Chapter 17: Regression

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Correlation vs regression

- Correlation measures the aspects of the linear relationship between two numerical variables
- **Regression** is a method that predicts values of one numerical variable from values of another numerical variable
- Fits a line through the data
  - Used for prediction
  - Measures how steeply one variable changes with the other

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Linear regression

- The most common type of regression
  - Although there are non-linear models (e.g., quadratic, logistic)

- Draws a straight line through the data to predict the response variable (Y, vertical axis) from the explanatory variable (X, horizontal axis)

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Ex 17.1: The lion's nose

- Trophy hunting of African lions typically aims to remove older males

- Is there a way to predict the age?

- Research has shown that the amount of black pigmentation on the nose of male lions increase with age

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Ex 17.1: The lion's nose

- Study used proportion black on nose (explanatory) to predict age (response)

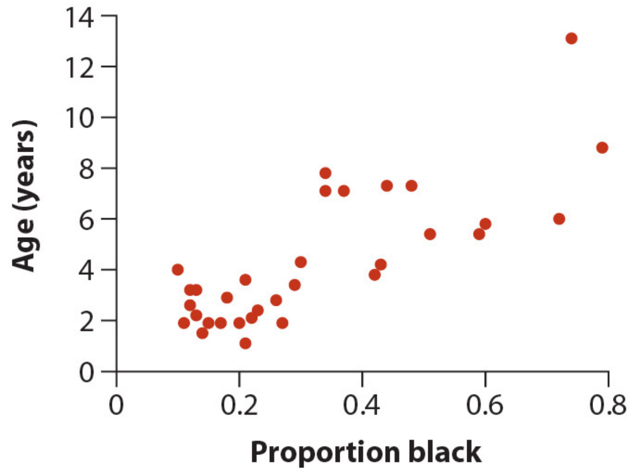

Fig 17.1-1

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Fitting the "best" line

- You want a line that gives the most accurate predictions of Y from X
- Least-squares regression: line for which the sum of all the squared deviations in Y is smallest

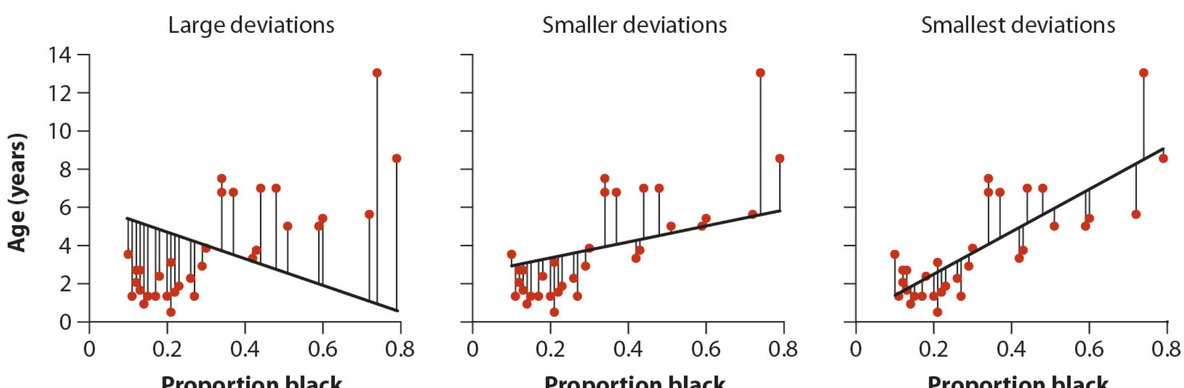

Fig 17.1-2

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Formula for the line

$$Y = a + bX$$

- a is the Y-**intercept**; b is the **slope**
- The **slope** of a linear regression is the rate of change in Y per unit X
- Also measures direction of prediction
  - Positive: as X increases Y increases
  - Negative: as X increases Y decreases

