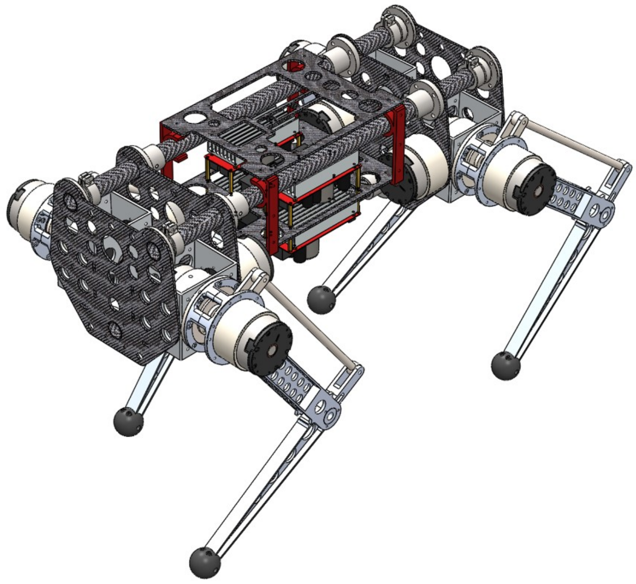
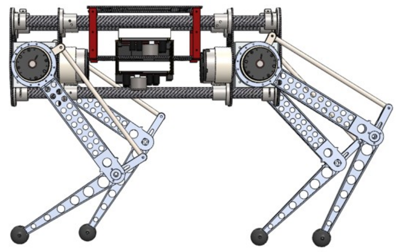
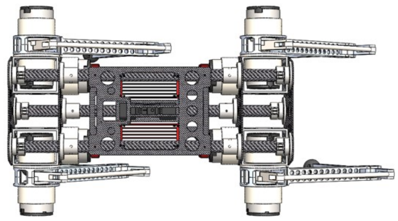
Note of Quadruped Robot -- 'Jun GO'

TODO List:

- read section of differential movement and derive jacobian matrix
- add introduction about JUN-GO

Kinematics

For the forward kinematics of the quadruped robot, we will use Denavit-Hartenberg Representation.

The main content of use Denavit-Hartenberg Representation is:

- All joints will be represented by z-axis. If the joint is revolute, the z-axis is in the direction of rotation as followed by the right-hand rule. And the rotation about the z-axis (θ) is the joint variable. And the index number for the z-axis of joint $n+1$ is z_n.
- The direction of x_n will be along the common normal between z_{n-1} and z_n.
    - If two z-axes are parallel, We will pick the common normal that is colinear with the common normal of the previous joint.
    - If two z-axes are intersecting, we will assign the x-axis along the direction of the cross-product of the two z-axes.

Four parameter in D-H Representation:

- θ represents a rotation about the z-axis
- d represents the distance on the z-axis between two successive common normals (or joint offset)

- a represents the length of each common normal (the length of a link)
- α represents the angle between two successive z-axes (also called joint twist angle). Commonly, only θ and d are joint variables.

The steps of necessary motions to transform from one reference frame to the next:

1. Rotate about the z_n-axis an angle of θ_{n+1} to make x_n and x_{n+1} parallel to each other.
2. Translate along the z_n-axis a distance of d_{n+1} to make x_n and x_{n+1} colinear.
3. Translate along the (already rotated) x_n-axis a distance of a_{n+1} to bring the origins of x_n and x_{n+1} together. At this point, the origins of the two reference frames will be at the same location.
4. Rotate z_n-axis about x_{n+1}-axis an angle of α_{n+1} to align z_n-axis with z_{n+1}-axis. At this point, frames n and $n+1$ will be exactly the same.

