Chapter 5

Other Parametric Distributions

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Other Parametric Distributions
Chapter 5 Objectives

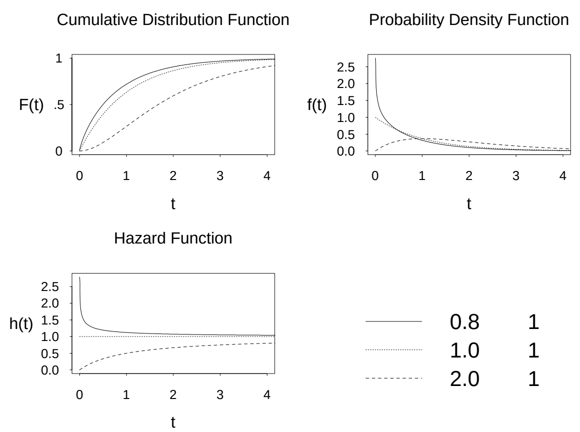
- Describe the properties and the importance of the following parametric distributions which cannot be transformed into a location-scale distribution:

  Gamma, Generalized Gamma, Extended Generalized Gamma, Generalized F, Inverse Gaussian, Birnbaum–Saunders, Gompertz–Makeham.

- Introduce the concept of a threshold-parameter distribution.

- Illustrate how other statistical models can be determined by applying basic ideas of probability theory to physical properties of a failure process, system, or population of units.

Examples of Gamma Distributions



Gamma Distribution

- T follows a gamma distribution, GAM(θ, κ), if

$$\begin{aligned} F(t;\theta,\kappa) &= \Gamma_{\mathrm{I}}\left(\frac{t}{\theta};\kappa\right) \\ f(t;\theta,\kappa) &= \frac{1}{\Gamma(\kappa)\,\theta}\left(\frac{t}{\theta}\right)^{\kappa-1}\exp\left(-\frac{t}{\theta}\right), \quad t>0 \end{aligned}$$

  $\theta > 0$ is a scale parameter and $\kappa > 0$ is a shape parameter. $\Gamma_{\mathrm{I}}(v;\kappa)$ is the incomplete gamma function defined by

$$\Gamma_{\mathrm{I}}(v;\kappa) = \frac{\int_0^v x^{\kappa-1}\exp(-x)\,dx}{\Gamma(\kappa)}, \quad v \geq 0.$$

- **Special case:** when $\kappa = 1$, GAM$(\theta,\kappa) \equiv$ EXP(θ).

- The hazard function $h(t;\theta,\kappa)$ is **decreasing** when $\kappa < 1$; **increasing** when $\kappa > 1$; and **approaches a constant** level late in life i.e.,

$$\lim_{t\to\infty} h(t;\theta,\kappa) = 1/\theta.$$

Moments and Quantiles of the Gamma Distribution

- **Moments:** For integer $m > 0$

$$\mathrm{E}(T^m) = \frac{\theta^m\,\Gamma(m+\kappa)}{\Gamma(\kappa)}.$$

  Then

$$\begin{aligned} \mathrm{E}(T) &= \theta\kappa \\ \mathrm{Var}(T) &= \theta^2\kappa \end{aligned}$$

- **Quantiles:** the p quantile of the distribution is given by

$$t_p = \theta\,\Gamma_{\mathrm{I}}^{-1}(p;\kappa).$$

Reparameterization of the Gamma Distribution

For accelerated time regression modeling, the cdf and pdf can be conveniently **reparameterized** as follows:

$$\begin{aligned} F(t;\theta,\kappa) &= \Phi_{\mathrm{lg}}\left[\log(t)-\mu;\kappa\right] \\ f(t;\theta,\kappa) &= \frac{1}{t}\,\phi_{\mathrm{lg}}\left[\log(t)-\mu;\kappa\right] \end{aligned}$$

where $\mu = \log(\theta)$, Φ_{lg} and ϕ_{lg} are the cdf and pdf for the **standardized** loggamma variable $Z = \log(T/\theta) = \log(T) - \mu$,

$$\begin{aligned} \Phi_{\mathrm{lg}}(z;\kappa) &= \Gamma_{\mathrm{I}}[\exp(z);\kappa] \\ \phi_{\mathrm{lg}}(z;\kappa) &= \frac{1}{\Gamma(\kappa)}\exp\left[\kappa z - \exp(z)\right]. \end{aligned}$$

Generalized Gamma Distribution

- T has a generalized gamma distribution if

$$F(t;\theta,\beta,\kappa) = \Gamma_{\mathrm{I}}\left[\left(\frac{t}{\theta}\right)^{\beta};\kappa\right]$$
$$f(t;\theta,\beta,\kappa) = \frac{\beta}{\Gamma(\kappa)\theta}\left(\frac{t}{\theta}\right)^{\kappa\beta-1}\exp\left[-\left(\frac{t}{\theta}\right)^{\beta}\right], \quad t>0$$

where $\theta > 0$ is a scale parameter, and $\kappa > 0$, $\beta > 0$ are shape parameters.

- If $\beta = 1$ the distribution becomes the GAM(θ, κ) distribution.

- If $\kappa = 1$ the distribution becomes the WEIB(μ, σ), where $\mu = \log(\theta)$ and $\sigma = 1/\beta$.

- If $\beta = 1$ and $\kappa = 1$ the distribution becomes the EXP(θ) distribution.

Generalized Gamma Distribution-Continued

- A more convenient parameterization is given by $\mu = \log(\theta) + (\sigma/\lambda)\log(\lambda^{-2})$, $\lambda = 1/\sqrt{\kappa}$, and $\sigma = 1/(\beta\sqrt{\kappa})$, in which case, we write $T \sim$ GENG(μ, σ, λ) and

$$F(t;\mu,\sigma,\lambda) = \Phi_{\mathrm{lg}}\left[\lambda\omega + \log(\lambda^{-2});\lambda^{-2}\right]$$
$$f(t;\mu,\sigma,\lambda) = \frac{\lambda}{\sigma t}\phi_{\mathrm{lg}}\left[\lambda\omega + \log(\lambda^{-2});\lambda^{-2}\right]$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\sigma > 0$, and $\lambda > 0$.

