Probabilistic Graphical Model

Session # 3 : Bayesian Model

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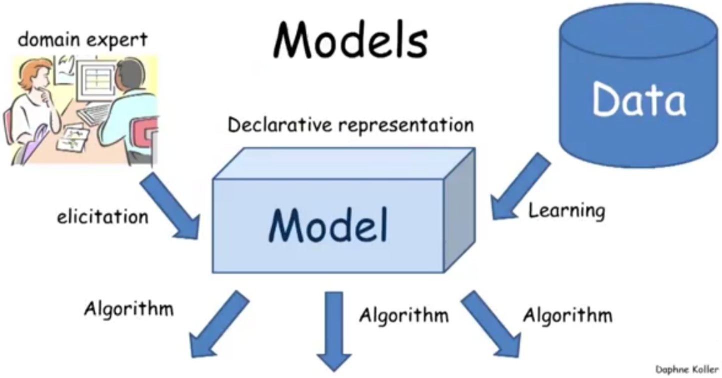
Table of Contents

Probabilistic Graphical Models

- Probabilistic Graphical Model is a declarative model that is standalone, where probability distributions and its semantics represent uncertainty about state of world.

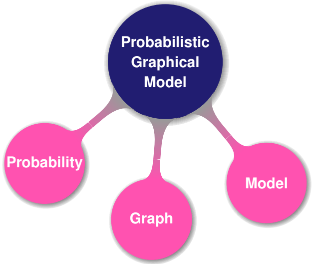


Probabilistic Graphical Models

Probabilistic : Nature of problems to be solved are probabilistic because of uncertainity. The type of queries are also probabilistic in nature.

Graphical : Use a graph to represent the participating features or variables and their interaction .

Model : declarative representation of the problem and not a derived representation. Use a mathematical equation or a graph.



Components of Probabilistic Graphical Model

Representation : declarative representation of the problem using graphs.

Inference : answer queries and explainations related to the problem.

Learning : learn the parameters of the model.

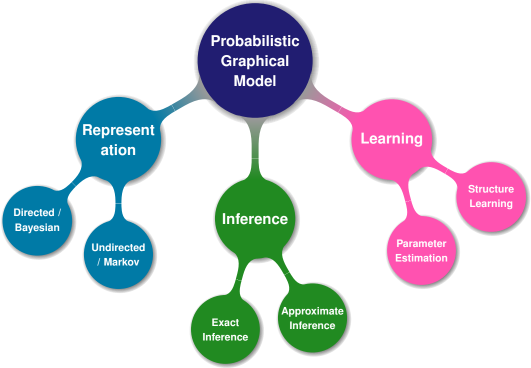


Table of Contents

Student Example

- Model the difficulty of a course, intelligence of students, Grade the students score in a particular course.
- Let D represent the difficulty of a course.

$$\textit{Domain of } D = \{easy, hard\} = \{d^0, d^1\}$$
$$P(D) = \{0.6, 0.4\}$$

- Let I represent the intelligence of a student.

$$\textit{Domain of } I = \{low, high\} = \{i^0, i^1\}$$
$$P(I) = \{0.7, 0.3\}$$

Student Example

- Let G represent the grade a student gets for a course.

$$\textit{Domain of } G = \{A, B, C\} = \{g^1, g^1, g^2\}$$

- How do we represent Joint distribution of the 3 random variables? How many parameters are required?
- $P(I, D, G)$ denotes the probabilities of all combinations of the values of the 3 random variables.
- These $2 * 2 * 3 = 12$ parameters can be represented using a Joint Distribution.

Student Example - Joint Distribution

I	D	G	$P(I, D, G)$
i^0	d^0	g^1	0.126
		g^2	0.168
		g^3	0.126
i^0	d^1	g^1	0.009
		g^2	0.045
		g^3	0.126
i^1	d^0	g^1	0.252
		g^2	0.0224
		g^3	0.0056
i^1	d^1	g^1	0.060
		g^2	0.036
		g^3	0.024

What is the sum of the joint distribution?

$$\sum P(I, D, G) = 1 \tag{1}$$

Operations on Joint Distribution

1. Conditioning
2. Renormalization
3. Marginalization

1. Conditioning on Joint Distribution

- Suppose a student score 'A' grade.
- Observation: $G = g^1$.
- This conditioning gives a reduced Joint distribution.
- Conditioning reduces Joint distribution.

