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Probabilistic Graphical Model

Session # 6 : Undirected Graphical Model

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The instructor is gratefully acknowledging the authors who made their course materials freely available online.

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Scenario 1

- Four people; Alice, Bob, Charlie, Diana; go out for dinner in different groups of two.
- Alice goes out with Bob, Bob goes out with Charlie, Charlie with Diana, and Diana with Alice.
- Bob doesn't go with Diana, and Alice doesn't go with Charlie.
- Let's think about the probability of them ordering food of the same cuisine.
- From our social experience, we know that people interacting with each other may influence each others choice of food.
- Alice can influence Bob's choice of cuisine. Bob can influence Charlie's choice of cuisine. But Alice and Charlie wont agree.
- How can we represent this in Bayesian Network?

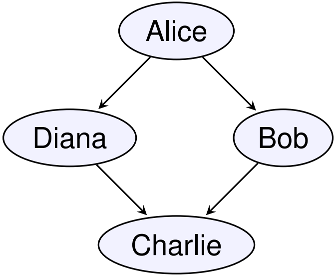
$$Alice \perp Charlie \mid Bob, Diana$$

$$Bob \perp Diana \mid Alice, Charlie$$

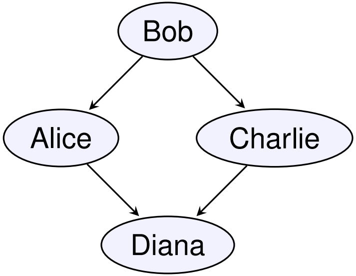
Scenario 1

$$Alice \perp Charlie \mid Bob, Diana \tag{1}$$

$$Bob \perp Diana \mid Alice, Charlie \tag{2}$$



Satisfies Eq(1) but not Eqn(2).



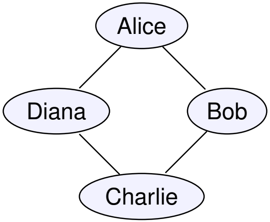
Satisfies Eq(2) but not Eqn(1).

Scenario 1

- Directed models have a limitation that they cannot represent symmetric interactions.
- Undirected graphical model to encode influence flows in both directions.
- Example:

$$\mathit{Alice} \perp \mathit{Charlie} \mid \mathit{Bob}, \mathit{Diana}$$

$$\mathit{Bob} \perp \mathit{Diana} \mid \mathit{Alice}, \mathit{Charlie}$$



Markov Network

Definition

Markov network is an undirected graph, where

- the nodes represent the random variables and
- the dependencies or direct probabilistic interaction between these random variables are represented with undirected edges.

- No parent-child relationship.
- So we do not use CPD.
- Use factor to represent how likely it is for some states of a variable to agree with the states of other variables.

Parameterizing Markov Network

- Markov Networks are parameterized using factors.
- Factors help in symmetric parameterization of random variables.
- Factors capture the affinities between related variables.
- Factors do not represent the probability.
- Factors are not constrained to sum up to 1 or to be in the range [0,1].
- The parameterization of the Markov network defines the local interactions between directly related variables.
- The scope of a factor to be the set of random variables over which it is defined.

FACTOR

- A factor Φ is a function or a table that maps a set of random variables to a real value.

$$\Phi : Val(X_1, \ldots, X_n) \rightarrow \mathrm{R} \tag{3}$$

- The argument of the factor is called scope of the factor.

$$Scope : \{X_1, \ldots, X_n\} \tag{4}$$

- Factors are building blocks used for defining high dimensional spaces and distributions.
- Factors are used to define an exponentially large probability distribution of N random variables.
- Factors are manipulated in the same way as probability distributions.

Operations on Factors

1. Factor Product
   - Factor product refers to the product of factors ϕ_1 with a scope X and ϕ_2 with scope Y to produce a factor ϕ_3 with a scope $X \cup Y$.
2. Factor Marginalization
   - Marginalize a factor ϕ whose scope is W with respect to a set of random variables X, sum out all the entries of X, to reduce its scope to $\{W - X\}$.
3. Factor Reduction
   - Reduction of a factor ϕ whose scope is W to the context $X = x^i$ means removing all the entries from the factor where $X = x^i$. This reduces the scope to $\{W - X\}$.

1. Factor Product

- Factor product is the cross product of two factors.

