1 Problem Formulation : The reflected case

<u>**Approximation of Backward RSDE's**</u>

$$\begin{aligned} Y_t &= g(X_1) - \int_t^1 Z_s \cdot dW_s + A_1 - A_t \\ Y_t &\geq g(X_t) \end{aligned}$$

where A is a non-decreasing continuous process satisfying

$$\int_0^1 (Y_t - g(X_t))\, dA_t = 0 \ ,$$

$$\text{and} \quad X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \quad X_0 \in \mathbb{R}^d$$

By the scheme : $\pi = \{t_i = i/n,\ 0 \leq i \leq n\}$

$$X^\pi_{t_{i+1}} = X^\pi_{t_i} + \frac{1}{n} b(X^\pi_{t_i}) + \sigma(X^\pi_{t_i})(W_{t_{i+1}} - W_{t_i}) \ , \quad X^\pi_0 = X_0 \ .$$

and

$$\begin{aligned} Y^\pi_1 &= g(X^\pi_1) \\ Y^\pi_{t_i} &= \max\left\{ g(X^\pi_{t_i}) \ , \ E\left[Y^\pi_{t_{i+1}} \mid X^\pi_{t_i} \right] \right\} \end{aligned}$$

Question : How to compute $E\left[Y^\pi_{t_{i+1}} \mid X^\pi_{t_i} \right]$?

∘ J. F. Carrière (1996), F. A. Longstaff & R. S. Schwartz (2001) and V. Bally and G. Pagès (2001)

⇒ **Convergence rate heavily depends on** d**.**

2 Monte-Carlo Estimation of conditional expectation :

2.1 The Gaussian Case $X = W$

1- Set $v(W_{t_i}) = Y^{\pi}_{t_i}$.
2- Reduction of the problem :

$$E\left[v(W_{t_{i+1}}) \mid W_{t_i} = w\right] = \frac{E\left[\delta_w(W_{t_i})v(W_{t_{i+1}})\right]}{E\left[\delta_w(W_{t_i})\right]}$$

δ_w Dirac mass at w.

3- Integration by parts argument :

$$\begin{gathered} E[\delta_w(W_{t_i})v(W_{t_{i+1}})] \\ = \\ \int\int \delta_w(x)\, v(x+y)\, f_{t_i}(x) f_{(t_{i+1}-t_i)}(y) dx dy \\ = \\ \int\int \mathbf{1}_{\{x\geq w\}}\left[v(x+y)\frac{x}{t_i} - v'(x+y)\right] f_{t_i}(x) f_{(t_{i+1}-t_i)}(y) dx dy \\ = \\ \int\int \mathbf{1}_{\{x\geq w\}} v(x+y)\left[\frac{x}{t_i} - \frac{y}{t_{i+1}-t_i}\right] f_{t_i}(x) f_{(t_{i+1}-t_i)}(y) dx dy\ , \\ = \\ E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}} v\left(W_{t_{i+1}}\right)\left(\frac{W_{t_i}}{t_i} - \frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)\right] \end{gathered}$$

⇒ **Alternative formulation :**

$$E\left[v(W_{t_{i+1}}) \mid W_{t_i} = w\right] \;=\; \frac{E\left[\mathbf{1}_{\{W_{t_i}\geq w\}} v\left(W_{t_{i+1}}\right)\left(\frac{W_{t_i}}{t_i} - \frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)\right]}{\underbrace{E\left[\mathbf{1}_{\{W_{t_i}\geq w\}}\dfrac{W_{t_i}}{t_i}\right]}_{E\left[\delta_w(W_{t_i})\right]=f_{W_{t_i}}(w)}}$$

⇒ **Monte-Carlo estimator :** $\{W^{(\ell)}\}_\ell$, N copies of W

$$\hat{E}\left[v(W_{t_{i+1}}) \mid W_{t_i} = w\right] \;:=\; \frac{\frac{1}{N}\sum_\ell \mathbf{1}_{\left\{W^{(\ell)}_{t_i}\geq w\right\}} v\left(W^{(\ell)}_{t_{i+1}}\right)\left(\frac{W^{(\ell)}_{t_i}}{t_i} - \frac{W^{(\ell)}_{t_{i+1}}-W^{(\ell)}_{t_i}}{t_{i+1}-t_i}\right)}{\frac{1}{N}\sum_\ell \mathbf{1}_{\left\{W^{(\ell)}_{t_i}\geq w\right\}}\frac{W^{(\ell)}_{t_i}}{t_i}}$$

Variance estimation

$$\mathrm{Var}\left[\frac{W^{(\ell)}_{t_{i+1}} - W^{(\ell)}_{t_i}}{t_{i+1}-t_i}\right]^{\frac{1}{2}} = \frac{(t_{i+1}-t_i)^{\frac{1}{2}}}{t_{i+1}-t_i} \;=\; n^{\frac{1}{2}}$$

⇒

$$\mathrm{Var}\left[\frac{1}{N}\sum_\ell \mathbf{1}_{\left\{W^{(\ell)}_{t_i}\geq w\right\}} v\left(W^{(\ell)}_{t_{i+1}}\right)\left(\frac{W^{(\ell)}_{t_i}}{t_i} - \frac{W^{(\ell)}_{t_{i+1}} - W^{(\ell)}_{t_i}}{t_{i+1}-t_i}\right)\right]^{\frac{1}{2}} \sim \frac{n^{\frac{1}{2}}}{N^{\frac{1}{2}}}\,.$$

1- Convergence rate in $N^{\frac{1}{2}}$?

