CSCI 3022 Intro to Data Science
Expectation

Opening Example:

The distribution of the amount of gravel (in tons) sold by a particular construction supply company in a given week is a continuous rv X with pdf

$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \leq x < 1 \\ 0 & else \end{cases}$$

1. What is the cdf of sales for any x?
2. Find the probability that X is less than .25?
3. X is greater than .75?
4. $P(.25 < X < .75)$?

Last Time...: the blocks of discrete probability

1. Bernoulli: *one* binary outcome experiment.
2. Binomial: binary outcome experiment success *count* in n tries.
3. Geometric: Total trials *until a success* of a binary outcome experiment.
4. Negative Binomial: Trials until *r* binary outcome experiment *successes*.
5. Poisson: *counting* outcomes with a fixed rate λ.

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   $f(x) = p^x(1-p)^{1-x}$
2. Binomial: binary outcome experiment success *count* in n tries.
   $f(x) = \binom{n}{x}p^x(1-p)^{(n-x)}$
3. Geometric: Total trials *until a success* of a binary outcome experiment.
   $f(x) = (1-p)^{x-1}p$
4. Negative Binomial: Trials until *r* binary outcome experiment *successes*.
   $f(x) = \binom{x-1}{r-1}p^r(1-p)^{(x-r)}$
5. Poisson: *counting* outcomes with a fixed rate λ.
   $f(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

Last Time...: the blocks of continuous probability

1. Exponential: time-until-event of a things that happen at a rate of $\lambda \frac{events}{time}$.

$$f(x) = \lambda e^{-\lambda x}; \quad x \geq 0$$

2. Uniform: all events form $[a, b]$ are equally likely:

$$f(x) = \frac{1}{b-a}; \qquad x \in [a, b]$$

For continuous distributions, we can't just add up a big list of outcomes and their probabilities. Instead, the probability of *single* outcomes is always zero. We add up *intervals*, which turns into an integral:

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx$$

tells us the probability of all outcomes from a to b of a continuous RV with pdf $f(x)$.

Percentiles of a Distribution

Definition: The median $\tilde{x}$ of a continuous distribution is the 50th percentile or $.5$ quantile of the distribution.

How can we express this in terms of $f(x), F(x)$?

Notation:

Visually:

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Notation:
$\tilde{x}$ satisfies $F(\tilde{x}) = .5$, or

Visually:

$$.5 = \int_{\infty}^{\tilde{x}} f(x)\,dx$$

Opening Solution

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1. What is the cdf of sales for any x?
   $F(x) = P(X \leq x) = \int_0^x \frac{3}{2}(1 - t^2)\, dt$
   $F(x) = \frac{3x}{2} - \frac{x^3}{2}$
2. Find the probability that X is less than .25? $F(.25)$
3. X is greater than .75? $1 - F(.75)$
4. $P(.25 < X < .75)$? $F(.75) - F(.25)$

Pops and Samples

Today marks the start of a large jump in how we approach data science problems:

1. We know about *sample statistics* like $\bar{X}$, s_X.
2. We've defined some *processes* that gives rise to distributions like the binomial, exponential, etc.
3. **Now:** we start bridging the gap! Given data and sample statistics, how do we estimate or infer properties of the underlying reality process? (parameters like p, λ).

   To do this, we need an understanding of centrality and dispersion of a process or density function might be.

Mean/Expected Value

Example:
Consider a university having 15,000 students and let X equal the number of courses for which a randomly selected student is registered.
The pdf of X is given to you as follows:

x	1	2	3	4	5	6	7
$f(x) = P(X = x)$	.01,	.03	.13	.25	.39	.17	.02

Students pay more money when enrolled in more courses, and so the university wants to know what the *average* number of courses students take per semester.

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What is $E[X]$?

$$E[X] = \sum_{x \in \Omega} x \cdot P(X = x) = 1 \cdot .01 + 2 \cdot .03 + 3 \cdot .13 + 4 \cdot .25 + 5 \cdot .39 + 6 \cdot .17 + 7 \cdot .02$$

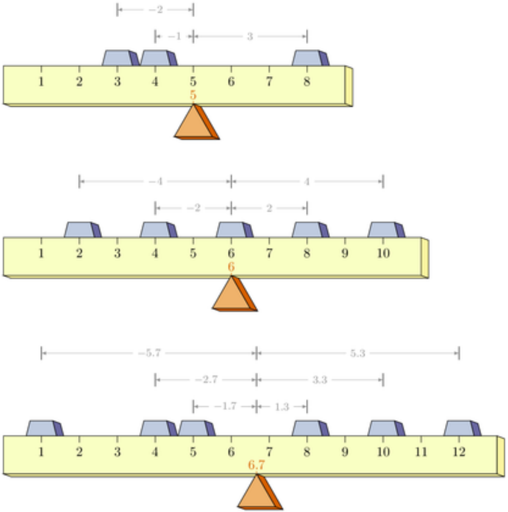
$$E[X] = 4.57$$

Interpreting Expected Value: Relative Frequency

One way to interpret expected value of a discrete distribution (especially on a finite support) is the sample mean if we managed to observe observations that *exactly* mirror the probability mass function.

