Write **clearly** and **in the box:**

CSCI 2824
Midterm Exam 2
Remote edition
Spring 2020

Name: Pourna Sengupta	
Student ID: 109086577	
Section number: 002	

Read the following:

- **RIGHT NOW!** Write your name, student ID and section number on the top of your exam. If you're handwriting your exam, include this information at the top of the first page!
- You may use the textbook, your notes, lecture materials, and Piazza as recourses. Piazza posts should not be about exact exam questions, but you may ask for technical clarifications and ask for help on review/past exam questions that might help you. You may not use external sources from the internet or collaborate with your peers.
- You may use a calculator.
- If you print a copy of the exam, clearly mark answers to multiple choice questions in the provided answer box. If you type or hand-write your exam answers, write each problem on their own line, clearly indicating both the problem number and answer letter.
- Mark only one answer for multiple choice questions. If you think two answers are correct, mark the answer that **best** answers the question. No justification is required for multiple choice questions. For handwriting multiple choice answers, clearly mark both the number of the problem and your answer for each and every problem.
- For free response questions you must clearly justify all conclusions to receive full credit. A correct answer with no supporting work will receive no credit.
- The Exam is due to Gradescope by midnight on Sunday, April 12.

Problems	Count	Points each	Points total
MCQ	12	3	36
Short Ans	2	10	20
Free Responses	3	15	45
Total	17		100+1 for free

[2]**Multiple choice problems: For a printed version of the exam, write your answers in the boxes. For typed or hand-written submissions, write each problem on their own line, clearly indicating both the problem number and answer letter.**

Problems 1, 2 and 3, refer to the following setup:
Suppose you have two dice, a red one and a blue one. The red die is a fair die so it lands on any of its 6 sides with equal probability, but the faces of that die are not all unique: they are {2,2,4,4,6,6}. The blue die is a loaded/unfair die and lands on 4, 5, or 6 twice as often as it lands on 1, 2, or 3. The numbers 1, 2 and 3 are equally likely among themselves, and the numbers 4, 5 and 6 are equally likely among themselves.

(1) What is the probability of rolling a 3 on the **blue** die?

A) 1/3
B) 1/6
C) 2/9
D) 1/12
E) 1/4
F) 1/9

4 5 6 1 2 3
11 11 11 1 1 1 = 9

1/9

f

(2) You roll both die and sum up their visible faces. Which of the following represents the probability of the sum being 3?

A) 1/27.
B) 1/3.
C) 1/6.
D) 1/9.
E) 2/9.
F) 1/36.

2 1
1/3 1/9

$1/3 \cdot 1/9 = 1/27$

a

(3) Denote the correct probability in question (2) by p. You roll both die and sum up their visible faces, then *repeat* the process until you've recorded the sum 10 times total. Which of the following represents the probability of never seeing a sum of 3 in any of those 10 rolls?

A) $10p$
B) p^{10}
C) $(1-p)^{10}$
D) 0
E) $1-(1-p)^{10}$
F) 1
G) $p/10$

$P = 1/27$

1 turn: $1-P$
10 turns: $(1-P)^{10}$

c

(4) Recall that to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ *multiplications.* The Google PageRank algorithm uses a square matrix that's filled with non-zero entries when pages link to one another. Suppose we have m web sites cataloged: this is then an $m \times m$ matrix. Denote this matrix by P. P is then run through an iterative algorithm that takes j loops to complete (for $5 < j < 100$), and *each* step of this loop an $m \times m$ matrix is multiplied by P.

What is the complexity order of the PageRank algorithm?

A) $mj \log m$
B) m^2
C) m^3
D) m^4
E) $m^2 j^2$
F) $m^2 \log m$

$m \times m \times m \times j = j \cdot m^3$

j constant so time $= m^3$

c

(5) Consider the set given by the following two rules:

- $1 \in E$
- If $x \in E$, then $\frac{x}{2} \in E$.

What is the sum of all of the elements of E?

A) The sum is infinitely large.
B) The sum approaches 0 as you include all elements.
C) The sum approaches 1 as you include all elements.
D) The sum approaches 2 as you include all elements.
E) $\sum_{i=0}^{\infty} \frac{i}{2}$
F) The sum approaches 1/2 as you include all elements.

$\boxed{d}$

(6) Suppose you run a small farm stand that sells Colorado tomatoes and spinach. Each of the two types of produce can be bought as one of three varieties: organic, greenhouse, or field-grown. What is the minimum number of shoppers who each purchase one item needed to guarantee that at least five shoppers buy the same produce in the same variety?

A) 31
B) 26
C) 30
D) 6
E) 25
F) 24
G) The number is not listed.

