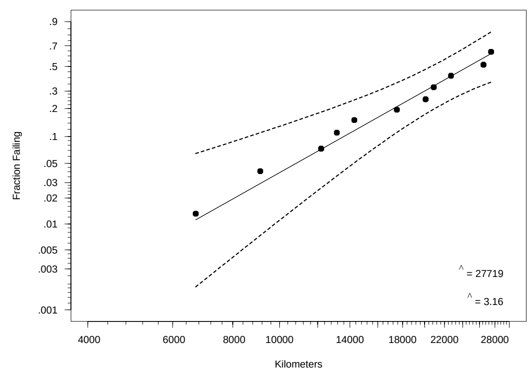
$$R(\mu, \sigma) > \exp\left[-\chi^2_{(1-\alpha;2)}/2\right] = 100\alpha\%$$



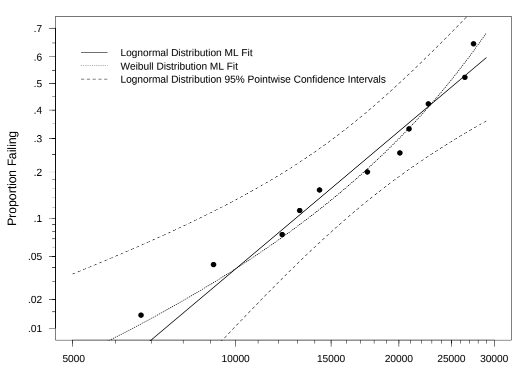
Six-Distribution ML Probability Plot of the Shock Absorber Data



Weibull Probability Plot of Shock Absorber Failure Times (Both Failure Modes) with Maximum Likelihood Estimates and Normal-Approximation 95% Pointwise Confidence Intervals for $F(t)$



Lognormal Probability Plots of Shock Absorber Data with ML Estimates and Normal-Approximation 95% Pointwise Confidence Intervals for $F(t)$. The Curved Line is the Weibull ML Estimate.



Large-Sample Approximate Theory for Likelihood Ratios for Parameter Vector

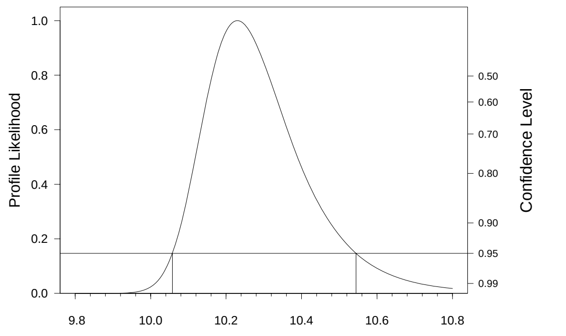
- Relative likelihood for (μ, σ) is

$$R(\mu, \sigma) = \frac{L(\mu, \sigma)}{L(\widehat{\mu}, \widehat{\sigma})}.$$

- If evaluated at the true (μ, σ), then, asymptotically, $-2\log[R(\mu, \sigma)]$ follows, a chisquare distribution with 2 degrees of freedom.
- General theory in the Appendix.

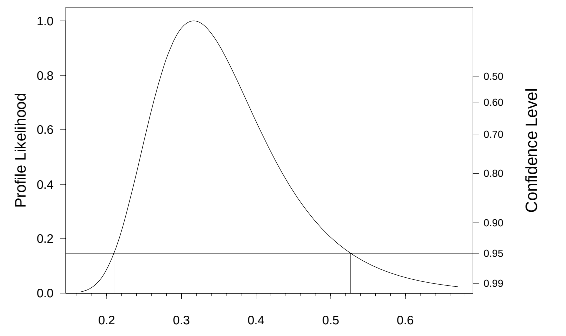
Weibull Profile Likelihood $R(\mu)$ ($\exp(\mu) \approx t_{.63}$) for the Shock Absorber Data

$$R(\mu) = \max_{\sigma} \left[\frac{L(\mu,\sigma)}{L(\widehat{\mu},\widehat{\sigma})}\right]$$



Weibull Profile Likelihood $R(\sigma)$ ($\sigma = 1/\beta$) for the Shock Absorber Data

$$R(\sigma) = \max_{\mu} \left[\frac{L(\mu,\sigma)}{L(\widehat{\mu},\widehat{\sigma})}\right]$$



Large-Sample Approximate Theory for Likelihood Ratios for Parameter Vector Subset

Need: Inferences on subset $\boldsymbol{\theta}_1$, from the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)'$.

