CSCI 3022 Intro to Data Science
Normals

The four big functions (`scipy.stats` as `stats`):

1. `stats.distribution.rvs(params, size=...)` generates random numbers from the named distribution.
2. `stats.distribution.pdf(x,params)` returns the pdf of the distribution at the x value input as the function's first argument. For a discrete random variable, this is $P(X = x)$.
3. `stats.distribution.cdf(x,params)` returns the cdf of the distribution at the x value input as the function's first argument. This is $P(X \leq x)$.
4. `stats.distribution.ppf(p,params)` returns the *inverse* of cdf of the probability p value input as the function's first argument. This is the value of x that satisfies $p = P(X \leq x)$.

`distribution` arguments we've seen include: `poisson`, `binomial`, `uniform`, `exponential`, and more to come!.

Announcements and Reminders

- Exam due Friday.
- Practicum posted: it's 2 longer homework problems; due Mar 19. Then we get a week with no HW!

The Normal Distribution

The normal distribution (sometimes called the Gaussian distribution) is probably the most important distribution in all of probability and statistics.

Many populations have distributions that can be fit very closely by an appropriate normal (or Gaussian, bell) curve.

Examples: height, weight, and other physical characteristics, scores on various tests, etc.

The Normal Distribution

Definition: *Normal Distribution:*

A continuous r.v. X is said to have a *normal distribution* with parameters _ and __ > 0, if the pdf of X is:

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

Notation: We write ____________

The Normal Distribution

Definition: *Normal Distribution:*

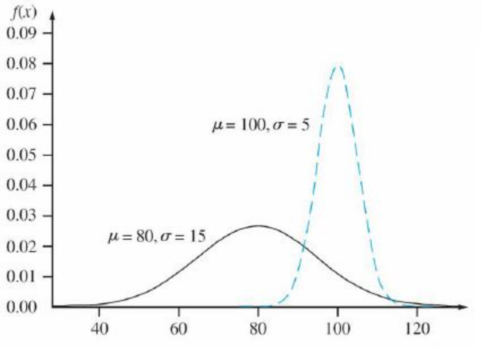
A continuous r.v. X is said to have a *normal distribution* with parameters $\underline{\mu}$ and $\underline{\sigma^2} > 0$, if the pdf of X is:

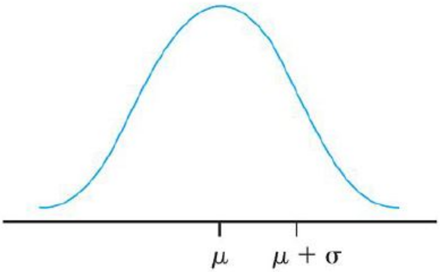
$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2}$$

Notation: We write $\underline{X \sim N(\mu,\sigma^2)}$

The Normal Distribution

The figure below presents graphs of f for different parameter pairs:





You can play with normals in any statistical software. See for example
`https://academo.org/demos/gaussian-distribution/`

The Standard Normal Distribution

Definition: *Standard Normal Distribution:*
The normal distribution with parameter values _____ and ______ is called the *standard normal distribution*.

A r.v. with this distribution is called a standard normal random variable and is denoted by Z. Its pdf is:

$$f(z) =$$

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$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

The normal cdf

Let's find the cdf of the standard normal distribution!
All we have to to is integrate:

$$\int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$$

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Should we try a substitution? IBP?... this may not go integreat for us.

The normal cdf

Let's find the cdf of the standard normal distribution!
All we have to to is integrate:

$$\int_{-\infty}^{Z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt$$

The CDF of the normal distribution has no closed form. But it's really important! So we give it it's own name.

The normal cdf

For a random variable $Z \sim N(0,1)$, the cdf of Z is given by

$$F(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt = \boxed{\Phi(z)}$$

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Old school statisticians used to carry around giant tables with values of $\Phi(z)$ in them. Actually, many current statisticians do that too, but that's a little silly. We have computers!