A	B	$\Phi_1(A,B)$
a^1	b^1	0.5
a^1	b^2	0.8
a^2	b^1	0.2
a^2	b^2	0

B	C	$\Phi_2(B,C)$
b^1	c^1	0.5
b^1	c^2	0.7
b^2	c^1	0.1
b^2	c^2	0.2

A	B	C	$\Phi_3(A,B,C) = \Phi_1 * \Phi_2$
a^1	b^1	c^1	$0.5 * 0.5 = 0.25$
a^1	b^1	c^2	$0.5 * 0.7 = 0.35$
a^1	b^2	c^1	$0.8 * 0.1 = 0.08$
a^1	b^2	c^2	$0.8 * 0.2 = 0.16$
a^2	b^1	c^1	$0.2 * 0.5 = 0.25$
a^2	b^1	c^2	$0.2 * 0.7 = 0.35$
a^2	b^2	c^1	$0 * 0.1 = 0$
a^2	b^2	c^2	$0 * 0.2 = 0$

2. Factor Marginalization

- Remove one random variable.

A	B	C	$\Phi_1(A,B,C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

A	C	$\Phi_2(A,C)$ marginalized on B
a^1	c^1	$0.25 + 0.08 = 0.33$
a^1	c^2	$0.35 + 0.16 = 0.51$
a^2	c^1	$0.25 + 0 = 0.25$
a^2	c^2	$0.35 + 0 = 0.35$

3. Factor Reduction

- Extract only one random variable.
- Observe $C = c^1$.

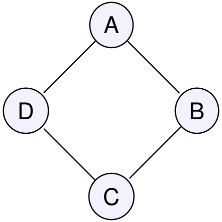
A	B	C	$\Phi_1(A, B, C)$
a^1	b^1	c^1	0.25
a^1	b^1	c^2	0.35
a^1	b^2	c^1	0.08
a^1	b^2	c^2	0.16
a^2	b^1	c^1	0.25
a^2	b^1	c^2	0.35
a^2	b^2	c^1	0
a^2	b^2	c^2	0

A	B	C	$\Phi_1(A, B, c^1)$
a^1	b^1	c^1	0.25
a^1	b^2	c^1	0.08
a^2	b^1	c^1	0.25
a^2	b^2	c^1	0

Factor

D	A	$\phi(D, A)$
d^0	a^0	100
d^0	a^1	1
d^1	a^0	1
d^1	a^1	100

A	B	$\phi(A, B)$
a^0	b^0	90
a^0	b^1	5
a^1	b^0	1
a^1	b^1	10



C	D	$\phi(C, D)$
c^0	d^0	1
c^0	d^1	100
c^1	d^0	100
c^1	d^1	1

B	C	$\phi(B, C)$
b^0	c^0	100
b^0	c^1	1
b^1	c^0	1
b^1	c^1	100

Queries using Factors

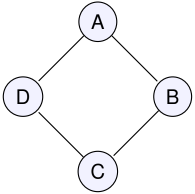
- Compute the probability corresponding to a^1, b^1, c^0, d^1.

$$\tilde{P}(a^1, b^1, c^0, d^1) = \phi_1(a^1, b^1) \times \phi_2(b^1, c^0) \times \phi_3(c^0, d^1) \times \phi_4(d^1, a^1)$$
$$= 10 * 1 * 100 * 100 = 700,000$$