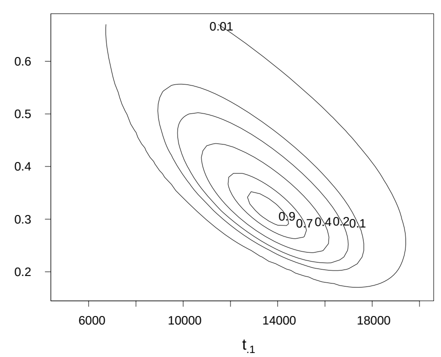
- $k_1 = \text{length}(\boldsymbol{\theta}_1)$.
- When $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)' = (\mu, \sigma)$, profile likelihood for $\boldsymbol{\theta}_1 = \mu$ is
$$R(\mu) = \max_{\sigma} \left[\frac{L(\mu,\sigma)}{L(\widehat{\mu},\widehat{\sigma})}\right].$$
- If evaluated at the true $\boldsymbol{\theta}_1 = \mu$, then, asymptotically, $-2\log[R(\mu)]$ follows, a chisquare distribution with $k_1 = 1$ degrees of freedom.
- General theory in the Appendix.

Asymptotic Theory of Likelihood Ratios – Continued

- An approximate $100(1-\alpha)\%$ likelihood-based confidence region for $\boldsymbol{\theta}_1$ is the set of all values of $\boldsymbol{\theta}_1$ such that
$$-2\log[R(\boldsymbol{\theta}_1)] < \chi^2_{(1-\alpha;k_1)}$$
or, equivalently, the set defined by
$$R(\boldsymbol{\theta}_1) > \exp\left[-\chi^2_{(1-\alpha;k_1)}/2\right].$$
- Transformation of $\boldsymbol{\theta}_1$ will not affect the confidence statement.
- Can improve the asymptotic approximation with simulation (only small effect except in very small samples).

Contour Plot of Weibull Relative Likelihood $R(t_{.1}, \sigma)$ for the Shock Absorber Data (Parameterized with $t_{.1}$ and σ)

$$R(t_{.1}, \sigma) = L(t_{.1}, \sigma)/L(\widehat{t}_{.1}, \widehat{\sigma})$$

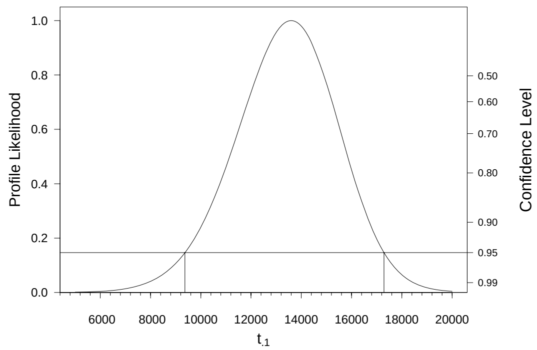


Confidence Regions and Intervals for Functions of μ and σ

- Likelihood approach can be applied to functions of parameters.
- Define the function of interest as one of the parameters, replacing one of the original parameters giving one-to-one reparameterization $\boldsymbol{g}(\mu, \sigma) = [g_1(\mu, \sigma), g_2(\mu, \sigma)]$.
- Then follow previous procedure.
- Simple to implement if function and its inverse are easy to compute.

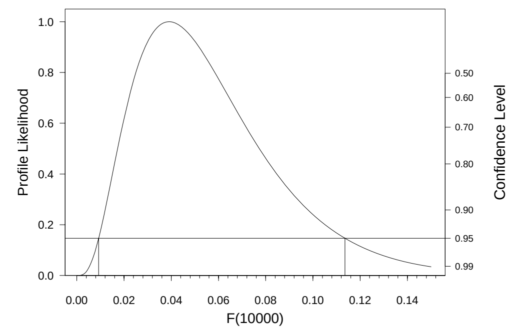
Weibull Profile Likelihood $R(t_{.1})$ for the Shock Absorber Data

$$R(t_{.1}) = \max_{\sigma} \left[\frac{L(t_{.1},\sigma)}{L(\widehat{t}_{.1},\widehat{\sigma})}\right]$$



Weibull Profile Likelihood $R[F(10000)]$ for the Shock Absorber Data

$$R\left[F(10000)\right] = \max_{\sigma} \left\{\frac{L[F(10000),\sigma]}{L\left[\widehat{F}(10000),\widehat{\sigma}\right]}\right\}$$



Asymptotic Theory of ML Estimation

Let $\widehat{\boldsymbol{\theta}}$ denote the ML estimator of $\boldsymbol{\theta}$.

