Chapter 9

Bootstrap Confidence Intervals

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Based on the authors' text *Statistical Methods for Reliability Data*, John Wiley & Sons Inc. 1998.

January 5, 2006
19h 14min

Bootstrap Confidence Intervals
Chapter 9 Objectives

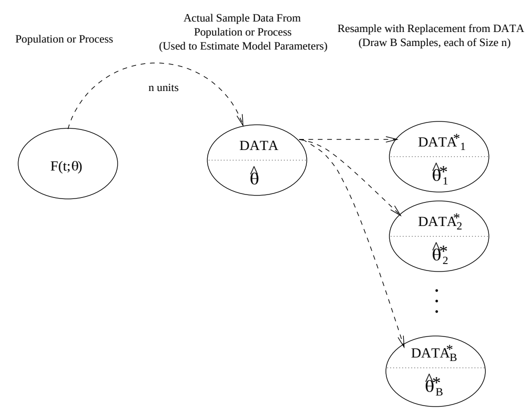
- Explain basic ideas behind the use of computer simulation to obtain bootstrap confidence intervals.
- Explain different methods for generating bootstrap samples.
- Obtain and interpret simulation-based pointwise parametric bootstrap confidence intervals.

Bootstrap Sampling and Bootstrap Confidence Intervals

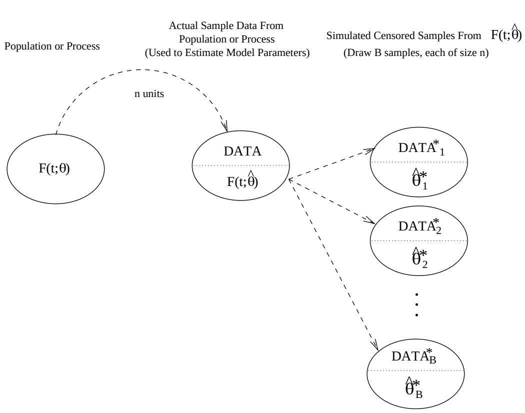
- Instead of assuming $Z_{\widehat{\mu}} = (\widehat{\mu} - \mu)/\widehat{\text{se}}_{\widehat{\mu}} \dot{\sim} \text{NOR}(0,1)$, use Monte Carlo simulation to approximate the distribution of $Z_{\widehat{\mu}}$.
- Simulate $B = 4000$ values of $Z_{\widehat{\mu}^*} = (\widehat{\mu}^* - \widehat{\mu})/\widehat{\text{se}}_{\widehat{\mu}^*}$.
- Some bootstrap approximations:
  - $Z_{\widehat{\mu}} \dot{\sim} Z_{\widehat{\mu}^*}$
  - $Z_{\log(\widehat{\sigma})} \dot{\sim} Z_{\log(\widehat{\sigma}^*)}$
  - $Z_{\text{logit}[\widehat{F}(t)]} \dot{\sim} Z_{\text{logit}[\widehat{F}^*(t)]}$

  when computing confidence intervals for μ, σ, and F.

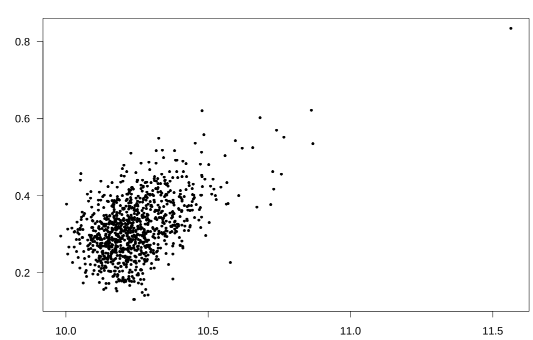
A Simple Bootstrap Re-Sampling Method



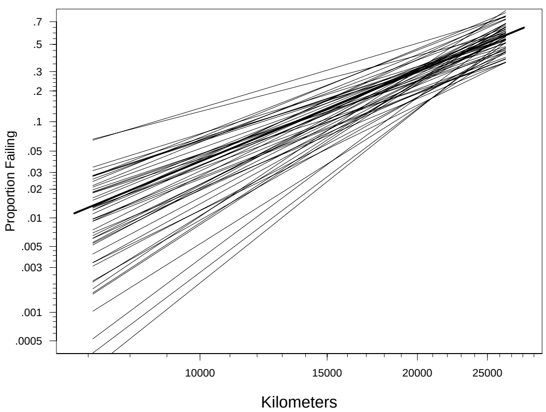
A Simple Parametric Bootstrap Sampling Method



Scatterplot of 1,000 (Out of B =10,000) Bootstrap Estimates $\widehat{\mu}^*$ and $\widehat{\sigma}^*$ for Shock Absorber



