Fuzzy Logic

Properties of Fuzzy Sets, Variations of FIS

n.h.reyes@massey.ac.nz

Properties of Fuzzy Sets

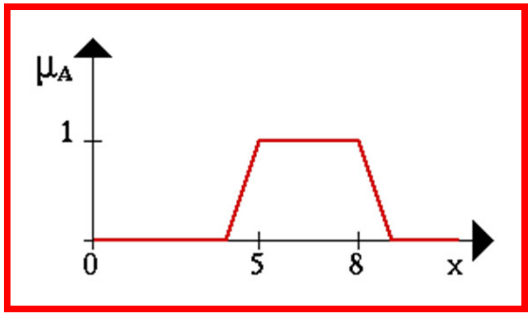
▶ back

menu

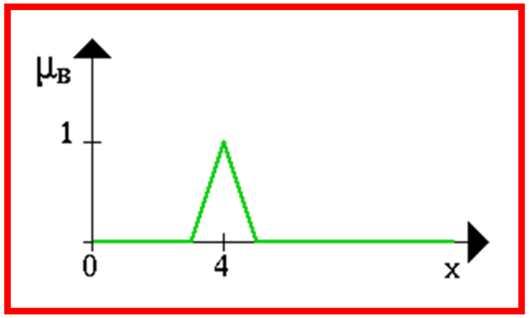
Fuzzy Sets

We will use the following fuzzy sets in explaining the different fuzzy operators that follows next.

Examples:



Fuzzy Set **A**



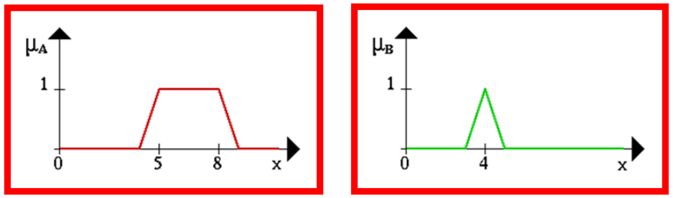
Fuzzy Set **B**

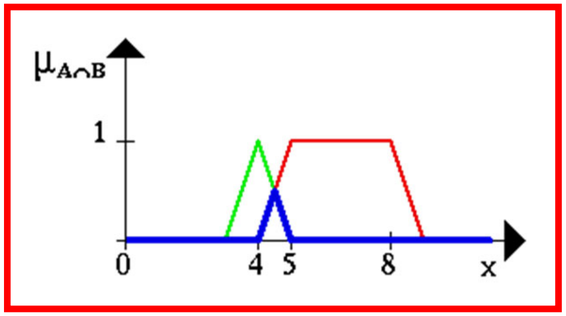
Fuzzy combinations (T-norms)

In making a fuzzy rule, we use the concept of “**and**”, “**or**”, and sometimes “**not**”. The sections below describe the most common definitions of these “fuzzy combination” operators. **Fuzzy intersections** are also referred to as “**T-norms**”.

Fuzzy “and”

Example:





Intersection of A and B

The fuzzy “and” is written as:

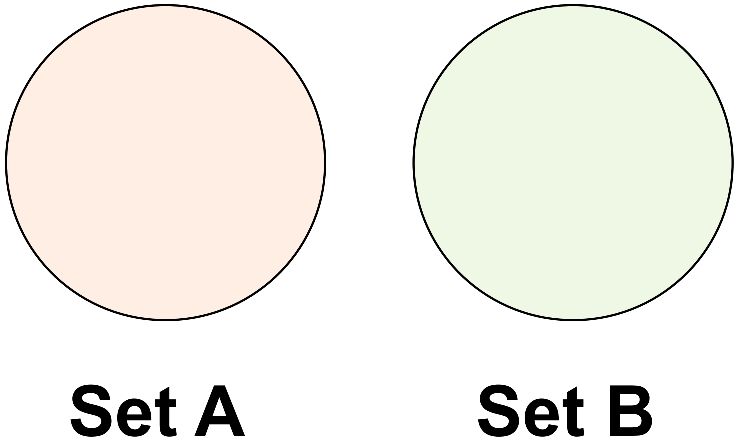
$$u_{A \cap B} = T(u_A(x), u_B(x))$$

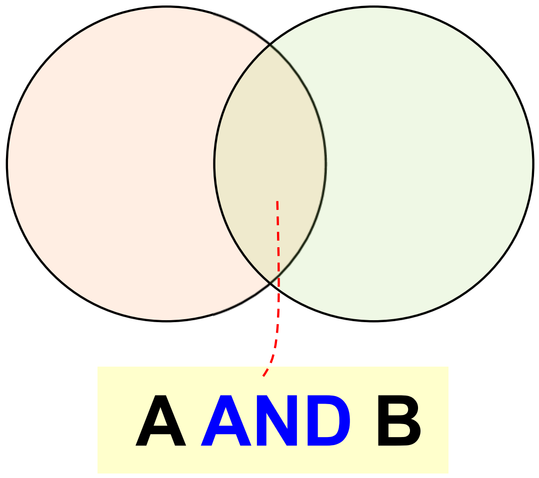
where μ_A is read as “the membership in class A” and μ_B is read as “the membership in class B”.

Recall

Example:

Boolean "AND"

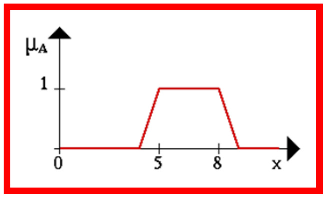




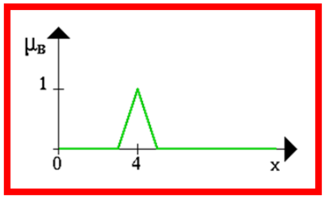
Fuzzy combinations (T-norms)

Example:

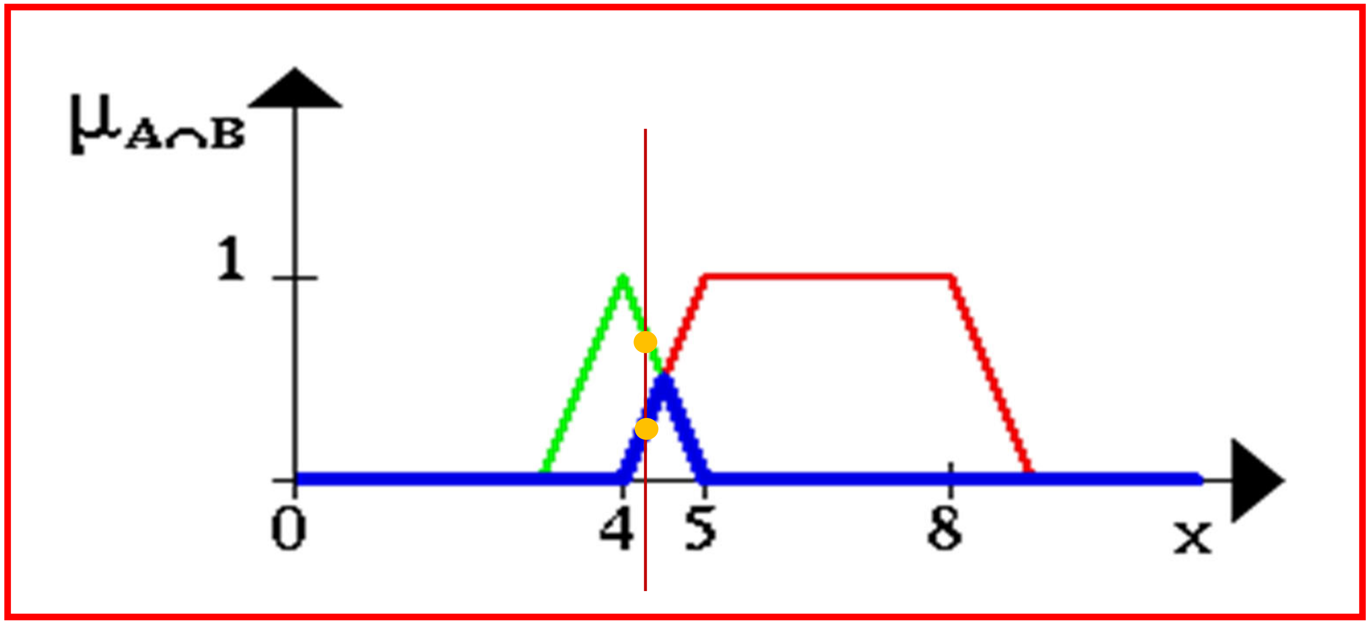
Fuzzy "and"



Fuzzy Set A



Fuzzy Set B



$$\min\left(u_A(x), u_B(x)\right)$$

Intersection of A and B

Fuzzy “and”

There are many ways to compute “**and**”. The two most common are:

Zadeh And - min($\mu_A(x)$, $\mu_B(x)$)

This technique, named after the inventor of fuzzy set Theory; it simply computes the “and” by taking the **minimum** of the two (or more) membership values. This is the most common definition of the fuzzy “and”.