- If evaluated at the true value of $\boldsymbol{\theta}$, then asymptotically, (large samples) $\widehat{\boldsymbol{\theta}}$ has a MVN$(\boldsymbol{\theta}, \Sigma_{\widehat{\boldsymbol{\theta}}})$ and thus the **Wald** statistic
$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \left[\Sigma_{\widehat{\boldsymbol{\theta}}}\right]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$
has a chisquare distribution with k degrees of freedom, where k is the length of $\boldsymbol{\theta}$.

- Here, $\Sigma_{\widehat{\boldsymbol{\theta}}} = I_{\boldsymbol{\theta}}^{-1}$ is the large sample approximate covariance matrix where the Fisher information matrix for $\boldsymbol{\theta}$ is
$$I_{\boldsymbol{\theta}} = \mathrm{E}\left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right].$$

Asymptotic Theory for Wald's Statistic

- Alternative asymptotic theory is based on the large-sample distribution of quadratic forms (Wald's statistic).

- Let $\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}$ be a consistent estimator of $\Sigma_{\widehat{\boldsymbol{\theta}}}$, the asymptotic covariance matrix of $\widehat{\boldsymbol{\theta}}$. For example,
$$\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}} = \left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]^{-1}$$
where the derivatives are evaluated at $\widehat{\boldsymbol{\theta}}$.

- Asymptotically, the Wald statistic
$$w(\boldsymbol{\theta}) = (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \left[\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}\right]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$$
when evaluated at the true $\boldsymbol{\theta}$, follows a chisquare distribution with k degrees of freedom, where k is the length of $\boldsymbol{\theta}$.

Asymptotic Theory for Wald's Statistic – Continued

- An approximate $100(1-\alpha)\%$ confidence region for $\boldsymbol{\theta}$ is the set of all values of $\boldsymbol{\theta}$ in the ellipsoid
$$(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})' \left[\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}\right]^{-1} (\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \le \chi^2_{(1-\alpha;k)}.$$

- This is sometimes known as the normal-theory confidence region.

- Can specialize to functions or subsets of $\boldsymbol{\theta}$.

- Can transform to improve asymptotic approximation. Try to get a log likelihood with approximate quadratic shape.

Normal-Approximation Confidence Intervals for Model Parameters

- Estimated variance matrix for the shock absorber data
$$\widehat{\Sigma}_{\widehat{\mu},\widehat{\sigma}} = \begin{bmatrix} \widehat{\mathrm{Var}}(\widehat{\mu}) & \widehat{\mathrm{Cov}}(\widehat{\mu},\widehat{\sigma}) \\ \widehat{\mathrm{Cov}}(\widehat{\mu},\widehat{\sigma}) & \widehat{\mathrm{Var}}(\widehat{\sigma}) \end{bmatrix} = \begin{bmatrix} .01208 & .00399 \\ .00399 & .00535 \end{bmatrix}$$

- Assuming that $Z_{\widehat{\mu}} = (\widehat{\mu} - \mu)/\widehat{\mathrm{se}}_{\widehat{\mu}} \mathrel{\dot\sim} \mathrm{NOR}(0,1)$ distribution, an approximate $100(1-\alpha)\%$ confidence interval for μ is
$$[\underset{\sim}{\mu}, \quad \widetilde{\mu}] = \widehat{\mu} \pm z_{(1-\alpha/2)}\widehat{\mathrm{se}}_{\widehat{\mu}}$$
where $\widehat{\mathrm{se}}_{\widehat{\mu}} = \sqrt{\widehat{\mathrm{Var}}(\widehat{\mu})}$.