2- Need to control the variance as $n \to \infty$.

2.2 Variance Reduction in the Gaussian Case :

<u>**Variance Reduction 1 : Control Variate**</u>

$$E\left[v\left(W_{t_{i+1}}\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)\right]$$
$$=$$
$$\int\int v\left(x+y\right)\left[\frac{x}{t_i}-\frac{y}{t_{i+1}-t_i}\right]f_{t_i}(x)f_{(t_{i+1}-t_i)}(y)dxdy$$
$$=$$
$$\int\int v'\left(x+y\right)f_{t_i}(x)f_{(t_{i+1}-t_i)}(y)dxdy$$
$$-$$
$$\int\int v'\left(x+y\right)f_{t_i}(x)f_{(t_{i+1}-t_i)}(y)dxdy$$
$$=$$
$$0\ .$$

$\Longrightarrow$ **Replace** $\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}$ by $(\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}-c(w))$

$$E\left[v(W_{t_{i+1}})\mid W_{t_i}=w\right]$$
$$=$$
$$\frac{E\left[\left(\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}-c(w)\right)v\left(W_{t_{i+1}}\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)\right]}{E\left[\left(\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}-\tilde{c}(w)\right)\frac{W_{t_i}}{t_i}\right]}$$

Variance Reduction 2 : Localization

Take φ smooth in L^2 with $\varphi(0) = 1$

$$E[\delta_w(W_{t_i})v(W_{t_{i+1}})]$$
$$=$$
$$\int\int \delta_w(x)\,\varphi(x-w)v(x+y)\,f_{t_i}(x)f_{(t_{i+1}-t_i)}(y)dxdy$$
$$=$$
$$\dots$$
$$=$$
$$E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)-\varphi'\left(W_{t_i}-w\right)\right\}\right]$$

$\Longrightarrow$

$$E\left[v(W_{t_{i+1}}) \mid W_{t_i}=w\right]$$
$$=$$
$$\frac{E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)-\varphi'\left(W_{t_i}-w\right)\right\}\right]}{E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}\left\{\tilde{\varphi}\left(W_{t_i}-w\right)\frac{W_{t_i}}{t_i}-\tilde{\varphi}'\left(W_{t_i}-w\right)\right\}\right]}$$

Variance Reduction 2 (bis) : Optimal Localization

$$E\left[v(W_{t_{i+1}})\mid W_{t_i}=w\right]$$
$$=$$
$$\frac{E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)-\varphi'\left(W_{t_i}-w\right)\right\}\right]}{E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}\left\{\tilde{\varphi}\left(W_{t_i}-w\right)\frac{W_{t_i}}{t_i}-\tilde{\varphi}'\left(W_{t_i}-w\right)\right\}\right]}$$

Problem Formulation : Optimizing the integrated variance

$$\min_{\varphi\in L^2,\,\varphi(0)=1}\int_{\mathbf{R}}E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}\left[F\varphi\left(W_{t_i}-w\right)-G\varphi'\left(W_{t_i}-w\right)\right]^2\right]dw$$

with

$$\begin{aligned}F &= v\left(W_{t_{i+1}}\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)\\ G &= v\left(W_{t_{i+1}}\right)\end{aligned}$$

Calculus of Variation : φ is optimal iif for all smooth ϕ with $\phi(0) = 0$ and compact support, and $\varepsilon > 0$

$$\int_{\mathbf{R}} E\left[\mathbf{1}_{\{W_{t_i} \geq w\}}\left[F\varphi\left(W_{t_i}-w\right)-G\varphi'\left(W_{t_i}-w\right)\right]^2\right]$$
$$\leq \int_{\mathbf{R}} E\left[\mathbf{1}_{\{W_{t_i} \geq w\}}\left[F(\varphi \pm \varepsilon\phi)\left(W_{t_i}-w\right)-G(\varphi' \pm \varepsilon\phi')\left(W_{t_i}-w\right)\right]^2\right]$$

Sending $\varepsilon \to 0$

$$\begin{aligned} 0 &= \int_{\mathbf{R}} E\Big[\mathbf{1}_{\{W_{t_i} \geq w\}}\left(F\varphi\left(W_{t_i}-w\right)-G\varphi'\left(W_{t_i}-w\right)\right) \\ &\qquad\qquad \left(F\phi\left(W_{t_i}-w\right)-G\phi'\left(W_{t_i}-w\right)\right)\Big]\, dw \\ &= E\left[\int_0^\infty \left(F\varphi(y)-G\varphi'(y)\right)\left(F\phi(y)-G\phi'(y)\right) dy\right] \\ &\quad \text{Fubini + change of variable } y = W_{t_i}(\omega) - w \\ &= E\left[\int_0^\infty \phi(y)\left(F^2\varphi(y)-G^2\varphi''(y)\right) dy\right] \\ &\quad \text{integration by parts} \\ &= \int_0^\infty \phi(y)\left(E\left[F^2\right]\varphi(y)-E\left[G^2\right]\varphi''(y)\right) dy \end{aligned}$$

