Chapter 14

Introduction to the Use of Bayesian Methods for Reliability Data

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Introduction to the Use of Bayesian Methods for Reliability Data Chapter 14 Objectives

- Describe the use of Bayesian statistical methods to combine **prior** information with data to make inferences.
- Explain the relationship between Bayesian methods and likelihood methods used in earlier chapters.
- Discuss sources of prior information.
- Describe useful computing methods for Bayesian methods.
- Illustrate Bayesian methods for estimating reliability.
- Illustrate Bayesian methods for prediction.
- Compare Bayesian and likelihood methods under different assumptions about prior information.
- Explain the dangers of using wishful thinking or expectations as prior information.

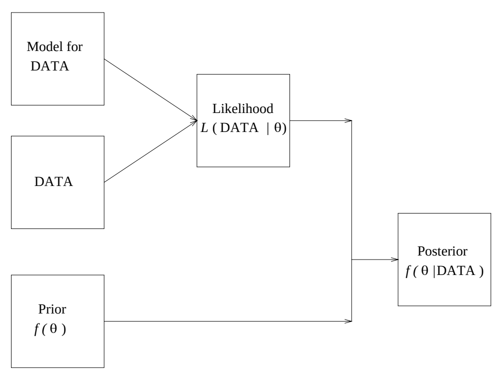
Introduction

- Bayes methods augment likelihood with **prior** information.
- A probability distribution is used to describe our **prior** beliefs about a parameter or set of parameters.
- Sources of prior information:

  Subjective Bayes: prior information subjective.

  Empirical Bayes: prior information from past data.
- Bayesian methods are closely related to likelihood methods.

Bayes Method for Inference



Updating Prior Information Using Bayes Theorem

Bayes Theorem provides a mechanism for combining *prior* information with sample data to make inferences on model parameters.

For a vector parameter $\boldsymbol{\theta}$ the procedure is as follows:

- Prior information on $\boldsymbol{\theta}$ is expressed in terms of a pdf $f(\boldsymbol{\theta})$.
- We observe some data which for the specified model has likelihood $L(\text{DATA}|\boldsymbol{\theta}) \equiv L(\boldsymbol{\theta};\text{DATA})$.
- Using Bayes Theorem, the conditional distribution of $\boldsymbol{\theta}$ given the data (also known as the **posterior** of $\boldsymbol{\theta}$) is

$$f(\boldsymbol{\theta}|\text{DATA}) = \frac{L(\text{DATA}|\boldsymbol{\theta})f(\boldsymbol{\theta})}{\int L(\text{DATA}|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}} = \frac{R(\boldsymbol{\theta})f(\boldsymbol{\theta})}{\int R(\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

where $R(\boldsymbol{\theta}) = L(\boldsymbol{\theta})/L(\widehat{\boldsymbol{\theta}})$ is the relative likelihood and the multiple integral is computed over the region $f(\boldsymbol{\theta}) > 0$.

Some Comments on Posterior Distributions

- The posterior $f(\boldsymbol{\theta}|\text{DATA})$ is function of the prior, the model, and the data.
- In general, it is impossible to compute the multiple integral $\int L(\text{DATA}|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$ in closed form.
- New statistical and numerical methods that take advantage of modern computing power are facilitating the computation of the posterior.

Differences Between Bayesian and Frequentist Inference

- Nuisance parameters
  - ▶ Bayes methods use marginals.
  - ▶ Large-sample likelihood theory suggest maximization.
- There are not important differences in large samples.
- Interpretation
  - ▶ Bayes methods justified in terms of probabilities.
  - ▶ Frequentist methods justified on repeated sampling and asymptotic theory.

Sources of Prior Information

- Informative.
  - ▶ Past data.
  - ▶ Physical, chemical, and mechanical theory.
  - ▶ Expert knowledge.
- Diffuse (or approximately non-informative).
  - ▶ Uniform over finite range of parameter (or function of parameter).
  - ▶ Uniform over infinite range of parameter (improper prior).
  - ▶ Other vague or diffuse priors.

Proper Prior Distributions

Any positive function defined on the parameter space that integrates to a finite value (usually 1).

- **Uniform prior:** $f(\theta) = 1/(b-a)$ for $a \le \theta \le b$.
  This prior does not express strong preference for specific values of θ in the interval.
- **Examples of non-uniform prior distributions:**
  - ▶ Normal with mean at a and and standard deviation b.
  - ▶ Beta between specified a and b with specified shape parameters (allows for a more general shape).
  - ▶ Isosceles triangle with base (range between) a and b.

