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Probabilistic Graphical Model

Session # 2 : Mathematical Preliminaries

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The instructor is gratefully acknowledging the authors who made their course materials freely available online.

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Graph

- Data structure used to represent the probability distribution of data.
- A graph is a data structure $\mathcal{K}$ consisting of a set of nodes and a set of edges.

$$\text{Graph} \qquad \mathcal{K} = (\mathcal{X}, \mathcal{E}) \tag{1}$$

- The set of nodes denote each random variable.

$$\text{Set of Nodes} \qquad \mathcal{X} = \{X_1 \dots X_n\} \tag{2}$$

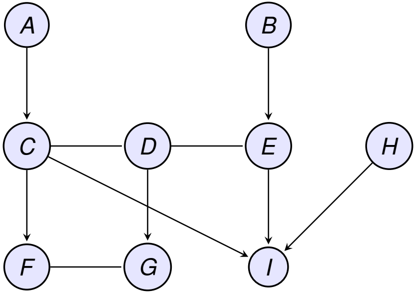
- A pair of nodes X_i, X_j can be connected by a directed edge $X_i \rightarrow X_j$ or an undirected edge $X_i - X_j$.

$$\text{Set of Edges} \qquad \mathcal{E} = X_i \rightarrow X_j \quad \text{or} \quad X_i - X_j \tag{3}$$

Directed Graph

- A graph is directed if all edges are directed. $X_i \rightarrow X_j$.

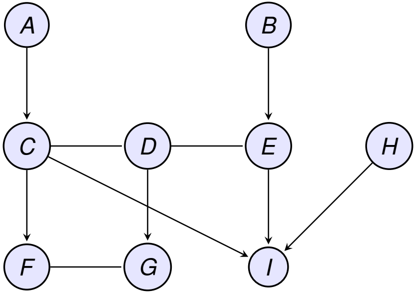
$$\mathcal{G} = (\mathcal{X}, \mathcal{E}) \quad \textit{where} \quad \mathcal{E} = \{X_i \rightarrow X_j\} \tag{4}$$



Undirected Graph

- A graph is undirected if all edges are undirected. $X_i - X_j$.

$$\mathcal{H} = (\mathcal{X}, \mathcal{E}') \quad where \quad \mathcal{E}' = \{X_i - X_j\} \tag{5}$$

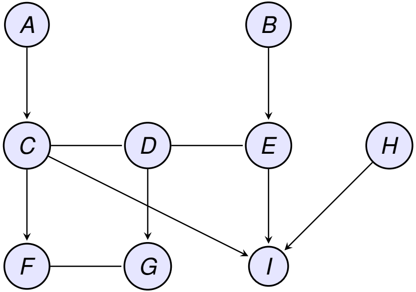


Parent and Child

Graph $\mathcal{K} = (\mathcal{X}, \mathcal{E})$ *where* $\mathcal{E} = \{X \rightleftharpoons Y\}$

- Parent
  - X is called the parent of Y.
  - Pa_X denote parents of X.
- Child
  - Y is called the child of X.
  - Ch_X denote children of X.
- Example: Identify the parents and children of Node E.

  Ans: $Pa_E = B$ $Ch_E = I$

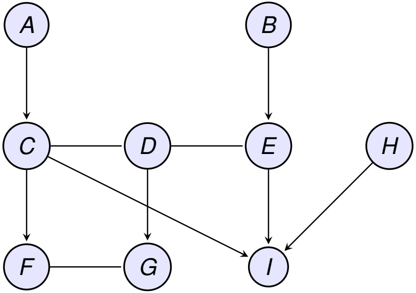


Neighbor and Boundary

- Neighbor
  - Whenever $X \rightleftharpoons Y \in \mathcal{E}$, X and Y are adjacent.
  - Nb_X denote neighbors of X.
- Boundary
  - In Directed graph, $Boundary_X = Pa_X$.
  - In Undirected graph, $Boundary_X = Nb_X$.