In the preceding example, the pmf was given at 7 values of X with a precision up to 1%. In this case, if we had exactly 100 students and their proportions *observed* exactly mirrored the probabilities given in the example, the sample mean would be identical to the population mean.

Interpreting Expected Value



- The "center of mass" of a set of point masses
- Each mass exerts an "$r \times f$" force on the balancing point.
- Same idea holds in continuous space: we're looking for a centroid of an object.

http://www.texample.net/media/tikz/examples/TEX/balance.tex

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Recall: Integration by Parts: $\int u\,dv = uv - \int v\,du$. Mental shortcuts: "integration product rule," "LIPET"

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How long, on average, will the battery last?
Start with $E[X] = \int_{\infty}^{\infty} xf(x)\,dx$, then use our known $f(x)$:
$E[X] = \int_0^{\infty} \lambda x e^{-\lambda x}\,dx$, now via IBP with $u = \lambda x$; $dv = e^{-\lambda x}$:
$E[X] = \lambda x(\frac{-1}{\lambda}e^{-\lambda x})|_0^{\infty} - \int_0^{\infty} \lambda(\frac{-1}{\lambda}e^{-\lambda x})\,dx$
Both xe^{-x} and $e^{-x} \to 0$ as $x \to \infty$, so we're left with:
$E[X] = \frac{-1}{\lambda}e^{-\lambda x}|_0^{\infty}$ which is just $1/\lambda$. This should come as no surprise, since we interpret λ as an average *rate* in events-per-time, but the exponential measures time-until-event, so the expected value of the exponential is the reciprocal of the rate!

Expected Value of a Function

If a discrete r.v. X has a a density $P(X = x)$, then the expected value of any function $g(X)$ is computed as:

1. Continuous:

2. Discrete:

Note that $E[g(X)]$ is computed in the same way that $E(X)$ itself is, except that $g(x)$ is substituted in place of x.

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$$E[X] = \sum_{x} xf(x)\ dx$$

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Review: What is $F(x)$?

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Review: What is $F(x)$?

$$F(x) = \int_{-1}^{x} f(t)\, dt = \frac{3t}{4} - \frac{3t^3}{12}|_{-1}^{x}$$

Expected Value of a Linear Function

If $g(X)$ is a linear function of X (i.e., $g(X) = aX + b$) then $E[g(X)]$ can be easily computed from $E(X)$.

Theorem:
Let $a, b \in \mathbb{R}$ and X be a random variable with density f. Then:

Proof:

Note: This works for continuous and discrete random variables.

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Theorem:
Let $a, b \in \mathbb{R}$ and X be a random variable with density f. Then:

$$E[g(X)] = g(E[X])$$
$$E[aX + b] = aE[X] + b$$

Proof:
$E[aX + b] = \int (ax + b) f(x)\, dx = a \int x f(x)\, dx + b \int f(x)\, dx = aE[X] + b$, since integration is also linear!
Note: This works for continuous and discrete random variables.

Linear Expectation

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Earlier, we calculated that $E(X)$ was 4.57. If students pay $500 per course plus a $100 per-semester registration fee, what is the average amount of money the university can expect a student to pay each a semester?

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Earlier, we calculated that $E(X)$ was 4.57. If students pay \$500 per course plus a \$100 per-semester registration fee, what is the average amount of money the university can expect a student to pay each a semester?
$Money = 500 \cdot Courses + 100 = 500X + 100 = g(X)$. Then,

$$E[g(X)] = g(E[X]) = 500 \cdot 4.57 + 100 = 2385.$$

Expectation and Spread

The idea of **Expected value** can be extended to describe all kind of notions of "what should happen if we have a (arbitrarily large) sample.

> Suppose we wish to know the variance or standard deviation of the population. For a *sample*, recall that
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We might ask: what is the *expected* value of how spread out x-value are?

Population variance is this idea expressed as an expectation:

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Population variance is this idea expressed as an expectation:

$$Var[X] = E[\underbrace{(X - E[X])^2}_{\text{squared deviations}}] = E[(X - \mu_X)^2]$$

Daily Recap

Today we learned

1. Expectation

Moving forward:

- nb day Friday!

Next time in lecture:

- Expected dispersion/spread: calculating variances from pdfs!