For a positive parameter θ, may want to specify the prior in terms of $\log(\theta)$.

Improper Prior Distributions

Positive function $f(\theta)$ over parameter space for which

$$\int f(\theta) d\theta = \infty,$$

- **Uniform** in an interval of infinite length: $f(\theta) = c$ for all θ.
- For a positive parameter θ the corresponding choice is $f[\log(\theta)] = c$ and $f(\theta) = (c/\theta)$, $\theta > 0$.

  To use an improper prior, one must have

  $$\int f(\theta) L(\theta|\text{DATA}) d\theta < \infty$$

  (a condition on the form of the likelihood and the DATA).
- These prior distributions can be made to be proper by specification of a finite interval for θ and choosing c such that the total probability is 1.

Effect of Using Vague (or Diffuse) Prior Distributions

- For a uniform prior $f(\theta)$ (possibly improper) across all possible values of θ

  $$f(\boldsymbol{\theta}|\text{DATA}) = \frac{R(\boldsymbol{\theta}) f(\boldsymbol{\theta})}{\int R(\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta}} = \frac{R(\boldsymbol{\theta})}{\int R(\boldsymbol{\theta}) d\boldsymbol{\theta}}$$

  which indicates that the posterior $f(\boldsymbol{\theta}|\text{DATA})$ is proportional to the likelihood.
- The posterior is approximately proportional to the likelihood for a proper (finite range) uniform if the range is large enough so that $R(\boldsymbol{\theta}) \approx 0$ where $f(\boldsymbol{\theta}) = 0$.
- Other diffuse priors also result in a posterior that is approximately proportional to the likelihood if $R(\boldsymbol{\theta})$ is large relative to $f(\boldsymbol{\theta})$.

Eliciting or Specifying a Prior Distribution

- The elicitation of a meaningful joint prior distribution for vector parameters may be difficult
  - ▶ The marginals may not completely determine the joint distribution.
  - ▶ Difficult to express/elicit dependences among parameters through a joint distribution.
  - ▶ The standard parameterization may not have practical meaning.
- General approach: choose an appropriate parameterization in which the priors for the parameters are approximately independent.

Expert Opinion and Eliciting Prior Information

- Identify parameters that, from past experience (or data), can be specified approximately independently (e.g., for high reliability applications a small quantile and the Weibull shape parameter).
- Determine for which parameters there is useful informative prior information.
- For parameters for which there in **no** useful informative prior information, determine the form and range of the vague prior (e.g., uniform over a wide interval).
- For parameters for which there is useful informative prior information, specify the form and range of the distribution (e.g., lognormal with 99.7% content between two specified points).

Example of Eliciting Prior Information: Bearing-Cage Time to Fracture Distribution

With appropriate questioning, engineers provided the following information:

- Time to fracture data can often be described by a Weibull distribution.
- From previous similar studies involving heavily censored data, (μ, σ) tend to be correlated (making it difficult to specify a joint prior for them).
- For small p (near the proportion failing in previous studies), (t_p, σ) are approximately independent (which allows for specification of approximately independent priors).

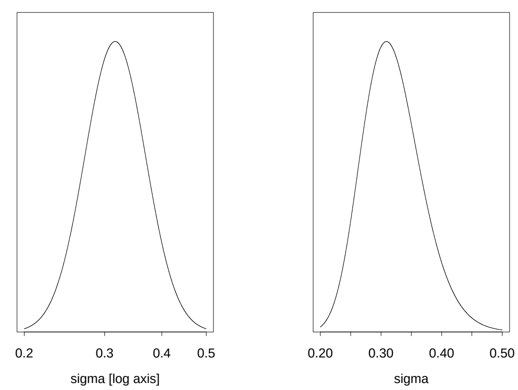
Example of Eliciting Prior Information: Bearing-Cage Fracture Field Data (Continued)

- Based on experience with previous products of the same material and knowledge of the failure mechanism, there is strong prior information about the Weibull shape parameter.
- The engineers did not have strong prior information on possible values for the distribution quantiles.
- For the Weibull shape parameter $\log(\sigma) \sim \text{NOR}(a_0, b_0)$, where a_0 and b_0 are obtained from the specification of two quantiles $\sigma_{\gamma/2}$ and $\sigma_{(1-\gamma/2)}$ of the prior distribution for σ. Then