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060

What is sum of the distribution now?

$$\sum P(I, D, g^1) \neq 1 \tag{2}$$

2. Renormalization of Conditioned JD

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060
			0.447

$\xrightarrow{\textit{normalize}}$

I	D	G	$P(I, D \mid g^1)$
i^0	d^0	g^1	$0.126/0.447 = 0.282$
i^0	d^1	g^1	$0.009/0.447 = 0.020$
i^1	d^0	g^1	$0.252/0.447 = 0.564$
i^1	d^1	g^1	$0.060/0.447 = 0.134$
			1

$$P(I, D, g^1) \xrightarrow{\textit{normalize}} P(I, D \mid g^1) \tag{3}$$

3. Marginalization on JD

Marginalization on JD = Summing Out

I	D	$P(I, D)$
i^0	d^0	0.282
i^0	d^1	0.020
i^1	d^0	0.564
i^1	d^1	0.134

D	$P(D)$
d^0	0.846
d^1	0.154

$$\sum_I P(I, D) = P(D) \tag{4}$$

Table of Contents

Factor

- A factor Φ is a function or a table that maps a set of random variables to a real value.

$$\Phi : Val(X_1, \dots, X_n) \rightarrow \mathrm{R} \tag{5}$$

- The argument of the factor is called scope of the factor.

$$Scope : \{X_1, \dots, X_n\} \tag{6}$$

- Factors are building blocks used for defining high dimensional spaces and distributions.
- Factors are used to define an exponentially large probability distribution of N random variables.
- Factors are manipulated in the same way as probability distributions.

Joint Distribution is a Factor

I	D	G	$P(I, D, G)$
i^0	d^0	g^1	0.126
		g^2	0.168
		g^3	0.126
i^0	d^1	g^1	0.009
		g^2	0.045
		g^3	0.126
i^1	d^0	g^1	0.252
		g^2	0.0224
		g^3	0.0056
i^1	d^1	g^1	0.060
		g^2	0.036
		g^3	0.024

$Scope : \{I, D, G\}$

Unnormalized Conditioned JD is a Factor

I	D	G	$P(I, D, g^1)$
i^0	d^0	g^1	0.126
i^0	d^1	g^1	0.009
i^1	d^0	g^1	0.252
i^1	d^1	g^1	0.060
			0.447

$Scope : \{I, D\}$

Conditional Probability Distribution

- CPD is a factor, which gives the conditional probability of a random variable, when other random variables are observed or known.
- For every combination of I and D, the value of G is observed.

$P(G|I, D)$

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

- Each row sums to 1.

$$\sum P i^1, d^1 = 1$$

Operations on Factors

1. Factor Product
2. Factor Marginalization
3. Factor Reduction

1. Factor Product

- Factor product is the cross product of two factors.

A	B	$\Phi_1(A,B)$
a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.2
a^2	b^2	0

B	C	$\Phi_2(B,C)$
b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

A	B	C	$\Phi_3(A,B,C) = \Phi_1 * \Phi_2$
a^1	b^1	c^1	$0.5 * 0.5 = 0.25$
a^1	b^1	c^2	$0.5 * 0.7 = 0.35$
a^1	b^2	c^1	$0.8 * 0.1 = 0.08$
a^1	b^2	c^2	$0.8 * 0.2 = 0.16$
a^2	b^1	c^1	$0.2 * 0.5 = 0.25$
a^2	b^1	c^2	$0.2 * 0.7 = 0.35$
a^2	b^2	c^1	$0 * 0.1 = 0$
a^2	b^2	c^2	$0 * 0.2 = 0$

2. Factor Marginalization

- Remove one random variable.

A	B	C	$\Phi_1(A, B, C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

A	C	$\Phi_2(A, C)$ marginalized on B
a^1	c^1	$0.25 + 0.08 = 0.33$
a^1	c^2	$0.35 + 0.16 = 0.51$
a^2	c^1	$0.25 + 0 = 0.25$
a^2	c^2	$0.35 + 0 = 0.35$

3. Factor Reduction

- Extract only one random variable.
- Observe $C = c^1$.