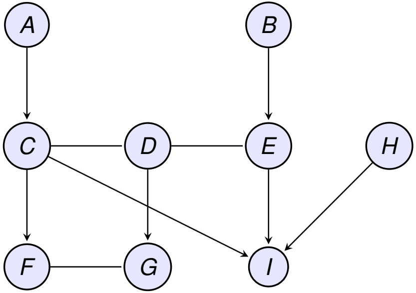
$$Boundary_X = Pa_X \cup Nb_X$$

- Example: Identify the neighbors and boundary of Node C.
  Ans: $Nb_C = F, D, I \qquad B_C = A, D, F, I$



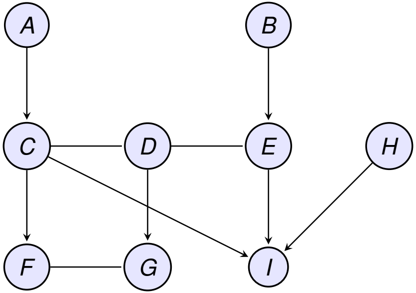
Degree of a Graph

- The degree of a node X is the number of edges in which it participates.
- The in-degree of a node is the number of directed edges $Y \rightarrow X$.
- The degree of a graph is the maximal degree of a node in the graph.
- Example: Identify the degree Node I. Ans=3



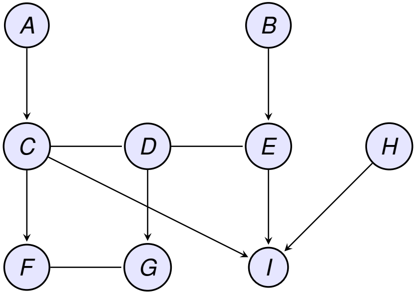
Path

- $X_1, \ldots, X_k$ form a path in the graph $\mathcal{K} = (\mathcal{X}, \mathcal{E})$, if for every $i = 1, \ldots, k-1$ we have either $X_i \rightarrow X_{i+1}$ or $X_i - X_{i+1}$.
- Example: Identify a path. Ans: $A \rightarrow C \rightarrow I$



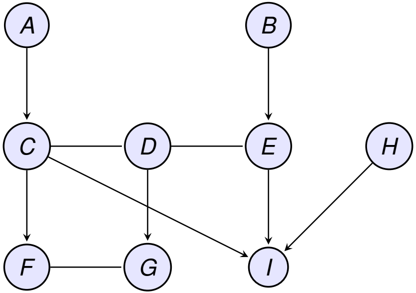
Trail

- $X_1, \ldots, X_k$ form a trail in the graph $\mathcal{K} = (\mathcal{X}, \mathcal{E})$, if for every $i = 1, \ldots, k-1$ we have either $X_i \rightleftharpoons X_{i+1}$.
- A graph is connected, if there is a trail between X_i and X_j.
- Example: Identify a trail. Ans: $A \rightarrow C - D - E \rightarrow I$



Ancestor and Descendant

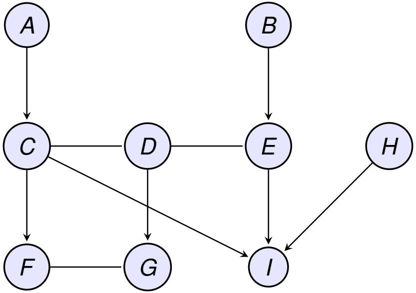
- X is an ancestor of Y in a graph $\mathcal{K}$ if there is a directed path $X_1, \ldots, X_k$ with $X_1 = X$ and $X_k = Y$.
- Y is the descendant of X.
- $Ancestor_X$ and $Descendant_X$



Cycle

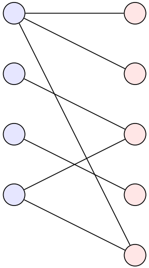
- A cycle in graph $\mathcal{K}$ is a directed path $X_1, \ldots, X_k$ with $X_1 = X_k$.
- Example: Identify a cycle.

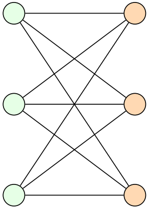
  Ans: No cycle



Bipartite Graphs

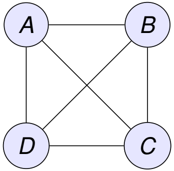
- If the vertex-set of a graph $\mathcal{G}$ can be split into two disjoint sets, V_1 and V_2 , in such a way that each edge in the graph joins a vertex in V_1 to a vertex in V_2 , and there are no edges in $\mathcal{G}$ that connect two vertices in V_1 or two vertices in V_2 , then the graph $\mathcal{G}$ is called a bipartite graph.
- Document-Terms, Student-Class, Movie preference of viewers

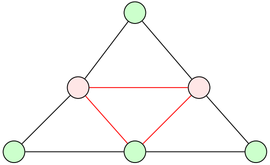




Clique

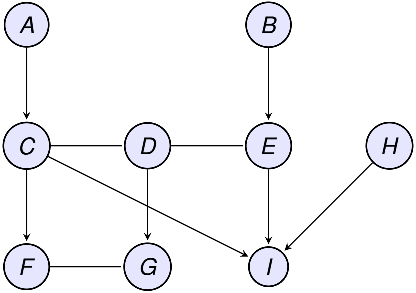
- Clique is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent.
- A maximum clique of a graph, is a clique, such that there is no clique with more vertices.