- Assuming that $Z_{\log(\widehat{\sigma})} = [\log(\widehat{\sigma}) - \log(\sigma)]/\widehat{\mathrm{se}}_{\log(\widehat{\sigma})} \mathrel{\dot\sim} \mathrm{NOR}(0,1)$ an approximate $100(1-\alpha)\%$ confidence interval for σ is
$$[\underset{\sim}{\sigma}, \quad \widetilde{\sigma}] = [\widehat{\sigma}/w, \quad \widehat{\sigma} \times w]$$
where $w = \exp\left[z_{(1-\alpha/2)}\widehat{\mathrm{se}}_{\widehat{\sigma}}/\widehat{\sigma}\right]$ and $\widehat{\mathrm{se}}_{\widehat{\sigma}} = \sqrt{\widehat{\mathrm{Var}}(\widehat{\sigma})}$.

Normal-Approximation Confidence Intervals for Function $g_1 = g_1(\mu, \sigma)$

- ML estimate $\widehat{g}_1 = g_1(\widehat{\mu}, \widehat{\sigma})$.
- Assuming $Z_{\widehat{g}_1} = (\widehat{g}_1 - g_1)/\widehat{\text{se}}_{\widehat{g}_1} \dot{\sim} \text{NOR}(0,1)$, an approximate $100(1-\alpha)\%$ confidence interval for g_1 is

$$[\underset{\sim}{g_1}, \quad \tilde{g}_1] \; = \; \widehat{g}_1 \pm z_{(1-\alpha/2)} \widehat{\text{se}}_{\widehat{g}_1},$$

where

$$\widehat{\text{se}}_{\widehat{g}_1} = \sqrt{\widehat{\text{Var}}(\widehat{g}_1)} = \left[\left(\frac{\partial g_1}{\partial \mu}\right)^2 \widehat{\text{Var}}(\widehat{\mu}) + \left(\frac{\partial g_1}{\partial \sigma}\right)^2 \widehat{\text{Var}}(\widehat{\sigma}) + 2\left(\frac{\partial g_1}{\partial \mu}\right)\left(\frac{\partial g_1}{\partial \sigma}\right)\widehat{\text{Cov}}(\widehat{\mu}, \widehat{\sigma})\right]$$

- Partial derivatives evaluated at $\widehat{\mu}, \widehat{\sigma}$.
- General theory in the appendix.

Normal-Approximation Confidence Interval for $F(t_e; \mu, \sigma)$

Objective: Obtain a point estimate and a confidence interval for $\Pr(T \le t_e) = F(t_e; \mu, \sigma)$ at a fixed and known point t_e.

- The ML estimates $\widehat{\boldsymbol{\theta}} = (\widehat{\mu}, \widehat{\sigma})$ and $\widehat{\Sigma}_{\widehat{\boldsymbol{\theta}}}$ are available.
- The ML estimate for $F(t_e; \mu, \sigma)$ is

$$\widehat{F} = F(t_e; \widehat{\mu}, \widehat{\sigma}) = \Phi(\widehat{\zeta_e})$$

where $\widehat{\zeta_e} = [\log(t_e) - \widehat{\mu}]/\widehat{\sigma}$.

- In the context of Wald's theory, however, there are many ways to obtain a confidence interval for $F(t_e; \mu, \sigma)$.

Confidence Interval for $F(t_e; \mu, \sigma)$–Continued

Note: Wald's confidence intervals depend on the parameterization used to derive the intervals.

For example, an approximate $100(1-\alpha)\%$ confidence interval for $F(t_e; \mu, \sigma)$ can be obtained using:

- The asymptotic normality of $Z_{\widehat{F}} = (\widehat{F} - F)/\widehat{\text{se}}_{\widehat{F}}$

$$[\underset{\sim}{F}, \quad \tilde{F}] = \widehat{F}(t_e) \pm z_{(1-\alpha/2)} \widehat{\text{se}}_{\widehat{F}}.$$

- The asymptotic normality of $Z_{\text{logit}(\widehat{F})} = [\text{logit}(\widehat{F}) - \text{logit}(F)]/\widehat{\text{se}}_{\text{logit}(\widehat{F})}$

$$[\underset{\sim}{F}, \quad \tilde{F}] = \left[\frac{\widehat{F}(t_e)}{\widehat{F}(t_e) + (1 - \widehat{F}(t_e)) \times w}, \quad \frac{\widehat{F}(t_e)}{\widehat{F}(t_e) + (1 - \widehat{F}(t_e))/w}\right]$$

where $w = \exp\{z_{(1-\alpha/2)} \widehat{\text{se}}_{\widehat{F}} / [\widehat{F}(t_e)(1 - \widehat{F}(t_e))]\}$.