$\Rightarrow E\left[F^2\right]\varphi(y) - E\left[G^2\right]\varphi''(y) = 0.$

Optimal Localizing Function : $\varphi(y) = e^{-\hat{\eta}y}$ with

$$\hat{\eta}^2 = E\left[F^2\right]/E\left[G^2\right]$$

Optimal Localizing Function : $\varphi(y) = e^{-\hat{\eta}y}$ with

$$\begin{aligned}
\hat{\eta}^2 &= E\left[F^2\right]/E\left[G^2\right] \\
&= \frac{E\left[v\left(W_{t_{i+1}}\right)^2\left(\frac{W_{t_i}}{t_i} - \frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)^2\right]}{E\left[v\left(W_{t_{i+1}}\right)^2\right]} \\
&\sim n
\end{aligned}$$

For $\varphi(y) = e^{-\eta y}$ **we have** $\varphi'(y) = -\eta\varphi(y)$

$$E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}} v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i} - \frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right) - \varphi'\left(W_{t_i}-w\right)\right\}\right]$$
$$=$$
$$E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}} v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i} - \frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i} + \eta\right)\right\}\right]$$

where

$$E\left[\left(\frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1}-t_i}\right)^2\right]^{\frac{1}{2}} = \sqrt{n} \sim \hat{\eta}$$

3 General case by Malliavin Calculus

3.1 Rewriting the Gaussian Case

1- Malliavin Derivative of W_s : $D_t W_s = \mathbf{1}_{t\leq s}$

2- Observe that for

$$h_t = \frac{1}{t_i}\mathbf{1}_{t\leq t_i} - \frac{1}{t_{i+1}-t_i}\mathbf{1}_{t_i<t\leq t_{i+1}}$$

we have

$$\int_0^{t_i} h_t D_t W_{t_i} = \int_0^{t_i}\left(\frac{1}{t_i}\mathbf{1}_{t\leq t_i} - \frac{1}{t_{i+1}-t_i}\mathbf{1}_{t_i<t\leq t_{i+1}}\right)\mathbf{1}_{t\leq t_i} = \int_0^{t_i}\frac{1}{t_i} = 1$$

$$\int_0^1 h_t D_t W_{t_{i+1}} = \int_0^1\left(\frac{1}{t_i}\mathbf{1}_{t\leq t_i} - \frac{1}{t_{i+1}-t_i}\mathbf{1}_{t_i<t\leq t_{i+1}}\right)\mathbf{1}_{t\leq t_{i+1}} = 0$$

3- Next compute the Skorohod integral

$$\begin{aligned}\mathcal{S}^h\left(\varphi\left(W_{t_i}-w\right)\right) &:= \int_0^1 \varphi\left(W_{t_i}-w\right)h_t\delta W_t\\ &= \varphi\left(W_{t_i}-w\right)\int_0^1 h_t dW_t - \int_0^1\varphi'\left(W_{t_i}-w\right)D_tW_{t_i}h_tdt\\ &= \varphi\left(W_{t_i}-w\right)\left(\frac{W_{t_i}}{t_i}-\frac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)-\varphi'\left(W_{t_i}-w\right)\end{aligned}$$

4- Rewrite the numerator in terms of $\mathcal{S}^h$

$$E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}v\left(W_{t_{i+1}}\right)\left\{\varphi\left(W_{t_i}-w\right)\left(\tfrac{W_{t_i}}{t_i}-\tfrac{W_{t_{i+1}}-W_{t_i}}{t_{i+1}-t_i}\right)-\varphi'\left(W_{t_i}-w\right)\right\}\right]$$

$$=$$

$$E\left[\mathbf{1}_{\left\{W_{t_i}\geq w\right\}}v\left(W_{t_{i+1}}\right)\mathcal{S}^h\left(\varphi\left(W_{t_i}-w\right)\right)\right]$$

3.2 The alternative representation in the general case : $d = 1$

1- X a process such that X_{t_i} and $X_{t_{i+1}}$ are "smooth".

2- Choose $h \in \mathbf{H}(X)$, i.e. a process in $L^2(\mathcal{F}_1)$ such that :

(i) $\int_0^{t_i} D_t X_{t_i} h_t dt = 1$, $\int_0^1 D_t X_{t_{i+1}} h_t dt = 0$.

(ii) Set

$$\mathcal{S}^h(F) := \int_0^1 F h_t \delta W_t \quad \left(= \; F \int_0^1 h_t \delta W_t - \int_0^1 D_t F h_t dt \right)$$

3- Problem Reduction

$$E\left[v(X_{t_{i+1}}) \mid X_{t_i} = x\right] \;=\; E\left[\delta_x(X_{t_i}) v(X_{t_{i+1}})\right] / E\left[\delta_x(X_{t_i})\right]$$

4- Chain rule formula : "$D_t \mathbf{1}_{x \leq X_{t_i}} = \delta_x(X_{t_i}) D_t X_{t_i}$".