- If $T \sim$ GENG(μ, σ, λ) and $c > 0$ then $cT \sim$ GENG$[\mu - \log(c), \lambda, \sigma]$.

- As $\lambda \to 0$, $T \,\dot\sim\,$ LOGNOR(μ, σ).

- Moments, quantiles, and other related distributions will follow as special cases of the more general extended generalized gamma distribution.

Extended Generalized Gamma Distribution

- T has an extended generalized gamma distribution, EGENG(μ, σ, λ), if

$$F(t;\mu,\sigma,\lambda) = \begin{cases} \Phi_{\mathrm{lg}}\left[\lambda\omega + \log(\lambda^{-2});\lambda^{-2}\right] & \text{if } \lambda > 0 \\ \Phi_{\mathrm{nor}}(\omega) & \text{if } \lambda = 0 \\ 1 - \Phi_{\mathrm{lg}}\left[\lambda\omega + \log(\lambda^{-2});\lambda^{-2}\right] & \text{if } \lambda < 0 \end{cases}$$
$$f(t;\mu,\sigma,\lambda) = \begin{cases} \frac{|\lambda|}{\sigma t}\phi_{\mathrm{lg}}\left[\lambda\omega + \log(\lambda^{-2});\lambda^{-2}\right] & \text{if } \lambda \neq 0 \\ \frac{1}{\sigma t}\phi_{\mathrm{nor}}(\omega) & \text{if } \lambda = 0 \end{cases}$$

where $\omega = [\log(t) - \mu]/\sigma$, $-\infty < \mu < \infty$, $\exp(\mu)$ is a scale parameter, $-\infty < \lambda < \infty$ and $\sigma > 0$ are shape parameters.

Comments on the EGENG Distribution

- The distribution at $\lambda = 0$ is defined by **continuity** (i.e., the limiting distribution when $\lambda \to 0$).

- If $T \sim$ EGENG(μ, σ, λ) and $c > 0$ then $cT \sim$ EGENG$[\mu - \log(c), \lambda, \sigma]$. Thus, $\exp(\mu)$ is a location-parameter for T.

- When $T \sim$ EGENG(μ, λ, σ) then the distribution of $W = [\log(T) - \mu]/\sigma$ depends only on λ.

- Note that for each fixed λ, $\log(T)$ is location-scale (μ, σ) with a standardized location-scale distribution equal to the distribution of W.

Extended Generalized Gamma Distribution–Continued

- **Moments:** For integer m and $\lambda \neq 0$

$$\mathrm{E}(T^m) = \begin{cases} \frac{\exp(m\mu)\,(\lambda^2)^{m\sigma/\lambda}\,\Gamma\left[\lambda^{-1}(m\sigma+\lambda^{-1})\right]}{\Gamma(\lambda^{-2})} & \text{if } m\lambda\sigma + 1 > 0 \\ \infty & \text{if } m\lambda\sigma + 1 \leq 0. \end{cases}$$

When $\lambda = 0$, the moments are

$$\mathrm{E}(T^m) = \exp\left[m\mu + (1/2)(m\sigma)^2\right].$$

- Thus when the mean and the variance are finite and $\lambda \neq 0$,

$$\mathrm{E}(T) = \frac{\theta\,\Gamma\left[\lambda^{-1}(\sigma + \lambda^{-1})\right]}{\Gamma(\lambda^{-2})}$$
$$\mathrm{Var}(T) = \theta^2\left[\frac{\Gamma\left[\lambda^{-1}(2\sigma+\lambda^{-1})\right]}{\Gamma(\lambda^{-2})} - \frac{\Gamma^2\left[\lambda^{-1}(\sigma+\lambda^{-1})\right]}{\Gamma^2(\lambda^{-2})}\right].$$

- When $\lambda = 0$, $\mathrm{E}(T) = \exp[\mu + (1/2)\sigma^2]$ and $\mathrm{Var}(T) = \exp(2\mu + \sigma^2) \times [\exp(\sigma^2) - 1]$.

Quantiles of the EGENG Distribution

The EGENG quantiles are

$$\log(t_p) = \mu + \sigma \times \omega(p;\lambda)$$

where $\omega(p;\lambda)$ is the p quantile of the distribution of W,

$$\omega(p;\lambda) = \begin{cases} \lambda^{-1}\log\left[\lambda^2\Gamma_{\mathrm{I}}^{-1}(p;\lambda^{-2})\right] & \text{if } \lambda > 0 \\ \Phi_{\mathrm{nor}}^{-1}(p) & \text{if } \lambda = 0 \\ \lambda^{-1}\log\left[\lambda^2\Gamma_{\mathrm{I}}^{-1}(1-p;\lambda^{-2})\right] & \text{if } \lambda < 0 \end{cases}$$

Distributions Related to EGENG

Special Cases:

- If $\lambda > 0$ then EGENG$(\mu, \sigma, \lambda) =$ GENG(μ, σ, λ).
- if $\lambda = 1$, $T \sim$ WEIB(μ, σ).
- if $\lambda = 0$, $T \sim$ LOGNOR(μ, σ).
- if $\lambda = -1$, $1/T \sim$ WEIB$(-\mu, \sigma)$, [i.e., T has a reciprocal Weibull (or Fréchet distribution of maxima)].
- When $\lambda = \sigma$, $T \sim$ GAM(θ, κ), where $\theta = \lambda^2 \exp(\mu)$ and $\kappa = \lambda^{-2}$.
- When $\lambda = \sigma = 1$, $T \sim$ EXP(θ), where $\theta = \lambda^2 \exp(\mu)$.