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Slopes and intercepts

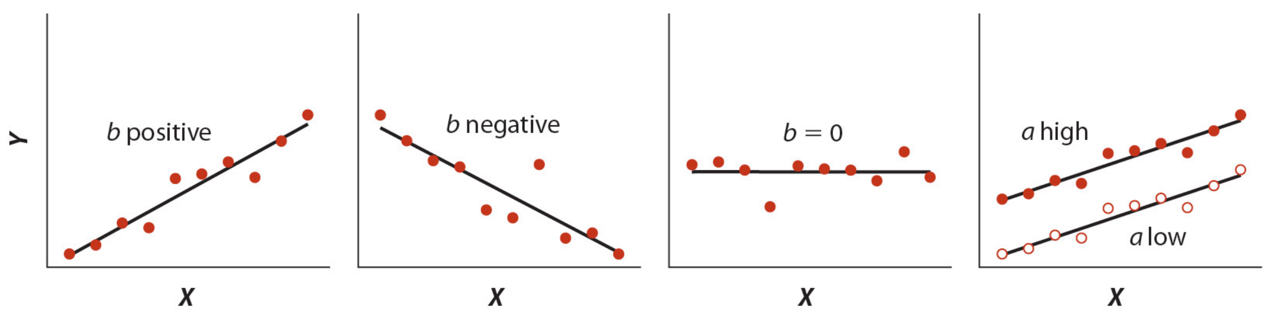

Fig 17.1-3

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Calculating slope

Measures how deviations in X and Y vary together

$$b = \frac{\sum_i \left(X_i - \overline{X}\right)\left(Y_i - \overline{Y}\right)}{\sum_i \left(X_i - \overline{X}\right)^2}$$

Sum of squares for X

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Calculating intercept

- Once slope is calculated, getting intercept is straightforward because the least-squares regression **always** goes through point $(\overline{X}, \overline{Y})$

Plug mean values into line formula: $\overline{Y} = a + b\overline{X}$

Rearrange to solve for intercept: $a = \overline{Y} - b\overline{X}$

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Ex 17.1: The lion's nose

$$b = \frac{\sum_i (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_i (X_i - \overline{X})^2} = \frac{13.0123}{1.2221} = 10.647$$

$$a = \overline{Y} - b\overline{X} = 4.3094 - 10.647(0.3222) = 0.879$$

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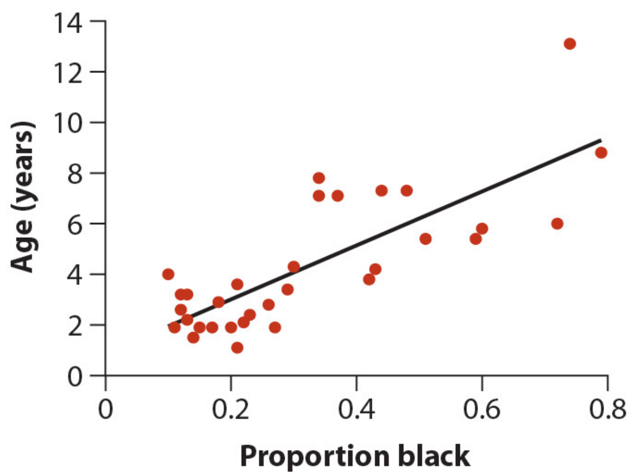
Ex 17.1: The lion's nose

$$Y = 0.88 + 10.65X$$



- On average, lion age increases by 10.65 years per unit of change in proportion of nose that is black
- Or, 1.065 years for every 0.1 increase of proportion of black on nose

Fig 17.1-4

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Samples vs populations

- The slope (b) and intercept (a) are estimated from a sample of measurements, hence these are estimates/statistics
- The true population slope (β) and intercept (α) are parameters
- Regression assumes that there is a population for every value of X, and the mean Y for each of these populations lies on the regression line

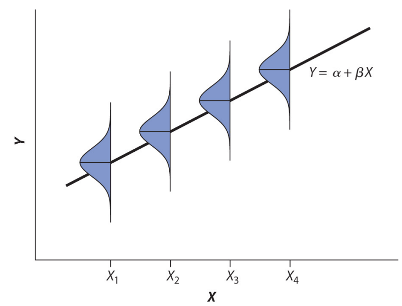

Fig 17.5-1

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Predicting values

- Now that you have the regression line you can predict values of Y for any specified value of X
- Predictions are mean Y for all individuals with value X
- Designated $\hat{Y}$, or "Y-hat"
- How old are lions with a proportion of black of 0.5?

$$\hat{Y} = 0.88 + 10.65(0.5) = 6.2$$

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How well do data fit line?

