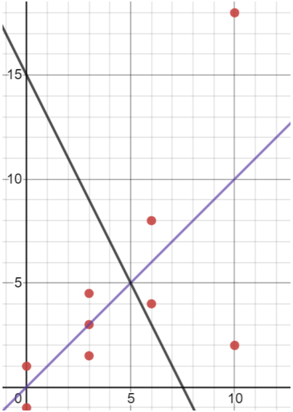
CSCI 3022 Intro to Data Science
Regression Inference

Consider the graph below. Do either of the candidate "best fit" lines violate the 4 big assumptions?



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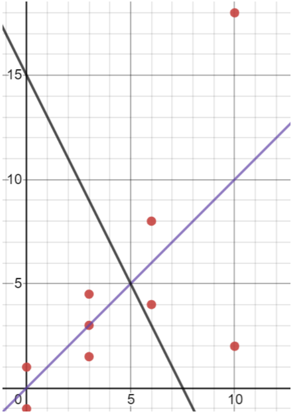
One thing to do: plot the errors *as a function of* X:



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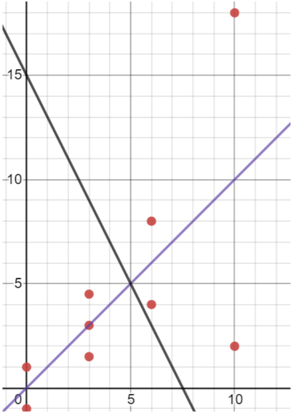

Errors are the values given by "estimated line minus data." Black line the errors clump and move up/down as X moves left-right.
Blue line the errors increase in *magnitude* as X goes right.

We've looked at the following test statistics for hypothesis testing.

1. To compare proportions against a baseline or against each other, we use Z-statistics.

$$\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \textbf{ OR } \frac{(\hat{p_1} - \hat{p_2}) - \Delta_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n_1} + \frac{\hat{p}(1-\hat{p})}{n_2}}}$$

2. To compare means when the samples are large **or** underlying normal with *known* variances, we also use Z-statistics.

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \textbf{ OR } \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \textbf{ OR } \frac{\left(\bar{X} - \bar{Y}\right) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \textbf{ OR } \frac{\left(\bar{X} - \bar{Y}\right) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{m}}}$$

3. To compare means when the samples are small **and** underlying normal, we use t-statistics.

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \textbf{ OR } \frac{\left(\bar{X} - \bar{Y}\right) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n} + \frac{s_2^2}{m}}}$$

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4. $$\varepsilon_i \sim N(0, 1)$$

Simple Linear Regression Model

The β estimators in the model are:

1. $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$
2. $\hat{\beta}_1 = \frac{Cov[X,Y]}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2} = \frac{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)}{\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2}$

Important Terminology:

- x: the independent variable, predictor, or explanatory variable (usually known). x is not random.
- Y: The dependent variable or response variable. For fixed x, Y is random.
- ε: The random deviation or random error term. For fixed x, ε is random. Has variance σ^2.
- β: the regression coefficients.
- r: the *residuals* or observed errors. Used to estimate σ^2.

Estimating SLR Parameters

Definitions:

1. The *fitted (or predicted) values* __ are obtained by plugging in __ to the equation of the estimated regression line:

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Residuals are estimates of the true error. Why?

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We don't have the true values of β_0, β_1, so when we estimate them we get variance and error in our estimates.

Estimating SLR Parameters: Results

For a model of the form $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$; $\varepsilon \sim N(0, \sigma^2)$

1. $\hat{\beta}_0 =$

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What happens if $\beta_0 \approx 0$? If $\beta_1 \approx 0$?

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What happens if $\beta_0 \approx 0$? If $\beta_1 \approx 0$?
One result: the regression line goes through $(0, \beta_0)$. It also goes through $(\bar{X}, \bar{Y})$!