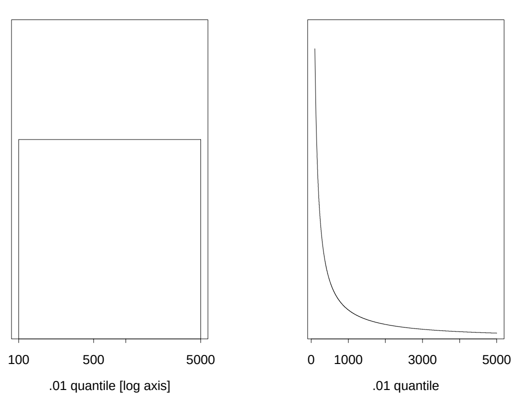
$$a_0 = \log\left[\sqrt{\sigma_{\gamma/2} \times \sigma_{(1-\gamma/2)}}\right], \quad b_0 = \log\left[\sqrt{\sigma_{(1-\gamma/2)}/\sigma_{\gamma/2}}\right]/z_{(1-\gamma/2)}$$

- Uncertainty in the Weibull 0.01 quantile will be described by UNIFORM$[\log(a_1), \log(b_1)]$ distribution where $a_1 = 100$ and $b_1 = 5000$ (wide range—not very informative).

Prior pdfs for $\log(\sigma)$ and σ when $\sigma_{0.005} = 0.2, \sigma_{0.995} = 0.5$



Prior pdfs for $\log(t_{0.01})$ and $t_{0.01}$ when $a_1 = 100, b_1 = 5000$



Joint Lognormal-Uniform Prior Distributions

- The prior for $\log(\sigma)$ is normal

$$f[\log(\sigma)] = \frac{1}{b_0}\phi_{\text{nor}}\left[\frac{\log(\sigma) - a_0}{b_0}\right], \quad \sigma > 0.$$

The corresponding density for σ is $f(\sigma) = (1/\sigma)f[\log(\sigma)]$.

- The prior for $\log(t_p)$ is uniform

$$f[\log(t_p)] = \frac{1}{\log(b_1/a_1)}, \quad a_1 \le t_p \le b_1.$$

The corresponding density for t_p is $f(t_p) = (1/t_p)f[\log(t_p)]$.

- Consequently, the joint prior distribution for (t_p, σ) is

$$f(t_p, \sigma) = \frac{f[\log(t_p)]}{t_p}\frac{f[\log(\sigma)]}{\sigma} \quad a_1 \le t_p \le b_1, \ \sigma > 0.$$

Joint Prior Distribution for (μ, σ)

- The transformation $\mu = \log(t_p) - \Phi_{\text{sev}}^{-1}(p)\sigma, \sigma = \sigma$ yields the prior for (μ, σ)

$$\begin{aligned} f(\mu,\sigma) &= \frac{f\left[\log(t_p)\right]}{t_p} \times \frac{f[\log(\sigma)]}{\sigma} \times t_p \\ &= f\left[\log(t_p)\right] \times \frac{f[\log(\sigma)]}{\sigma} \\ &= \frac{1}{\log(b_1/a_1)} \times \frac{\phi_{\text{nor}}\left\{\left[\log(\sigma) - a_0\right]/b_0\right\}}{\sigma b_0} \end{aligned}$$

  where $\log(a_1) - \Phi_{\text{sev}}^{-1}(p)\sigma \le \mu \le \log(b_1) - \Phi_{\text{sev}}^{-1}(p)\sigma,\ \sigma > 0$.

- The region in which $f(\mu,\sigma) > 0$ is South-West to North-East oriented because $\text{Cov}(\mu,\sigma) = -\Phi_{\text{sev}}^{-1}(p)\text{Var}(\sigma) > 0$.

Joint Posterior Distribution for (μ, σ)

- The likelihood is

$$L(\mu,\sigma) = \prod_{i=1}^{2003} \left\{\frac{1}{\sigma t_i}\phi_{\text{sev}}\left[\frac{\log(t_i)-\mu}{\sigma}\right]\right\}^{\delta_i} \times \left\{1 - \Phi_{\text{sev}}\left[\frac{\log(t_i)-\mu}{\sigma}\right]\right\}^{1-\delta_i}$$

  where δ_i indicates whether the observation i is a failure or a right censored observation.

- The posterior distribution is

$$f(\mu,\sigma|\text{DATA}) = \frac{L(\mu,\sigma)f(\mu,\sigma)}{\int\int L(v,w)f(v,w)dvdw} = \frac{R(\mu,\sigma)f(\mu,\sigma)}{\int\int R(v,w)f(v,w)dvdw}.$$

Methods to Compute the Posterior

- **Numerical integration**: to obtain the posterior, one needs to evaluate the integral $f(\boldsymbol{\theta}|\text{DATA}) = \int R(\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$ over the region on which $f(\boldsymbol{\theta}) > 0$.

