Chapter 10

Planning Life Tests

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19h 14min

Planning Life Tests
Chapter 10 Objectives

- Explain the basic ideas behind planning a life test.
- Use simulation to anticipate the results, analysis, and precision for a proposed test plan.
- Explain large-sample approximate methods to assess precision of future results from a reliability study.
- Compute sample size needed to achieve a specified degree of precision.
- Assess tradeoffs between sample size and length of a study.
- Illustrate the use of simulation to calibrate the easier-to-use large-sample approximate methods.
- Planning minimum size demonstration tests.

Basic Ideas in Test Planning

- The enormous cost of reliability studies makes it essential to do careful planning. Frequently asked **questions** include:
  - ▶ How many units do I need to test in order to estimate the .1 quantile of life?
  - ▶ How long do I need to run the life test?

  Clearly, more test units and more time will buy more information and thus more precision in estimation.

- To anticipate the results from a test plan and to respond to the questions above, it is necessary to have some **planning** information about the life distribution to be estimated.

Engineering Planning Values and Assumed Distribution for Planning an Insulation Life Test

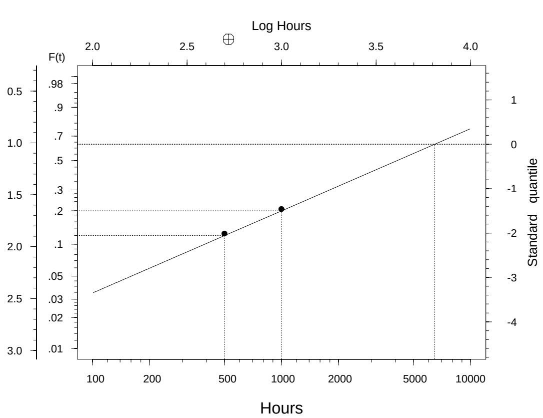
Want to estimate $t_{.1}$ of the life distribution of a newly developed insulation. Tests are run at higher than usual volts/thickness to cause failures to occur more quickly.

Information (planning values) from engineering

- Expect about 20% failures in the 1000 hour test and about 12% failures in the first 500 hours of the test.
- Willing to assume a Weibull distribution to describe failure-time.
- Equivalent information for **planning values**: $\eta^{\square} = 6464$ hours (or $\mu^{\square} = \log(6464) = 8.774$), $\beta^{\square} = .8037$ (or $\sigma^{\square} = 1/\beta^{\square} = 1.244$).

  Starting point: Use simulated data to assess precision.

Weibull Probability Paper Showing the Insulation Life cdf Corresponding to the Test Planning Values $\eta^{\square} = 6464$ and $\beta^{\square} = .8037$



Assessing the Variability of the Estimates For an Unrestricted Quantity

- For an unrestricted quantile y_p an approximate $100(1-\alpha)\%$ confidence interval is given by

$$\begin{aligned}[\underset{\sim}{y_p}, \quad \tilde{y_p}] &= \hat{y}_p \pm z_{(1-\alpha/2)}\widehat{\text{se}}_{\hat{y}_p} \\ &= [\hat{y}_p - D, \quad \hat{y}_p + D]\end{aligned}$$

  where $D = z_{(1-\alpha/2)}\widehat{\text{se}}_{\hat{y}_p}$.

- The half-width D is an indication of the width of the interval and can be used to assess the variability in the estimates $\hat{y}_p$.

Assessing the Variability of the Estimates For a Positive Quantity

- For a positive quantile t_p an approximate $100(1-\alpha)\%$ confidence interval for $\log(t_p)$ is given by

$$\left[\log(\underset{\sim}{t_p}),\quad \log(\tilde{t}_p)\right] = \log(\widehat{t}_p) \pm z_{(1-\alpha/2)}\widehat{\mathsf{se}}_{\log(\widehat{t}_p)}$$

Exponentiation yields a confidence interval for t_p

$$[\underset{\sim}{t_p},\quad \tilde{t}_p] = [\widehat{t}_p/R,\quad \widehat{t}_p R]$$

where $R = \exp\left[z_{(1-\alpha/2)}\widehat{\mathsf{se}}_{\log(\widehat{t}_p)}\right]$.

- The factor $R > 1$ is an indication of the width of the interval and can be used to assess the variability in the estimates $\widehat{t}_p$.