A	B	C	$\Phi_1(A, B, C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

A	B	C	$\Phi_1(A, B, c^1)$
a^1	b^1	c^1	0.25
a^1	b^2	c^1	0.08
a^2	b^1	c^1	0.25
a^2	b^2	c^1	0

Table of Contents

INDEPENDENCE

- Independent parameters are parameters whose values are not completely determined by the values of the other parameters.
- Random variables $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$ can be considered independent if

$$P(\{X_1, X_2, \ldots, X_n\}) = P(X_1)P(X_2)\ldots P(X_n) \tag{7}$$

$$P(\{X_1, X_2, \ldots, X_n\}) = \prod_{i=1}^{n} P(X_i) \tag{8}$$

- A set of random variables are independent of each other, if their joint probability distribution is equal to the product of probabilities of each individual random variable.

Student Example

- A company is trying to hire a recent intelligent college graduate. The company has access to the student's SAT scores.
- The probability space is induced by Intelligence I and SAT score S.

$$I = \{high, low\} = \{i^1, i^0\}$$
$$S = \{high, low\} = \{s^1, s^0\}$$

Student Example - Joint Distribution

The joint distribution of $P(I, S)$ is given as

I	S	$P(I, S)$
i^0	s^0	0.665
i^0	s^1	0.035
i^1	s^0	0.06
i^1	s^1	0.24

Student Example - Conditional Distribution

- The student's SAT score is determined by his intelligence. This represents causality.

$$I \longrightarrow S$$

- Joint distribution $P(I, S)$ can be computed by using chain rule.

$$P(I, S) = P(I)P(S \mid I)$$

$P(I)$

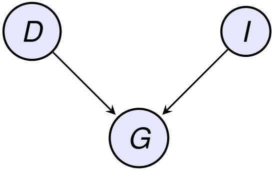
i^0	i^1
0.7	0.3

$P(S \mid I)$

I	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

Student Example - Conditional Distribution

- The grade student score depends on her intelligence and the difficulty of the course. (by intuition)



- Joint distribution $P(I, D, G)$ can be computed by using chain rule.

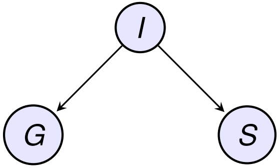
$$P(I, D, G) = P(I)P(D)P(G \mid D, I)$$

Student Example - Conditional Independence

- With 3 random variables, Intelligence I, Grade G and SAT score S, the JD has 12 entries.
- Both the SAT score and the grade are highly correlated on student's intelligence.
- If I is known, knowing $Grade = A$ no longer gives information that $S = high$.
- If I is known, knowing $S = high$ no longer gives information that $Grade = A$.

$$S \perp G \mid I$$

- The student's intelligence is the only reason why his grade and SAT score might be correlated.



Student Example - Conditional Independence

- Joint distribution $P(I, S, G)$ can be computed by using chain rule.

$$P(I, S, G) = P(I)P(S, G \mid I)$$
$$P(S, G|I) = P(S|I)P(G \mid I)$$
$$P(I, S, G) = P(I)P(S \mid I)P(G \mid I)$$

- 3 CPDs fully specify the JD.

$P(I)$

i^0	i^1
0.7	0.3

$P(S \mid I)$

I	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

$P(G \mid I)$

I	g^1	g^2	g^3
i^0	0.2	0.34	0.46
i^1	0.74	0.17	0.09

Table of Contents

Student Example

Difficulty of course D	$Val(D) = \{hard, easy\}$	$\{d^1, d^0\}$
Intelligence I	$Val(I) = \{high, low\}$	$\{i^1, i^0\}$
Grade G	$Val(G) = \{A, B, C\}$	$\{g^1, g^2, g^3\}$
SAT score S	$Val(S) = \{high, low\}$	$\{s^1, s^0\}$
Recommendation Letter L	$Val(l) = \{strong, weak\}$	$\{l^1, l^0\}$

- Joint distribution is given by

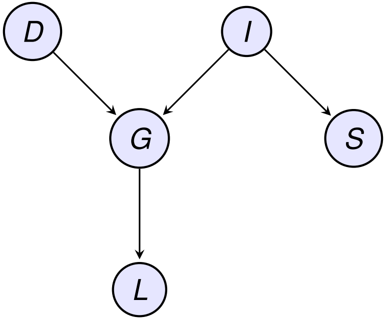
$$P(D, I, G, S, L)$$

- JD $= 2 * 3 * 2 * 2 * 2 = 48$ entries.