$4 \cdot 6 = 24$
$+1$
25

$\boxed{e}$

(7) Which of the following is the greatest common divisor of 354144 and 182160?

A) 2
B) 24
C) 48
D) 36
E) 2304
F) 182160
G) The number is not listed.

354144/2
177072/2
88536/2
44268/2
22134/2
11067/3
3689/7
527

$2^5 \cdot 3 \cdot 7 \cdot 527$

182160/2
91080/2
45540/2
22770/2
11385/5
2277/3
759/3
253

$2^4 \cdot 5 \cdot 3^2 \cdot 253$

gcd: $2^4 \cdot 3^1$
$= 48$

$\boxed{c}$

(8) Consider the function $g(n) = 2n! + 18n^n + 5n^3 \log(n^3) - n^2$ which represents the complexity of some algorithm. What is the order of g?

A) $n!$
B) n^3
C) $n^3 \log n$
D) n^n
E) $-n^2$

$2 + 18 + 5 + 1 = 26$

$\boxed{d}$

(9) Suppose the United States decides to abolish the penny and replace it with a 3 cent piece, leaving common coin denominations of 3, 5, 10, and 25 cents. Consider the claim $P(k)$: "it is possible to make *any* nonnegative integer k amount of change using only nonnegative integer counts of these denominations."

A) This claim is valid and its proof is an example of *strong* induction.
B) This claim is valid and its proof is an example of *weak* induction.
C) This claim is invalid because it lacks a proper base case.
D) This claim is invalid because there are *no* assumptions where you could prove the property holds for $P(k+1)$ making change on $k+1$ cents.

$\boxed{c}$

(10) Select the answer that is a closed form solution to this recurrence relation:

$$a_n = (n+3)a_{n-1}; \quad a_0 = 2$$

A) $a_n = P(n, n-4)$
B) $a_n = n!$
C) $a_n = 2^n$
D) $a_n = \frac{(n+3)!}{3}$
E) $a_n = 2n+3$

$a_1 = 4(2) = 8$
$a_2 = 5(8) = 40$
$a_3 = 6(40) = 240$

$a_3 = \frac{(3+3)!}{3} = \frac{720}{3}$
$= 240$ ✓

$\boxed{d}$

(11) Zach's cooking is sometimes delicious and sometimes nutritious. Suppose in the month of March Zach cooked himself dinner 25 times. 14of those meals were nutritious and 17 were delicious. 8 meals were both nutritious and delicious. How many meals were *neither* nutritious *nor* delicious?

A) 23
B) 2
C) 31
D) 8
E) 0

n - 14
d - 17
b - 8

$14 + 17 - 8 = 23$

$25 - 23 = 2$

$\boxed{b}$

(12) Consider creating a 9-digit number out of the digits 1-9, using each of those digits exactly once. How many such numbers contain the digits 12345 - Zach's luggage code - in that order somewhere within them?

A) 9!
B) $5 \cdot 5!$
C) 6!
D) $4 \cdot 5!$
E) 5!
F) 9^9

12345 = 1#

$\underline{5} \cdot \underline{4} \cdot \underline{3} \cdot \underline{2} \cdot \underline{1}$

5 places to put 12345
rest are 4!

$5 \cdot 4! = 5!$

$\boxed{e}$

Short Answers. If your answers do not fit in the given box, MAKE A NOTE of where the work is continued or it will NOT be graded!

(13) You've joined the CSCI Data Science team, and now you're in charge of how to distribute projects for students to work on in teams. The 4 projects are: Data Science, Machine Learning, Algorithmic Theory, and Computational Biology. There are 22 students on the team, each of which needs a project.

Please leave your answer unsimplified, in terms of multiplications, sums, powers and factorials. Do not leave the answer in terms of $P(n,r)$ or $C(n,r)$ notations.

(a) How many different ways can you allocate the 22 students to the projects?

(b) For Part (b) only, you realize that you want to make sure that each of the 4 teams has at least 3 students. Now how many different ways can you allocate the 22 students to the projects?

For each part, only count the ways to allocate **numbers** of students: putting 21 students on Data Science and 1 on Biology represents *one* way, not many. Each student should get a project.