The Standard Normal

Note:

1. The standard normal distribution rarely occurs naturally.
2. Instead, it is a reference distribution from which information about other normal distributions can be obtained via a simple formula.
3. These probabilities can then be found "normal tables".
4. This can also be computed with a single command... (`scipy.stats.norm.cdf`, for example)

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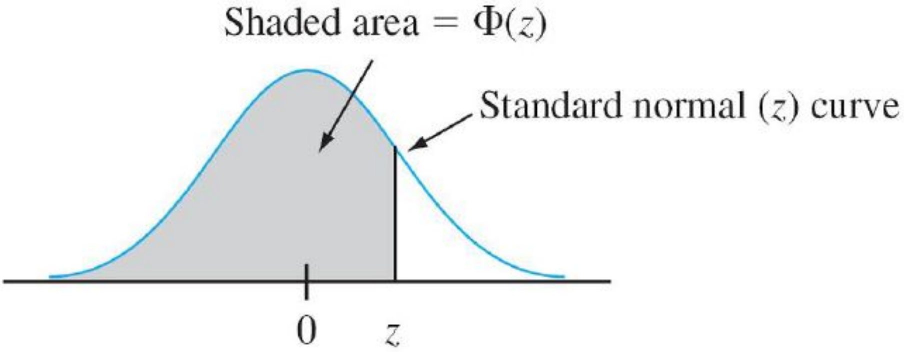
Recall: one example from HW1: if we take a data set, and *subtract the mean* from each of the data values, then we *divide by the standard deviation*, we ended up with a new data set that was mean of 0 and variance/standard deviation of 1. The new data set had the same **shape** as the original, but now it was "centered" at 0 and "scaled" to be of a known (average) spread.

The Standard Distribution

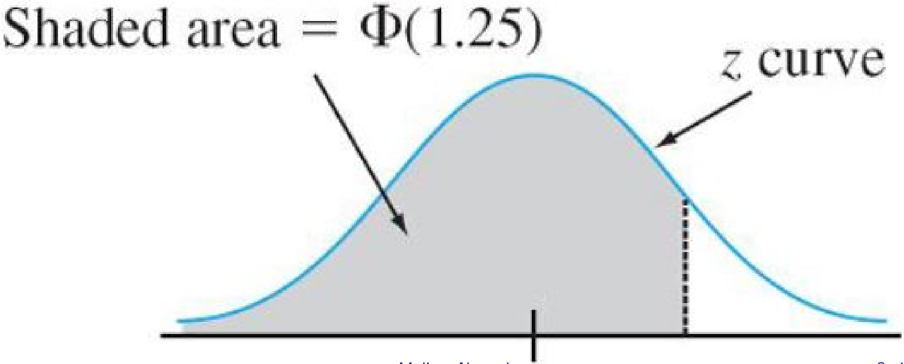
The figure below illustrates the probabilities found in a normal table (such a table can easily be found online):



The Standard Distribution

$P(Z \leq 1.25) = \Phi(1.25)$, a probability that is tabulated in a normal table. What is this probability?

The figure below illustrates this probability:



The Standard Distribution

Some quick examples:

1. $P(Z \geq 1.25)$

2. Why does $P(Z < -1.25) = P(Z > 1.25)$? What is $\Phi(-1.25)$?

3. How do we calculate $P(-.38 \leq Z \leq 1.25)$?

The Standard Distribution

Some quick examples:

1. $P(Z \geq 1.25)$
   It's `1-scipy.stats.norm.cdf(1.25)`. Or as a picture:

2. Why does $P(Z < -1.25) = P(Z > 1.25)$? What is $\Phi(-1.25)$?
   Symmetry! Same as above.

3. How do we calculate $P(-.38 \leq Z \leq 1.25)$?
   As an integral, this is $\int_{-.38}^{1.25} f(z)\,dz$. We could split this into 2:
   $\int_{-\infty}^{1.25} f(z)\,dz + \int_{-.38}^{-\infty} f(z)\,dz =$

$$\Phi(1.25) - \Phi(-.38)$$

Standard Quantiles

The 99th *percentile* of the standard normal distribution is that value of z such that the area under the z curve to the left of the value is 0.99.

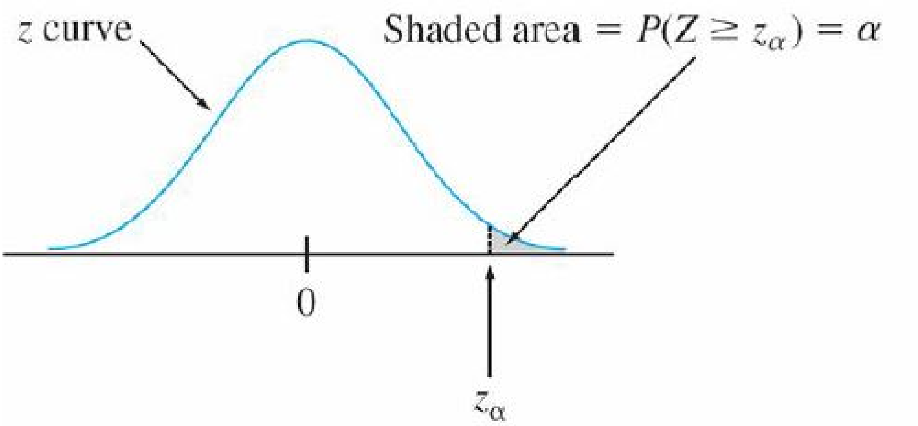
Tables and cdf functions give, for fixed z, the area under the standard normal curve to the left of z; now we have the area and want the value of z.