Student Example

- Assume that the grade depends on *Difficulty* of the course and *Intelligence* of the student.
- The *SAT* score depends on *Intelligence* of the student
- Assume that the quality of the Recommendation *Letter* depends on *Grade*.

Student Example

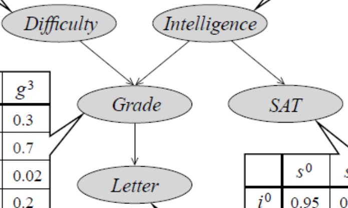


$$P(I, D, G, S, L) = P(I)P(D)P(G \mid I, D)P(S \mid I)P(L \mid G)$$

How many parameters?

- Parameters $= 1 + 1 + 8 + 2 + 3 = 15$ entries.

Student Example - $\mathcal{B}^{Student}$



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^3	0.99	0.01

Bayesian Network

- A Bayesian Network is a data structure to represent dependencies among random variables.
- Compact and natural representation.
- Represented using Directed acyclic graph (DAG) $\mathcal{G}$
  - Each node is a random variable.
  - A set of directed edge connects pairs of nodes. Edges correspond to direct influence of one node on another.
- A data structure that provides the skeleton for representing a joint distribution compactly in a factorized way.
- A compact representation for a set of conditional independence assumptions about a distribution.

Bayesian Network - Topology

- Topology specifies the conditional independencies.

$$Cause = Parent(Effects)$$

- A Bayesian network represents the joint distribution of all random variables.
- Network structure together with its CPDs is called a Bayesian network or local probability model.

$$P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P\left(X_i \middle| Pa(X_i)\right) \tag{9}$$

Bayesian Network - Construction

1. Nodes
   - Determine the set of random variables that are required to model the domain.
   - Order them such that the causes precedes the effects.

$$\{X_1, \ldots, X_n\}$$

2. Links: For each node X_i,
   - Choose a set of parents $Pa(X_i)$.
   - For each parent, insert a link from the Parent to the node X_i.
   - Write down the conditional probability table $P(X_i \mid Pa(X_i))$.

Table of Contents

Restaurant Example

- Let Q represent the random variable for the quality of food.

Q	Good	Average	Bad
$P(Q)$	0.3	0.5	0.2

- Let L represent the random variable for the location of restaurant.

L	Good	Bad
$P(L)$	0.6	0.4

- Random variables Q and L are independent of each other.

Restaurant Example

- Let C represent the cost of food.

$$C = \{high, low\}$$

- Cost C is dependent on the quality Q of food and the location L of the restaurant.
- Let N represent the number of people visiting the restaurant.

$$N = \{high, low\}$$

- N is affected by C which in turn is affected by Q .

Restaurant Example

- What is the size of joint distribution $P(Q, L, C, N)$?
- List all the independencies and conditionally dependencies.
- Draw the Bayesian Network.
- How many parameters are required to represent $P(Q, L, C, N)$?
- Write the expression for $P(Q, L, C, N)$.

Restaurant Example

- What is the size of joint distribution $P(Q, L, C, N)$?

$$3 * 2 * 2 * 2 = 24$$

- How many parameters are required to represent $P(Q, L, C, N)$?

$$(3-1) + (2-1) + (6-2) + (4-1) = 10$$

- Write the expression for $P(Q, L, C, N)$.
  According to Bayesian Network ,

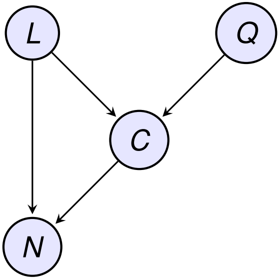
$$P(Q, L, C, N) = P(Q)P(L)P(C|L, Q)P(N|C, L)$$

Restaurant Example

- List all the independencies and conditionally dependencies.

$$Q \perp L$$
$$C|Q, L$$
$$N|C, L$$
$$Q \perp N|C$$

- Draw the Bayesian Network.



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Thank You !!!