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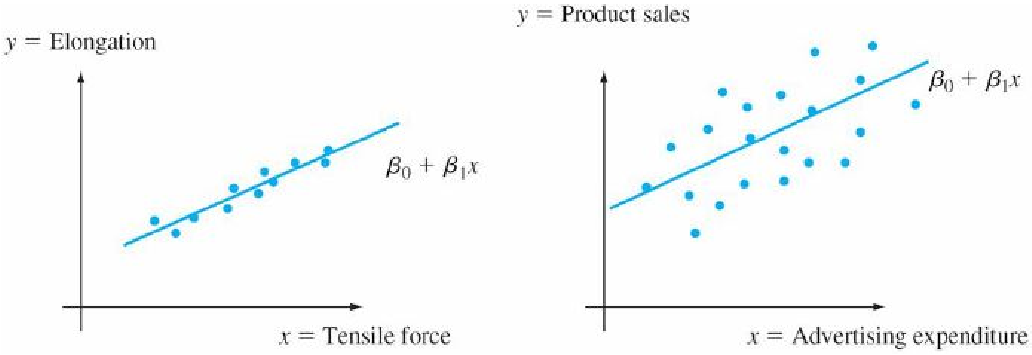
We don't have the true values of β_0, β_1, so when we estimate them we get variance and error in our estimates.

Estimating SLR Parameters: σ^2

The parameter σ^2 determines the amount of spread about the true regression line. Two separate examples:



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An estimate of σ^2 will be used in confidence interval formulas and hypothesis testing procedures presented in the next days. Recall that the residual sum of squares or sum of squared errors (SSE) is:

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So, our estimate of the variance of the model is like a measure for an average of this summand:

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$$\hat{\sigma^2} = \frac{SSE}{n-2}$$

Wait, what? Why the $n-2$??

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These are again *degrees of freedom*.

Degrees of Freedom Intuition

Suppose you have 3 (random) points on the XY plane.

1. Can you draw a line through them?

2. Can you draw a parabola through them?

3. Can you draw a cubic function through them?

4. Can you draw a quartic function through them?

Degrees of Freedom Intuition

Suppose you have 3 (random) points on the XY plane.

1. Can you draw a line through them?
   It's very unlikely. In fact, for truly random (normal) points, this result has probability zero!
2. Can you draw a parabola through them?
   Yes, but there's only one such parabola.
3. Can you draw a cubic function through them?
   Yes. Not only that, you could choose *any one* of a, b, c, d in the $ax^3 + bx^2 + cx + d = 0$ and then solve for the others. You have **one degree of freedom**.
4. Can you draw a quartic function through them?
   Yes. Not only that, you could choose *any two* of a, b, c, d, e in the $ax^4 + bx^3 + cx^2 + dx + e = 0$ and then solve for the others. You have **two degrees of freedom**.

Degrees of Freedom

The takeaway?

One property of mathematical estimation: the more you estimate, the more you risk *overfitting*. In this model we've estimated **2** "means" ($\hat{\beta}_0$,$\hat{\beta}_1$) before we got to σ, which "costs" us two degrees of freedom.

The more we estimate, the less options - degrees of freedom - we get for the remaining terms.

Estimating SLR Parameters: σ^2

Some properties of our estimate:

1. The divisor $n–2$ in is the number of degrees of freedom (df) associated with SSE and $\hat{\sigma}^2$.

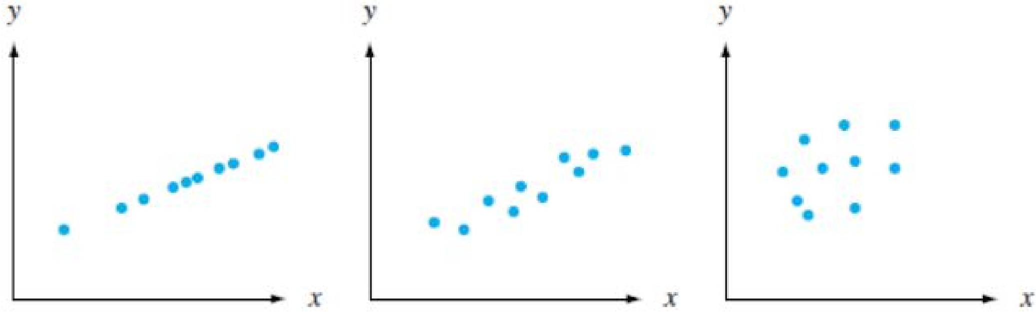
2. This is because to obtain $\hat{\sigma}^2$, two parameters must first be estimated, which results in a loss of 2 df.