  In general there is not a closed form for the integral and the computation has to done numerically using fixed quadrature or adaptive integration algorithms.

- **Simulation methods**: the posterior can be approximated using Monte Carlo simulation resampling methods.

Computing the Posterior Using Simulation

Using simulation, one can draw a sample from the posterior using only the likelihood and the prior. The procedure for a general parameter $\boldsymbol{\theta}$ and prior distribution $f(\boldsymbol{\theta})$ is as follows:

- Let $\boldsymbol{\theta}_i,\ i = 1, \ldots, M$ be a random sample from $f(\boldsymbol{\theta})$.

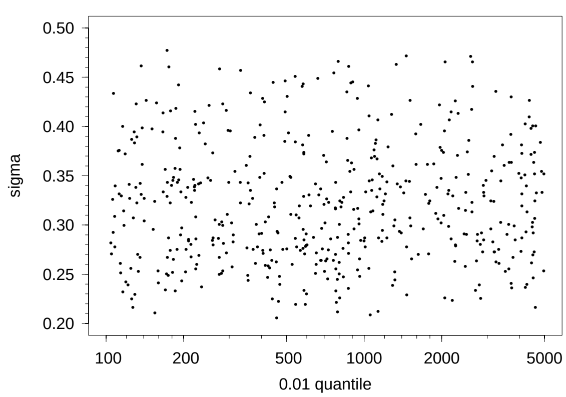
- The ith observation, $\boldsymbol{\theta}_i$, is retained with probability $R(\boldsymbol{\theta}_i)$.

  Then if U_i is a random observation from a uniform $(0, 1)$, $\boldsymbol{\theta}_i$ is retained if

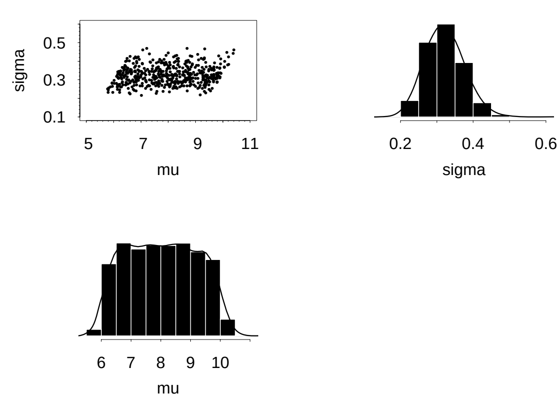
$$U_i \le R(\boldsymbol{\theta}_i).$$

- It can be shown that the retained observations, say $\boldsymbol{\theta}_1^\star, \ldots \boldsymbol{\theta}_{M^\star}^\star$ $(M^\star \le M)$ are observations from the posterior $f(\boldsymbol{\theta}|\text{DATA})$.

Simulated Joint Prior for $t_{0.01}$ and σ



Simulated Joint and Marginal Prior Distributions for μ and σ



Sampling from the Prior

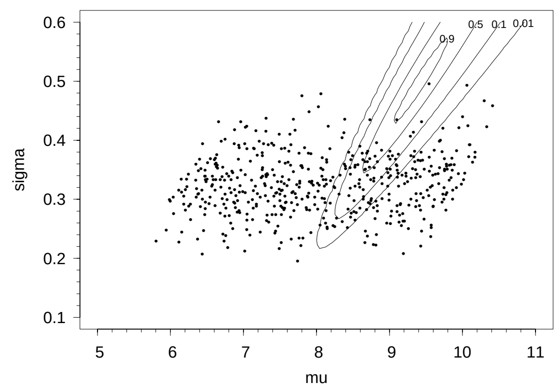
The joint prior for $\boldsymbol{\theta} = (\mu, \sigma)$, is generated as follows:

- Use the inverse cdf method (see Chapter 4) to obtain a pseudorandom sample for t_p, say
$$(t_p)_i = a_1 \times b_1^{U_{1i}}, \quad i = 1, \ldots, M$$
where $U_{11}, \ldots, U_{1M}$ are a pseudorandom sample from a uniform $(0, 1)$.