Comment on EGENG(μ, σ, λ) Parameterization

- The (μ, σ, λ) parameterization is due to Farewell and Prentice (1977). Observe that

$$F[\exp(\mu); \mu, \sigma, \lambda] = \begin{cases} \Gamma_{\mathrm{I}}(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda > 0 \\ .5 & \text{if } \lambda = 0 \\ 1 - \Gamma_{\mathrm{I}}(\lambda^{-2}; \lambda^{-2}) & \text{if } \lambda < 0 \end{cases}$$

  This value of $F[\exp(\mu); \mu, \sigma, \lambda]$, as a function of λ, is always in the interval $[.5, 1)$. Thus $\exp(\mu)$ equals a quantile t_p with $p \geq .5$.

- The parameterization is stable when there is not much censoring. It tends to be unstable when there is heavy censoring.
- When there is heavy censoring a different parameterization is needed for ML estimation.

EGENG Stable Parameterization

- **Parameterization for Numerical Stability:** with $p_1 < p_2$, an stable parameterization can be obtained using two quantiles (t_{p_1}, t_{p_2}), and λ, i.e.,

$$\log(t_{p_1}) = \mu + \sigma\omega(p_1, \lambda)$$
$$\log(t_{p_2}) = \mu + \sigma\omega(p_2, \lambda)$$

  and solving for μ and σ,

$$\mu = \frac{\omega(p_2, \lambda) \times \log(t_{p_1}) - \omega(p_1, \lambda) \times \log(t_{p_2})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)}$$
$$\sigma = \frac{\log(t_{p_2}) - \log(t_{p_1})}{\omega(p_2, \lambda) - \omega(p_1, \lambda)}.$$

Generalized F Distribution

T has a generalized F distribution with parameters (μ, σ, κ, r), say GENF(μ, σ, κ, r), if

$$F_T(t; \mu, \sigma, \kappa, r) = \Phi_{\mathrm{lf}}\left[\frac{\log(t) - \mu}{\sigma}; \kappa, r\right]$$
$$f_T(t; \mu, \sigma, \kappa, r) = \frac{1}{\sigma t}\phi_{\mathrm{lf}}\left[\frac{\log(t) - \mu}{\sigma}; \kappa, r\right], \quad t > 0$$

where

$$\phi_{\mathrm{lf}}(z; \kappa, r) = \frac{\Gamma(\kappa + r)}{\Gamma(\kappa)\,\Gamma(r)} \frac{(\kappa/r)^\kappa \exp(\kappa z)}{[1 + (\kappa/r)\exp(z)]^{\kappa + r}}$$

is the pdf of the central log F distribution with 2κ and $2r$ degrees of freedom and Φ_{lf} is the corresponding cdf.

It follows that $\phi_{\mathrm{lf}}(z; \kappa, r)$ and $\Phi_{\mathrm{lf}}(z; \kappa, r)$ are the pdf and cdf of $Z = [\log(T) - \mu]/\sigma$.

$\exp(\mu)$ is a scale parameter and $\sigma > 0$, $\kappa > 0$, $r > 0$ are shape parameters.

Generalized F Distribution-Continued

- **Moments:** For integer $m \geq 0$,

$$\mathrm{E}(T^m) = \begin{cases} \frac{\exp(m\mu)\,\Gamma(\kappa + m\sigma)\,\Gamma(r - m\sigma)}{\Gamma(\kappa)\,\Gamma(r)} \left(\frac{r}{\kappa}\right)^{m\sigma}, & \text{if } r > m\sigma \\ \infty & \text{otherwise.} \end{cases}$$

  Then

$$\mathrm{E}(T) = \frac{\Gamma(\kappa + \sigma)\,\Gamma(r - \sigma)}{\Gamma(\kappa)\Gamma(r)} \exp(\mu) \left(\frac{r}{\kappa}\right)^\sigma$$
$$\mathrm{Var}(T) = \left\{\frac{\Gamma(\kappa + 2\sigma)\,\Gamma(r - 2\sigma)}{\Gamma(\kappa)\Gamma(r)} - \frac{\Gamma^2(\kappa + \sigma)\,\Gamma^2(r - \sigma)}{\Gamma^2(\kappa)\Gamma^2(r)}\right\} \exp(2\mu) \left(\frac{r}{\kappa}\right)^{2\sigma}$$

  where $r > \sigma$ for the mean and $r > 2\sigma$ for the variance.

- **Quantiles:** The p quantile of the distribution is

$$t_p = \exp(\mu)\left[\mathcal{F}_{(p, 2\kappa, 2r)}\right]^\sigma$$

  where $\mathcal{F}_{(p, 2\kappa, 2r)}$ is the p quantile of an F distribution with $(2\kappa, 2r)$ degrees of freedom.

  The expression for t_p follows directly from the fact that $T = \exp(\mu)V^\sigma$ where V has an F distribution with $(2\kappa, 2r)$ degrees of freedom.

Generalized F Distribution–Special Cases

- $1/T \sim$ GENF$(-\mu, \sigma, r, \kappa)$.
- When $(\mu, \sigma) = (0, 1)$ then T follows an F distribution with 2κ numerator and $2r$ denominator degrees of freedom.
- When $(\kappa, r) = (1, 1)$, GENF$(\mu, \sigma, \kappa, r) \equiv$ LOGLOGIS(μ, σ).
- When $r \to \infty$, $T \;\dot\sim\;$ GENG$[\exp(\mu)/\kappa^\sigma, 1/\sigma, \kappa]$.
- When $(\kappa, r) = (1, \infty)$, $T \sim$ WEIB(μ, σ).
- When $\kappa = 1$, T follows a Burr type XII distribution with cdf

$$F(t; \mu, \sigma, r) = 1 - \frac{1}{\left[1 + \frac{1}{r}\left(\frac{t}{\theta}\right)^{\frac{1}{\sigma}}\right]^r}, \quad t > 0$$

  where $r > 0$, $\sigma > 0$ are shape parameters, and $\theta = \exp(\mu)$ is a scale parameter.