This is the "inverse" problem to $P(Z \leq z) = ?$

How can the table be used for this?

Standard Quantiles

In statistical inference, we need the z values that give certain tail areas under the standard normal curve. There, this notation will be standard: __ will denote the z value for which _ of the area under the z curve lies to the right of __.



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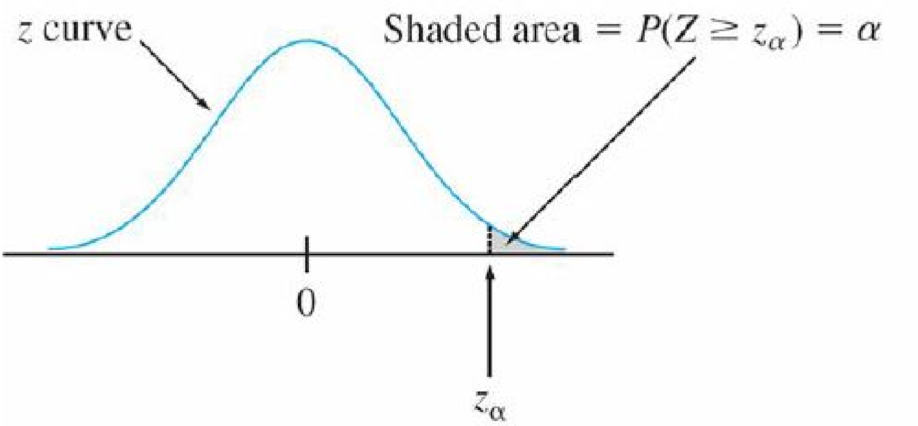

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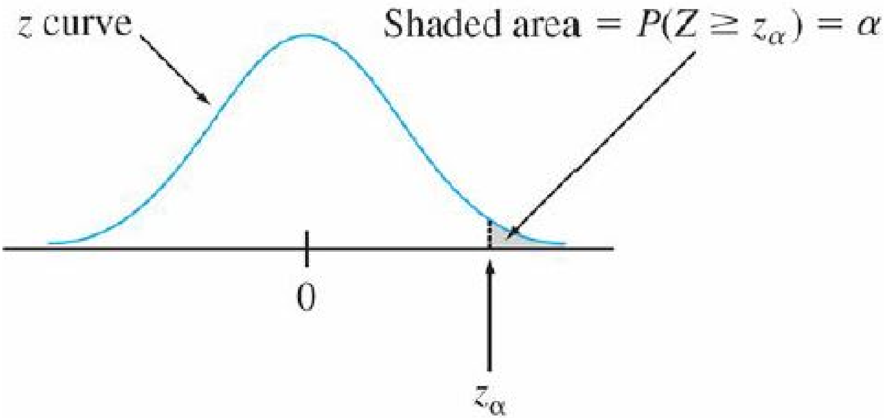

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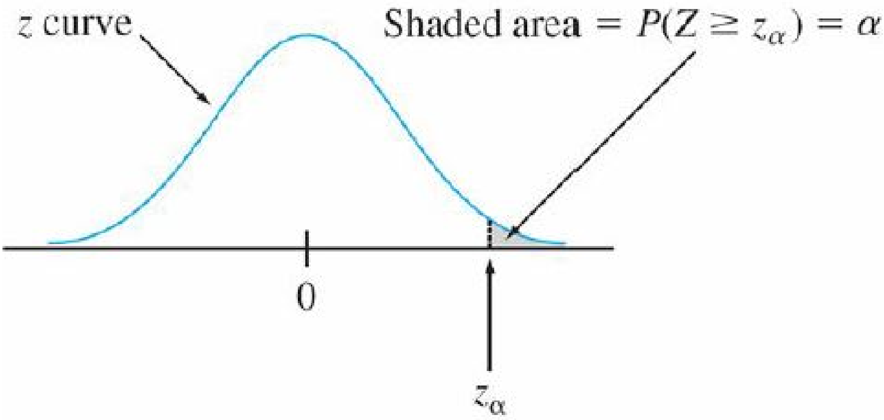

Non-Standard Normals

When $X \sim N(\mu, \sigma^2)$, probabilities involving X are computed by "standardizing." The standardized variable is:

Proposition: If X has a normal distribution with mean _ and standard deviation _, then

is distributed standard normal.