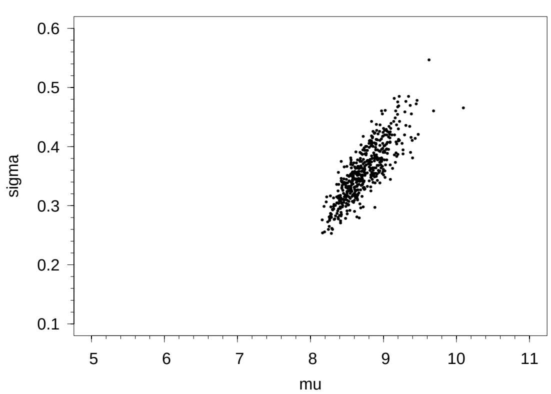
- Similarly, obtain a pseudorandom sample for σ, say
$$\sigma_i = \exp\left[a_0 + b_0 \Phi_{\text{nor}}^{-1}(U_{2i})\right]$$
where $U_{21}, \ldots, U_{2M}$ are another independent pseudorandom sample from a uniform $(0, 1)$.

- Then $\boldsymbol{\theta}_i = (\mu_i, \sigma_i)$ with $\mu_i = \log\left[(t_p)_i\right] - \Phi_{\text{sev}}^{-1}(p)\sigma_i$ is a pseudorandom sample from the (μ, σ) prior.

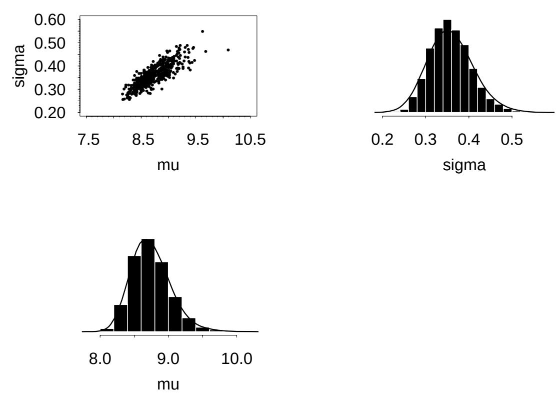
Simulated Joint Prior Distribution with μ and σ Relative Likelihood



Joint Posterior for μ and σ



Joint Posterior and Marginals for μ and σ for the Bearing Cage Data



Comments on Computing Posteriors Using Resampling

The number of observations $M^{\star}$ from the posterior is random with an expected value of
$$\mathrm{E}(M^{\star}) = M \int f(\boldsymbol{\theta}) R(\boldsymbol{\theta}) d\boldsymbol{\theta}$$
Consequently,

- When the prior and the data do not agree well, $M^{\star} << M$ otherwise and a larger prior sample will be required.

- Can add to the posterior by sequentially filtering groups of prior points until a sufficient number is available in the posterior.

Posterior and Marginal Posterior Distributions for the Model Parameters

- Inferences on individual parameters are obtained by using the marginal posterior distribution of the parameter of interest. The marginal posterior of θ_j is
$$f[\boldsymbol{\theta}_j|\text{DATA}] = \int f(\boldsymbol{\theta}|\text{DATA}) d\boldsymbol{\theta}'.$$
where θ' is the subset of the parameters excluding θ_j.

- Using the general resampling method described above, one gets a sample for the posterior for $\boldsymbol{\theta}$, say $\boldsymbol{\theta}_i^{\star} = (\mu_i^{\star}, \sigma_i^{\star})$, $i = 1, \ldots, M^{\star}$.

- Inferences for μ or σ alone are based on the corresponding **marginal** distributions $\mu_i^{\star}$ and $\sigma_i^{\star}$, respectively.

Posterior and Marginal Posterior Distributions for the Functions of Model Parameters

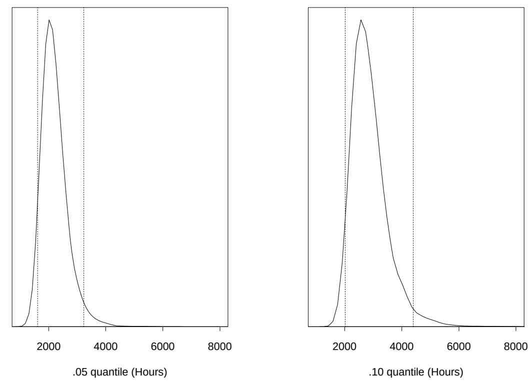
- Inferences on a scalar function of the parameters $g(\boldsymbol{\theta})$ are obtained by using the marginal posterior distribution of the functions of the parameters of interest, $f[g(\boldsymbol{\theta})|\text{DATA}]$.