5- Integration by parts argument

$$\begin{aligned}
& E\left[\delta_x(X_{t_i}) v(X_{t_{i+1}})\right] = E\left[\delta_x(X_{t_i}) v(X_{t_{i+1}}) \int_0^{t_i} D_t X_{t_i} h_t dt\right] \\
& \textit{for } h \textit{ such that } \int_0^{t_i} D_t X_{t_i} h_t dt = 1 \\
= \; & E\left[\int_0^{t_i} v(X_{t_{i+1}}) \delta_x(X_{t_i}) D_t X_{t_i} h_t dt\right] = E\left[\int_0^{t_i} v(X_{t_{i+1}}) D_t \mathbf{1}_{x \leq X_{t_i}} h_t dt\right] \\
= \; & E\left[\mathbf{1}_{x \leq X_{t_i}} \int_0^{t_i} v(X_{t_{i+1}}) h_t \delta W_t\right] \quad \textit{integration by parts}
\end{aligned}$$

$$= \underbrace{E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \int_0^{t_i} h_t \delta W_t\right]}_{A} \underbrace{-E\left[\mathbf{1}_{x\leq X_{t_i}} \int_0^{t_i} D_t v(X_{t_{i+1}}) h_t dt\right]}_{B}$$

decomposition of the Skorohod integral

$$\begin{aligned}
&-E\left[\mathbf{1}_{x\leq X_{t_i}} \int_0^{t_i} D_t v(X_{t_{i+1}}) h_t dt\right] \\
&= -E\left[\mathbf{1}_{x\leq X_{t_i}} \int_0^{t_i} v'(X_{t_{i+1}}) D_t X_{t_{i+1}} h_t dt\right] \quad \textit{chain rule} \\
&= E\left[\mathbf{1}_{x\leq X_{t_i}} \int_{t_i}^{1} v'(X_{t_{i+1}}) D_t X_{t_{i+1}} h_t dt\right] \quad \textit{for } h \textit{ such that } \int_0^1 D_t X_{t_{i+1}} h_t dt = 0 \\
&= E\left[\mathbf{1}_{x\leq X_{t_i}} \int_{t_i}^{1} D_t v(X_{t_{i+1}}) h_t dt\right] \quad \textit{chain rule} \\
&= \underbrace{E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \int_{t_i}^{1} h_t \delta W_t\right]}_{B} \quad \textit{integration by parts}
\end{aligned}$$

6- Conclusion

$$\begin{aligned}
E\left[\delta_x(X_{t_i}) v(X_{t_{i+1}})\right] &= A + B \\
&= E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \int_0^{t_i} h_t \delta W_t\right] \\
&+ E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \int_{t_i}^{1} h_t \delta W_t\right] \\
&= E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \int_0^{1} h_t \delta W_t\right] \\
&= E\left[\mathbf{1}_{x\leq X_{t_i}} v(X_{t_{i+1}}) \mathcal{S}^h(1)\right]
\end{aligned}$$

since $\mathcal{S}^h(F) := \displaystyle\int_0^1 F h_t \delta W_t$.

7- Same thing with $\varphi \in \mathcal{L}_+$, the set of functions of $C^0(\mathbb{R}_+)$ such that $\varphi(0) = 1$, with φ and its first derivative in L^2.

$$\begin{aligned}
& E\left[\delta_x(X_{t_i})v(X_{t_{i+1}})\right] \\
& = E\left[\delta_x(X_{t_i})\varphi(X_{t_i} - x)v(X_{t_{i+1}})\right] \\
& = E\left[\delta_x(X_{t_i})\varphi(X_{t_i} - x)v(X_{t_{i+1}}) \int_0^{t_i} D_t X_{t_i} h_t dt\right] \\
& \textit{for } h \textit{ such that } \int_0^{t_i} D_t X_{t_i} h_t dt = 1 \\
& \ldots
\end{aligned}$$

Theorem : For $v \in L^2(X_{t_{i+1}})$, $\varphi \in \mathcal{L}_+$, $h \in \mathbf{H}(X)$ and $c \in \mathbb{R}$

$$E\left[v(X_{t_{i+1}}) \mid X_{t_i} = x\right] = \frac{E\left[\ \left(\mathbf{1}_{x \leq X_{t_i}} - c\right)\ v(X_{t_{i+1}})\, \mathcal{S}^h(\varphi(X_{t_i} - x))\ \right]}{E\left[\ \left(\mathbf{1}_{x \leq X_{t_i}} - c\right)\ \mathcal{S}^h(\varphi(X_{t_i} - x))\ \right]}$$

3.3 The alternative representation in the general case : $d \in \mathbb{N}$

1- X a process such that X_{t_i} and $X_{t_{i+1}}$ are "smooth".