Confidence Interval for $F(t_e; \mu, \sigma)$—Continued

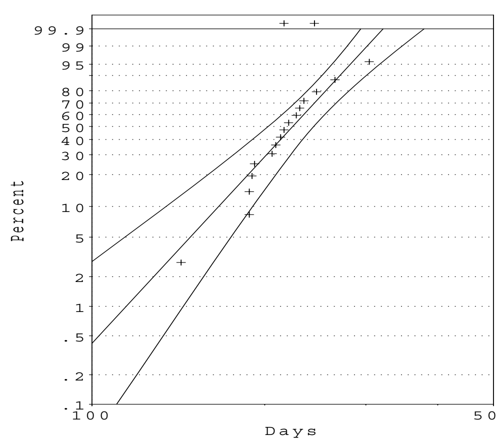
Comments:

- Often the confidence interval based on the asymptotic normality of $Z_{\widehat{F}}$ has poor statistical properties caused by the slow convergence toward normality of $Z_{\widehat{F}}$.
- The confidence interval based on the transformation $Z_{\text{logit}(\widehat{F})}$ can have better statistical properties if $Z_{\text{logit}(\widehat{F})}$ converges to normality faster than $Z_{\widehat{F}}$.

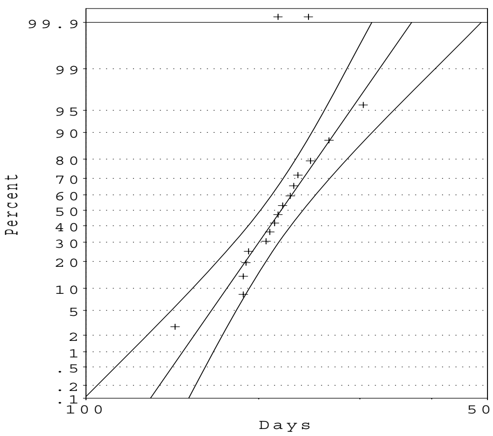
ML Estimates for Biomedical Data

Here we display SAS® Proc Reliability ML estimates (Weibull and lognormal) for the DMBA and the IUD data.

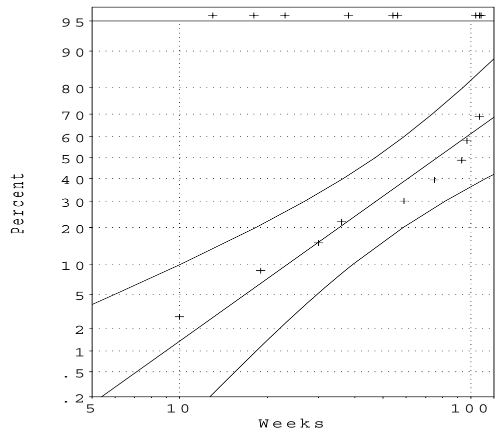
SAS® Proc Reliability Nonparametric and Weibull ML Estimate for DMBA Data with Parametric Pointwise Approximate 95% Confidence Intervals



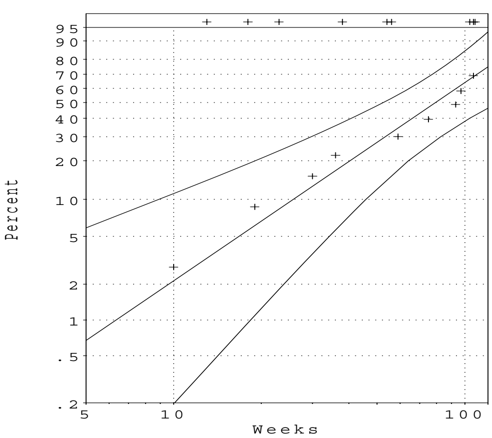
SAS® Proc Reliability
Nonparametric and Lognormal ML Estimate for DMBA Data with Parametric Pointwise Approximate 95% Confidence Intervals



SAS® Proc Reliability
Lognormal ML Estimate for IUD Data with a set of Pointwise Approximate 95% Confidence Intervals



SAS® Proc Reliability
Weibull ML Estimate for IUD Data with a set of Pointwise Approximate 95% Confidence Intervals