- Using the simulation method, inferences are based on the simulated posterior marginal distributions. For example:

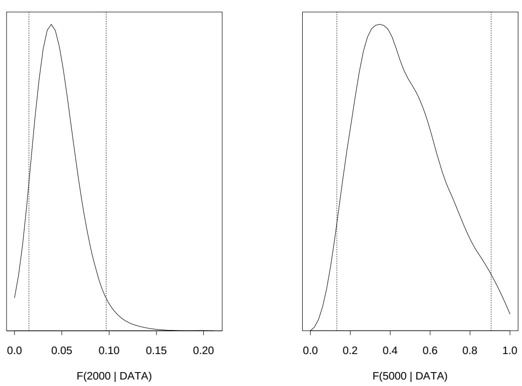
  - ▶ The marginal posterior distribution of $f(t_p|\text{DATA})$ for inference on quantiles is obtained from the empirical distribution of $\mu_i^\star + \Phi_{\text{sev}}^{-1}(p)\sigma_i^\star$.

  - ▶ The marginal posterior distribution of $f[F(t_e)|\text{DATA}]$ for inference for failure probabilities at t_e is obtained from the empirical distribution of $\Phi_{\text{sev}}\left[\frac{\log(t_e)-\mu_i^\star}{\sigma_i^\star}\right]$.

Simulated Marginal Posterior Distributions for $t_{0.05}$ and $t_{0.10}$



Simulated Marginal Posterior Distributions for $F(2000)$ and $F(5000)$



Bayes Point Estimation

Bayesian inference for $\boldsymbol{\theta}$ and functions of the parameters $\boldsymbol{g}(\boldsymbol{\theta})$ are entirely based on their posterior distributions $f(\boldsymbol{\theta}|\text{DATA})$ and $f[\boldsymbol{g}(\boldsymbol{\theta})|\text{DATA}]$.

Point Estimation:

- If $g(\boldsymbol{\theta})$ is a scalar, a common Bayesian estimate of $g(\boldsymbol{\theta})$ is its posterior mean, which is given by
$$\widehat{g}(\boldsymbol{\theta}) = \text{E}[g(\boldsymbol{\theta})|\text{DATA}] = \int g(\boldsymbol{\theta})f(\boldsymbol{\theta}|\text{DATA})d\boldsymbol{\theta}.$$
In particular, for the ith component of $\boldsymbol{\theta}$, $\widehat{\theta}_i$ is the posterior mean of θ_i. This estimate is the the Bayes estimate that minimizes the square error loss.

- Other possible choices to estimate $g(\boldsymbol{\theta})$ include (a) the posterior mode, which is very similar to the ML estimate and (b) the posterior median.

One-Sided Bayes Confidence Bounds

- A $100(1-\alpha)\%$ Bayes lower confidence bound (or credible bound) for a scalar function $g(\boldsymbol{\theta})$ is value $\underset{\sim}{g}$ satisfying
$$\int_{\underset{\sim}{g}}^{\infty} f[g(\boldsymbol{\theta})|\text{DATA}]dg(\boldsymbol{\theta}) = 1-\alpha$$

- A $100(1-\alpha)\%$ Bayes upper confidence bound (or credible bound) for a scalar function $g(\boldsymbol{\theta})$ is value $\tilde{g}$ satisfying
$$\int_{-\infty}^{\tilde{g}} f[g(\boldsymbol{\theta})|\text{DATA}]dg(\boldsymbol{\theta}) = 1-\alpha$$

Two-Sided Bayes Confidence Intervals

- A $100(1-\alpha)\%$ Bayes confidence interval (or credible interval) for a scalar function $g(\boldsymbol{\theta})$ is any interval $[\underset{\sim}{g}, \quad \tilde{g}]$ satisfying
$$\int_{\underset{\sim}{g}}^{\tilde{g}} f[g(\boldsymbol{\theta})|\text{DATA}]dg(\boldsymbol{\theta}) = 1-\alpha \qquad (1)$$
- The interval $[\underset{\sim}{g}, \quad \tilde{g}]$ can be chosen in different ways

  - ▶ Combining two $100(1-\alpha/2)\%$ intervals puts equal probability in each tail (preferable when there is more concern for being incorrect in one direction than the other).

  - ▶ A $100(1-\alpha)\%$ Highest Posterior Density (HPD) confidence interval chooses $[\underset{\sim}{g}, \quad \tilde{g}]$ to consist of all values of g with $f(g|\text{DATA}) > c$ where c is chosen such that (1) holds. HPD intervals are similar to likelihood-based confidence intervals. Also, when $f[g(\boldsymbol{\theta})|\text{DATA}]$ is unimodal the HPD is the narrowest Bayes interval.