Sample Size Formulas

- **Approximate sample size to estimate the mean of a normal distribution with complete data and precision D_T.**

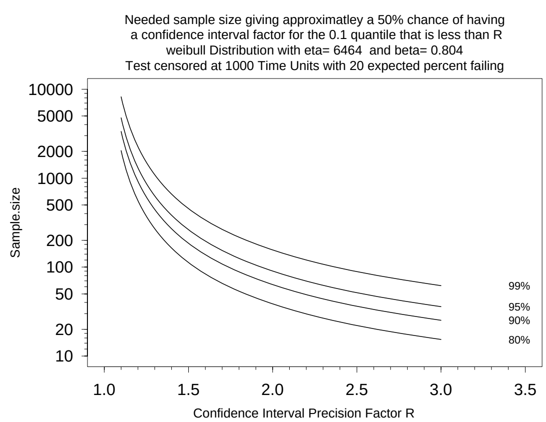
$$n = \frac{z^2_{(1-\alpha/2)}(\sigma^\square)^2}{D_T^2}$$

$(\sigma^\square)^2$ is a planning value for the variance σ^2 and D_T is the target half width of a $100(1-\alpha)\%$ confidence interval for μ.

- **To estimate a quantile of a positive response with censored data and precision R_T.**

$$n = \frac{z^2_{(1-\alpha/2)}\mathsf{V}^\square_{\log(\widehat{t}_p)}}{[\log(R_T)]^2}$$

where $\mathsf{V}^\square_{\log(\widehat{t}_p)}$ is a planning value of the variance factor $\mathsf{V}_{\log(\widehat{t}_p)}$ which may depend on $t_p^\square$ and the amount of censoring. R_T is the target precision factor for a $100(1-\alpha)\%$ confidence interval for t_p.

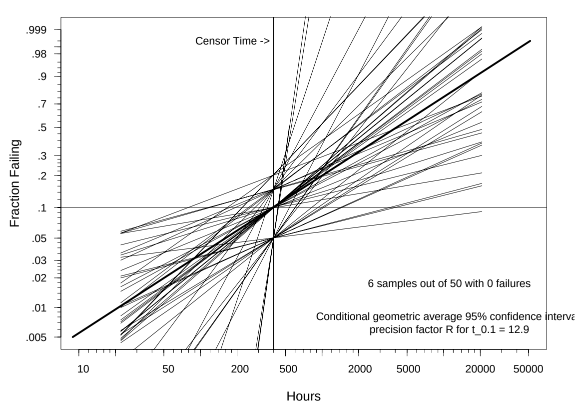


Simulation as a Tool for Test Planning

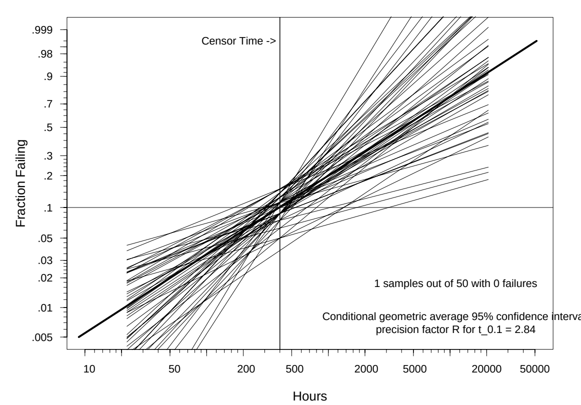
- Use assumed model and planning values of model parameters to simulate data from the proposed study.
- Analyze the data perhaps under different assumed models.
- Assess precision provided.
- Simulate many times to assess actual sample-to-sample differences.
- Repeat with different sample sizes to gauge needs.
- Repeat with different input planning values to assess sensitivity to these inputs.

Any surprises?

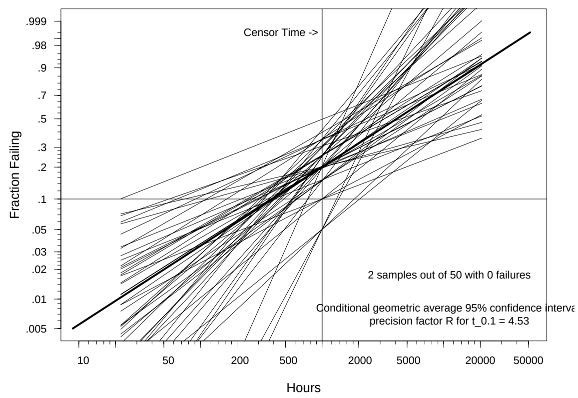
ML Estimates from 50 Simulated Samples of Size $n = 20$, $t_c = 400$ from a Weibull Distribution with $\mu^\square = 8.774$ and $\sigma^\square = 1.244$