Directed Acyclic Graph (DAG)

- A graph is acyclic if it contains no cycles.
- A directed acyclic graph is a graph that has directed edges but no cycles.
- DAG is the basic graphical representation of Bayesian Networks.



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Uncertainty

- We select a course of actions among many possibilities.
- Decisions may be based on the information obtained from the environment, previous knowledge and the objectives.
- Eg: It looks cloudy. Should I carry umbrella?
- The information and knowledge is incomplete or unreliable. So the decisions made are not certain. We make decisions under uncertainty.
- One of the goals of AI is to develop systems that can reason and make decisions under uncertainty.

Uncertainty

- Due to
  - Partial observability
  - Non-determinism
- Complexity increases
  - Each piece of knowledge may not be independently used to arrive at decisions.
  - Deduced facts are maintained along with new facts. This increases the knowledge base.
- Examples
  - A medical doctor in an emergency.
  - An autonomous vehicle that detects what might be an obstacle in its way.
  - A financial agent needs to select the best investment.

Uncertain Reasoning

Example

- Diagnosing a dental patients' toothache.
- Toothache may be causes of various reasons.
  Equation using propositional logic:

$$Toothache \implies Cavity \lor GumProblem \lor Abscess \lor \ldots$$

- Change to a causal rule.

$$Cavity \implies Toothache$$

  But not all cavity cause toothache.
- So make logically exhaustive.

Uncertain Reasoning

3 reasons for failure when using logic in Judgmental domains [medical diagnosis, law, business, design, automobile repair, gardening, ...]

- Laziness – complete set of antecedents and consequences
- Theoretical ignorance – no complete theory
- Practical ignorance – not all tests can be run

Belief and Degree of Belief

- Belief State is a representation of a set of all possible world states.
- Agent's knowledge can provide only a degree of belief.
- Tool to deal with Degree of Belief is Probability Theory.

Belief is derived from

1. statistical data.
2. some general knowledge.
3. combination of evidence sources.

Probabilistic Statements

- Probability statements instead of propositional logic.
- Probability statements are made with respect to knowledge state.

Example

1. Probability that a patient has a cavity, given that she has toothache is 0.8.
2. Probability that a patient has a cavity, given that she has toothache and a history of gum disease is 0.4.

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Sample Space

- A sample space Ω specifies set of all possible outcomes that we want to consider.

$$\text{Coin toss} \qquad \Omega = \{H, T\}$$

$$\text{Die Roll} \qquad \Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}$$

- Probability of an outcome $P(\omega)$ specifies the chance or probability with each possible outcome.

$$P(H) = 0.5$$

$$P(⚄) = \frac{1}{6}$$

Measurable Event

- An event space $\mathcal{S}$ or Φ is a subset of outcomes, where probabilities can be assigned.
- We are interested in the set of outcomes.

Even die roll $E = \{$⚁, ⚃, ⚅$\}$

Prime die roll $M = \{$⚁, ⚂, ⚄$\}$

- Properties of Event Space
  - Event space contains empty event ϕ and the trivial event Ω.
  - It is closed under union.

$$\text{If} \quad \alpha, \beta \in \mathcal{S}, \quad \text{then} \quad \alpha \cup \beta \in \mathcal{S}$$

  - It is closed under complementary.

$$\text{If} \quad \alpha \in \mathcal{S}, \quad \text{then} \quad \Omega - \alpha \in \mathcal{S}$$

Probability of Event

- Probability of an event is given by the sum of the probabilities of the outcomes it contains.

$$P(\alpha) = \sum_{\omega \in \alpha} P(\omega) \tag{6}$$

$$\text{Even die roll} \qquad P(E) = \frac{3}{6} = 0.5$$

$$\text{Prime die roll} \qquad P(M) = \frac{3}{6} = 0.5$$

Prior Probability

- Prior or Unconditional probabilities refer to degree of belief in the absence of any other information.

$$P(DieTotal = 11) = P((5, 6)) + P((6, 5)) = 1/18$$

Evidence

- The information that has already been revealed is called evidence.
  - She is having toothache. $Toothache = True$ *or* $toothache$
  - We roll a dice and we get 5. $Die_1 = 5$