Bayesian Joint Confidence Regions

The same procedure generalizes to confidence regions for vector functions $\boldsymbol{g}(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$.

- A $100(1-\alpha)\%$ Bayes confidence region (or credible region) for a vector valued function $\boldsymbol{g}(\boldsymbol{\theta})$ is defined as
$$\text{CR}_\text{B} = \{\boldsymbol{g}(\boldsymbol{\theta}) | f[g|\text{DATA}] \geq c\}$$
where c is chosen such that
$$\int_{\text{CR}_\text{B}} f[\boldsymbol{g}(\boldsymbol{\theta})|\text{DATA})d\boldsymbol{g}(\boldsymbol{\theta}) = 1-\alpha$$

- In this case the presentation of the confidence region is difficult when $\boldsymbol{\theta}$ has more than 2 components.

Bayes Versus Likelihood

- Summary table or plots to compare the Likelihood versus the Bayes Methods to compare confidence intervals for μ, σ, and $t_{0.1}$ for the Bearing-cage data example.

Prediction of Future Events

- Future events can be predicted by using the Bayes predictive distribution.

- If X [with pdf $f(\cdot|\boldsymbol{\theta})$] represents a future random variable
  - the posterior predictive pdf of X is
$$\begin{aligned} f(x|\text{DATA}) &= \int f(x|\boldsymbol{\theta}) f(\boldsymbol{\theta}|\text{DATA}) d\boldsymbol{\theta} \\ &= \text{E}_{\boldsymbol{\theta}|\text{DATA}}\left[f(x|\boldsymbol{\theta})\right] \end{aligned}$$
  - the posterior predictive cdf of X is
$$\begin{aligned} F(x|\text{DATA}) &= \int_{-\infty}^{x} f(u|\boldsymbol{\theta})du = \int F(x|\boldsymbol{\theta}) f(\boldsymbol{\theta}|\text{DATA}) d\boldsymbol{\theta} \\ &= \text{E}_{\boldsymbol{\theta}|\text{DATA}}\left[F(x|\boldsymbol{\theta})\right] \end{aligned}$$

  where the expectations are computed with respect to the posterior distribution of $\boldsymbol{\theta}$.

Approximating Predictive Distributions

- $f(x|\text{DATA})$ can be approximated by the average of the posterior pdfs $f(x|\boldsymbol{\theta}_i^\star)$. Then
$$f(x|\text{DATA}) \approx \frac{1}{M^\star} \sum_{i=1}^{M^\star} f(x|\boldsymbol{\theta}_i^\star).$$

- Similarly, $F(x|\text{DATA})$ can be approximated by the average of the posterior cdfs $F(x|\boldsymbol{\theta}_i^\star)$. Then
$$F(x|\text{DATA}) \approx \frac{1}{M^\star} \sum_{i=1}^{M^\star} F(x|\boldsymbol{\theta}_i^\star).$$

- A two-sided $100(1-\alpha)\%$ Bayesian prediction interval for a new observation is given by the $\alpha/2$ and $(1-\alpha/2)$ quantiles of $F(x|\text{DATA})$.

Location-Scale Based Prediction Problems

Here we consider prediction problems when $\log(T)$ has a location-scale distribution.

- Predicting a future value of T. In this case, $X = T$ and $x = t$, then
$$f(t|\boldsymbol{\theta}) = \frac{1}{\sigma t}\phi(\zeta), \quad F(t|\boldsymbol{\theta}) = \Phi(\zeta)$$
where $\zeta = [\log(t) - \mu]/\sigma$.

- Thus, for the Bearing-cage fracture data, approximations of the predictive pdf and cdf for a **new** observation are:
$$\begin{aligned} f(t|\text{DATA}) &\approx \frac{1}{M^\star} \sum_{i=1}^{M^\star} \frac{1}{\sigma_i^\star t} \phi_{\text{sev}}(\zeta_i^\star) \\ F(t|\text{DATA}) &\approx \frac{1}{M^\star} \sum_{i=1}^{M^\star} \Phi_{\text{sev}}(\zeta_i^\star) \end{aligned}$$
where $\zeta_i^\star = [\log(t) - \mu_i^\star]/\sigma_i^\star$.

Prediction of an Order Statistic

Here we consider prediction of the kth order statistic in a future sample of size m from the distribution of T when $\log(T)$ has a location-scale distribution.