ML Estimates from 50 Simulated Samples of Size $n = 80$, $t_c = 400$ from a Weibull Distribution with $\mu^\square = 8.774$ and $\sigma^\square = 1.244$



ML Estimates from 50 Simulated Samples of Size $n = 20$, $t_c = 1000$ from a Weibull Distribution with $\mu^{\square} = 8.774$ and $\sigma^{\square} = 1.244$



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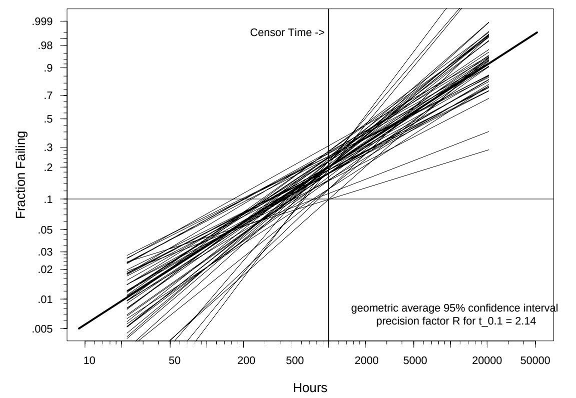


Simulations of Insulation Life Tests

- ML estimates obtained from 50 simulated samples of size $n = 20, 80$, from a Weibull distribution with $\mu^{\square} = 8.774$, $\sigma^{\square} = 1.244$ ($\beta^{\square} = .8037$).
- The vertical lines at $t_c = 400, 1000$ hours (shown with the thicker line) indicates the censoring time (end of the test).
- The horizontal line is drawn at $p = .1$ so to provide a better visualization of the distribution of estimates of $t_{.1}$.
- Results at $t_c = 400$ and $n = 20$ are highly variable.

Trade-offs Between Test Length and Sample Size

Geometric average $\widehat{R}$ factor from 50 simulated exponential samples ($\theta = 5$) for combinations of sample size n and test length t_c (conditional on $r \geq 1$ failures)

Test Length t_c	Sample Size n 20	80
400	12.9 (2)	2.84 (8)
1000	4.53 (4)	2.14 (16)

Numbers within parenthesis are the expected number of failures at each test condition.

Simulations of Insulation Life Tests-Continued

Some important points about the effect that sample size will have on our ability to make inferences:

- For the $t_c = 400$ and $n = 5$ simulation
  - Enormous amount of variability in the ML estimates.
  - For several of the simulated data sets, no ML estimates exist because all units were censored.
- Increasing the experiment length to $t_c = 1000$ and the sample size to $n = 80$ provides
  - A more stable estimation process.
  - A substantial improvement in precision.

Motivation for Use of Large-Sample Approximations of Test Plan Properties

Asymptotic methods provide:

- Simple expressions giving precision of a specified estimator as a function of sample size.
- Simple expressions giving needed sample size as a function of specified precision of a specified estimator.
- Simple tables or graphs that will allow easy assessments of tradeoffs in test planning decisions like sample size and test length.
- Can be fine tuned with simulation evaluation.

Asymptotic Variances

Under certain regularity conditions the following results hold asymptotically (large sample)

- $\widehat{\boldsymbol{\theta}} \,\dot\sim\, \text{MVN}(\boldsymbol{\theta}, \Sigma_{\widehat{\boldsymbol{\theta}}})$, where $\Sigma_{\widehat{\boldsymbol{\theta}}} = I_{\boldsymbol{\theta}}^{-1}$, and
$$I_{\boldsymbol{\theta}} = \text{E}\left[-\frac{\partial^2 \mathcal{L}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right] = \sum_{i=1}^{n} \text{E}\left[-\frac{\partial^2 \mathcal{L}_i(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right].$$

- For a scalar $g = g(\widehat{\boldsymbol{\theta}}) \,\dot\sim\, \text{NOR}[g(\boldsymbol{\theta}), \text{Avar}(\widehat{g})]$, where
$$\text{Avar}(\widehat{g}) = \left[\frac{\partial g(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right]' \Sigma_{\widehat{\boldsymbol{\theta}}} \left[\frac{\partial g(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\right].$$

- When $g(\boldsymbol{\theta})$ is **positive** for all $\boldsymbol{\theta}$, then $\log[g(\widehat{\boldsymbol{\theta}})] \,\dot\sim\, \text{NOR}\{\log[g(\boldsymbol{\theta})], \text{Avar}[\log(\widehat{g})]\}$, where
$$\text{Avar}[\log(\widehat{g})] = \left(\frac{1}{g}\right)^2 \text{Avar}(\widehat{g}).$$

Asymptotic Approximate Standard Errors for a Function of the Parameters $g(\boldsymbol{\theta})$

Given an assumed model, parameter values (but not sample size), one can compute scaled asymptotic variances.