3. Replacing each y_i in the formula for $\hat{\sigma}^2$ by the r.v. Y_i gives a random variable.

4. It can be shown that the r.v. $\hat{\sigma}^2$ is an unbiased estimator for σ^2.

The Coefficient of Determination

The residual sum of squares SSR can be interpreted as a measure of how much variation in y is left unexplained by the model—that is, how much cannot be attributed to a linear relationship. In the first plot, $SSE = 0$, and there is no unexplained variation, whereas unexplained variation is small for second, and large for the third plot.



Picturing Sums of Squares

The goodness-of-fit of a regressive model is often decomposed into three components based on squared deviations. These are:

1. SSE: Sum of squared errors: (vertical) distances from the regression line to the data values.
2. SST: Sum of squares, total: total deviation in Y. Looks like $Var[Y]$.
3. SSR: Sum of squares of regression line: the amount of variability tied to the model.

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$$\sum_i \left(\hat{Y} - Y_i\right)^2$$
2. SST: Sum of squares, total: total deviation in Y. Looks like $Var[Y]$.
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3. SSR: Sum of squares of regression line: the amount of variability tied to the model.
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Picturing Sums of Squares

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The sum of squared deviations about the least squares line is smaller than the sum of squared deviations about any other line, i.e. $SSE < SST$ unless the horizontal line itself is the least squares line.

The ratio SSE/SST is the proportion of total variation that cannot be explained by the simple linear regression model. The coefficient of determination is:

This coefficient is a number between 0 and 1 and is the *proportion of observed y variation explained by the model*.

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$$R^2 = 1 - \frac{SSE}{SST} = \frac{SSR}{SST}$$

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Again, R^2 is the proportion of observed y variation explained by the model.

The higher the value of R^2, the more successful is the simple linear regression model in explaining y variation, assuming the linear model is correct.

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Crucially, R^2 is a measure of *linear* dependence between X and Y. If $R^2 = 0$, X and Y may still be related! Ex: $Y = X^2(+\varepsilon)$.

Inferences about Parameters

The parameters in SLR have distributions. From these distributions, we can conduct hypothesis tests (e.g., __________), compute confidence intervals, etc.

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Distributions:

$$\hat{\beta}_0 \sim N\left(\beta_0, \frac{\sigma^2}{n} + \frac{\sigma^2 \bar{X}^2}{\left(X_i - \bar{X}\right)^2}\right)$$

$$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{\left(X_i - \bar{X}\right)^2}\right)$$

... but of course, we don't know σ^2, so we estimate with $SSE/(n-2)$.

Inferences about Parameters

Confidence Intervals: The CIs for regression are two-sided, and because $\varepsilon \sim N(0, \sigma^2)$, we may use t statistics. Since we have written down the variances of the βs, we can also write down their standard errors:

$$s.e.(\hat{\beta}_0) = \sigma\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\left(X_i - \bar{X}\right)^2}}; \qquad s.e.(\hat{\beta}_1) = \sigma\sqrt{\frac{1}{\left(X_i - \bar{X}\right)^2}}$$

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Tests then result from comparing $t = \frac{\hat{\beta}_i}{s.e.(\hat{\beta}_i)}$ to the corresponding critical t values for a one or two-tailed test.

Inferences about Y

There are more types on confidence intervals we may care about!

1. Last slide was how to perform inference on the **parameters** of the *line* β. We also might care about inference on values of Y!
2. A **confidence band** is how sure we are about the mean of Y at specific values of X, or $E[Y|X]$.
3. A **prediction band** is how we estimate the distribution of new Y observations at specific values of X. It's the same as the confidence band, but also includes our estimate for ε.

See: nb accompanying lecture: `SLR Inference`

Daily Recap

Today we learned

1. Regression Inference!

Moving forward:

- nb day Friday

Next time in lecture:

- More Regression! More predictor!