CSCI 2824
Spring 2020

Write **clearly**:

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Section number:002
Assignment: Homework 8

Read the following:

- This cover sheet must be included as the first page for all written homework submissions to CSCI 2824.
- Fill out all of the fields above.
- Submit your written homework assignment to the electronic dropbox. You will receive graded feedback through the same mechanism.
- If you type up your homework assignment using MS Word or LaTeX, then you can earn two extra credit points per homework assignment. You **must** use properly formatted equations and nice-looking text in order to be eligible for this extra credit point. If you type it up and do not format equations properly or do not use the cover sheet (for example), you might still lose the style/neatness points.
- By submitting this assignment, you are agreeing that you have abided by the **CU Honor Code**, and that this submission constitutes your own original work.

CSCI 2824 – Spring 2020 **Homework 8**

This assignment is due on Friday, Mar 13 to Gradescope by 11:59pm. You are expected to write up your solutions neatly and **use the coverpage**. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: On the **official CSCI 2824 cover page** of your assignment clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number. You may **neatly** type your solutions for +2 extra credit on the assignment. You will lose *all* 5 style/neatness points if you fail to use the official cover page.

(1) The following are riffs on exam one questions that many students struggled with.

(a) Name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **true**? Next, name two domains - with the same domain for both x and y - such that *both* the statements $\forall x \exists y \left(x^2 \geq y\right)$ and $\exists x \forall y \left(x^2 > y\right)$ are **false**?

(b) Suppose sets A and B satisfy $A \cup B = A$. What can you conclude about sets A and B? Explain.

(c) Suppose sets A and B satisfy $A - B = A$. What can you conclude about sets A and B? Explain.

(a) Domain that satisfies the statements (statements are true)
ANSWER:

(i) Digits from {1, 2, 3, ... 9}

(ii) Digits from {-1, -2, -3, ... -9}

Domains that do not satisfy the statements (statements are false)
ANSWER:

(i) all real numbers (false for (-1, 1))

(ii) all integers (false for {-1, 0, 1})

(b) **ANSWER:**
A and B have all elements of A while B has elements that are not in A. Therefore we can conclude that, $A \subseteq B$, where A is a subset of B.

(c) **ANSWER:**
B has no elements of A and therefore there are no intersections between sets A and B, $A \cap B = \emptyset$.

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(2) Use divisibility and modular arithmetic to answer the following, showing all work:

(a) Find the greatest common divisor of $a = 8640$ and $b = 102816$.

(b) Determine whether $c = 733$ is prime or not by checking its divisibility by prime numbers up to $\sqrt{c}$.

(a) **GCD(8640):**

$$\begin{aligned}
8640 &= \mathbf{2} * 4320\\
4320 &= \mathbf{2} * 2160\\
2160 &= \mathbf{2} * 1080\\
1080 &= \mathbf{2} * 540\\
540 &= \mathbf{2} * 370\\
370 &= 2 * 135\\
135 &= 5 * 27\\
27 &= \mathbf{3} * 9\\
9 &= \mathbf{3} * 3\\
3 &= \mathbf{3} * 1
\end{aligned}$$

GCD(102816):

$$\begin{aligned}
102816 &= \mathbf{2} * 51408\\
51408 &= \mathbf{2} * 25704\\
25704 &= \mathbf{2} * 12852\\
12852 &= \mathbf{2} * 6426\\
6426 &= \mathbf{2} *3213\\
3213 &= \mathbf{3} * 1071\\
1071 &= \mathbf{3} * 357\\
357 &= \mathbf{3} * 119\\
119 &= 7 * 17\\
17 &= 17 * 1
\end{aligned}$$

ANSWER: GCD(8640, 102816) = 864

(b) $\sqrt{733} = 27.07397$

$$\begin{aligned}
733 \mod 2 &= 1\\
733 \mod 3 &= 1\\
733 \mod 5 &= 3\\
733 \mod 7 &= 5\\
733 \mod 11 &= 7\\
733 \mod 13 &= 5\\
733 \mod 17 &= 2\\
733 \mod 19 &= 11\\
733 \mod 23 &= 20
\end{aligned}$$

ANSWER: No modulus of 733 and prime numbers from 1 to $\sqrt{733}$ equal 0 so 733 is prime.

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(3) Consider the function $f(n) = 7n^4 + 22n^4 \log(n) - 5n^2 \log(n^2)$ which represents the complexity of some algorithm.

(a) Find a tight big-**O** bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-**O** definition?

(b) Find a tight big-$\boldsymbol{\Omega}$ bound of the form $g(n) = n^p$ for the given function f with some natural number p. What are the constants C and k from the big-$\boldsymbol{\Omega}$ definition?