- The variance factors $\text{V}_{\widehat{g}} = n\text{Avar}(\widehat{g})$ and $\text{V}_{\log(\widehat{g})} = n\text{Avar}[\log(\widehat{g})]$ may depend on the actual value of $\boldsymbol{\theta}$ but they do **not** depend on n.

  To compute these variance factors one uses planning values for $\boldsymbol{\theta}$ (denoted by $\boldsymbol{\theta}^{\square}$) as discussed later.

- The asymptotic standard error for $\widehat{g}$ and $\log(\widehat{g})$ are
$$\begin{aligned}\text{Ase}(\widehat{g}) &= \frac{1}{\sqrt{n}}\sqrt{\text{V}_{\widehat{g}}}\\ \text{Ase}[\log(\widehat{g})] &= \frac{1}{\sqrt{n}}\sqrt{\text{V}_{\log(\widehat{g})}}.\end{aligned}$$

- Easy to choose n to control Ase.

Sample Size Determination for Positive Functions of the Parameters

- When $g(\boldsymbol{\theta}) > 0$ for all $\boldsymbol{\theta}$, an approximate $100(1-\alpha)\%$ confidence interval for $\log[g(\boldsymbol{\theta})]$ is
$$\left[\log(\underset{\sim}{g}), \log(\tilde{g})\right] = \log(\widehat{g}) \pm (1/\sqrt{n}) z_{(1-\alpha/2)}\sqrt{\widehat{\text{V}}_{\log(\widehat{g})}} = \log(\widehat{g}) \pm \log(R)$$
  Exponentiation yields a confidence interval for g
$$[\underset{\sim}{g}, \quad \tilde{g}] = [\widehat{g}/R, \quad \widehat{g}R]$$
$$R = \exp\left[(1/\sqrt{n}) z_{(1-\alpha/2)}\sqrt{\widehat{\text{V}}_{\log(\widehat{g})}}\right] = \tilde{g}/\widehat{g} = \widehat{g}/\underset{\sim}{g} = \sqrt{\tilde{g}/\underset{\sim}{g}}.$$

- Replace $\widehat{\text{V}}_{\log(\widehat{g})}$ with $\text{V}^{\square}_{\log(\widehat{g})}$ and solve for n to compute the needed sample size giving
$$n = \frac{z^2_{(1-\alpha/2)}\text{V}^{\square}_{\log(\widehat{g})}}{[\log(R_T)]^2}.$$

Sample Size Determination for Positive Functions of the Parameters-Continued

Test plans with a sample size of
$$n = \frac{z^2_{(1-\alpha/2)}\text{V}^{\square}_{\log(\widehat{g})}}{[\log(R_T)]^2}.$$
provides confidence intervals for $g(\boldsymbol{\theta})$ with the following characteristics:

- In repeated samples approximately $100(1-\alpha)\%$ of the intervals will contain $g(\boldsymbol{\theta})$.

- In repeated samples $\widehat{\text{V}}_{\log(\widehat{g})}$ is random and if $\widehat{\text{V}}_{\log(\widehat{g})} > \text{V}^{\square}_{\log(\widehat{g})}$ then the ratio $R = \sqrt{\tilde{g}/\underset{\sim}{g}}$ will be greater than R_T.

- The ratio $R = \sqrt{\tilde{g}/\underset{\sim}{g}}$ will be greater than R_T with a probability of order .5.

Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life

- Need a test plan that will estimate the mean life of insulation specimens at highly-accelerated (i.e., higher than usual voltage to get failure information quickly) conditions.

- Desire a 95% confidence interval with endpoints that are approximately 50% away from the estimated mean (so $R_T = 1.5$).

- Can assume an exponential distribution with a mean $\theta^{\square} = 1000$ hours.

- Simultaneous testing of all units; must terminate test at 500 hours.