- The **residual** of a point is the difference between its measured *Y* value and the value of *Y* predicted by the regression line

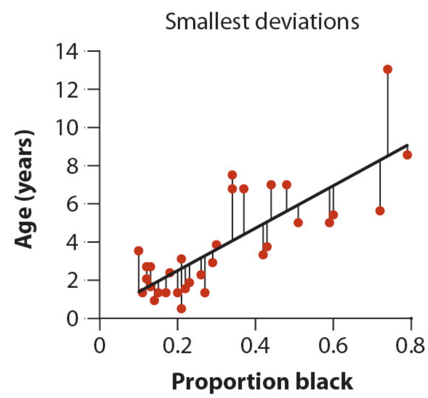

Fig 17.1-2

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How well do data fit line?

- **Residuals** measure the scatter of points above and below the least-squares regression line
- Can be positive or negative
- Variance in residuals ($MS_{residual}$) quantifies the spread of the scatter
  - **Residual mean square**
  - Analogous to error mean square in ANOVA
- Used to quantify the uncertainty of the slope

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Residual mean square

$$MS_{residual} = \frac{\sum_i (Y_i - \overline{Y})^2 - b \sum_i (X_i - \overline{X})(Y_i - \overline{Y})}{n - 2}$$

$$= \frac{222.0872 - 10.647(13.0123)}{32 - 2} = 2.785$$

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Standard error of slope

- Uncertainty (precision) with the sample estimate (b) of the population slope (β)

$$SE_b = \sqrt{\frac{MS_{residual}}{\sum_i \left(X_i - \overline{X}\right)^2}} = \sqrt{\frac{2.785}{1.2221}} = 1.510$$

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Confidence interval of the slope

$$b - t_{\alpha(2),df} SE_b > \beta > b + t_{\alpha(2),df} SE_b$$

$$10.647 - 2.042(1.510) > \beta > 10.647 + 2.042(1.510)$$

$$7.56 > \beta > 13.73$$

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Two types of predictions

- Predict **mean** Y for a given X
  - e.g., what is the mean age of all male lions whose noses are 60% black?
- Predict **single** Y for a given X
  - e.g., how old is that lion over there with a 60% black nose?
- Both predictions give the same value of $\hat{Y}$, but they **differ in precision**
  - Can predict mean with more certainty than a single value

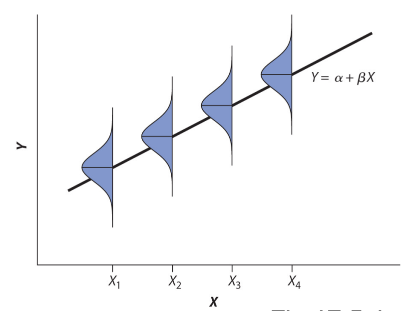

Fig 17.5-1

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Two types of predictions

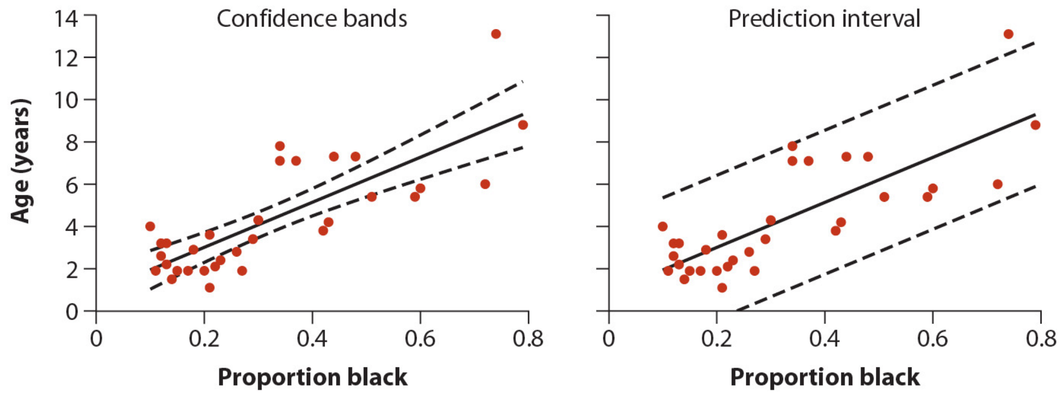

Fig 17.2-1

- **Confidence bands** measure the precision of the predicted mean Y for each value of X. **Prediction intervals** measure the precision of the predicted single Y-values for each X

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Extrapolation

- Regressions should be used to predict Y for any value of X lying between the smallest and largest values of X (**interpolation**)

- **Extrapolation** is the prediction of the value of a response variable outside the range of X-values in the data

- No way to ensure the relationship continues to be linear beyond the range of the data!