- When $\kappa \to \infty$, and $r \to \infty$, $T \;\dot\sim\;$ LOGNOR$\left(\mu, \sigma\sqrt{(\kappa + r)/\kappa r}\right)$.

Inverse Gaussian Distribution

- A common parameterization for the cdf of this distribution is (see Chhikara and Folks 1989) is

$$\Pr(T \le t; \theta, \lambda) = \Phi_{\text{nor}}\left[\frac{(t-\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right] + \exp\left(\frac{2\lambda}{\theta}\right)\Phi_{\text{nor}}\left[-\frac{(t+\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right].$$

  $t > 0$; $\theta > 0$ and $\lambda > 0$ are parameters in the same units of T.

- Wald (1947) derived this distribution as a limiting form for the distribution of sample size in sequential probability ratio test.

Inverse Gaussian Distribution–Origin

- The inverse Gaussian distribution was originally given by Schrödinger (1915) as the distribution of the first passage time in Brownian motion. The parameters θ and λ relate to the Brownian motion parameters as follows:

- Consider a Brownian process

$$B(t) = ct + dW(t), \quad t > 0$$

  where c, d are constants and $W(t)$ is a Wiener process. Let T be the first passage time of a specified level b_0, say

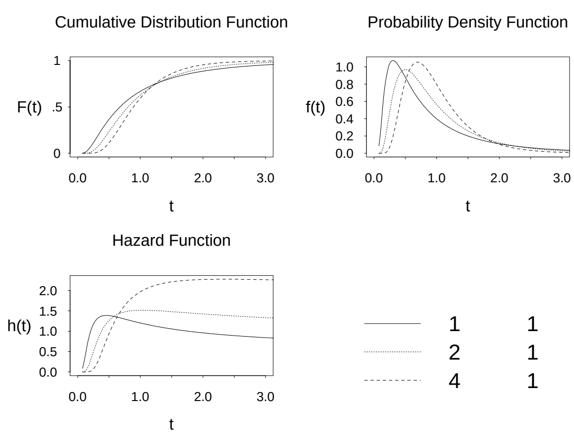
$$T = \inf\{t; B(t) \ge b_0\}.$$

  Then

$$\Pr(T \le t) = \Phi_{\text{nor}}\left[\frac{(t-\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right] + \exp\left(\frac{2\lambda}{\theta}\right)\Phi_{\text{nor}}\left[-\frac{(t+\theta)\sqrt{\lambda}}{\theta\sqrt{t}}\right]$$

  where $\theta = b_0/c$ and $\sqrt{\lambda} = b_0/d$. Tweedie (1945) gives more details on this approach.

Examples of Inverse Gaussian Distributions



Inverse Gaussian Distribution–Continued

- The reparameterization $(\theta, \beta = \lambda/\theta)$ separates the location and scale parameters. We say that $T \sim \text{IGAU}(\theta, \beta)$ if

$$\begin{aligned} F_T(t; \theta, \beta) &= \Phi_{\text{ligau}}[\log(t/\theta); \beta] \\ f_T(t; \theta, \beta) &= \frac{1}{t}\phi_{\text{ligau}}[\log(t/\theta); \beta], \quad t > 0 \end{aligned}$$

  where $\theta > 0$ is a scale parameter, $\beta > 0$ is at unit less shape parameter, and

$$\begin{aligned} \Phi_{\text{ligau}}(z; \beta) &= \Phi_{\text{nor}}\left\{\sqrt{\beta}\left[\frac{\exp(z) - 1}{\exp(z/2)}\right]\right\} + \\ &\quad \exp(2\beta)\,\Phi_{\text{nor}}\left\{-\sqrt{\beta}\left[\frac{\exp(z) + 1}{\exp(z/2)}\right]\right\} \\ \phi_{\text{ligau}}(z; \beta) &= \frac{\sqrt{\beta}}{\exp(z/2)}\,\phi_{\text{nor}}\left\{\sqrt{\beta}\left[\frac{\exp(z) - 1}{\exp(z/2)}\right]\right\}, \quad -\infty < z < \infty. \end{aligned}$$

- The hazard function has the following behavior: $h_T(0; \theta, \beta) = 0$, $h_T(t; \theta, \beta)$ is unimodal, and $\lim_{t\to\infty} h_T(t; \theta, \beta) = \beta/(2\theta)$.

Inverse Gaussian Distribution-Continued

- **Moments:** For integer $m > 0$

$$\text{E}(T^m) = \theta^m \sum_{i=0}^{m-1} \frac{(m-1+i)!}{i!\,(m-1-i)!}\left(\frac{1}{2\beta}\right)^i.$$

  From this it follows that

$$\text{E}(T) = \theta \quad \text{and} \quad \text{Var}(T) = \theta^2/\beta.$$

- **Quantiles:** the p quantile of the IGAU distribution is

$$t_p = \theta\,\Phi^{-1}_{\text{ligau}}(p; \beta).$$

  There is no simple closed form equation for $\Phi^{-1}_{\text{ligau}}(p; \beta)$, so it must be computed by inverting $p = \Phi_{\text{ligau}}(z; \beta)$ numerically.

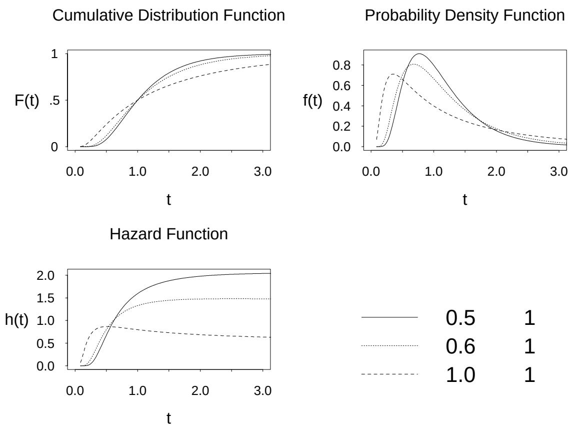
Inverse Gaussian Distribution–Continued

Special cases:

- If $T \sim \text{IGAU}(\theta, \beta)$ and $c > 0$ then $cT \sim \text{IGAU}(c\theta, \beta)$.