Sample Size Needed to Estimate the Mean of an Exponential Distribution Used to Describe Insulation Life-Continued

- ML estimate of the exponential mean is $\widehat{\theta} = TTT/r$, where TTT is the total time on test and r is the number of failures. It follows that
$$\text{V}_{\widehat{\theta}} = n\text{Avar}(\widehat{\theta}) = \frac{n}{\text{E}\left[-\frac{\partial^2\mathcal{L}(\theta)}{\partial\theta^2}\right]} = \frac{\theta^2}{1 - \exp\left(-\frac{t_c}{\theta}\right)}$$
  from which
$$\text{V}^{\square}_{\log(\widehat{\theta})} = \frac{\text{V}^{\square}_{\widehat{\theta}}}{[\theta^{\square}]^2} = \frac{1}{1 - \exp\left(-\frac{500}{1000}\right)} = 2.5415.$$
  Thus the number of needed specimens is
$$n \;=\; \frac{z^2_{(1-\alpha/2)}\text{V}^{\square}_{\log(\widehat{\theta})}}{[\log(R_T)]^2} = \frac{(1.96)^2 2.5415}{[\log(1.5)]^2} \approx 60.$$

Location-Scale Distributions and Single Right Censoring Asymptotic Variance-Covariance

Here we specialize the computation of sample sizes to situations in which

- $\log(T)$ is location-scale Φ with parameters (μ, σ).
- When the data are Type I singly right censored at t_c. In this case,

$$\frac{n}{\sigma^2}\Sigma_{(\widehat{\mu},\widehat{\sigma})} = \frac{1}{\sigma^2}\begin{bmatrix} V_{\widehat{\mu}} & V_{(\widehat{\mu},\widehat{\sigma})} \\ V_{(\widehat{\mu},\widehat{\sigma})} & V_{\widehat{\sigma}} \end{bmatrix} = \left[\frac{\sigma^2}{n} I_{(\mu,\sigma)}\right]^{-1} = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}^{-1}$$
$$= \left(\frac{1}{f_{11}f_{22} - f_{12}^2}\right)\begin{bmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{bmatrix}$$

where the f_{ij} values depend only on Φ and the standardized censoring time $\zeta_c = [\log(t_c) - \mu]/\sigma$ [or equivalently, the proportion failing by t_c, $\Phi(\zeta_c)$].

Location-Scale Distributions and Single Right Censoring Fisher Information Elements

The f_{ij} values are defined as:

$$f_{11} = f_{11}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \mu^2}\right]$$
$$f_{22} = f_{22}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \sigma^2}\right]$$
$$f_{12} = f_{12}(\zeta_c) = \frac{\sigma^2}{n}\mathrm{E}\left[-\frac{\partial^2 \mathcal{L}_i(\mu,\sigma)}{\partial \mu \partial \sigma}\right]$$

The f_{ij} values are available from tables or algorithm LSINF for the SEV (Weibull), normal (lognormal), and logistic (loglogistic) distributions.

For a single fixed censoring time, the asymptotic variance-covariance factors $\frac{1}{\sigma^2}V_{\widehat{\mu}}$, $\frac{1}{\sigma^2}V_{\widehat{\sigma}}$, and $\frac{1}{\sigma^2}V_{(\widehat{\mu},\widehat{\sigma})}$ are easily tabulated as a function of ζ_c.

Table of Information Matrix Elements and Variance Factors

Table C.20 provides for the normal/lognormal distributions, as functions of the standardized censoring time ζ_c:

- $100\Phi(\zeta_c)$, the percentage in the population failing by the standardized censoring time.
- Fisher information matrix elements f_{11}, f_{22}, and f_{12}.
- The asymptotic variance-covariance factors $\frac{1}{\sigma^2}V_{\widehat{\mu}}$, $\frac{1}{\sigma^2}V_{\widehat{\sigma}}$, and $\frac{1}{\sigma^2}V_{(\widehat{\mu},\widehat{\sigma})}$.
- Asymptotic correlation $\rho_{(\widehat{\mu},\widehat{\sigma})}$ between $\widehat{\mu}$ and $\widehat{\sigma}$.
- The σ-known asymptotic variance factor $\frac{1}{\sigma^2}V_{\widehat{\mu}|\sigma} = n\mathrm{Avar}(\widehat{\mu})$, and the μ-known factor $\frac{1}{\sigma^2}V_{\widehat{\sigma}|\mu} = n\mathrm{Avar}(\widehat{\sigma})$.