Factor Product

D	A	$\phi_4(D, A)$
d^0	a^0	80
d^0	a^1	60
d^1	a^0	20
d^1	a^1	10

C	D	$\phi_3(C, D)$
c^0	d^0	10
c^0	d^1	1
c^1	d^0	100
c^1	d^1	90



A	B	$\phi_1(A, B)$
a^0	b^0	90
a^0	b^1	100
a^1	b^0	1
a^1	b^1	10

B	C	$\phi_2(B, C)$
b^0	c^0	10
b^0	c^1	80
b^1	c^0	70
b^1	c^1	30

A	B	C	D	$\tilde{P}(A, B, C, D) = \Phi(A, B, C, D)$
a^0	b^0	c^0	d^0	$90 \cdot 10 \cdot 10 \cdot 80 = 720,000$
a^0	b^0	c^0	d^1	$90 \cdot 10 \cdot 1 \cdot 20 = 18,000$
a^0	b^0	c^1	d^0	$90 \cdot 80 \cdot 100 \cdot 80 = 57600,000$
a^0	b^0	c^1	d^1	$90 \cdot 80 \cdot 90 \cdot 20 = 12960,000$
a^0	b^1	c^0	d^0	$100 \cdot 70 \cdot 10 \cdot 80 = 5600,000$
a^0	b^1	c^0	d^1	$100 \cdot 70 \cdot 1 \cdot 20 = 140,000$
a^0	b^1	c^1	d^0	$100 \cdot 30 \cdot 100 \cdot 80 = 24000,000$
a^0	b^1	c^1	d^1	$100 \cdot 30 \cdot 90 \cdot 20 = 5400,000$
a^1	b^0	c^0	d^0	$1 \cdot 10 \cdot 10 \cdot 60 = 6,000$
a^1	b^0	c^0	d^1	$1 \cdot 10 \cdot 1 \cdot 10 = 100$
a^1	b^0	c^1	d^0	$1 \cdot 80 \cdot 100 \cdot 60 = 480,000$
a^1	b^0	c^1	d^1	$1 \cdot 80 \cdot 90 \cdot 10 = 72,000$
a^1	b^1	c^0	d^0	$10 \cdot 70 \cdot 10 \cdot 60 = 420,000$
a^1	b^1	c^0	d^1	$10 \cdot 70 \cdot 1 \cdot 10 = 70,000$
a^1	b^1	c^1	d^0	$10 \cdot 30 \cdot 100 \cdot 60 = 1800,000$
a^1	b^1	c^1	d^1	$10 \cdot 30 \cdot 90 \cdot 10 = 270,000$

Factor Product

- Factor Product

$$\tilde{P}(A, B, C, D) = \phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A) \tag{5}$$

  is un-normalized. It is not probability distribution.
- Normalize $\tilde{P}(A, B, C, D)$ using partition function Z. Z is called the partition function and is function of the parameters.

$$Z = \sum_{A,B,C,D} \tilde{P}(A, B, C, D) \tag{6}$$

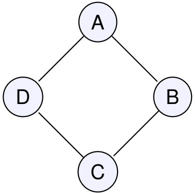
- Normalized factor product

$$P(A, B, C, D) = \frac{1}{Z}\tilde{P}(A, B, C, D) \tag{7}$$

Normalized Factor Product

D	A	$\phi_4(D, A)$
d^0	a^0	80
d^0	a^1	60
d^1	a^0	20
d^1	a^1	10

A	B	$\phi_1(A, B)$
a^0	b^0	90
a^0	b^1	100
a^1	b^0	1
a^1	b^1	10



C	D	$\phi_3(C, D)$
c^0	d^0	10
c^0	d^1	1
c^1	d^0	100
c^1	d^1	90

B	C	$\phi_2(B, C)$
b^0	c^0	10
b^0	c^1	80
b^1	c^0	70
b^1	c^1	30

A	B	C	D	$\tilde{P}(A, B, C, D)$	$P(A, B, C, D)$
a^0	b^0	c^0	d^0	720, 000	0.0055
a^0	b^0	c^0	d^1	18, 000	0.0001
a^0	b^0	c^1	d^0	57600, 000	0.4365
a^0	b^0	c^1	d^1	12960, 000	0.0982
a^0	b^1	c^0	d^0	5600, 000	0.0424
a^0	b^1	c^0	d^1	140, 000	0.0011
a^0	b^1	c^1	d^0	24000, 000	0.1819
a^0	b^1	c^1	d^1	5400, 000	0.0409
a^1	b^0	c^0	d^0	6, 000	0.0000
a^1	b^0	c^0	d^1	100	0.0000
a^1	b^0	c^1	d^0	480, 000	0.0036
a^1	b^0	c^1	d^1	72, 000	0.0005
a^1	b^1	c^0	d^0	420, 000	0.0318
a^1	b^1	c^0	d^1	70, 000	0.0005
a^1	b^1	c^1	d^0	1800, 000	0.1364
a^1	b^1	c^1	d^1	270, 000	0.0205
				109493,100	

Queries using Factor Product

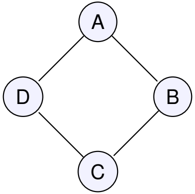
- Compute the probability of B.
  Marginalize wrt A,C,D

$$P(b^1) = 0.4555$$
$$P(b^0) = 0.5445$$

- Compute the probability of B agreeing with C given c^0.