2- Choose $h \in \mathbf{H}(X)$, i.e. a matrix-valued process in $L^2(\mathcal{F}_1)$ such that :
$\int_0^{t_i} D_t X_{t_i} h_t dt = I_d$, $\int_0^1 D_t X_{t_{i+1}} h_t dt = 0$.

3- Choose $\varphi \in \mathcal{L}_+$, the set of functions of $C^0(\mathbb{R}_+^d)$ such that $\varphi(0) = 1$, with φ and all cross derivatives in L^2.

Theorem : For $v \in L^2(X_{t_{i+1}})$, $\varphi \in \mathcal{L}_+$, $h \in \mathbf{H}(X)$ and $c \in \mathbb{R}$

$$E\left[v(X_{t_{i+1}}) \mid X_{t_i} = x\right] = \frac{E\left[\ (H_x(X_{t_i}) - c)\ v(X_{t_{i+1}})\ \mathcal{S}^h(\varphi(X_{t_i} - x))\ \right]}{E\left[\ (H_x(X_{t_i}) - c)\ \mathcal{S}^h(\varphi(X_{t_i} - x))\ \right]}$$

where $H_x(X_{t_i}) = \prod_{j=1}^{d} \mathbf{1}_{X_{t_i}^j \geq x^j}$ and, h^i denoting the ith column of h,

$$\mathcal{S}^h(F)$$
$$=$$
$$\underbrace{\int_0^1 \left(\int_0^1 \left(\dots \int_0^1 \left(\int_0^1 F(h_t^1)^* \delta W_t\right)(h_t^2)^* \delta W_t \dots\right)(h_t^{d-1})^* \delta W_t\right)(h_t^d)^* \delta W_t}_{\text{d iterated Skorohod integrals}}$$

3.4 The integrated MSE minimization problem

3.4.1 Separable functions : $\varphi(x) = \prod_{i=1}^{d} \varphi^i(x^i)$

$$\min_{\varphi} \int E\left[\; H_x(X_{t_i})\; v(X_{t_{i+1}})^2\; \mathcal{S}^h(\varphi(X_{t_i}-x))^2 \;\right] dx$$

admits a unique solution

$$\varphi(x) := \exp\left(-\sum_{i=1}^{d} \hat{\eta}^i x^i\right)$$

in the class of separable functions in $\mathcal{L}_+$.

The $\hat{\eta}^i > 0$ are solution of the system :

$$(\hat{\eta}^i)^2 \;=\; \frac{E\left[v(X_{t_{i+1}})^2\left(\sum_{k=0}^{d-1}(-1)^k \sum_{I\in\mathcal{J}_k^{-i}} \mathcal{S}^h_{-I}(1)\prod_{j\in I}\hat{\eta}^j\right)^2\right]}{E\left[v(X_{t_{i+1}})^2\left(\sum_{k=0}^{d-1}(-1)^k \sum_{I\in\mathcal{J}_k^{-i}} \mathcal{S}^h_{-(I\vee i)}(1)\prod_{j\in I}\hat{\eta}^j\right)^2\right]}\;.$$

where

$\mathcal{J}_k^{-i} = \{I \in \mathcal{J}_k \;:\; i \notin I\}$, for $I \in \mathcal{J}_k$, $-I$ denotes the element of $\mathcal{J}_{d-k}$ such that $(-I)\vee I = \{1,\ldots,d\}$.

Remark : $\hat{\eta}^i \sim \sqrt{n}$.

3.4.2 General localizing functions

By a simple change of variable ($\xi = X_{t_i}(\omega) - x$)

$$\int E\left[\ H_x(X_{t_i})\ v(X_{t_{i+1}})^2\ \mathcal{S}^h(\varphi(X_{t_i} - x))^2\ \right] dx$$
$$=$$
$$\int_{\mathbb{R}^d_+} \partial\varphi(\xi)^* E[Q_h Q_h^*] \partial\varphi(\xi) d\xi$$

where

$$\begin{array}{rcll} \partial\varphi & := & (\ \partial_I \varphi\ )_{I \in \mathcal{J}_k,\ k \leq d} & \text{vector of all cross derivatives} \\ \text{and} \quad Q_h & := & \left(\ (-1)^k v(X_{t_{i+1}}) \mathcal{S}^h_{-I}(1)\ \right)_{I \in \mathcal{J}_k,\ k \leq d} & . \end{array}$$

Bounded Cross Derivatives Sobolev space $\mathtt{BCD}$ **:**

1- $\mathtt{BCD}_0(\mathbb{R}^d_+)$: the set of functions $\varphi\ : \mathbb{R}^d_+ \to \mathbb{R}$ such that all partial derivatives $\partial_I \varphi$, $I \in \mathcal{I}_k, k = 0, ..., d$, exist and are continuous on the interior of $\mathbb{R}^d_+$ and can be extended continuously to the boundary.

2- $\mathtt{BCD}(\mathbb{R}^d_+)$: Completion of $\mathtt{BCD}_0(\mathbb{R}^d_+)$ for

$$< \varphi, \psi >_{\mathtt{BCD}_0} := \int_{\mathbb{R}^d_+} \partial\varphi^* \partial\psi dx$$

Proposition 3.1 *There is a continuous map i : $\mathtt{BCD}(\mathbb{R}^d_+) \to C^0\left(\mathbb{R}^d_+\right)$ such that $u = i(u)$ almost everywhere.*

$\Longrightarrow$ Provides a sense to $\varphi(0) = 1$.