Large-Sample Asymptotic Variance for Estimators of Functions of Location-Scale Parameters

It is straightforward to compute asymptotic variance factors for functions of parameters. For example, when $\widehat{g} = g(\widehat{\mu}, \widehat{\sigma})$

$$\mathrm{Avar}(\widehat{g}) = \left[\frac{\partial g}{\partial \mu}\right]^2 \mathrm{Avar}(\widehat{\mu}) + \left[\frac{\partial g}{\partial \sigma}\right]^2 \mathrm{Avar}(\widehat{\sigma}) + 2\left[\frac{\partial g}{\partial \mu}\right]\left[\frac{\partial g}{\partial \sigma}\right]\mathrm{Acov}(\widehat{\mu},\widehat{\sigma})$$

$$\mathrm{Avar}[\log(\widehat{g})] = \left(\frac{1}{g}\right)^2 \mathrm{Avar}(\widehat{g}).$$

Thus

$$V_{\widehat{g}} = \left[\frac{\partial g}{\partial \mu}\right]^2 V_{\widehat{\mu}} + \left[\frac{\partial g}{\partial \sigma}\right]^2 V_{\widehat{\sigma}} + 2\left[\frac{\partial g}{\partial \mu}\right]\left[\frac{\partial g}{\partial \sigma}\right] V_{(\widehat{\mu},\widehat{\sigma})}$$
$$V_{\log(\widehat{g})} = \left(\frac{1}{g}\right)^2 V_{\widehat{g}}; \quad V_{\exp(\widehat{g})} = \exp(2g)V_{\widehat{g}}$$

Sample Size to Estimate a Quantile of T when $\log(T)$ is Location-Scale (μ, σ)

- Let $g(\boldsymbol{\theta}) = t_p$ be the p quantile of T. Then $\log(t_p) = \mu + \Phi^{-1}(p)\sigma$, where $\Phi^{-1}(p)$ is the p quantile of the standardized random variable $Z = [\log(T) - \mu]/\sigma$.
- From the previous results, n is given by

$$n = \frac{z^2_{(1-\alpha/2)} V^{\Box}_{\log(\widehat{t}_p)}}{[\log(R_T)]^2}$$

where $V^{\Box}_{\log(\widehat{t}_p)}$ is obtained by evaluating

$$V_{\log(\widehat{t}_p)} = \left\{V_{\widehat{\mu}} + \left[\Phi^{-1}(p)\right]^2 V_{\widehat{\sigma}} + 2\left[\Phi^{-1}(p)\right] V_{(\widehat{\mu},\widehat{\sigma})}\right\}$$

at $\boldsymbol{\theta}^{\Box} = (\mu^{\Box}, \sigma^{\Box}), \zeta^{\Box}_c = [\log(t_c) - \mu^{\Box}]/\sigma^{\Box}$.

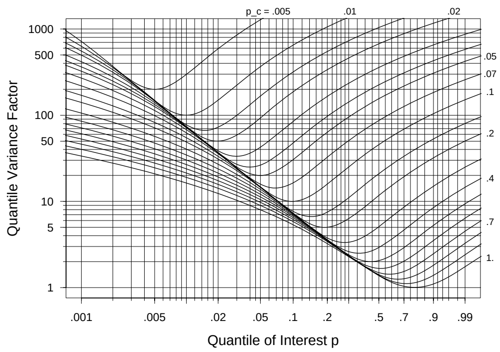
- Figure 10.5 gives $\frac{1}{\sigma^2}V_{\log(\widehat{t}_p)}$ as a function of $p_c = \Pr(Z \le \zeta_c)$ for the Weibull distribution. To obtain n one also needs to specify Φ and a target value R_T for $R = \tilde{g}/\widehat{g} = \widehat{g}/\underset{\sim}{g} = \sqrt{\tilde{g}/\underset{\sim}{g}}$.

Sample Size Needed to Estimate $t_{.1}$ of a Weibull Distribution Used to Describe Insulation Life