(a) 22 students
4 projects $\quad P(22,4) = \boxed{4^{22} \text{ ways}}$

(b) $C(22,12) \to C(12,3) \to C(9,3) \to C(6,3) \to C(3,3) \cdot \frac{1}{4!}$

$C(12,3) \to$ 12 students, 3 per group, 4 groups

$C(9,3) \to$ 9 students left to assign to a group

$C(6,3) \to$ 6 "

$C(3,3) \to$ 3 "

$$\frac{22!}{12!(22-12)!} \cdot \frac{12!}{3!(12-3)!} \cdot \frac{9!}{3!(9-3)!} \cdot \frac{6!}{3!(6-3)!} \cdot \frac{3!}{\cancel{3!}(3-3)!} \cdot \frac{1}{4!} \cdot 4^{10}$$

$$= \frac{22!}{\cancel{12!} \cdot 10!} \cdot \frac{\cancel{12!}}{3! \cdot \cancel{9!}} \cdot \frac{\cancel{9!}}{3! \cdot \cancel{6!}} \cdot \frac{\cancel{6!}}{3! \cdot 3!} \cdot \frac{1}{4!} \cdot 4^{10}$$

$$= \boxed{\frac{22!}{10!} \cdot \left(\frac{1}{3!}\right)^4 \cdot \frac{1}{4!} \cdot 4^{10} \text{ ways}}$$

(14) Find a closed form of the recurrence given by $a_n = -6 + 12 \cdot a_{n-1}$; $a_0 = 2$. Show work.

$a_0 = 2 \qquad a_n = -6 + 12 \cdot a_{n-1}$

$a_1 = -6 + 12(2) = -6 + 24 = 18$

$a_2 = -6 + 12(18) = -6 + 216 = 210$

$a_3 = -6 + 12(210) = -6 + 2520 = 2514$

$a_0 = 2$

$a_1 = 12a_0 - 6$

$a_2 = 12a_1 - 6 = 12(12a_0 - 6) - 6$

$a_3 = 12a_2 - 6 = 12[12(12a_0 - 6) - 6] - 6$

$\vdots \qquad \vdots$

$a_n = 12(a_{n-1}) - 6$

$= 12(12^{n-1} a_0 - 6 \cdot 12^{n-1} - \cdots - 6) - 6$

$= 12^n a_0 - 6 \cdot 12^{n-1} - \cdots - 6 \cdot 12 - 6$

$= 2 \cdot 12^n - 6\underbrace{(12^{n-1} + 12^{n-2} - \cdots 12^0)}_{\text{geometric sum}}$

$= (2 \cdot 12^n) + \sum_{k=1}^{n} -6 \cdot 12^k$

$= (2 \cdot 12^n) + \left[-6 \cdot \frac{1-12^n}{1-12}\right]$

$= (2 \cdot 12^n) + \frac{6(1-12^n)}{11}$

$$\boxed{a_n = [2 \cdot 12^n] + \left[\frac{6(1-12^n)}{11}\right]}$$

Free response problems. If your answers do not fit in one page, MAKE A NOTE of where the work is continued! Show all work.

(15) Many variants of poker are played with both cards in players' hands and shared community cards. Players' hand are some combination of the two sets of cards. For parts (a) and (b), consider playing such that Anna, Brad, Charlie, and Dre each have 2 cards for themselves, and build a 5 card hand out of those 2 cards and 3 shared cards. Assume a standard 52-card deck is being used.

Please leave your answer unsimplified, in terms of multiplications, sums, powers and factorials. Do not leave the answer in terms of $P(n, r)$ or $C(n, r)$ notations.

(a) What is the probability that Anna has a flush, where her 2 cards and the 3 community cards share a suit?

(b) What is the probability that Brad also has a flush *given* that Anna has a flush?

(c) Suppose for the next round, when the cards are dealt a 4th community card is added to the center, and play. What is the probability that Charlie has a flush? For this variant, a flush means at least 5 of the 6 cards a player has access to - in their hand or the community - share a suit.

(d) For this round in part (c) with a 4th community card, what is the probability that Dre also has a flush *given* that Charlie has a flush?

(a) $C(13,5) * C(4,1) \;/\; C(52.5)$

$$\frac{13!}{5!(13-5)!} \cdot \frac{4!}{1!\cdot(4-1)!} \cdot \frac{5!\cdot(52-5)!}{52!}$$

$$\boxed{\frac{13!}{5!\cdot 8!} \cdot \frac{4!}{3!} \cdot \frac{5!\cdot 47!}{52!}}$$

13 in the suit
3 in community

(b) $C(8,2) \;/\; C(52,5)$

$$\frac{8!}{2!\,(8-2)!} \cdot \frac{5!\,(52-5)!}{52!}$$

$$\boxed{\frac{8!}{2!\cdot 6!} \cdot \frac{5!\cdot 47!}{52!}}$$

8 left in suit
3 in community
2 in hand must be suit

(c) $C(6,1) * C(5,1) \;/\; C(52,5)$

$$\frac{6!}{1!\,(6-1)!} \cdot \frac{5!}{1!(5-1)!} \cdot \frac{5!\,(52-5)!}{52!}$$

$$\boxed{\frac{6!}{5!} \cdot \frac{5!}{4!} \cdot \frac{5!\cdot 47!}{52!}}$$

6 left in suit
1 in hand must be suit
4 in community
5 left in suit

(d) $C(4,1) \;/\; C(52,5)$

$$\boxed{\frac{4!}{3!} \cdot \frac{5!\cdot 47!}{52!}}$$

1 hand must be suit
4 in community
4 left in suit

(16) Use induction to prove the following rule. Be sure to state all relevant parts of the inductive proof, and mention whether you are using strong or weak induction.