Theorem *If $E[Q_h Q_h^*]$ is positive definite, there exists a unique solution $\hat{\varphi}$ in $\mathtt{BCD}(\mathbb{R}^d_+)$.*

$\Longrightarrow$ **PDE characterization.**

4 Numerical Applications

$$dX_t = \mathrm{diag}[X_t]\sigma dW_t \ , \ \ X_0^1 = X_0^2 = X_0^3 = 1$$

with

$$\sigma = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.08 & 0.4 & 0 \\ 0.03 & -0.15 & 0.32 \end{bmatrix}.$$

Density $f_{X_1}(x) = E[\delta_x(X_1)]$ $\quad x^1 = 1.0$

$x^3 \backslash x^2$		0.7	1.0	1.3
0.7	True value	1.78	2.44	1.65
	Reduction by φ, c	1.80[0.10]	2.44[0.07]	1.65[0.04]
	Reduction by φ	1.80[0.11]	2.44[0.08]	1.65[0.04]
	Reduction by c	1.78[0.26]	2.45[0.26]	1.67[0.27]
	No Reduction	1.79[0.30]	2.45[0.31]	1.68[0.32]
1.0	True value	2.72	2.33	1.12
	Reduction by φ, c	2.73[0.07]	2.34[0.04]	1.12[0.02]
	Reduction by φ	2.73[0.08]	2.34[0.04]	1.13[0.71]
	Reduction by c	2.73[0.27]	2.35[0.27]	1.15[0.29]
	No Reduction	2.74[0.34]	2.36[0.35]	1.16[0.37]
1.3	True value	1.68	1.02	0.38
	Reduction by φ, c	1.69[0.03]	1.02[0.01]	0.38[0.01]
	Reduction by φ	1.69[0.03]	1.02[0.01]	0.38[0.01]
	Reduction by c	1.69[0.27]	1.05[0.27]	0.41[0.28]
	No Reduction	1.70[0.35]	1.06[0.37]	0.43[0.39]

Regression $r(x) = 100 * E\left[\left(\frac{X_2^1+X_2^2}{2} - X_2^3\right)^+ \mid X_1 = x\right]$ $x^1 = 1.0$

$x^3 \backslash x^2$		0.9	1.0	1.1
0.9	True value	20.08	23.58	27.24
	Reduction by φ	19.93[1.01]	23.40[1.06]	27.08[1.16]
	No Reduction	20.59[8.80]	23.94[32.24]	30.95[63.28]
1.0	True value	16.08	19.18	22.47
	Reduction by φ	15.94[0.85]	19.00[0.92]]	22.27[0.95]
	No Reduction	16.04[11.48]	20.25[32.05]	23.48[68.62]
1.1	True value	12.87	15.58	18.50
	Reduction by φ	12.77[0.76]	15.57[0.85]	18.50[0.96]
	No Reduction	11.26[55.83]	14.11[30.39]	25.46[325.15]

5 Approximation of RBSDE's

$$\begin{aligned} Y_t &= g(X_1) - \int_t^1 Z_s \cdot dW_s + A_1 - A_t \\ Y_t &\geq g(X_t) \end{aligned}$$

where A is a non-decreasing continuous process satisfying

$$\int_0^1 (Y_t - g(X_t))\, dA_t = 0 \ ,$$

$$\text{and} \quad X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \quad X_0 \in \mathbb{R}^d$$

By the scheme : $\pi = \{t_i = i/n,\ 0 \leq i \leq n\}$

$$X^\pi_{t_{i+1}} = X^\pi_{t_i} + \frac{1}{n} b(X^\pi_{t_i}) + \sigma(X^\pi_{t_i})(W_{t_{i+1}} - W_{t_i}) \ , \quad X^\pi_0 = X_0 \ .$$

and

$$\begin{aligned} Y^\pi_1 &= g(X^\pi_1) \\ Y^\pi_{t_i} &= \max\left\{ g(X^\pi_{t_i}) \ , \ E\left[Y^\pi_{t_{i+1}} \mid X^\pi_{t_i} \right] \right\} \end{aligned}$$

Approximate

$$\begin{aligned} Y_1^\pi &= g(X_1^\pi) \\ Y_{t_i}^\pi &= \max\left\{g(X_{t_i}^\pi)\ ,\ E\left[Y_{t_{i+1}}^\pi \mid X_{t_i}^\pi\right]\right\} \end{aligned}$$

by

$$\begin{aligned} \hat{Y}_1^\pi &= g(X_1^\pi) \\ \hat{Y}_{t_i}^\pi &= \max\left\{g(X_{t_i}^\pi)\ ,\ \hat{E}\left[\hat{Y}_{t_{i+1}}^\pi \mid X_{t_i}^\pi\right]\right\} \end{aligned}$$

where $\hat{E}\left[\hat{Y}_{t_{i+1}}^\pi \mid X_{t_i}^\pi\right]$ is the estimator of $E\left[\hat{Y}_{t_{i+1}}^\pi \mid X_{t_i}^\pi\right]$ given by the Malliavin calculus.