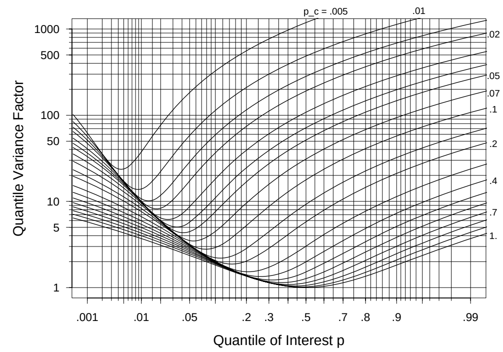
- Again expect about 20% failures in the 1000 hour test and 12% failures in the first 500 hours. Equivalent information: $\mu^{\Box} = 8.774$, $\sigma^{\Box} = 1.244$ (or $\beta^{\Box} = 1/1.244 = .8037$).
- Need a test plan that will estimate the Weibull .1 quantile (so $p = .1$) such that a 95% confidence interval will have endpoints that are approximately 50% away from the estimated mean (so $R_T = 1.5$). For a 1000-hour test, $p_c = .2$.
- By computing from tables and formula or from Figure 10.5, $\frac{1}{\sigma^2}V_{\log(\widehat{t}_p)} = 7.28$ so $V^{\Box}_{\log(\widehat{t}_p)} = 7.28 \times (1.244)^2 = 11.266$.

$$\text{Thus,} \quad n = \frac{z^2_{(1-\alpha/2)} V^{\Box}_{\log(\widehat{t}_{.1})}}{[\log(R_T)]^2} = \frac{(1.96)^2(11.266)}{[\log(1.5)]^2} \approx 263.$$

Variance Factor $\frac{1}{\sigma^2}\mathsf{V}_{\log(\widehat{t}_p)}$ for ML Estimation of Weibull Distribution Quantiles as a Function of p_c, the Population Proportion Failing by Time t_c and p, the Quantile of Interest (Figure 10.5)



Variance Factor $\frac{1}{\sigma^2}\mathsf{V}_{\log(\widehat{t}_p)}$ for ML Estimation of Lognormal Distribution Quantiles as a Function of p_c, the Population Proportion Failing by Time t_c and p, the Quantile of Interest (Figure 10.6)



Figures for Sample Sizes to Estimate Weibull, Lognormal, and Loglogistic Quantiles

Figures give plots of the factor $\frac{1}{\sigma^2}\mathsf{V}_{\log(\widehat{t}_p)}$ for quantile of interest p as a function of $p = \Pr(Z \le \zeta_c)$ for the Weibull, lognormal, and loglogistic distributions. Close inspection of the plots indicates the following:

- Increasing the length of a life test (increasing the expected proportion of failures) will always reduce the asymptotic variance. After a point, however, the returns are diminishing.
- Estimating quantiles with p large or p small generally results in larger asymptotic variances than quantiles near to the expected proportion failing.

Generalization: Location-Scale Parameters and Multiple Censoring

In some applications, a life test may run in parts, each part having a different censoring time (e.g., testing at two different locations or beginning as lots of units to be tested are received). In this case we need to generalize the single-censoring formula. Assume that a proportion δ_i ($\sum_{i=1}^{k}\delta_i = 1$) of data are to be run until right censoring time t_{c_i} or failure (which ever comes first). In this case,

$$\begin{aligned}\frac{n}{\sigma^2}\Sigma_{(\widehat{\mu},\widehat{\sigma})} &= \frac{1}{\sigma^2}\begin{bmatrix} \mathsf{V}_{\widehat{\mu}} & \mathsf{V}_{(\widehat{\mu},\widehat{\sigma})} \\ \mathsf{V}_{(\widehat{\mu},\widehat{\sigma})} & \mathsf{V}_{\widehat{\sigma}} \end{bmatrix} = \left[\frac{\sigma^2}{n} I_{(\mu,\sigma)}\right]^{-1} \\ &= \left(\frac{1}{J_{11}J_{22} - J_{12}^2}\right)\begin{bmatrix} J_{22} & -J_{12} \\ -J_{12} & J_{11} \end{bmatrix}\end{aligned}$$

where $J_{11} = \sum_{i=1}^{k}\delta_i f_{11}(z_{c_i}), J_{22} = \sum_{i=1}^{k}\delta_i f_{22}(z_{c_i})$, and $J_{12} = \sum_{i=1}^{k}\delta_i f_{12}(z_{c_i})$ where $z_{c_i} = (\log(t_{c_i}) - \mu)/\sigma$.

In this case, the asymptotic variance-covariance factors $\frac{1}{\sigma^2}\mathsf{V}_{\widehat{\mu}}$, $\frac{1}{\sigma^2}\mathsf{V}_{\widehat{\sigma}}$, and $\frac{1}{\sigma^2}\mathsf{V}_{(\widehat{\mu},\widehat{\sigma})}$ depend on Φ, the standardized censoring times z_{c_i}, and the proportions $\delta_i, i = 1, \ldots k$.

Test Plans to Demonstrate Conformity with a Reliability Standard

Objective: to find a sample size to **demonstrate** with some level of confidence that reliability exceeds a given standard.