Weibull Plot of $F(t;\widehat{\mu},\widehat{\sigma})$ from the Original Sample (dark line) and 50 (Out of B =10,000) $F(t;\widehat{\mu}^*,\widehat{\sigma}^*)$ Computed from Bootstrap Samples for the Shock Absorber



Bootstrap Confidence Interval for μ

- With complete data or Type II censoring,

$$Z_{\widehat{\mu}_j^*} = \frac{\widehat{\mu}_j^* - \widehat{\mu}}{\widehat{\text{se}}_{\widehat{\mu}_j^*}}$$

has a distribution that does not depend on any unknown parameters. Such a quantity is called a **pivotal** quantity.

- By the definition of quantiles, then

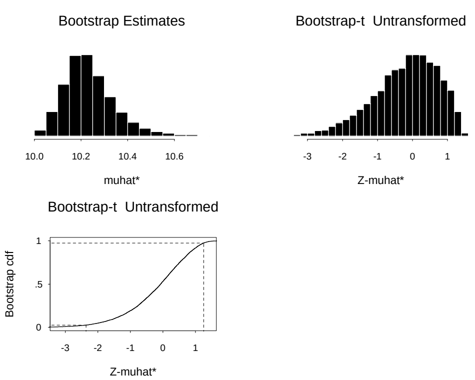
$$\Pr\left(z_{\widehat{\mu}^*_{(\alpha/2)}} < Z_{\widehat{\mu}_j^*} \le z_{\widehat{\mu}^*_{(1-\alpha/2)}}\right) = 1-\alpha$$

- Simple algebra shows that

$$[\underset{\sim}{\mu}, \quad \tilde{\mu}] = [\widehat{\mu} - z_{\widehat{\mu}^*_{(1-\alpha/2)}}\widehat{\text{se}}_{\widehat{\mu}}, \quad \widehat{\mu} - z_{\widehat{\mu}^*_{(\alpha/2)}}\widehat{\text{se}}_{\widehat{\mu}}]$$

provides an exact 95% confidence interval for μ. With other kinds of censoring, the interval is, in general, only **approximate**.

Bootstrap Distributions of Weibull $\widehat{\mu}^*$ and $Z_{\widehat{\mu}^*}$ Based on B=10,000 Bootstrap Samples for the Shock Absorber



Bootstrap Confidence Interval for σ

- With complete data or Type II censoring,

$$Z_{\log(\widehat{\sigma}^*)} = \frac{\log(\widehat{\sigma}^*) - \log(\widehat{\sigma})}{\widehat{\text{se}}_{\log(\widehat{\sigma}^*)}}$$

has a distribution that does not depend on any unknown parameters. Such a quantity is called a **pivotal** quantity.

- By the definition of quantiles, then

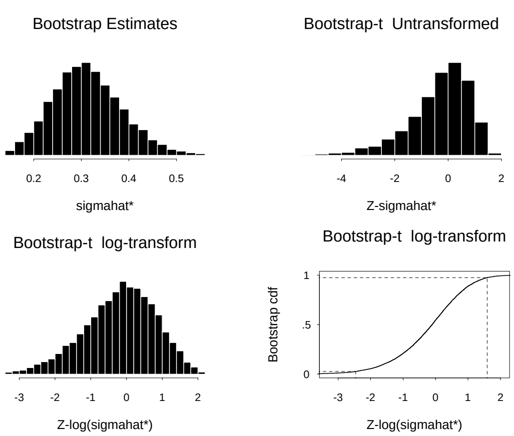
$$\Pr\left(z_{\log(\widehat{\sigma}^*)_{(\alpha/2)}} < Z_{\log(\widehat{\sigma}_j^*)} \le z_{\log(\widehat{\sigma}^*)_{(1-\alpha/2)}}\right) = 1-\alpha$$

- Simple algebra shows that

$$[\underset{\sim}{\sigma}, \quad \tilde{\sigma}] = [\widehat{\sigma}/\underset{\sim}{w}, \quad \widehat{\sigma}/\tilde{w}]$$

provides an exact 95% confidence interval for σ, where $\underset{\sim}{w} =$ $\exp\left[z_{\log(\widehat{\sigma}^*)_{(1-\alpha/2)}}\widehat{\text{se}}_{\log(\widehat{\sigma})}\right]$ and $\tilde{w} = \exp\left[z_{\log(\widehat{\sigma}^*)_{(\alpha/2)}}\widehat{\text{se}}_{\log(\widehat{\sigma})}\right]$ With other kinds of censoring, the interval is, in general, only **approximate**.