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When $\underline{X \sim N(\mu, \sigma^2)}$, probabilities involving X are computed by "standardizing." The standardized variable is:

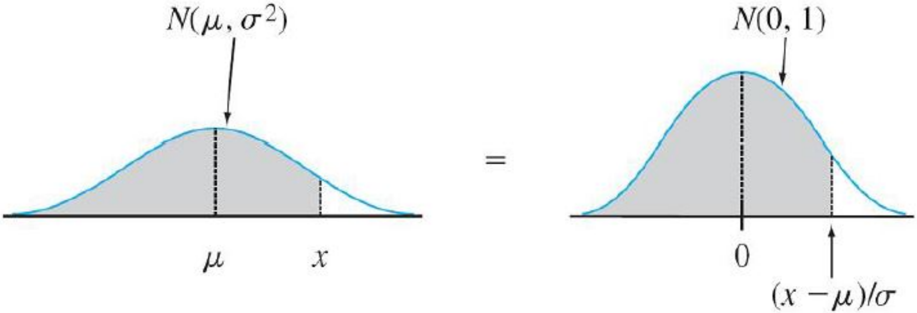
$$Z = \frac{X - \mu}{\sigma}$$

Proposition: If X has a normal distribution with mean $\underline{\mu}$ and standard deviation $\underline{\sigma}$, then

is distributed standard normal.

Non-Standard Normals

Why do we standardize normal random variables?



Equality of nonstandard and standard normal curve areas

Using Normals

Example:
The time that it takes a driver to react to the brake lights on a decelerating vehicle is critical in helping to avoid rear-end collisions.

Research suggests that reaction time for an in-traffic response to a brake signal from standard brake lights can be modeled with a normal distribution having mean value 1.25 sec and standard deviation of 0.46 sec.

What is the probability that reaction time is between 1.00 sec and 1.75 sec?

Solution:

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$$X \sim N(1.25, .46)$$

What is the probability that reaction time is between 1.00 sec and 1.75 sec?
We want $P(1 < X < 1.75)$... but we can't compute these probabilities unless the r.v. in the middle of the inequality is *standard* normal. So we normalize!

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$$P(1 < X < 1.75) = P(1 - 1.25 < X - 1.25 < 1.75 - 1.25)$$

$$= P(\frac{-.25}{.46} < \frac{X - 1.25}{.46} < \frac{.5}{.46}) = P(\frac{-.25}{.46} < Z < \frac{.5}{.46})$$

$$= \Phi(\frac{-.25}{.46}) - \Phi(\frac{.5}{.46})$$

iid

Definition: *Random Sample:*

The r.v.'s $X_1, X_2, \ldots, X_n$ are said to form a (simple) random sample of size n if:

1.

2.

We say that these X_i's are:

iid

Definition: *Random Sample:*
The r.v.'s $X_1, X_2, \ldots, X_n$ are said to form a (simple) random sample of size n if:

1. $X_1, X_2, \ldots X_n$ are independent.

2.No value in the population has a higher chance of being included than any other.

We say that these X_i's are: *independent* and *identically distributed.*
and we write:

$$X_1, X_2, \ldots X_n \overset{iid}{\sim} f(x;\theta)$$

Estimators and Their Distributions

We use estimators to summarize our i.i.d. sample.

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1. *Sample* Mean might estimate a population mean.
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3. *Sample* Quantiles
4. $\hat{p}$ for p
5. etc., etc.

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4. $\hat{p}$ for p
5. etc., etc.

Why use one estimator over another?

Estimators and Their Distributions

We use estimators to summarize our i.i.d. sample. Any estimator, including the sample mean ___ is a random variable (since it is based on a random sample).

This means that ___ has a distribution of it's own, which is referred to as sampling distribution of the sample mean. This sampling distribution depends on:

Definition: The standard deviation of this distribution is called the *standard error* of the estimator.

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2. population distribution
3. method of sampling

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Distribution of the Sample Mean

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with known mean value and standard deviation . Then:

$$E[\bar{X}] =$$

$$Var[\bar{X}] =$$

The standard deviation of the sample mean is:

This is also called the standard error of the mean.

Distribution of the Sample Mean

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with known mean value and standard deviation . Then:

$$E[\bar{X}] = \mu$$

$$Var[\bar{X}] = \frac{\sigma^2}{n}$$

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Distribution of the Sample Mean

What does this mean? Why is it true?