- In this case, $X = T_{(k)}$ and $x = t_{(k)}$, then
$$\begin{aligned} f[t_{(k)}|\boldsymbol{\theta}] &= \frac{m!}{(k-1)!\,(m-k)!} \times [\Phi(\zeta)]^{k-1} \times \frac{1}{\sigma t_{(k)}}\phi(\zeta) \\ &\quad \times [1-\Phi(\zeta)]^{m-k} \\ F[t_{(k)}|\boldsymbol{\theta}] &= \sum_{j=k}^{m} \frac{m!}{j!\,(m-j)!} [\Phi(\zeta)]^{j} \times [1-\Phi(\zeta)]^{m-j} \end{aligned}$$
where $\zeta = [\log(t_{(k)}) - \mu]/\sigma$.

Predicting the 1st Order Statistic

When $k = 1$ (predicting the 1st order statistic), the formulas simplify to

- Predictive pdf

$$f[t_{(1)}|\boldsymbol{\theta}] \;=\; m \times \frac{1}{\sigma t_{(1)}}\phi(\zeta) \times [1 - \Phi(\zeta)]^{m-1}$$

- Predictive cdf

$$F[t_{(1)}|\boldsymbol{\theta}] \;=\; 1 - [1 - \Phi(\zeta)]^{m}$$

where $\zeta = [\log(t_{(1)}) - \mu]/\sigma$.

Predicting the 1st Order Statistic for the Bearing-Cage Fracture Data

For the Bearing-cage fracture data:

- An approximation for the predictive pdf for the 1st order statistic is

$$f[t_{(1)}|\text{DATA}] \;\approx\; \frac{1}{M^{\star}}\sum_{i=1}^{M^{\star}}\left\{m \times \frac{1}{\sigma_i^{\star} t}\phi(\zeta_i^{\star}) \times [1 - \Phi(\zeta_i^{\star})]^{m-1}\right\}$$

- The corresponding predictive cdf is

$$F[t_{(k)}|\text{DATA}] \;\approx\; \frac{1}{M^{\star}}\sum_{i=1}^{M^{\star}}\left\{1 - [1 - \Phi(\zeta_i^{\star})]^{m}\right\}$$

where $\zeta_i^{\star} = [\log(t) - \mu_i^{\star}]/\sigma_i^{\star}$.

Predicting a New Observation

- $F(t|\text{DATA})$ can be approximated by the average of the posterior probabilities $F(t|\boldsymbol{\theta}_i^{\star}),\; i = 1,\ldots,M^{\star}$.

- Similarly, $f(t|\text{DATA})$ can be approximated by the average of the posterior densities $f(t|\boldsymbol{\theta}_i^{\star}),\; i = 1,\ldots,M^{\star}$.

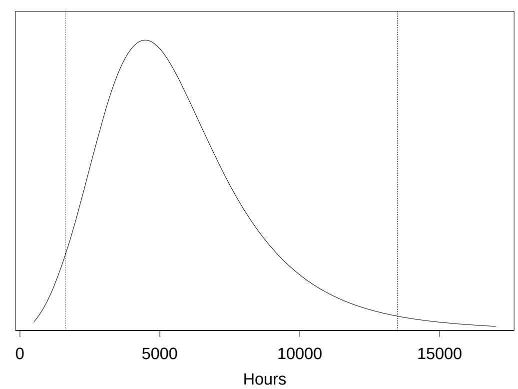
- In particular for the Bearing-cage fracture data, an approximation for the predictive pdf and cdf are

$$f(t|\text{DATA}) \;\approx\; \frac{1}{M^{\star}}\sum_{i=1}^{M^{\star}}\frac{1}{\sigma_i^{\star} t}\phi_{\text{sev}}\left[\frac{\log(t) - \mu_i^{\star}}{\sigma_i^{\star}}\right]$$

$$F(t|\text{DATA}) \;\approx\; \frac{1}{M^{\star}}\sum_{i=1}^{M^{\star}}\Phi_{\text{sev}}\left[\frac{\log(t) - \mu_i^{\star}}{\sigma_i^{\star}}\right].$$

- A $100(1-\alpha)\%$ Bayesian prediction interval for a new observation is given by the percentiles of this distribution.

Predictive Density and Prediction Intervals for a Future Observation from the Bearing Cage Population



Caution on the Use of Prior Information

- In many applications, engineers really have useful, indisputable prior information. In such cases, the information should be integrated into the analysis.

- We must beware of the use of **wishful thinking** as prior information. The potential for generating seriously misleading conclusions is high.

- As with other inferential methods, when using Bayesian methods, it is important to do sensitivity analyses with respect to uncertain inputs to ones model (including the inputted prior information)