Inference when σ (or Weibull β) is Given

- Simplifies problem. Only one parameter with r failures and $t_1, \ldots, t_n$ failures and censor times

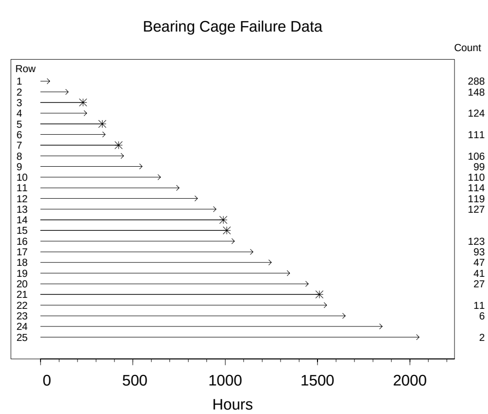
$$\widehat{\eta} = \left(\frac{\sum_{i=1}^{n} t_i^{\beta}}{r} \right)^{1/\beta}, \quad \widehat{\text{se}}_{\widehat{\eta}} = \frac{\widehat{\eta}}{\beta}\sqrt{\frac{1}{r}}.$$

- Provides much more precision, especially with small r.
- If 0 failures can provide
  - ▶ Upper confidence bound on $F(t)$.
  - ▶ Lower confidence bound on t_p.
- Requires sensitivity analysis because β is in doubt.
- Danger of misleading inferences.

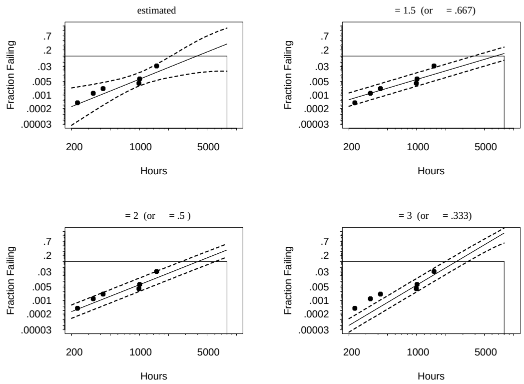
Bearing-Cage Fracture Field Data

- A population of $n = 1703$ units had been introduced into service over time and 6 failures have been observed.
- There is concern that the B10 design life specification of $t_{.1} = 8$ thousand hours was not being met.
- ML estimate is $\widehat{t}_{.1}$= 3.903 thousand hours and an approximate 95% likelihood-ratio confidence interval for $t_{.1}$ is [2.093, 22.144] thousand hours.
- Management also wanted to know how many additional failures could be expected in the next year.

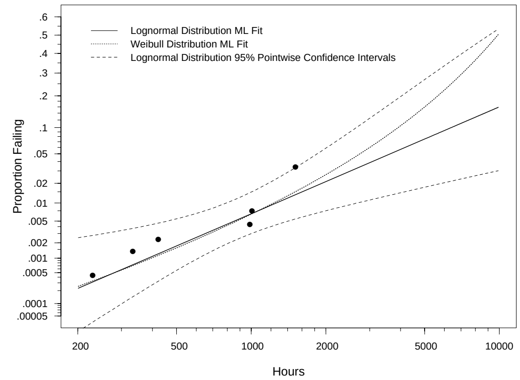
Bearing-Cage Fracture Data Event Plot



Weibull Probability Plots Bearing-Cage Fracture Data with Weibull ML Estimates and Sets of 95% Pointwise Confidence Intervals for $F(t)$ with β Estimated, and Assumed Known Values of β= 1.5, 2, and 3.



Lognormal and Weibull Comparison Bearing-Cage Fracture Field Data Lognormal Probability Paper



Weibull/SEV Distribution with Given $\beta = 1/\sigma$ and Zero Failures

- ML Estimate for the Weibull Scale Parameter η Cannot be Computed Unless the Available Data Contains One or More Failures.
- For a sample of n units with running times $t_1, \ldots, t_n$ and no failures, a conservative $100(1-\alpha)\%$ lower confidence bound for η is

$$\underset{\sim}{\eta} = \left(\frac{2 \sum_{i=1}^{n} t_i^{\beta}}{\chi^2_{(1-\alpha;2)}} \right)^{\frac{1}{\beta}}.$$

- The lower bound $\underset{\sim}{\eta}$ can be translated into an lower confidence bound for functions like t_p for specified p or a upper confidence bound for $F(t_e)$ for a specified t_e.