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Also, what do we know about the *distribution* of the sample mean?

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$$Var[\bar{X}] = Var[\sum X_i/n] = \frac{1}{n^2}\sum Var[X_i] = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Also, what do we know about the *distribution* of the sample mean?

Distribution of the Sample Mean (Normal Population)

Proposition:
If $X_1, X_2, \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then

We know everything there is to know about the distribution of the sample mean when the population distribution is normal.
This happens to be a result of that "a sum of normal random variables is still normal."

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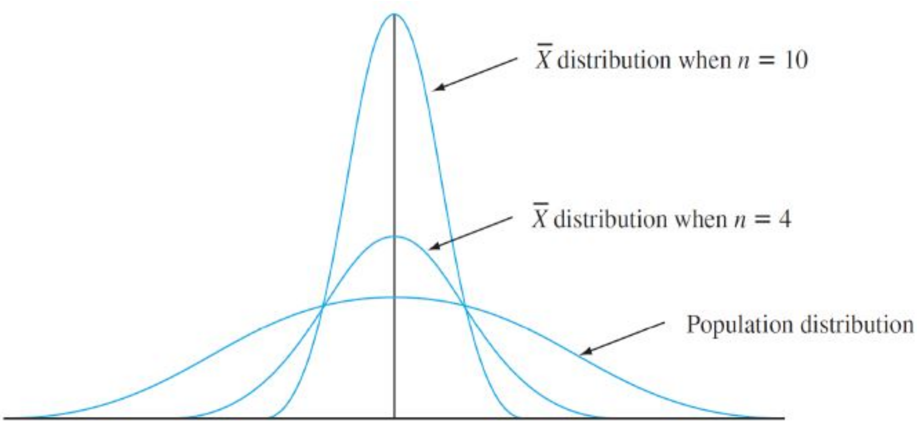

Central Limit Theorem

But what if the underlying distribution of the X_i's is not normal?

Central Limit Theorem

Important: When the population distribution is nonnormal, averaging produces a distribution more bellshaped than the one being sampled.

A reasonable conjecture is that if n is large, a suitable normal curve will approximate the actual distribution of the sample mean.

The formal statement of this result is one of the most important theorems in probability: *Central Limit Theorem!*

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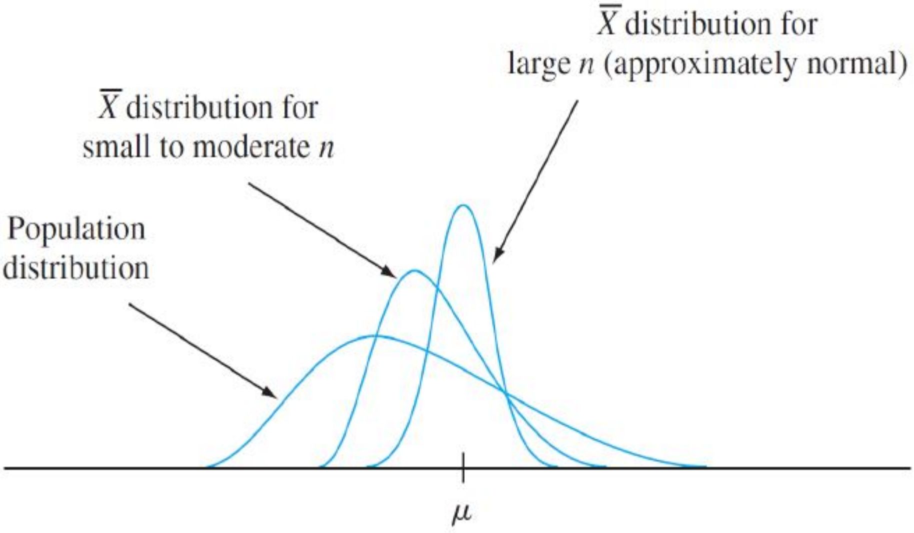
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

The larger the value of n, the better the approximation! Typical rule of thumb: $n > 30$.

Central Limit Theorem



Central Limit Theorem

The CLT provides insight into why many random variables have probability distributions that are approximately normal.

For example, the measurement error in a scientific experiment can be thought of as the sum of a number of underlying perturbations and errors of small magnitude.

A practical difficulty in applying the CLT is in knowing when n is sufficiently large. The problem is that the accuracy of the approximation for a particular n depends on the shape of the original underlying distribution being sampled.

Daily Recap

Today we learned

1. The Normal Distribution... and why we care!

Moving forward:

- nb day Friday!

Next time in lecture:

- Using Normals to estimate *population* means based on *sample* means