Theorem

$$\left\|\hat{Y}_{t_i}^\pi - Y_{t_i}^\pi\right\|_{L^p} \leq nC(p) \max_{0\leq j\leq n-1} \mathcal{E}_{j,p}(\hat{E}, n)$$

with

$$\mathcal{E}_{j,p}(\hat{E}, n) := \left\|\hat{E}\left[\hat{Y}_{t_{j+1}}^\pi \mid X_{t_j}^\pi\right] - E\left[\hat{Y}_{t_{j+1}}^\pi \mid X_{t_j}^\pi\right]\right\|_{L^p}$$

Remark :

1- At best $\mathcal{E}_{j,p}(\hat{E}, n) \sim N^{-1/2}$ if computed by pure Monte-Carlo.

2- $\left\|Y_{t_i} - Y_{t_i}^\pi\right\|_{L^p} \sim n^{-1/2}$, hence to get $\left\|\hat{Y}_{t_i}^\pi - Y_{t_i}^\pi\right\|_{L^p} \sim n^{-1/2}$ we need to take at least $N = n^3$.

5.1 Naive numerical scheme

We consider N copies $(X^{\pi^{(1)}}, \ldots, X^{\pi^{(N)}})$ of X^π.

Initialization : For all j : $\hat{Y}_1^{\pi^{(j)}} = g\left(X_1^{\pi^{(j)}}\right)$.

Backward induction : For $i = n, \ldots, 2$, we set, for all j :

$$\hat{Y}_{t_{i-1}}^{\pi^{(j)}} \;=\; \max\left\{g(X_{t_i})\ ,\ \hat{E}^{(j)}\left[\hat{Y}_{t_i}^{\pi^{(j)}} \mid X_{t_i}^{\pi^{(j)}}\right]\right\}$$

where $\hat{E}^{(j)} = \tilde{E}^{(j)}$ defined by

$$\tilde{E}^{(j)}\left[\hat{Y}_{t_i}^{\pi^{(j)}} \mid X_{t_i}^{\pi^{(j)}}\right] \;=\; \frac{\sum_{\ell \le N} \mathbf{1}_{X_{t_{i-1}}^{\pi^{(j)}} \le X_{t_{i-1}}^{\pi^{(\ell)}}} \hat{Y}_{t_i}^{\pi^{(\ell)}} \mathcal{S}^{h^{(\ell)}}\left(\varphi(X_{t_{i-1}}^{\pi^{(\ell)}} - X_{t_{i-1}}^{\pi^{(j)}})\right)}{\sum_{\ell \le N} \mathbf{1}_{X_{t_{i-1}}^{n^{(j)}} \le X_{t_{i-1}}^{\pi^{(\ell)}}} \mathcal{S}^{h^{(\ell)}}\left(\varphi(X_{t_{i-1}}^{\pi^{(\ell)}} - X_{t_{i-1}}^{\pi^{(j)}})\right)}$$

In the Gaussian case : $X = W$ and

$$\begin{aligned}\mathcal{S}^{h^{(\ell)}}\left(\varphi(W_{t_{i-1}}^{(\ell)} - W_{t_{i-1}}^{(j)})\right) \;&=\; \varphi\left(W_{t_{i-1}}^{(\ell)} - W_{t_{i-1}}^{(j)}\right)\left(\frac{W_{t_{i-1}}^{(\ell)}}{t_{i-1}} - \frac{W_{t_i}^{(\ell)} - W_{t_{i-1}}^{(\ell)}}{t_i - t_{i-1}}\right) \\ &\quad -\varphi'\left(W_{t_{i-1}}^{(\ell)} - W_{t_{i-1}}^{(j)}\right)\end{aligned}$$

$\hat{E} = \tilde{E}$ **in** L^p **???**

5.2 Truncated scheme

Lemma *There exists some* (α_i^n, β_i^n) *such that*

$$\left|Y_{t_i}^\pi\right| \;\leq\; \alpha_i^n + \beta_i^n \left|X_{t_i}^\pi\right|^2 \; .$$

Moreover,

$$\limsup_{n\to\infty} \max_{0\leq i\leq n} \{\alpha_i^n + \beta_i^n\} \;\; < \;\; \infty \; .$$

$\Rightarrow$ There exists sequences of polynomials (with uniformly bounded coefficients) $\wp^n = \left\{\left(\underline{\wp}_i^n, \overline{\wp}_i^n\right)\right\}$, $\Re^n = \left\{\left(\underline{\Re}_i^n, \overline{\Re}_i^n\right)\right\}$ such that :

$$\underline{\wp}_i^n(X_{t_i}^\pi) \leq Y_{t_i}^\pi \leq \overline{\wp}_i^n(X_{t_i}^\pi)$$

and

$$\underline{\Re}_{i-1}^n(X_{t_{i-1}}^\pi) \leq E[\underline{\wp}_i^n(X_{t_i}^\pi) \mid X_{t_{i-1}}^\pi] \;\; \leq \;\; E[\overline{\wp}_i^n(X_{t_i}^\pi) \mid X_{t_{i-1}}^\pi] \leq \overline{\Re}_{i-1}^n(X_{t_{i-1}}^\pi)$$

Initialization : For all j : $\hat{Y}_1^{\pi(j)} = g\left(X_1^{\pi(j)}\right)$.