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Testing hypotheses about a slope

- H_0: age cannot be predicted by proportion of black on nose ($\beta = 0$)
- H_A: age can be predicted by proportion of black on nose ($\beta \neq 0$)

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t-test of regression slope

- t-statistic $t = \frac{b - \beta_0}{SE_b}$ $df = n - 2$

$$t = \frac{10.65 - 0}{1.510} = 7.053 \qquad df = 32 - 2 = 30$$

- P-value for $t_{30} = 7.053$?
- P = 7.68 x 10^{-8} (reject the null hypothesis that $\beta = 0$)

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ANOVA (F) approach

- Recall two source of variation in ANOVA
  - Among groups (MS_{groups})
  - Within groups (MS_{error})
- In regression framework:
- Deviations between the predicted values $\hat{Y}_i$ and $\overline{Y}$
  - Analogous to MS_{groups}
- Deviations between each Y_i and its predictive value $\hat{Y}_i$
  - Analogous to MS_{error}

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ANOVA (F) approach

- Using ANOVA approach will generate the same P-value as the t-test approach

- Can be used to measure R^2: the fraction of the variation in Y that is "explained" by X

$$R^2 = \frac{SS_{regression}}{SS_{total}}$$

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summary(regression) in R

```
summary(lionRegression)
##
## Call:
## lm(formula = ageInYears ~ proportionBlack, data = lion)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.5449 -1.1117 -0.5285  0.9635  4.3421
##
## Coefficients:
##                 Estimate Std. Error t value Pr(>|t|)
## (Intercept)       0.8790     0.5688   1.545    0.133
## proportionBlack  10.6471     1.5095   7.053 7.68e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.669 on 30 degrees of freedom
## Multiple R-squared:  0.6238, Adjusted R-squared:  0.6113
## F-statistic: 49.75 on 1 and 30 DF,  p-value: 7.677e-08
```

Y-intercept

slope

R^2

P-value
H_0: $\alpha = 0$

P-value
H_0: $\beta = 0$

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Regression toward the mean

- **Regression toward the mean** results when two variables measured on a sample of individuals have a correlation less than one. Individuals that are far from the mean for one of the measurements will, on average, lie closer to the mean for the other measurement

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Regression toward the mean

- Cholesterol measurements before and after drug
- Solid line: linear regression
- Dashed line: one-to-one line with slope of 1

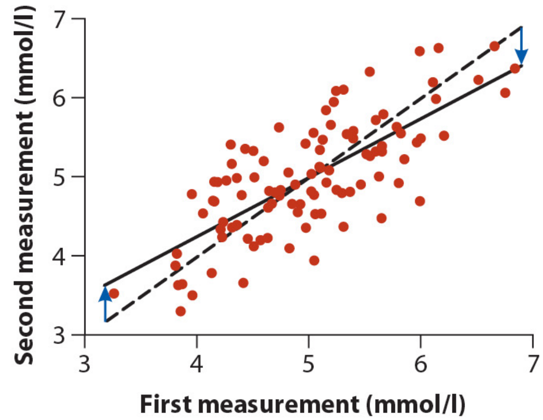

Fig 17.4-1

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Assumptions of linear regression

- At each value of X:
  - there is a population of Y-values whose mean lies on the regression line
  - the distribution of possible Y-values is normal (with same variance)
  - The variance of Y-values is the same at all values of X
  - the Y-measurements represent a random sample from the possible Y-values

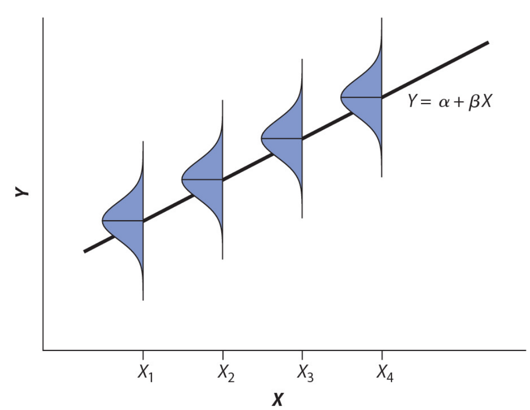

Fig 17.5-1

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Detecting issues

- **Outliers**
- If only one (or a low number) then it may be reasonable to report regression with and without outlier