(c) Can we conclude that f is big-$\boldsymbol{\Theta}(n^p)$ for some natural number p?

(a) $\text{f(n)} = 7\text{n}^4 + 22n^4\text{log(n)} - 5n^2\text{log(n}^2\text{)}$
$\text{f(n)} = 7\text{n}^4 + 22n^4\text{log(n)} - 10n^2\text{log(n)}$
$f(n) = 7n^4 + 2n^2\text{log(n)}(11n^2 - 5)$

$|\ f(n)\ | = |\ 7n^4 + 22n^4\text{log(n)} - 10n^2\text{log(n)}|$
$|\ f(n)\ | \leq |\ 7n^4\ | + |\ 22n^4\text{log(n)}| + |\ 10n^2\text{log(n)}|$
$|\ f(n)\ | \leq 7n^4 + 22n^4\text{log(n)} + 10n^2\text{log(n)}$

$|\ f(n)\ | \leq 7n^5 + (22n^4 * n) + (10n^2 * n^3)$
$|\ f(n)\ | \leq 39n^5$

ANSWER: $\text{O(n}^5\text{)}$ where C = 39 and k = 1

(b) $\text{f(n)} = 7\text{n}^4 + 2n^2\text{log(n)}(11n^2 - 5)$
$(\text{n}^2 \geq 0), (log(n) \geq 0), ((11n^2 - 5) \geq 0)$ therefore,

$2\text{n}^2\text{log(n)}(11n^2 - 5) \geq 0$
$f(n) \geq 7n^4$
ANSWER: $\Omega(n^4)$ where C = 7 and k = 1

(c) We cannot conclude that f is big - $\Theta(n^p)$. We can say that f(n) = $\Theta(n^4\text{log(n)})$

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(4) Consider the function $g(n) = 2^n + \dfrac{n(n+1)}{2} - \log\left(n^{n^n}\right)$ which represents the complexity of some algorithm.

(a) Between 2^n and $\frac{n(n+1)}{2}$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(b) Between 2^n and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(c) Between $\frac{n(n+1)}{2}$ and $\log\left(n^{n^n}\right)$, which function grows *asymptotically faster* as $n \to \infty$? Justify by computing an appropriate limit.

(d) What is the order of g?

(a) $\lim_{x\to\infty}(\frac{2^n}{0.5n^2+0.5n}) = \frac{\infty}{\infty} = L'Hopitals = \frac{ln(2)2^n}{n+0.5} = \frac{\infty}{\infty} = L'Hopitals = \frac{ln^2(2)2^n}{1} = \infty$

ANSWER: as n $\to \infty$,2^n grows asymptotically faster

(b) $\lim_{x\to\infty}(\frac{2^n}{log(n^{n^n})}) = \frac{\infty}{\infty} = L'Hopitals = \frac{ln(2)2^n}{2nln(n)+n} = \frac{\infty}{\infty} = L'Hopitals = \frac{ln^2(2)2^n}{2ln(n)+3} = \frac{\infty}{\infty} = L'Hopitals = \frac{ln^3(2)2n^2}{2/n} = \frac{nln^3(2)2^n}{2} = \infty$

ANSWER: as n $\to \infty$,2^n grows asymptotically faster

(c) $\lim_{x\to\infty}(\frac{0.5n^2+0.5n}{log(n^{n^n})}) = \frac{\infty}{\infty} = L'Hopitals = \frac{n+0.5}{2nln(n)+n} = \frac{\infty}{\infty} = L'Hopitals = \frac{1}{2ln(n)+3} = 0$

ANSWER: as n $\to \infty$,log(n$^{n^n}$) grows asymptotically faster

(d) g(n) $= 2^n + \frac{n(n+1)}{2} - log(n^{n^n})$

$2^n \leq 2^n$
$\frac{n(n+1}{2} \leq 2^n$
$log(n^{n^n}) \leq 2^n$
g(n) $\leq 3(2^n)$ so $O(n^2)$ where C = 3 and k = 1

$g(n) \geq 12(2^n)$ so $\Omega(2^n)$ where C = $\frac{1}{2}$ and k = 1

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(5) Consider the following matrices.

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

For this problem, we will calculate product $P = ABC$. Note that matrix multiplication is associative, which means we can calculate the product P by first computing the matrix (AB), then multiplying this by C to obtain $P = (AB)C$. **Or** we could first compute the matrix (BC), then multiply it by A to obtain $P = A(BC)$. Now recall that to multiply an $m \times n$ matrix by an $n \times k$ matrix requires $m \times n \times k$ *multiplications.*

(a) Suppose A is $m \times n$, B is $n \times k$ and C is $k \times p$. How many multiplications are needed to calculate P in the order $(AB)C$? Do not just write down an expression; show your work/justification!