Posterior Probability

- Conditional or Posterior probability refer to the probability of some event occurring given a particular condition.

$$P(cavity \mid toothache) = 0.6$$

- Condition on all evidences that has been observed.

$$P(\alpha|\beta) = \frac{P(\alpha \wedge \beta)}{P(\beta)} \qquad where\ P(\beta) > 0 \tag{7}$$

Probability Model

- Associate a numerical probability $P(\omega)$ with each event $\mathcal{S}$.
- Axioms of probability theory

$$P(\omega) \geq 0 \tag{8}$$

$$P(\Omega) = \sum_{\omega \in \Omega} P(\omega) = 1 \tag{9}$$

$$P(\phi) = 0 \tag{10}$$

$$P(\alpha \mid \beta) = \frac{P(\alpha \wedge \beta)}{P(\beta)} \qquad \textit{where } P(\beta) > 0 \tag{11}$$

$$P(\alpha \vee \beta) = P(\alpha) + P(\beta) - P(\alpha \wedge \beta) \tag{12}$$

Bayes Rule

- Conditional probabilities can be derived from the prior given the evidence.

$$P(\alpha|\beta) = \frac{P(\beta|\alpha)P(\alpha)}{P(\beta)} \qquad \textit{where } P(\beta) > 0 \tag{13}$$

Example 1

- Consider the student population, and let Smart denote smart students and GradeA denote students who got grade A. Based on estimates from past statistics assume that $P(GradeA|Smart) = 0.6$, the probability for students being smart is 0.3 and the prior probability of students receiving high grades is 0.2. Estimate the probability that the student is smart.

$$\text{Given,} P(Smart) = 0.3$$
$$P(GradeA) = 0.2$$
$$P(GradeA|Smart) = 0.6$$

According to Bayes' rule

$$P(Smart|GradeA) = \frac{0.6 * 0.3}{0.2} = 0.9$$

Example 2

- Suppose that a tuberculosis (TB) skin test is 95 percent accurate. Suppose that 1 in 1000 of the subjects who get tested is infected. What is the probability of getting a positive test result?

$$\begin{aligned}
\text{Given}, P(TB) &= 0.001 \\
P(\text{infected subjects get a positive result}) &= 0.001 * 0.95 \\
P(\text{uninfected subjects get a positive result}) &= 0.999 * 0.05 \\
P(Positive) &= 0.001 * 0.95 + 0.999 * 0.05 = 0.0509 \\
\text{According to Bayes' rule,} & \\
P(TB|Positive) &= \frac{0.001 * 0.95}{0.0509} = 0.0187 \\
&< 2\%
\end{aligned}$$

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Random Variable

- Variables used in probability theory.
- Uppercase letter
- A random variable X is defined by a function that associates with each outcome in Ω a value or a state.

$$\mathbf{P}(attribute = value) = \mathbf{P}(\omega \in \Omega : f_{attribute} = value)$$

$$\mathbf{P}(X = x) = \mathbf{P}(\omega \in \Omega : X(\omega) = x) \tag{14}$$

- Use $P(x)$ as a shorthand for $P(X = x)$.

$$\sum_{x \in Val(X)} P(X = x) = \sum_{x} P(x) = 1 \tag{15}$$

Domain of Random Variable

- Every random variable has a domain, the set of possible values it can take.
- Finite random variable or Infinite random variable.
- Domain can be discrete or continuous (integer or real).
  Weather random variable – Domain $= \{sunny, overcast, rainy, cloudy, snow\}$
- Boolean Random variable
  - $Domain = Val(X) = \{true, false\}$
  - x^1 to denote *true* and x^0 to denote *false*.
  - The distribution of binary random variable is called a Bernoulli distribution.