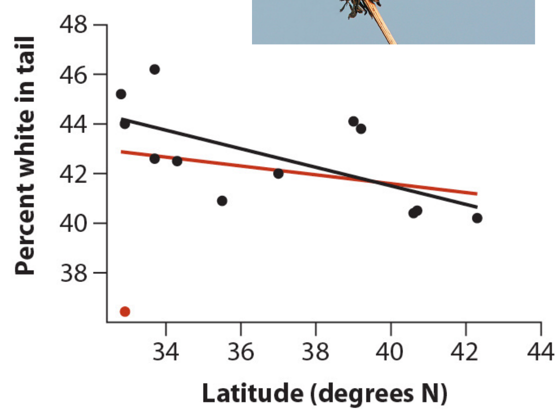

Fig 17.5-2

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Detecting issues

- **Nonlinearity** can be detected by inspecting graphs

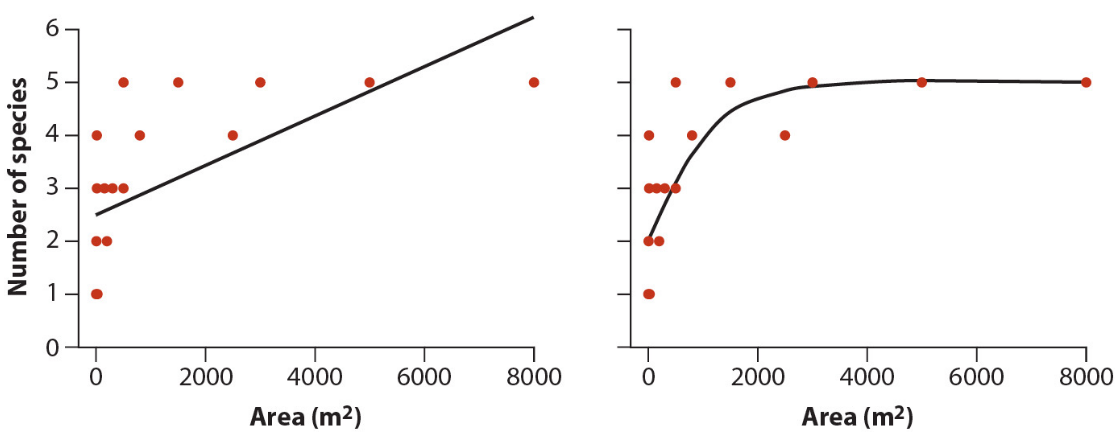

Fig 17.5-3

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Detecting issues

- **Non-normality and unequal variances** can be inspected with a residual plot
- Residual plot: residual of every data point ($Y_i - \hat{Y}_i$) is plotted against X_i
- If assumptions of normality and equal variances are met then there should be a roughly symmetric cloud above/below horizontal line at 0

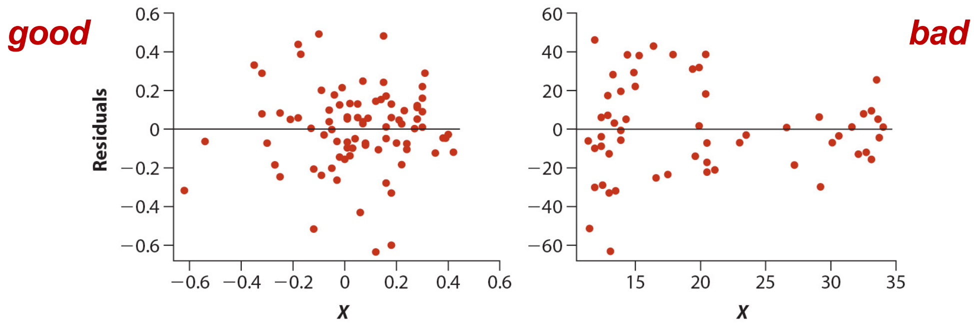

Fig 17.5-4

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Notes

- Skipping sections 17.6 through 17.9

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