Bootstrap Distributions of $\widehat{\sigma}^*$, $Z_{\widehat{\sigma}^*}$, and $Z_{\log(\widehat{\sigma}^*)}$ Based on B=10,000 Bootstrap Samples



Bootstrap Confidence Interval for $F(t_e)$

- With complete data or Type II censoring [using $F = F(t_e)$],

$$Z_{\text{logit}(\widehat{F}^*)} = \frac{\text{logit}(\widehat{F}^*) - \text{logit}(\widehat{F})}{\widehat{\text{se}}_{\text{logit}(\widehat{F}^*)}}$$

has a distribution that does not depend on any unknown parameters. Such a quantity is called a **pivotal** quantity.

- By the definition of quantiles, then

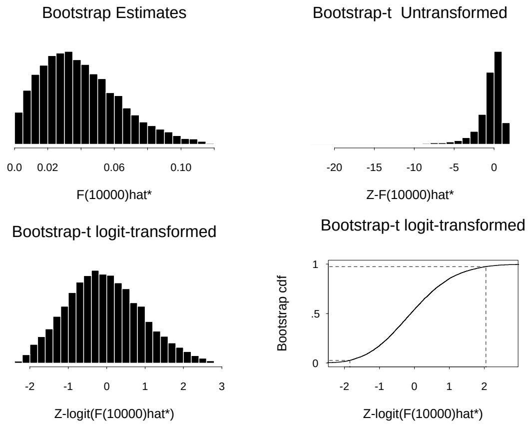
$$\Pr\left(z_{\text{logit}(\widehat{F}^*)_{(\alpha/2)}} < Z_{\text{logit}(\widehat{F}_j^*)} \le z_{\text{logit}(\widehat{F}^*)_{(1-\alpha/2)}}\right) = 1-\alpha$$

- Simple algebra shows that

$$[\underset{\sim}{F}, \quad \tilde{F}] = \left[\frac{\widehat{F}}{\widehat{F} + (1-\widehat{F}) \times \underset{\sim}{w}}, \quad \frac{\widehat{F}}{\widehat{F} + (1-\widehat{F}) \times \tilde{w}}\right]$$

where provides an exact 95% confidence interval for F, where $\underset{\sim}{w} =$ $\exp\left[z_{\text{logit}(\widehat{F}^*)_{(1-\alpha/2)}}\widehat{\text{se}}_{\text{logit}(\widehat{F})}\right]$ and $\tilde{w} = \exp\left[z_{\text{logit}(\widehat{F}^*)_{(\alpha/2)}}\widehat{\text{se}}_{\text{logit}(\widehat{F})}\right]$ With other kinds of censoring, the interval is, in general, only **approximate**.

Bootstrap Distributions of $\widehat{F}(t_e)^*$, $Z_{\widehat{F}(t_e)^*}$, and $Z_{\text{logit}[\widehat{F}(t_e)^*]}$ for t_e=10,000 km Based on B=10,000 Bootstrap Samples



Bootstrap Confidence Interval for t_p

- With complete data or Type II censoring,

$$Z_{\log(\widehat{t}_p^*)} = \frac{\log(\widehat{t}_p^*) - \log(\widehat{t}_p)]}{\widehat{\text{se}}_{\log(\widehat{t}_p^*)}}$$

has a distribution that does not depend on any unknown parameters. Such a quantity is called a **pivotal** quantity.

- By the definition of quantiles, then

$$\Pr\left(z_{\log(\widehat{t}_p^*)_{(\alpha/2)}} < Z_{\log(\widehat{t}_p^*)_j} \le z_{\log(\widehat{t}_p^*)_{(1-\alpha/2)}}\right) = 1 - \alpha$$

- Simple algebra shows that

$$[\underset{\sim}{t_p}, \quad \tilde{t_p}] = [\widehat{t}_p/\underset{\sim}{w}, \quad \widehat{t}_p/\widetilde{w}]$$

provides an exact 95% confidence interval for t_p, where $\underset{\sim}{w} = \exp\left[z_{\log(\widehat{t}_p^*)_{(1-\alpha/2)}} \widehat{\text{se}}_{\log(\widehat{t}_p)}\right]$ and $\widetilde{w} = \exp\left[z_{\log(\widehat{t}_p^*)_{(\alpha/2)}} \widehat{\text{se}}_{\log(\widehat{t}_p)}\right]$ With other kinds of censoring, the interval is, in general, only **approximate**.

Bootstrap Distributions of $\widehat{t}_p^*$, $Z_{\widehat{t}_p^*}$, and $Z_{\log[\widehat{t}_p^*]}$ for t_e=10,000 km Based on B=10,000 Bootstrap Samples