- For large values of β, the distribution is very similar to a $\text{NOR}(\theta, \theta/\sqrt{\beta})$.

Examples of Birnbaum–Saunders Distributions



Birnbaum–Saunders Distribution

- For a variable T with Birnbaum–Saunders distribution, BISA(θ, β),

$$\begin{aligned} F_T(t;\beta,\theta) &= \Phi_{\text{nor}}(\zeta) \\ f_T(t;\beta,\theta) &= \frac{\sqrt{\frac{t}{\theta}}+\sqrt{\frac{\theta}{t}}}{2\beta t}\,\phi_{\text{nor}}(\zeta) \end{aligned}$$

where $t \geq 0$, $\theta > 0$ is a scale parameter, $\beta > 0$ is a shape parameter, and

$$\zeta = \frac{1}{\beta}\left(\sqrt{\frac{t}{\theta}} - \sqrt{\frac{\theta}{t}}\right)$$

- **Moments:** For an integer $m > 0$,

$$\text{E}(T^m) = \theta^m \sum_{i=0}^{m} \beta^{2(m-i)} \frac{[2(m-i)]!}{\left[2^{3(m-i)}\right]\,(m-i)!} \sum_{k=0}^{m-i} \begin{pmatrix} 2m \\ 2k \end{pmatrix} \begin{pmatrix} m-k \\ i \end{pmatrix}.$$

Then

$$\text{E}(T) = \theta\left(1+\frac{\beta^2}{2}\right) \quad \text{and} \quad \text{Var}(T) = (\theta\beta)^2\left(1+\frac{5\beta^2}{4}\right).$$

- **Quantiles:** The p quantile is

$$t_p = \frac{\theta}{4}\left\{\beta\,\Phi^{-1}_{\text{nor}}(p) + \sqrt{4+\left[\beta\,\Phi^{-1}_{\text{nor}}(p)\right]^2}\right\}^2.$$

Birnbaum–Saunders Distribution–Continued

To isolate the scale parameter θ and the unitless shape parameter β, we write the cdf and pdf as follows

$$\begin{aligned} F_T(t;\beta,\theta) &= \Phi_{\text{lbisa}}\left[\log(t/\theta);\beta\right] \\ f_T(t;\beta,\theta) &= \frac{1}{t}\,\phi_{\text{lbisa}}\left[\log(t/\theta);\beta\right] \end{aligned}$$

where

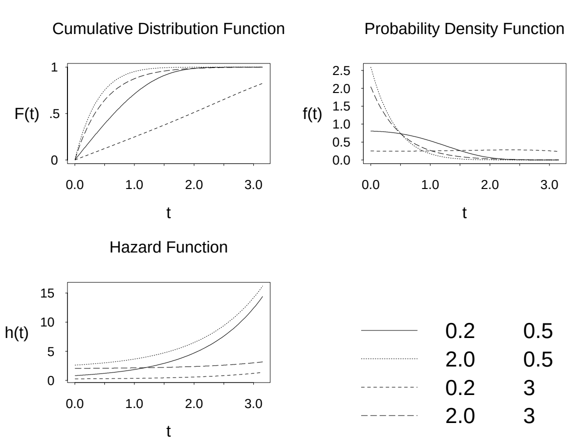
$$\begin{aligned} \Phi_{\text{lbisa}}(z;\beta) &= \Phi_{\text{nor}}(\nu) \\ \phi_{\text{lbisa}}(z;\beta) &= \left[\frac{\exp(z/2)+\exp(-z/2)}{2\beta}\right]\phi_{\text{nor}}(\nu)\,, \; -\infty < z < \infty \\ \nu &= \frac{1}{\beta}\left[\exp(z/2)-\exp(-z/2)\right]. \end{aligned}$$

Birnbaum–Saunders Distribution–Continued

Notes:

- If $T \sim \text{BISA}(\theta,\beta)$ and $c > 0$ then $cT \sim \text{BISA}(c\theta,\beta)$.
- If $T \sim \text{BISA}(\theta,\beta)$ then $1/T \sim \text{BISA}(\theta^{-1},\beta)$.
- The hazard function BISA $h(t;\theta,\beta)$ is not always increasing.
  - $h(0;\theta,\beta) = 0$.
  - $\lim_{t\to\infty} h(t;\theta,\beta) = 1/(2\theta\beta^2)$.
  - extensive numerical experiments indicate that $h(t;\theta,\beta)$ is always unimodal.
- This distribution was derived by Birnbaum and Saunders (1969) in the modeling of fatigue crack extension.

Examples of Gompertz-Makeham Distributions



Gompertz–Makeham Distribution

- A common parameterization for this distribution is

$$\Pr(T \leq t;\gamma,\kappa,\lambda) = 1 - \exp\left[-\frac{\lambda\kappa t + \gamma\exp(\kappa t) - \gamma}{\kappa}\right], \quad t > 0.$$

$\gamma > 0, \kappa > 0, \lambda \geq 0$ and all the parameters have units that are the reciprocal of the units of t.

- This distribution originated from the need of a positive random variable with a hazard function similar to the hazard of the SEV. It can be shown that

$$\Pr(T \leq t;\gamma,\kappa,\lambda) = 1 - \left[\frac{1-\Phi_{\text{sev}}\left(\frac{t-\mu}{\sigma}\right)}{1-\Phi_{\text{sev}}\left(\frac{-\mu}{\sigma}\right)}\right]\exp(-\lambda t)$$

where $\mu = -(1/\kappa)\log(\gamma/\kappa)$, $\sigma = 1/\kappa$.

- When $\lambda = 0$, one gets Gompertz–distribution which corresponds to a truncated SEV at the origin.