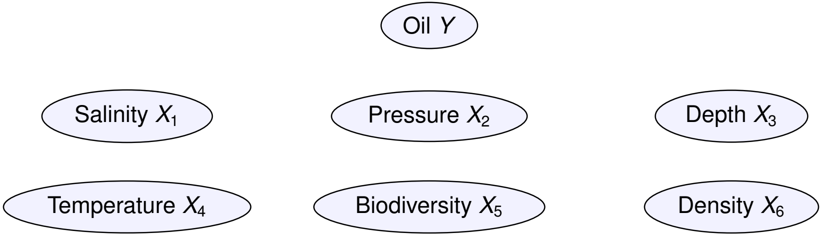
Joint Distribution

- The distribution over several random variables are described using joint distribution.
- Set of random variables $\mathcal{X} = \{X_1, X_2, \ldots, X_n\}$
- Joint Distribution $\mathbf{P}(\mathcal{X}) = P(X_1, X_2, \ldots, X_n)$.
- Eg: Suppose random variable *Grade* reports the final grade of a student and the student's intelligence is given by *Intelligence*. Then the joint distribution $\mathbf{P}(Intelligence, Grade)$

		Intelligence	
		low	high
	A	0.07	0.18
Grade	B	0.28	0.09
	C	0.35	0.03

Why Joint Distribution?

- Once we know the joint distribution of the participating random variables, all inference queries regarding Marginal Distribution and Conditional Probability Distribution can be answered.



$$JD = P(Y, X_1, X_2, X_3, X_4, X_5, X_6)$$

Marginal Distribution

- The distribution over events that can be described using X is often referred to as the marginal distribution over the random variable X.
- Summing out
- Marginal distribution is denoted by $P(X)$.
- Row-wise or Column-wise summations in the JD gives MD.

$$\mathbf{P}(Y) = \sum_{z \in Z} \mathbf{P}(Y, z) \tag{16}$$

- Sum over all the possible combinations of values of the set of varaibles of Z.

Marginal Distribution

- Eg: Suppose random variable *Grade* reports the final grade of a student . Then the marginal distribution of *Grade*

$$P(Grade = A) = 0.25$$
$$P(Grade = B) = 0.37$$
$$P(Grade = C) = 0.38$$
$$\sum P(Grade) = 1$$

Conditional Probability Distribution

- The conditional distribution over a random variable given an observation of the value of another random variable is referred to as Conditional Probability Distribution.
- Compute conditional probability of some variable given evidence about others.

$$\mathbf{P}(Y) = \sum_{z} \mathbf{P}(Y \mid z) P(z) \tag{17}$$

- Eg: What is the probability for the student to have high intelligence given that the grade scored is A.

$$\begin{aligned} P(Intelligence = high | Grade = A) &= \frac{0.18}{0.25} = 0.72 \\ &\neq P(Intelligence) \\ &\neq \text{Marginal} \quad \text{Distribution} \end{aligned}$$

Exercise

Given Joint Distribution $\mathbf{P}(Cavity, Toothache, Catch)$ of 3 binary random variables.

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

1. Compute $P(cavity \lor toothache)$.
2. Compute $P(cavity)$.
3. Compute $P(cavity)$ given the evidence of toothache .

Exercise - Answer to Q1

	toothache		$\neg$*toothache*	
	catch	$\neg$*catch*	*catch*	$\neg$*catch*
cavity	0.108	0.012	0.072	0.008
$\neg$*cavity*	0.016	0.064	0.144	0.576

Compute $P(cavity \lor toothache)$.

$$
\begin{aligned}
P(cavity \lor toothache) &= 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 \\
&= 0.28
\end{aligned}
$$

Exercise - Answer to Q2

	toothache		*¬toothache*	
	catch	*¬catch*	*catch*	*¬catch*
cavity	0.108	0.012	0.072	0.008
¬cavity	0.016	0.064	0.144	0.576

Compute $P(cavity)$.

$$\begin{aligned} P(cavity) &= 0.108 + 0.012 + 0.072 + 0.008 \\ &= 0.2 \end{aligned}$$

This is the unconditional or marginal probability.

Exercise - Answer to Q3

Compute $P(cavity)$ given the evidence of toothache.

$$
\begin{aligned}
\mathbf{P}(Cavity \mid toothache) &= \frac{\mathbf{P}(Cavity \vee toothache)}{\mathbf{P}(toothache)} \\
&= \alpha \mathbf{P}(Cavity \vee toothache) \qquad \text{Let } \alpha = 1/\mathbf{P}(toothache) \\
&= \alpha \left[P(Cavity, toothache, catch) + P(Cavity, toothache, \neg catch) \right] \\
&= \alpha [< 0.108, 0.016 > + < 0.012, 0.064 >] \\
&= \alpha < 0.12, 0.08 > \qquad \alpha = \frac{1}{0.12 + 0.08} = \frac{1}{0.2} \\
&= < 0.6, 0.4 > \\
P(cavity) &= 0.6
\end{aligned}
$$

Challenges with Joint Distribution

- Computational: Expensive to manipulate and too large to to store.
- Cognitive: Impossible to acquire so many numbers from a human.
- Statistical: Need huge amounts of data to learn the parameters.