Backward induction : For $i = n, \ldots, 2$, we set, for j :

$$\hat{E}^{(j)}\left[\hat{Y}_{t_i}^{\pi(j)} \mid X_{t_{i-1}}^{\pi(j)}\right] \;=\; \underline{\Re}_{i-1}^n(X_{t_{i-1}}^{\pi(j)}) \vee \tilde{E}^{(j)}\left[\hat{Y}_{t_i}^{\pi(j)} \mid X_{t_{i-1}}^{\pi(j)}\right] \wedge \overline{\Re}_{i-1}^n(X_{t_{i-1}}^{\pi(j)})$$

$$\check{Y}_{t_{i-1}}^{\pi(j)} \;=\; \max\left\{g(X_{t_{i-1}}^{\pi(j)}) \, , \; \hat{E}^{(j)}\left[\hat{Y}_{t_i}^{\pi(j)} \mid X_{t_{i-1}}^{\pi(j)}\right]\right\}$$

$$\hat{Y}_{t_{i-1}}^{\pi(j)} \;=\; \underline{\wp}_{i-1}^n(X_{t_{i-1}}^{\pi(j)}) \vee \check{Y}_{t_{i-1}}^{\pi(j)} \wedge \overline{\wp}_{i-1}^n(X_{t_{i-1}}^{\pi(j)})$$

5.3 Simulation error in the truncated scheme

Theorem

$$\left\|\hat{Y}^{\pi}_{t_i} - Y^{\pi}_{t_i}\right\|_{L^p} \;\leq\; nC(p) \max_{0\leq j\leq n-1} \mathcal{E}_{j,p}(\hat{E},n)$$

with

$$\mathcal{E}_{j,p}(\hat{E},n) \;:=\; \left\|\hat{E}\left[\hat{Y}^{\pi}_{t_{j+1}} \mid X^{\pi}_{t_j}\right] - E\left[\hat{Y}^{\pi}_{t_{j+1}} \mid X^{\pi}_{t_j}\right]\right\|_{L^p}$$

Theorem *If we choose $\varphi^n = \varphi(\sqrt{n}x)$ for $\varphi \in \mathcal{L}_+$. Then :*

$$\max_{1\leq i\leq n} \mathcal{E}_{i,p}(\hat{E},n) \;\leq\; Cn^{\frac{d}{4p}} N^{-\frac{1}{2p}} \;.$$

$\Rightarrow$ **Global error :**

$$\max_{0\leq i\leq n} \left\|\hat{Y}^{\pi}_{t_i} - Y_{t_i}\right\|_{L^p} \;\leq\; C(p)\left(n^{-\frac{1}{2}} + n\frac{n^{\frac{d}{4p}}}{N^{\frac{1}{2p}}}\right)$$

- $n^{-\frac{1}{2}}$: discretization error
- n : number of regression estimations
- $N^{-\frac{1}{2p}}$: convergence rate of the regression estimator in terms of the number of simulations N.
- $n^{\frac{d}{4p}}$: "variance" of the regression operator.

$\Rightarrow$ **For an L^p error of order of $n^{-\frac{1}{2}}$:** $N \sim n^{3p+\frac{d}{2}}$.

Impact of the dimension through the time step : for $p = 1$, $N \sim n^{3+\frac{d}{2}} \gg n^3$.

5.4 Numerical experiment : American Option in the BS model

$$dX_t \;=\; 0.05\,X_t + \mathrm{diag}[X_t]\sigma dW_t \;\;,\;\; X_0 = 100 I_d$$

$$g(X_t) \;=\; \left[100 - \left(\prod_{i=1}^{d} X_t^i\right)^{\frac{1}{d}}\right]^+ .$$

- $\sigma = 0.15,\ n = 50,\ N = 4096$:

$$\underbrace{\hat{Y}_0^\pi = 4.21}_{\text{estimation}} \quad \underbrace{(0.32\%)}_{\text{std in \% of } \hat{Y}_0^\pi} \quad , \underbrace{Y_0 = 4.23}_{\text{true value}}$$

- $\sigma = \begin{bmatrix} 0.15 & 0 \\ -0.05 & 0.1 \end{bmatrix},\ n = 20,\ N = 4096$:

$$\hat{Y}_0^\pi = 1.53\ (1.13\%)\ ,\ Y_0 = 1.54$$

- $\sigma = \begin{bmatrix} 0.15 & 0 & 0 \\ -0.05 & 0.1 & 0 \\ 0.03 & -0.04 & 0.15 \end{bmatrix},\ n = 20,\ N = 16384$:

$$\hat{Y}_0^\pi = 1.53\ (0.52\%)\ ,\ Y_0 = 1.53$$