Gompertz–Makeham Continued

The parameterization in terms of $[\theta, \psi, \eta] = [1/\kappa, \log(\kappa/\gamma), \lambda/\kappa]$ isolates the scale parameter from the shape parameter and we say that $T \sim \text{GOMA}(\theta, \psi, \eta)$, if

$$\begin{aligned} F_T(t;\theta,\psi,\eta) &= \Phi_{\text{lgoma}}[\log(t/\theta);\psi,\eta] \\ f_T(t;\theta,\psi,\eta) &= \frac{1}{t}\,\phi_{\text{lgoma}}[\log(t/\theta);\psi,\eta] \\ h_T(t;\theta,\psi,\eta) &= \frac{\eta}{\theta} + \frac{\exp(-\psi)}{\theta}\exp\left(\frac{t}{\theta}\right), \quad t > 0 \end{aligned}$$

here θ is a scale parameter, ψ and η are unitless shape parameters, and

$$\begin{aligned} \Phi_{\text{lgoma}}(z;\psi,\eta) &= 1 - \exp\{\exp(-\psi) - \exp[\exp(z) - \psi] - \eta\exp(z)\} \\ \phi_{\text{lgoma}}(z;\psi,\eta) &= \exp(z)\{\eta + \exp[\exp(z) - \psi]\}[1 - \Phi_{\text{lgoma}}(z;\psi,\eta)] \end{aligned}$$

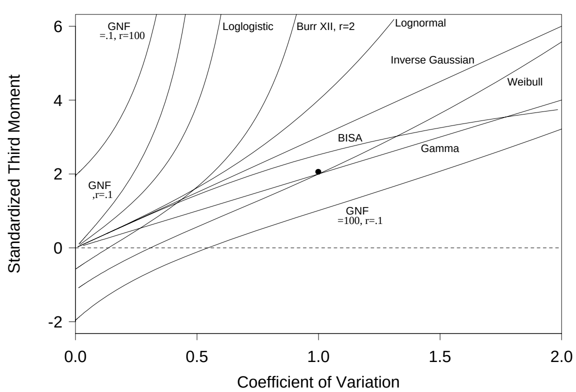
are, respectively, the standardized cdf and pdf of $Z = \log(t/\theta)$.

Gompertz–Makeham Distribution–Continued

Notes:

- $h_T(0;\theta,\psi,\eta) = (1/\theta)[\eta + \exp(-\psi)]$.
- $h_T(t;\theta,\psi,\eta)$ increases with t at an exponential rate.
- If $T \sim \text{GOMA}(\theta,\psi,\eta)$ and $c > 0$ then $cT \sim \text{GOMA}(c\theta,\psi,\eta)$.

Standardized Third Moment Versus Coefficient of Variation



Comparison of Spread and Skewness Parameters

- The **standardized** third central moment of T defined by

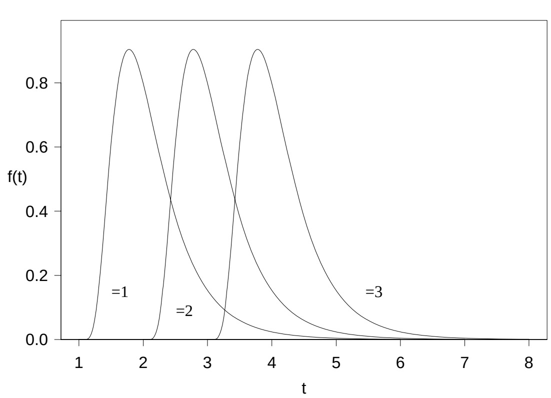
$$\gamma_3 = \frac{\int_0^\infty [t - \text{E}(T)]^3 f(t;\boldsymbol{\theta})\,dt}{[\text{Var}(T)]^{\frac{3}{2}}}$$

is a measure of the skewness in the distribution of T. This parameter is unitless and it has the these properties:

  - Distributions with $\gamma_3 > 0$ will tend to be skewed to the right.
  - Distributions with $\gamma_3 < 0$ will tend to be skewed to the left (e.g., the Weibull distribution with large β).

- The unitless **coefficient** of variation of T, $\gamma_2 = \sqrt{\text{Var}(T)}/\text{E}(T)$, is useful for comparing the relative amount of variability in the distributions of random variables having different units.

pdfs for Three-Parameter Lognormal Distributions for $\mu = 0$ and $\sigma = .5$ with $\gamma = $ 1,2,3.



Distributions with a Threshold Parameter

- So far we have discussed nonnegative random variables with cdfs that begin increasing at $t = 0$.
- One can generalize these and similar distributions by adding a **threshold**, γ, to shift the beginning of the distribution away from 0.
- Distributions with a threshold are particularly useful for fitting skewed distributions that are shifted far to the right of 0.
- The cdf for location-scale log-based threshold distributions is

$$\begin{aligned} F(t;\mu,\sigma,\gamma) &= \Phi\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right] \\ \text{or} \quad F(t;\eta,\sigma,\gamma) &= \Phi\left[\log\left(\frac{t-\gamma}{\eta}\right)^{1/\sigma}\right], \quad t > \gamma \end{aligned}$$

where $\eta = \exp(\mu)$, $-\infty < \gamma < \infty$, $-\infty < \mu < \infty$, $\sigma > 0$, $\eta > 0$, and Φ is a completely specified cdf.

Examples of Distributions with a Threshold Parameter

- Three-parameter lognormal distribution

$$F(t;\mu,\sigma,\gamma) = \Phi_{\text{nor}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t > \gamma.$$

- Three-parameter Weibull distribution

$$\begin{aligned} F(t;\eta,\beta,\gamma) &= 1-\exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right] \\ &= \Phi_{\text{sev}}\left[\frac{\log(t-\gamma)-\mu}{\sigma}\right],\ t>\gamma \end{aligned}$$

where $\sigma = 1/\beta$ and $\mu = \log(\eta)$.