(b) For the same matrix dimensions specified in (a), how many multiplications are needed to calculate P in the order $A(BC)$? Again, do not just write down an expression.

(c) Based on the specific dimensions of A, B, and C in the problem description, which multiplication order would be the most efficient?

(d) Calculate $P = ABC$ using whichever order you specified in part (c).

(a) (AB)C

$$\text{(AB)} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+4 & 2+4 & 2+4 & 1+8 \\ 2+1 & 2+1 & 2+1 & 1+2 \\ 2+4 & 2+4 & 2+4 & 1+8 \\ 2+1 & 2+1 & 2+1 & 1+2 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 5 & 9 \\ 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 9 \\ 3 & 3 & 3 & 3 \end{bmatrix}$$

32 multiplications

$$(AB)C = \begin{bmatrix} 6 & 6 & 5 & 9 \\ 3 & 3 & 3 & 3 \\ 6 & 6 & 6 & 9 \\ 3 & 3 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (6+12+6+18) & (6+12+6+0) \\ (3+6+3+6) & (3+6+3+0) \\ (6+12+6+18) & (6+12+6+0) \\ (3+6+3+6) & (3+6+3+0) \end{bmatrix} = \begin{bmatrix} 42 & 24 \\ 18 & 12 \\ 42 & 24 \\ 18 & 12 \end{bmatrix}$$

ANSWER: Total of 64 multiplications

(b) A(BC)

$$\text{(BC)} = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (2+4+2+2) & (2+4+2+0) \\ (1+2+1+4) & (1+2+1+0) \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$$

16 multiplications

$$\text{A(BC)} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} (10+32) & (8+16) \\ (10+8) & (8+4) \\ (10+32) & (8+16) \\ (10+8) & (8+4) \end{bmatrix} = \begin{bmatrix} 42 & 24 \\ 18 & 12 \\ 42 & 24 \\ 18 & 12 \end{bmatrix}$$

ANSWER: Total of 32 multiplications

(c) **ANSWER:** (AB)C took 64 multiplications while A(BC) took 32 multiplications. A(BC) is more efficient because it took half the number of multiplications at (AB)C.

(d) P = ABC = A(BC)

$$(BC) = \begin{bmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} (2+4+2+2) & (2+4+2+0) \\ (1+2+1+4) & (1+2+1+0) \end{bmatrix} = \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix}$$

16 multiplications

$$A(BC) = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 10 & 8 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} (10+32) & (8+16) \\ (10+8) & (8+4) \\ (10+32) & (8+16) \\ (10+8) & (8+4) \end{bmatrix} = \begin{bmatrix} 42 & 24 \\ 18 & 12 \\ 42 & 24 \\ 18 & 12 \end{bmatrix}$$

ANSWER: Total of 32 multiplications

(6) Use induction to show that $\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. Be sure to state whether you're using weak or strong induction.

Using Weak Induction:

Base Case:
n = 0
$0 = \frac{(0^2)(0+1)^2}{4}$

Induction: Sum of the first n positive cubic integers is
$i^3 = \frac{n^2(n+1)^2}{4}$

Goal is to show that by adding the next positive cubic integer, the new total sum is now

$$\frac{(n+1)^2(n+2)^2}{4}$$

k $\geq 0 : (0)^3 + (1)^3 + (2)^3 + ... + (k)^3 = \frac{k^2(k+1)^2}{4}$
$0 + 1 + 8 + 27 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$
$= \frac{k^4+2k^3+k^2}{4} + (k^3 + 3k^2 + 3k + 1)$
$= \frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4}$
$= \frac{k^4+16k^3+13k^2+12k+4}{4}$
$= \frac{(k+1)^2(k+2)^2}{4}$

ANSWER: Through weak induction we have proven that we can show that adding the next positive cubic integer results in the new total sum $\frac{(k+1)^2(k+2)^2}{4}$.

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(7) Let $A_1, A_2, \ldots A_n$ be sets. Use induction to show that for $n \geq 2$, the cardinality of the union of n sets is always less than or equal to the sum of the cardinalities of those sets. In other words, show:

$$\left| \bigcup_{i=1}^{n} A_i \right| \leq \sum_{i=1}^{n} |A_i|$$

Be sure to state whether you're using weak or strong induction.
Hint: use the same rule that HW 4 #6 was based around.