- **Non-Differentiable expression**

Product - $\mu_A(x) * \mu_B(x)$

This technique computes the fuzzy “and” by multiplying the two membership values.

- **Differentiable expression – suitable for Neuro-Fuzzy Systems**

▶ back

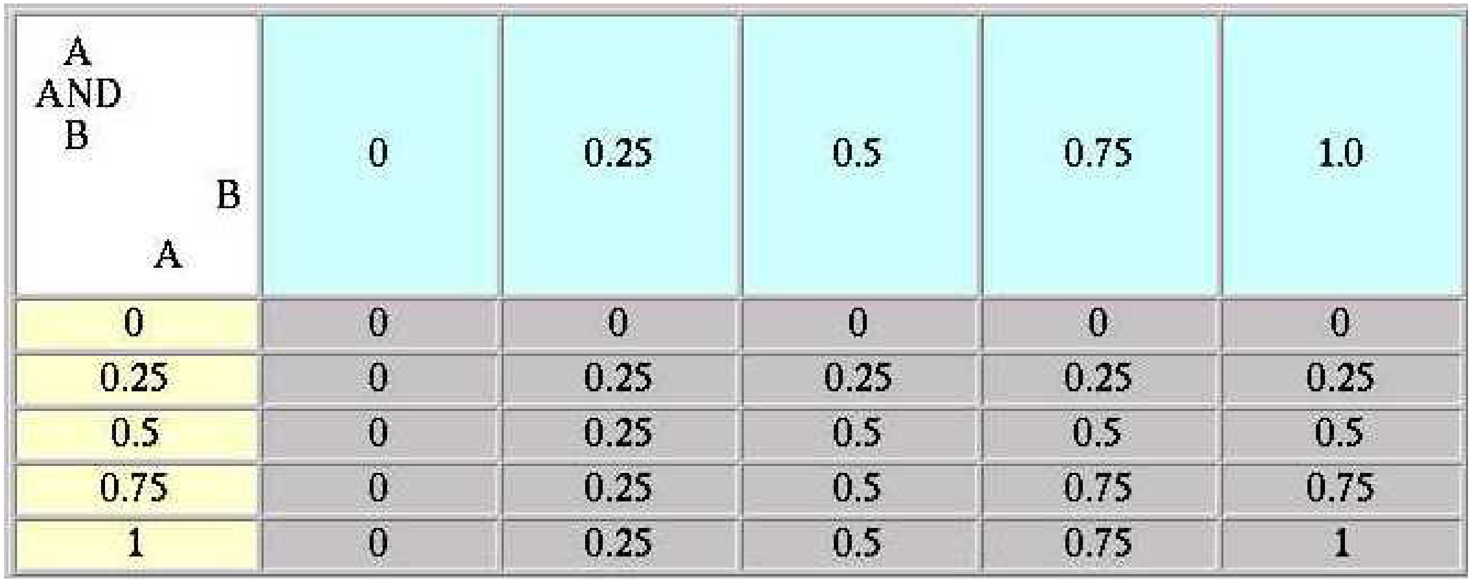
menu

Fuzzy “and”

Both techniques have the following two properties:

$$T(0,0) = T(a,0) = T(0,a) = 0$$
$$T(a,1) = T(1,a) = a$$

One of the nice things about both definitions is that they also can be used to compute the Boolean “and”. The **fuzzy “and”** is an extension of the **Boolean “and”** to numbers that are not just 0 or 1, but between 0 and 1.