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$$

* Using weak induction

base case

$n=1$
$i=1$

$\frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$

$\frac{1}{1(1+1)} = 1 - \frac{1}{1+1}$

$\frac{1}{2} = 1 - \frac{1}{2} \rightarrow \frac{1}{2} = \frac{1}{2}$ ✓

induction step

The sum the first n positive fractions equals $1 - \frac{1}{n+1}$

goal

By adding the next positive integer fraction, the new total sum equals $1 - \frac{1}{(n+1)+1}$

For $k \geq 1$,

$S = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = 1 - \frac{1}{k+1}$

$S = 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$

$S = \frac{(k+1)(k+2)}{(k+1)(k+2)} - \frac{(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$

$S = \frac{(k+1)(k+2) - (k+2) + 1}{(k+1)(k+2)}$

$S = \frac{k^2 + \overset{2}{\cancel{3}}k + \cancel{2} - \cancel{k} - \cancel{2} + 1}{(k+1)(k+2)}$

$S = \frac{k^2 + 2k + 1}{(k+1)(k+2)}$

$S = \frac{(k+1)\cancel{(k+1)}}{\cancel{(k+1)}(k+2)}$

$S = \frac{(k+1)}{(k+2)}$

$S = \frac{(k+2) - 1}{(k+2)} = 1 - \frac{1}{k+2} = 1 - \frac{1}{(k+1)+1}$ ✓

$\boxed{\text{The conjecture can be proven by weak induction}}$

(17) Consider the function $f(n) = 18n^2 - 2n^2 \log(n) + 5n^3$ which represents the complexity of some algorithm.

(a) Find the smallest nonnegative integer p for which n^p is a tight big-**O** bound on $f(n)$. Be sure to justify any inequalities you use and provide the C and k from the big-**O** definition.

(b) Find the largest nonnegative integer p for which n^p is a tight big-**Ω** bound on $f(n)$. Be sure to justify any inequalities you use and provide the C and k from the definition.

(c) Based on your work in parts (a) and (b), what is the order of f?

(d) Verify that your answer in part (c) is correct by computing any relevant limits. Show all work.

$f(n) = 18n^2 - 2n^2\log(n) + 5n^3$

(a) tight big-O bound is $O(n^3)$

$18n^2 \leq 18n^3$
$2n^2\log(n) \leq 2n^3$
$5n^3 \leq 5n^3$

$18n^3 + 2n^3 + 5n^3 = 25n^3$
$18n^2 - 2n^2\log(n) + 5n^3 \leq 25n^3$

$\boxed{O(n^3) \quad C=25 \quad k=1}$

(b) tight big-Ω bound is $\Omega(n^3)$

$18n^2 \geq 0$
$2n^2\log(n) \geq 0$
$5n^3 \geq 5n^3$

$0 + 0 + 5n^3 = 5n^3$
$18n^2 + 2n^2\log(n) + 5n^3 \geq 5n^3$

$\boxed{\Omega(n^3) \quad C=5 \quad k=1}$

(c) order of f

f(n) is $\Theta(n^3)$ because both both big O and big Ω bounds are $n^3 \rightarrow$ f(n) is $O(n^3)$ and $\Omega(n^3)$

d) show that $5n^3$ grows asymptotically faster than both $18n^2$ and $2n^2\log(n)$

(1) $\lim_{n\to\infty} \frac{5n^3}{18n^2} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopitals}} \frac{15n^2}{36n} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopitals}} \frac{30n}{36} = \infty$

as n goes to infinity, $5n^3$ grows asymptotically faster than $18n^2$.

(2) $\lim_{n\to\infty} \frac{5n^3}{2n^2\log(n)} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopitals}} \frac{15n^2}{2(2n\ln(n)+n)} = \frac{\infty}{\infty} \xrightarrow{\text{L'Hopitals}} \frac{30n}{2(2\ln(n)+3)} = \frac{\infty}{\infty}$

$\xrightarrow{\text{L'Hopitals}} \frac{30}{4/n} = \frac{30n}{4} = \infty$

as n goes to infinity, $5n^3$ grows asymptoncally faster than $2n^2\log(n)$