Need to compactly represent joint distribution.
Solution: Exploit independence or conditional independence among random variables.

Independence of Events

- An event α is independent of event β in P,

$$P(\alpha \perp \beta) \quad \text{if} \quad \begin{cases} P(\alpha \mid \beta) = P(\alpha) & \text{or} \\ P(\beta \mid \alpha) = P(\beta) & \text{or} \\ P(\alpha \wedge \beta) = P(\alpha)P(\beta) \end{cases} \tag{18}$$

- $\alpha \perp \beta$ implies $\beta \perp \alpha$
- Eg: Tossing two coins, Rolling a die.

Independence of Random Variables

- Two random variables X and Y can be independent of each other.
- A random variable X is independent of another random variable Y,

$$P(X \perp Y) \quad \text{if} \quad \begin{cases} P(X \mid Y) = P(X) & \text{or} \\ P(Y \mid X) = P(Y) & \text{or} \\ P(X \wedge Y) = P(X)P(Y) \end{cases} \tag{19}$$

- $X \perp Y$ implies $Y \perp X$
- Eg: Weather is independent of dental problems.

Conditional Independence of Events

- An event α is conditionally independent of event β given event γ in P

$$P(\alpha \perp \beta \mid \gamma) \quad \text{if} \quad \begin{cases} P(\alpha \mid \beta \wedge \gamma) = P(\alpha \mid \gamma) \\ P(\alpha \mid \beta \wedge \gamma) = P(\beta \mid \gamma) = 0 \\ P(\alpha \wedge \beta \mid \gamma) = P(\alpha \mid \gamma)P(\beta \mid \gamma) \end{cases} \tag{20}$$

- Eg: Getting Admission in MIT is independent of getting admission in Stanford, given the student has scored Grade A.

Conditional Independence of Random Variables

- A random variable X is conditionally independent of random variable Y given random variable Z

$$P(X \perp Y \mid Z) \quad \text{if} \quad \begin{cases} P(X \mid Y, Z) = P(X \mid Z) \\ P(X \mid Y, Z) = P(Y \mid Z) \\ P(X, Y \mid Z) = P(X \mid Z)P(Y \mid Z) \end{cases} \tag{21}$$

- Eg: The random variables Toothache and Catch are independent given the presence or absence of Cavity.

Student Example

- Model the difficulty of a course, intelligence of students, Grade the students score in a particular course.
- Let D represent the difficulty of a course.

$$
\begin{aligned}
\textit{Domain of } \mathbf{D} &= \{easy, hard\} \\
\mathbf{P}(\mathbf{D}) &= \{d^0, d^1\} \\
&= \{0.6, 0.4\}
\end{aligned}
$$

- Let I represent the intelligence of a student.

$$
\begin{aligned}
\textit{Domain of } \mathbf{I} &= \{low, high\} \\
\mathbf{P}(\mathbf{I}) &= \{i^0, i^1\} \\
&= \{0.7, 0.3\}
\end{aligned}
$$

Student Example

- Let G represent the grade a student gets for a course.

$$\textit{Domain of } \mathbf{G} = \{A, B, C\}$$
$$\mathbf{P}(\mathbf{G}) = \{g^1, g^2, g^3\}$$

- $\mathbf{P}(\mathbf{D}, \mathbf{I}, \mathbf{G})$ denotes the probabilities of all combinations of the values of the 3 random variables.
- These $2 * 3 * 3 = 12$ values can be represented using a Joint Distribution Table.

Joint Probability Distribution

- Joint Probability Distribution completely represents the joint distribution for all random variables.
- In the students example, the $\mathbf{P}(\mathbf{D}, \mathbf{I}, \mathbf{G})$, the 12 parameters cannot be determined by the value of the other parameters. Hence called Independent parameters.
- Independent parameters are parameters whose values are not completely determined by the values of the other parameters.

Student Example - Joint Distribution

I	D	G	P
i^0	d^0	g^1	0.126
		g^2	0.168
		g^3	0.126
i^0	d^1	g^1	0.009
		g^2	0.045
		g^3	0.126
i^1	d^0	g^1	0.052
		g^2	0.0224
		g^3	0.0056
i^1	d^1	g^1	0.069
		g^2	0.036
		g^3	0.024

$$\sum \mathbf{P(D, I, G)} = 1$$

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Thank You !!!