$$P(b^1 | c^0) = 0.0759$$

Factors vs Probability Distribution

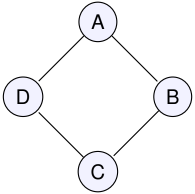


A	B	$\phi_1(A, B)$
a^0	b^0	90
a^0	b^1	100
a^1	b^0	1
a^1	b^1	10

Marginal Probability of A and B

A	B	C	D	$P(A, B, C, D)$	$P_\phi(A, B)$
a^0	b^0	c^0	d^0	0.0055	
a^0	b^0	c^0	d^1	0.0001	
a^0	b^0	c^1	d^0	0.4365	
a^0	b^0	c^1	d^1	0.0982	0.5403
a^0	b^1	c^0	d^0	0.0424	
a^0	b^1	c^0	d^1	0.0011	
a^0	b^1	c^1	d^0	0.1819	
a^0	b^1	c^1	d^1	0.0409	0.2663
a^1	b^0	c^0	d^0	0.0000	
a^1	b^0	c^0	d^1	0.0000	
a^1	b^0	c^1	d^0	0.0036	
a^1	b^0	c^1	d^1	0.0005	0.0042
a^1	b^1	c^0	d^0	0.0318	
a^1	b^1	c^0	d^1	0.0005	
a^1	b^1	c^1	d^0	0.1364	
a^1	b^1	c^1	d^1	0.0205	0.1892

Factors vs Probability Distribution



A	B	$\phi_1(A,B)$
a^0	b^0	90
a^0	b^1	100
a^1	b^0	1
a^1	b^1	10

A	B	$P_\phi(A,B)$
a^0	b^0	0.5403
a^0	b^1	0.2663
a^1	b^0	0.0042
a^1	b^1	0.1892

There is no natural mapping between factors and probability distribution.

Marginal Probability of A and B

A	B	C	D	$P(A,B,C,D)$	$P_\phi(A,B)$
a^0	b^0	c^0	d^0	0.0055	
a^0	b^0	c^0	d^1	0.0001	
a^0	b^0	c^1	d^0	0.4365	
a^0	b^0	c^1	d^1	0.0982	0.5403
a^0	b^1	c^0	d^0	0.0424	
a^0	b^1	c^0	d^1	0.0011	
a^0	b^1	c^1	d^0	0.1819	
a^0	b^1	c^1	d^1	0.0409	0.2663
a^1	b^0	c^0	d^0	0.0000	
a^1	b^0	c^0	d^1	0.0000	
a^1	b^0	c^1	d^0	0.0036	
a^1	b^0	c^1	d^1	0.0005	0.0042
a^1	b^1	c^0	d^0	0.0318	
a^1	b^1	c^0	d^1	0.0005	
a^1	b^1	c^1	d^0	0.1364	
a^1	b^1	c^1	d^1	0.0205	0.1892

Factorization and Independencies

- $P \models (B \perp D | A, C)$ should have a decomposition

$$P = \frac{1}{Z}[\phi_1(A, B) \times \phi_2(B, C)] \times \phi_3(C, D) \times \phi_4(D, A)$$

B and D are separated given A and C.

- $P \models (A \perp C | B, D)$ should have a decomposition

$$P = \frac{1}{Z}[\phi_4(D, A) \times \phi_1(A, B)] \times \phi_2(B, C) \times \phi_3(C, D)$$

A and C are separated given B and D.

Factorization and Independencies

$$P \models (X \perp Y | Z) \quad \text{if and only if} \quad P = \phi_1(X, Z)\phi_2(Y, Z) \tag{8}$$

- Independence properties of the distribution P correspond directly to separation properties in the graph over which P factorizes.

Gibbs Distribution

Definition

A distribution P_Φ is called a Gibbs distribution parameterized by a set of factors $\Phi = \{\phi_1(D_1), \ldots, \phi_k(D_k)\}$ if it can be expressed as product of the factors.

$$P_\Phi(X_i, \ldots, X_n) = \frac{1}{Z_\Phi}[\phi_1(D_1) \times \ldots \times \phi_k(D_k)]$$

$$\tilde{P}(X_i, \ldots, X_n) = \prod_{i=1}^{k} \phi_i(D_i) \tag{9}$$

$$Z_\Phi = \sum_{X_i, \ldots, X_n} \tilde{P}(X_i, \ldots, X_n) \tag{10}$$

$$P_\Phi(X_i, \ldots, X_n) = \frac{1}{Z_\Phi}\tilde{P}(X_i, \ldots, X_n) \tag{11}$$

(12)

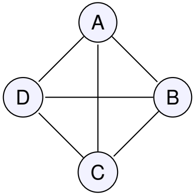
Gibbs Distribution

Definition

A distribution P_Φ with $\Phi = \{\phi_1(D_1), \ldots, \phi_k(D_k)\}$ factorizes over a Markov Network $\mathcal{H}$ if each D_k is a complete subgraph of $\mathcal{H}$.