Component A Safe Data

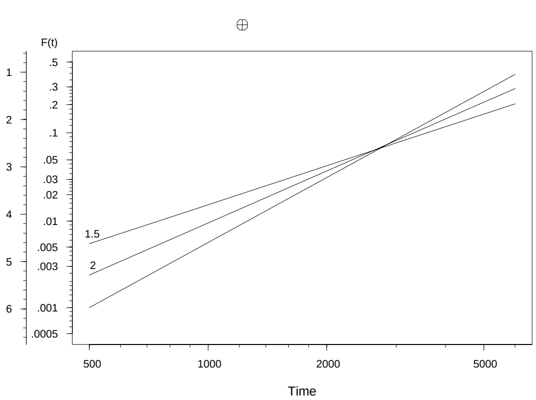
- A metal component in a ship's propulsion system fails from fatigue-caused fracture.
- Because of persistent reliability problems, the component was redesigned to have a longer service life.
- Previous experience suggests that the Weibull shape parameter is near $\beta = 2$, and almost certainly between 1.5 and 2.5.
- Newly designed components were put into service during the past year and no failures have been reported.

Hours:	500	1000	1500	2000	2500	3000	3500	4000
Number of Units:	10	12	8	9	7	9	6	3

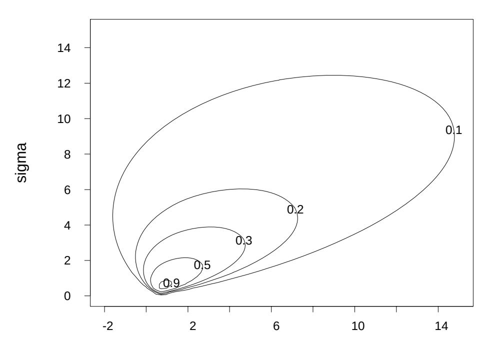
Staggered entry data, with no reported failures.

- Can replacement be increased from 2000 hours to 4000 hours?

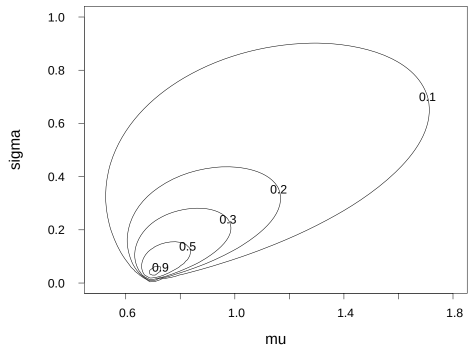
Weibull Model 95% Upper Confidence Bounds on $F(t)$ for Component-A with Different Fixed Values for the Weibull Shape Parameter



Relative Weibull Likelihood with One Failure at 1 and One Survivor at 2



Relative Weibull Likelihood with One Failure at 1.9 and One Survivor at 2



Regularity Conditions

- Each technical result (e.g., asymptotic distribution of an estimator) has its own set of conditions on the model (see Lehmann 1983, Rao 1973).
- Frequent reference to Regularity Conditions which give rise to simple results.
- For special cases the regularity conditions are easy to state and check. For example, for some location-scale distributions the needed conditions are:

$$\lim_{z \to -\infty} \frac{z^2 \phi^2(z)}{\Phi(z)} = 0$$
$$\lim_{z \to +\infty} \frac{z^2 \phi^2(z)}{1 - \Phi(z)} = 0.$$

- In **non-regular** models, asymptotic behavior is more complicated (e.g., behavior depends on $\boldsymbol{\theta}$), but there are still useful asymptotic results.

Regularity Conditions – Continued

Some **typical** regularity conditions include:

- Support does not depend on unknown parameters.
- Number of parameters does not grow too fast with n.
- Continuous derivatives of log likelihood (w.r.t. $\boldsymbol{\theta}$).
- Bounded derivatives of likelihood.
- Can exchange the order of differentiation of log likelihood w.r.t. $\boldsymbol{\theta}$ and integration w.r.t. data.
- Identifiability.

Other Topics Related to Parametric Likelihood Covered in the Book

- Truncated data (Chapter 11).
- Threshold parameters (Chapter 11).
- Other distributions (e.g., gamma) (Chapter 11).
- Bayesian methods (Chapter 14).
- Multiple failure modes (Chapter 15).