Properties of Distributions with a Threshold

- When the distribution of T has a threshold, γ, then the distribution of $W = T - \gamma$ has a distribution with 0 threshold.
- The properties of the distribution of T are **closely** related to the properties of the distribution of W.
- In general, $\mathrm{E}(T) = \gamma + \mathrm{E}(W)$ and $t_p = \gamma + w_p$, where w_p is the p quantile of the distribution of W.
- Changing γ simply shifts the distribution on the time axis, there is no effect on the distribution's spread or shape. Thus $\mathrm{Var}(T) = \mathrm{Var}(W)$.
- There are, however, some very specific issues in the estimation of γ because the points at which the cdf is positive depends on γ.

Embedded Models

- For some values of (μ,σ,γ), the model is very similar to a two-parameter location-scale model, as described below.
- **Embedded models:** Using the **reparameterization**, $\alpha = \gamma + \eta$, $\varsigma = \sigma\eta$, the model becomes

$$\begin{aligned} F(t;\alpha,\sigma,\varsigma) &= \Phi\left[\log\left(1+\sigma\times\frac{t-\alpha}{\varsigma}\right)^{1/\sigma}\right] \\ &= \Phi\left[\log\left(1+\sigma z\right)^{1/\sigma}\right],\quad \text{for } z > -1/\sigma \end{aligned}$$

where $z = (t-\alpha)/\varsigma$.

When $\sigma \to 0^{+}$, $(1+\sigma z)^{1/\sigma} \to \exp(z)$, and the **limiting** distribution is

$$F(t;\alpha,0,\varsigma) = \Phi\left(z\right),\quad \text{for } -\infty < t < \infty.$$

- For example, if $\Phi = \Phi_{\text{sev}}$ the limiting distribution is the SEV and if $\Phi = \Phi_{\text{nor}}$ the limiting distribution is normal.

Some Comments on the Embedded Models

- The limiting distribution arises when

a. $1/\sigma$ and η are going to ∞ at the same rate, and

b. γ is going to $-\infty$ at the same rate that η is going to ∞.

- Precisely, if $F(t;\eta_i,\sigma_i,\gamma_i)$ is a sequence of cdfs such that

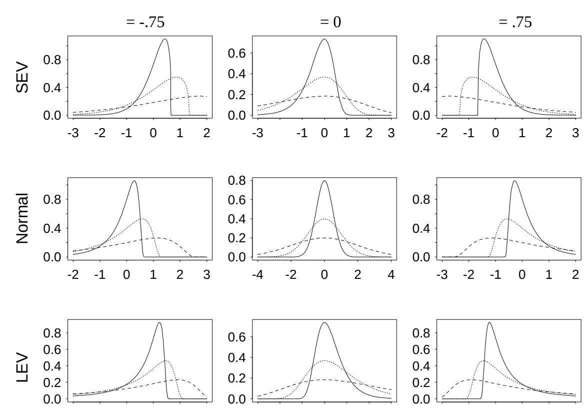
$$\begin{aligned} \sigma_i &\to 0 \\ \varsigma &= \lim_{i\to\infty}(\sigma_i\eta_i) \quad \text{with } 0<\varsigma<\infty \\ \alpha &= \lim_{i\to\infty}(\gamma_i+\eta_i) \quad \text{with } -\infty<\alpha<\infty \end{aligned}$$

then $F(t;\eta_i,\sigma_i,\gamma_i) \to \Phi(z)$, where $z = (t-\alpha)/\varsigma$

Generalized Threshold Scale (GETS) Models

- The original threshold parameter space $(\alpha,\sigma,\varsigma)$ (with $\sigma > 0$) does not contain the limiting distributions.
- It is convenient to enlarge the parameter space such that the limiting distributions are interior points of the parameter space.
- This is achieved by allowing σ to take values in $(-\infty,\infty)$.
- The family of distributions corresponding to this enlarged parameter space is known as the generalized threshold scale (GETS) family .

SEV-GETS, NOR-GETS, and LEV-GETS pdfs with $\alpha = 0$, $\sigma = -.75, 0, .75$, and $\varsigma = .5$ (Least Disperse), 1, and 2 (Most Disperse)



GETS MODEL

- The **cdf** for the GETS model is

$$F(t;\alpha,\sigma,\varsigma)=\begin{cases}\Phi\left[\log\left(1+\sigma z\right)^{1/\sigma}\right], & \text{for } \sigma>0,\ z>-1/\sigma\\ \Phi\left(z\right), & \text{for } \sigma=0,\ -\infty<t<\infty\\ 1-\Phi\left[\log\left(1+\sigma z\right)^{1/|\sigma|}\right], & \text{for } \sigma<0,\ z<-1/\sigma\end{cases}$$

  where $z=(t-\alpha)/\varsigma$.

- The corresponding **pdf** is

$$f(t;\alpha,\sigma,\varsigma)=\begin{cases}\phi\left[\log\left(1+\sigma z\right)^{1/|\sigma|}\right]\times\frac{1}{\varsigma(1+\sigma z)}, & \text{for } \sigma\neq 0\\ \phi\left(z\right)\times\frac{1}{\varsigma}, & \text{for } \sigma=0,\ -\infty<t<\end{cases}$$

  Note: for $\sigma>0, z>-1/\sigma$ and for $\sigma<0, z<-1/\sigma$.

- If $T\sim$ GETS$(\alpha,\sigma,\varsigma)$ and $a\neq 0$ then $(aT+b)\sim$ GETS$(a\alpha+b, a\sigma/|a|, \varsigma|a|)$.

Some Special Cases

- The GETS model includes all the location-scales distributions. These are obtained when $\sigma=0$, as

$$F(t;\alpha,0,\varsigma)=\Phi[(t-\alpha)/\varsigma].$$

  This includes the normal, logistic, SEV, LEV, etc.