The transformation ${}^nT_{n+1}$ (called A_{n+1}) between two successive frames representing the preceding four movements is the product of the four matrices representing them. Since all transformations are relative to the current frame (they are measured and performed relative to the axes of the current local frame), all matrices are post-multiplied. The result is:

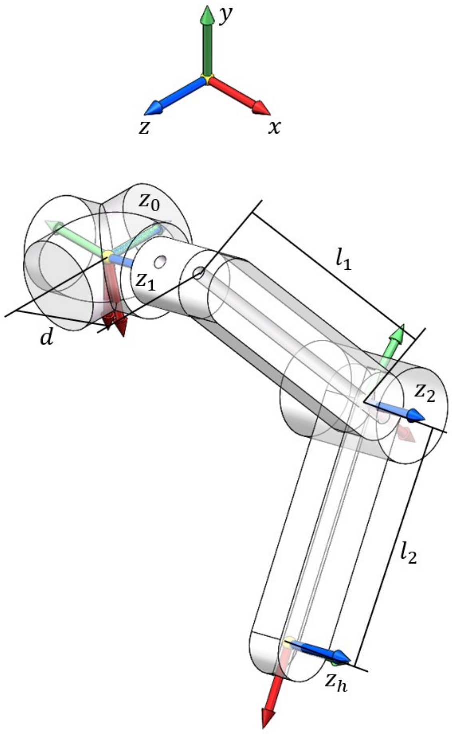
$$
\begin{aligned}
{}^nT_{n+1} = A_{n+1} &= Rot(z,\theta_{n+1}) \times Trans(0,0,d_{n+1}) \times Trans(a_{n+1},0) \times Rot(x,\alpha_{n+1}) \\
&= \begin{bmatrix} C\theta_{n+1} & -S\theta_{n+1}C\alpha_{n+1} & S\theta_{n+1}S\alpha_{n+1} & a_{n+1}C\theta_{n+1} \\ S\theta_{n+1} & C\theta_{n+1}C\alpha_{n+1} & -C\theta_{n+1}S\alpha_{n+1} & a_{n+1}S\theta_{n+1} \\ 0 & S\alpha_{n+1} & C\alpha_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

For convinence, we can write the D-H parameters into a table. With that we can get the transformation matrix between joints easily.

#	θ	d	a	α
$0-1$				
$1-2$				
$2-3$				
$3-4$				
$4-5$				
$5-6$				

e.g. For a 2-DoF robot, there will be 2 joints(Joint1, Joint2). And there will be 3 coordinates(z_0, z_1, z_H) fixed on the arm.

Forward Kinematics Modeling



#	θ	d	a	α
$0-1$	θ_1	0	0	90°
$1-2$	θ_2	d	l_1	0
$2-h$	θ_3	0	l_2	0

But we can revise the above D-H table to facilitate the Unify the direction of joint Angle change.

For the legs in the left side, we will get the following table:

#	θ	d	a	α
$0-1$	θ_1	0	0	90°
$1-2$	θ_2	d	l_1	0
$2-h$	$-90^\circ-\theta_3$	0	l_2	0

$$A_1 = \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) & 0 \\ sin(\theta_1) & 0 & -cos(\theta_1) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 & a_1 cos(\theta_2) \\ sin(\theta_2) & cos(\theta_2) & 0 & a_1 sin(\theta_2) \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & a_2 cos(\theta_3) \\ sin(\theta_3) & cos(\theta_3) & 0 & a_2 sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
{}^{0}T_H = A_1 A_2 A_3 = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & dS_1 + a_1C_1(C_2 + C_{23}) \\ S_1C_{23} & -S_1S_{23} & -C_1 & -dC_1 + S_1(a_1C_2 + a_2C_{23}) \\ S_{23} & C_{23} & 0 & a_1S_2 + a_2S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

For the legs in the right side, we will get the following table:

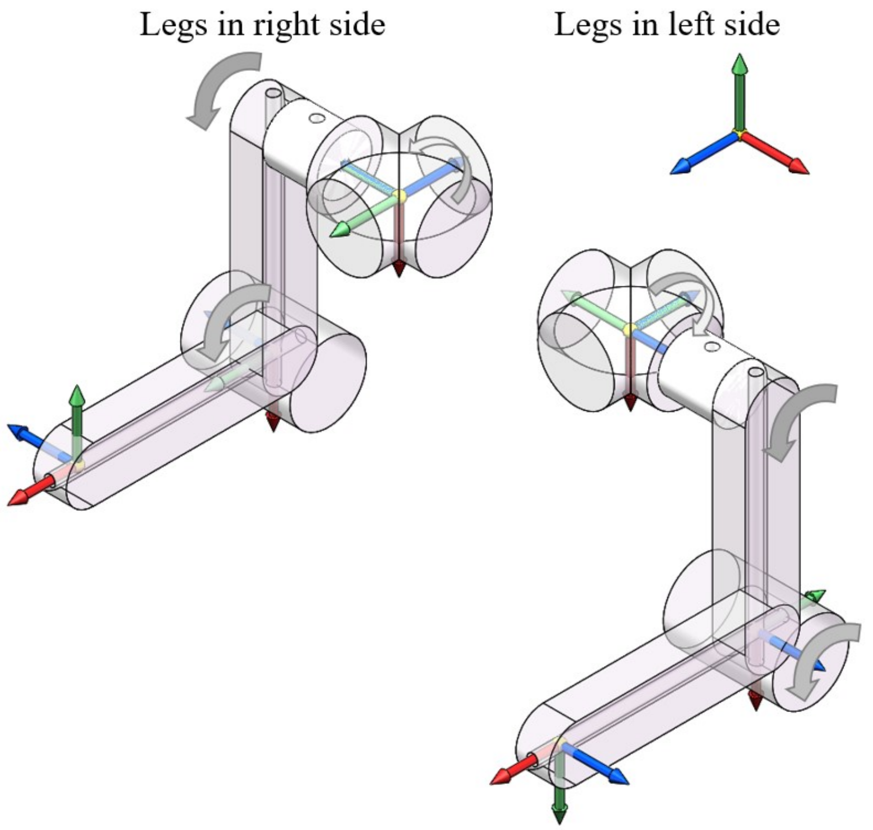
$\#$	θ	d	a	α
$0-1$	θ_1	0	0	-90°
$1-2$	$-\theta_2$	d	l_1	0
$2-h$	$90^\circ + \theta_3$	0	l_2	0

$$
A_1 = \begin{bmatrix} cos(\theta_1) & 0 & -sin(\theta_1) & 0 \\ sin(\theta_1) & 0 & cos(\theta_1) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_2 = \begin{bmatrix} cos(\theta_2) & -sin(\theta_2) & 0 & a_1cos(\theta_2) \\ sin(\theta_2) & cos(\theta_2) & 0 & a_1sin(\theta_2) \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
A_3 = \begin{bmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & a_2cos(\theta_3) \\ sin(\theta_3) & cos(\theta_3) & 0 & a_2sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Based on the above configuartion, the zero position and positive rotation direction of the joints is:



Zero position and positive rotation direction

Inverse Kinematics Modeling

1. θ_1

$$A_1^{-1} \cdot {}^0 T_H = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ S_1 & -C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x C_1 + n_y S_1 & o_x C_1 + o_y S_1 & a_x C_1 + a_y S_1 & p_x C_1 + p_y S_1 \\ n_z & o_z & a_z & p_z \\ n_x S_1 - n_y C_1 & o_x S_1 - o_y C_1 & a_x S_1 - a_y C_1 & p_x S_1 - p_y C_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_{23} & -S_{23} & 0 & a_1 C_2 + a_2 C_{23} \\ S_{23} & C_{23} & 0 & a_1 S_2 + a_2 S_{23} \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the legs in the left side:

$$d = p_x S_1 - p_y C_1$$

$$p_x = \rho cos\phi \qquad p_y = \rho sin\phi$$
$$\rho = \sqrt{p_x^2 + p_y^2} \quad \phi = arctan2(p_y, p_x)$$

$$d = \rho cos\phi sin\theta_1 - \rho sin\phi cos\theta_1 = \rho sin(\theta_1 - \phi)$$

$$sin(\theta_1 - \phi) = \frac{d}{\rho} \quad cos(\theta_1 - \phi) = \pm\sqrt{1 - (\frac{d}{\rho})^2}$$

$$\theta_1 - \phi = arctan2(\frac{d}{\rho}, \pm\sqrt{1 - \frac{d}{\rho}^2})$$

$$\theta_1 = arctan2(\frac{d}{\rho}, \pm\sqrt{1 - \frac{d}{\rho}^2} + arctan2(p_y, p_x)$$