- The reliability is specified in terms of a quantile, say t_p.

  The customer requires demonstration that

  $$t_p > t_p^{\dagger}$$

  where $t_p^{\dagger}$ is a specified value.

  For example, for a component to be installed in a system with a 1-year warranty, a vendor may have to demonstrate that $t_{.01}$ exceeds $24 \times 365 = 8760$ hours.
- Equivalently, in terms of failure probabilities the reliability requirement could be specified as

  $$F(t_e) < p^{\dagger}.$$

  For the example, $t_e = 8760$ and $p^{\dagger} = .01$.

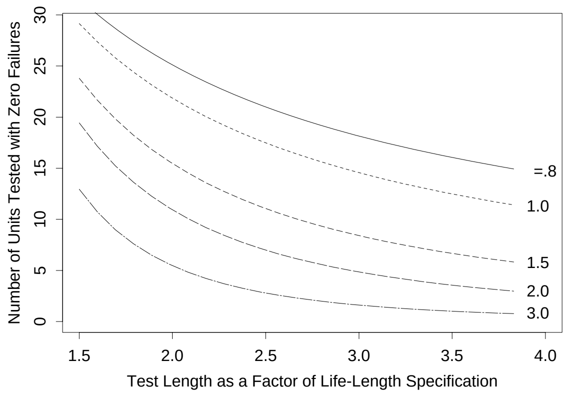
Minimum Sample Size Reliability Demonstration Test Plans

- In general the demonstration that $t_p > t_p^{\dagger}$ is successful at the $100(1-\alpha)\%$ level of confidence if $\underset{\sim}{t_p} > t_p^{\dagger}$.
- Suppose that failure-times are Weibull with a given β. A **minimum sample size** test plan is one that has a particular sample size n (depending on β, α, p and amount of time available for testing).
- The minimum sample size test plan is: Test n units until t_c where n is the smallest integer greater than

  $$\frac{1}{k^{\beta}} \times \frac{\log(\alpha)}{\log(1-p)}.$$

  and $k = t_c/t_p^{\dagger}$.
- If there is zero failures during the test the demonstration is successful.

Minimum Sample Size for a 99% Reliability Demonstration for $t_{.1}$ with Given β



Justification for the Weibull Zero-Failures Test Plan

Suppose that failure-times are Weibull with a given β and zero failures during a test in which n units are tested until t_c. Using the results in Chapter 8, to obtain $100(1-\alpha)\%$ lower bounds for η and t_p are

$$\underset{\sim}{\eta} = \left[\frac{2nt_c^{\beta}}{\chi^2_{(1-\alpha;2)}}\right]^{\frac{1}{\beta}} = \left[\frac{nt_c^{\beta}}{-\log(\alpha)}\right]^{\frac{1}{\beta}}$$

$$\underset{\sim}{t_p} = \underset{\sim}{\eta} \times [-\log(1-p)]^{\frac{1}{\beta}}.$$

- Using the inequality $\underset{\sim}{t_p} > t_p^{\dagger}$ and solving for the smallest integer n such that
$$n \geq \frac{1}{k^{\beta}} \times \frac{\log(\alpha)}{\log(1-p)}$$
gives the needed minimum sample size, where $k = t_c/t_p^{\dagger}$.

Justification for the Weibull Zero-Failures Test Plan (Continued)

- For tests with $k < 1$, which implies extrapolation in time, having a specified value of β **greater** than the true value is conservative (the confidence level is greater than the nominal).
- For tests with $k > 1$ having a specified value of β **less** than the true value is conservative (in the sense that the demonstration is still valid).
- When $k = 1$ the value of β does not effect the sample size.

Additional Comments on Zero-Failure Test Plans

- The inequality $\underset{\sim}{t_p} > t_p^{\dagger}$ can be solved for n, k, β, or α.
- Zero-failure test plans can be obtained for any distribution that has one unknown parameter.
- The ideas here can be extended to test plans with one or more failures. Such test plans require more units but provide a higher probability of successful demonstration for a given $t_p^{\dagger} > t_p$.

Other Topics in Chapter 10

- Uncertainty in planning values and sensitivity analysis.
- Location-scale distributions and limited test positions.
- Variance factors for location-scale parameters and batch testing.
- Test planning for non-location-scale distributions.
- Sample size to estimate: unrestricted functions of the parameters, the mean of an exponential, the hazard function of a location-scale distribution.