- The GETS includes all the threshold, log-based location-scale distributions. These are obtained with $\sigma>0$ which gives

$$F(t;\alpha,\sigma,\varsigma)=\Phi\{[\log(t-\gamma)-\mu]/\sigma\},\quad t>\gamma$$

  where $\gamma=\alpha-\varsigma/\sigma$, $\mu=\log(\varsigma/\sigma)$.

  - With $\Phi=\Phi_{\text{nor}}$ this gives the lognormal with a threshold.
  - With $\Phi=\Phi_{\text{sev}}$ this gives the Weibull (also known as Weibull-type for **minima**) with a threshold.
  - And with $\Phi=\Phi_{\text{lev}}$ one obtains the Fréchet for **maxima** with a threshold.

Some Special Cases-Continued

- The GETS includes the reflection (negative) of the threshold, log-based location-scale distributions. These are obtained with $\sigma<0$, giving

$$F(t;\alpha,\sigma,\varsigma)=\Phi\{[\log(-t-\gamma)-\mu]/\sigma\},\quad t<-\gamma$$

  where $\gamma=-(\alpha-\varsigma/\sigma)$, $\mu=\log(-\varsigma/\sigma)$.

- With $\Phi=\Phi_{\text{nor}}$ this gives the negative of a lognormal with a threshold.

- With $\Phi=\Phi_{\text{sev}}$ this gives the negative of a Weibull with a threshold. Or equivalently a Weibull-type distribution for **maxima**.

- With with $\Phi=\Phi_{\text{lev}}$ one obtains the negative of a Fréchet for **maxima** with a threshold. Or equivalently, a Fréchet-type distribution for **minima**.

Quantiles for the GETS Distribution

- **Quantiles:** the p quantile of the GETS distribution is

$$t_p \;=\; \alpha+\varsigma\times w(\sigma,p)$$

  where

$$w(\sigma,p)\;=\;\begin{cases}\frac{\exp[\sigma\Phi^{-1}(p)]-1}{\sigma}, & \text{for } \sigma>0\\ \Phi^{-1}(p), & \text{for } \sigma=0\\ \frac{\exp\{|\sigma|\Phi^{-1}(1-p)\}-1}{\sigma}, & \text{for } \sigma<0\end{cases}$$

- Then for fixed σ, t_p versus $w(\sigma,p)$ plots as a straight line.

GETS Stable Parameterization

- **Parameterization for Numerical Stability:** with $p_1<p_2$, a stable parameterization can be obtained using two quantiles and σ, i.e., $(t_{p_1}, t_{p_2}, \sigma)$.

- Using the expression for the quantiles

$$\begin{aligned}t_{p_1} &= \alpha+\varsigma\times w(\sigma,p_1)\\ t_{p_2} &= \alpha+\varsigma\times w(\sigma,p_2).\end{aligned}$$

  Solving for α and ς

$$\begin{aligned}\alpha &= \frac{w(\sigma,p_1)\times t_{p_2}-w(\sigma,p_2)\times t_{p_1}}{w(\sigma,p_1)-w(\sigma,p_2)}\\ \varsigma &= \frac{t_{p_1}-t_{p_2}}{w(\sigma,p_1)-w(\sigma,p_2)}.\end{aligned}$$

Finite (Discrete) Mixture Distributions

- The cdf of units in a population consisting of a mixture of units from k different populations can be expressed as

$$F(t;\boldsymbol{\theta})=\sum_i \xi_i F_i(t;\boldsymbol{\theta}_i)$$

  where $\boldsymbol{\theta}=(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2,\ldots,\xi_1,\xi_2,\ldots)$, $\xi_i\geq 0$, and $\sum_i\xi_i=1$.

- Mixtures tend to have a large number of parameters and estimation can be complicated. But estimation is facilitated by:

  - identification of the individual population from which sample units originated.
  - considerable **separation** in the components and/or enormous amounts of data.

- Sometimes it is sufficient to fit a simpler distribution to describe the overall mixture.

Continuous Mixture (Compound Distributions)

- These probability models arise from distributions in which one or more of the parameters are continuous random variable.
- These distributions are called **compound** distributions and correspond to continuous mixture of a family of distributions, as follows:

  Assume that for a fixed value of a scalar parameter θ_1, $T|\theta_1 \sim f_{T|\theta_1}(t;\boldsymbol{\theta})$ with $\boldsymbol{\theta} = (\theta_1, \boldsymbol{\theta}_2)$. Assuming that θ_1 is random from unit to unit with $\theta_1 \sim f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3)$, where $\boldsymbol{\theta}_3$ does not have elements in common with $\boldsymbol{\theta}$, then

$$\begin{aligned} F(t;\boldsymbol{\theta}_2,\boldsymbol{\theta}_3) = \Pr(T \le t) &= \int_{-\infty}^{\infty} \Pr(T \le t|\theta_1 = \vartheta) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3)d\vartheta \\ &= \int_{-\infty}^{\infty} F_{T|\theta_1=\vartheta}(t;\boldsymbol{\theta}) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3)d\vartheta \end{aligned}$$

  and the corresponding pdf is

$$f(t;\boldsymbol{\theta}_2,\boldsymbol{\theta}_3) = \int_{-\infty}^{\infty} f_{T|\theta_1=\vartheta}(t;\boldsymbol{\theta}) f_{\theta_1}(\vartheta;\boldsymbol{\theta}_3)d\vartheta.$$

Pareto Distribution as a Compound Distribution

- If life of the ith unit in a population can be modeled by

$$T|\eta \sim \text{EXP}(\eta).$$

- But the failure rate varies from unit to unit in the population according to a GAM(θ,κ), i.e,

$$\frac{1}{\eta} \sim \text{GAM}(\theta,\kappa).$$

- Then the unconditional failure time of a unit selected at random from the population follows a Pareto distribution of the form

$$F(t;\theta,\kappa) = 1 - \frac{1}{(1+\theta t)^{\kappa}}, \quad t > 0.$$

Other Distributions

- Power distributions.
- Distributions based on stochastic components of physical/chemical degradation models.
- Multivariate failure time distributions.