For the legs in the right side:

$$d = p_y C_1 - p_x S_1$$

$$\theta_1 = arctan2(-\frac{d}{\rho}, \pm\sqrt{1 - \frac{d}{\rho}^2} + arctan2(p_y, p_x)$$

2. θ_3

For the legs in the left side:

$$p_x C_1 + p_y S_1 = a_1 C_2 + a_2 C_{23} \tag{1}$$

$$p_z = a_1 S_2 + a_2 S_{23} \tag{2}$$

$$p_x S_1 - p_y C_1 = d \tag{3}$$

For the legs in the right side:

$$p_x C_1 + p_y S_1 = a_1 C_2 + a_2 C_{23} \tag{1}$$

$$-p_z = a_1 S_2 + a_2 S_{23} \tag{2}$$

$$p_y C_1 - p_x S_1 = d \tag{3}$$

The sum of $equ(1), (2), (3)$ is:

$$p_x^2 + p_y^2 + p_z^2 = a_1^2 + a_2^2 + d^2 + 2a_1 a_2 C_3$$

$$C_3 = \frac{p_x^2 + p_y^2 + p_z^2 - a_1^2 - a_2^2 - d^2}{2a_1 a_2}$$

$$S_3 = \pm\sqrt{1 - C_3^2}$$

$$\theta_3 = arctan\frac{S_3}{C_3}$$

3. θ_2

For the legs in the left side:

$$p_z = a_1 S_2 + a_2 S_{23} = a_1 S_2 + a_2 S_2 C_3 + a_2 C_2 S_3 = (a_1 + a_2 C_3) S_2 + a_2 S_3 C_2$$

$$a_1 + a_2 C_3 = \rho' sin\phi'$$

$$a_2 S_3 = \rho' cos\phi'$$

$$p_z = \rho' sin\phi' sin\theta_2 + \rho' cos\phi' cos\theta_2$$

$$p_z = \rho' cos(\phi' - \theta_2)$$

$$\rho' = \sqrt{(a_1 + a_2C_3)^2 + (a_2S_3)^2} \quad \phi' = arctan2(a_1 + a_2C_3, a_2S_3)$$

$$cos(\phi' - \theta_2) = \frac{p_z}{\rho'} \qquad sin(\phi' - \theta_2) = \pm\sqrt{1 - \frac{p_z}{\rho}^2}$$

$$\phi' - \theta_2 = arctan2\left[\pm\sqrt{1 - \frac{p_z}{\rho'}^2}, \frac{p_z}{\rho'}\right]$$

$$\theta_2 = arctan2(a_1 + a_2C_3, a_2S_3) - arctan2\left[\pm\sqrt{1 - \frac{p_z}{\rho'}^2}, \frac{p_z}{\rho'}\right]$$

For the legs in the right side:

$$-p_z = a_1S_2 + a_2S_{23} = a_1S_2 + a_2S_2C_3 + a_2C_2S_3 = (a_1 + a_2C_3)S_2 + a_2S_3C_2$$

$$a_1 + a_2C_3 = \rho' sin\phi'$$

$$a_2S_3 = \rho' cos\phi'$$

$$-p_z = \rho' sin\phi' sin\theta_2 + \rho' cos\phi' cos\theta_2$$

$$-p_z = \rho' cos(\phi' - \theta_2)$$

$$\rho' = \sqrt{(a_1 + a_2C_3)^2 + (a_2S_3)^2} \quad \phi' = arctan2(a_1 + a_2C_3, a_2S_3)$$

$$cos(\phi' - \theta_2) = \frac{p_z}{\rho'} \qquad sin(\phi' - \theta_2) = \pm\sqrt{1 - \frac{p_z}{\rho}^2}$$

$$\phi' - \theta_2 = arctan2\left[\pm\sqrt{1 - \frac{p_z}{\rho'}^2}, \frac{-p_z}{\rho'}\right]$$

$$\theta_2 = arctan2(a_1 + a_2C_3, a_2S_3) - arctan2\left[\pm\sqrt{1 - \frac{p_z}{\rho'}^2}, \frac{-p_z}{\rho'}\right]$$