- The factors that parametrize a Markov network are often called clique potentials.
- Reduce the number of factors in the parameterization by allowing factors only for maximal cliques.
- Let $C_1, \ldots, C_k$ be the cliques in $\mathcal{H}$.
- Parametrize P using a set of factors $\phi_1(C_1), \ldots, \phi_l(C_l)$.

Gibbs Distribution Example



- Cliques (Option 1): $\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}, \{D, B\}, \{A, C\}$
- Cliques (Option 2): $\{A, B, D\}, \{B, C, D\}$
- Cliques (Option 3): $\{A, B, C\}, \{A, C, D\}$

Pairwise Markov Network

Definition

Pairwise Markov Network is an undirected graph whose nodes $X_i, \ldots, X_n$ and edges $X_i - X_j$ are associated with a factor $\phi_{ij}(X_i, X_j)$.

- A subclass of Markov networks.
- Eg:

$$P(A, B, C, D) = \frac{1}{Z}[\phi_1(A, B) \times \phi_2(B, C) \times \phi_3(C, D) \times \phi_4(D, A)]$$

- How many parameters for n RV with d values each?

$$\text{Number of parameters in Pairwise Markov Network} = O(n^2 d^2) \tag{13}$$

Induced Markov Network

Definition

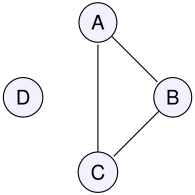
For a set of factors ϕ_i, with a scope D_i, the Induced Markov Network H_Φ, has an edge between a pair of variables X_i and X_j whenever there exits a factor $\phi_m \in \Phi$ such that $X_i, X_j \in D_m$.

- X and Y will have an undirected edge
  - if they appear together in some factor ϕ
  - if there exists a factor $\phi(X, Y)$.

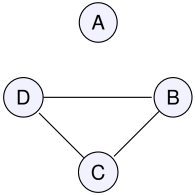
Induced Markov Network

Consider 4 RVs A,B,C,and D. The factor and its induced Markov Network is given below.

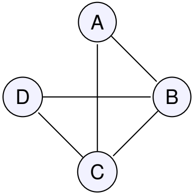
$\phi_1(A, B, C)$



$\phi_2(B, C, D)$



$$\Phi = \phi_1(A, B, C) \times \phi_2(B, C, D)$$



P factorizes H

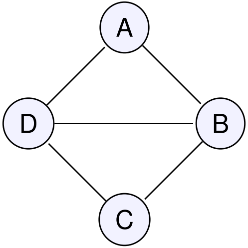
Definition

Gibbs distribution P factorizes a Markov Network H if there exists $\Phi = \{\phi_1(D_1), \ldots \phi_k(D_k)\}$ such that

- $P = P_\Phi$, normalized product of factors ϕ_i
- H is the induced graph for Φ.

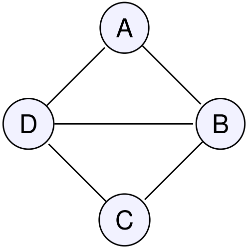
P FACTORIZES H

- From an induced Markov network H, we cannot read the factorization P_Φ from the graph, as there can be multiple possible factorization.



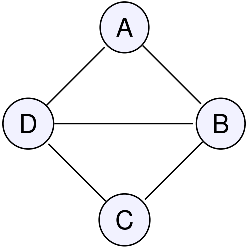
- $\phi_1(A, B, D), \phi_2(B, C, D)$
- $\phi_5(A, B), \phi_6(B, C), \phi_7(C, D), \phi_8(D, A), \phi_9(A, B$

Flow of Influence



- $\phi_1(A, B, D), \phi_2(B, C, D)$
- $\phi_5(A, B), \phi_6(B, C), \phi_7(C, D), \phi_8(D, A), \phi_9(B, D)$
- When can B influence D?
- When can A influence C?

Flow of Influence



- $\phi_1(A, B, D), \phi_2(B, C, D)$
- $\phi_5(A, B), \phi_6(B, C), \phi_7(C, D), \phi_8(D, A), \phi_9(B, D)$
- When can B influence D?
  - Direct influence
  - $\phi_1(A, B, D)$
  - $\phi_9(B, D)$
- When can A influence C?
  - Indirect influence
  - Through B or D
  - $\phi_1(B, C, D)$
  - $\phi_1(A, B, D)\phi_2(B, C, D)$
  - $\phi_5(A, B), \phi_6(B, C)$
  - $\phi_7(C, D), \phi_8(D, A)$

Flow of Influence

- Parameterization of the distributions are different.
- The trails in the graph through which influence can flow are the same.
- Active trails depend only on the graph structure.

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Thank You !!!