A AND B B A	0	0.25	0.5	0.75	1.0
0	0	0	0	0	0
0.25	0	0.25	0.25	0.25	0.25
0.5	0	0.25	0.5	0.5	0.5
0.75	0	0.25	0.5	0.75	0.75
1	0	0.25	0.5	0.75	1

▶ back

menu

Fuzzy “or”

Fuzzy “or”

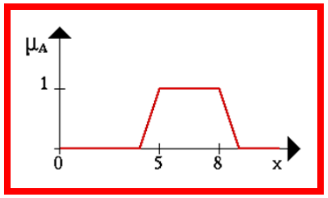
The fuzzy “or” is written as:

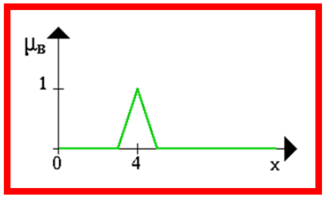
$$u_{A \cup B} = T(u_A(x), u_B(x))$$

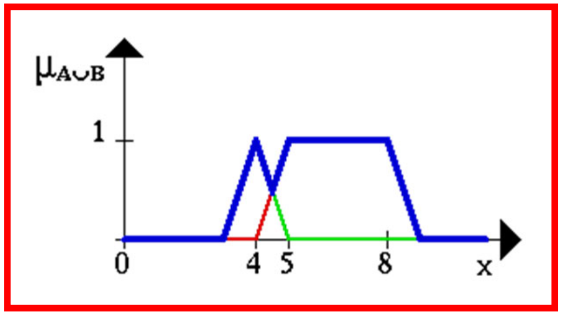
where μ_A is read as “the membership in class A” and μ_B is read as “the membership in class B”.

Fuzzy unions are also referred to as “**T-conorms**” or “**S-norms**”.

Example:







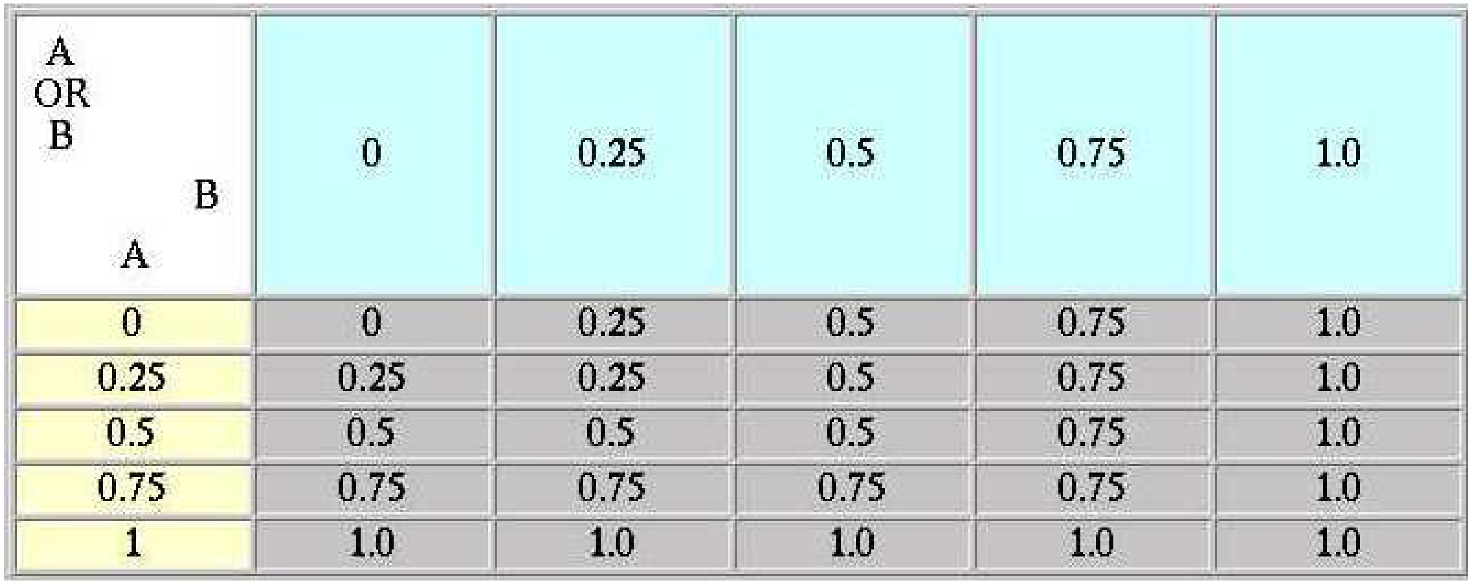
Union of **A** and **B**

▶ back

menu

Fuzzy “or”

The fuzzy “or” is an extension of the Boolean “or” to numbers that are not just 0 or 1, but between 0 and 1.



A OR B B A	0	0.25	0.5	0.75	1.0
0	0	0.25	0.5	0.75	1.0
0.25	0.25	0.25	0.5	0.75	1.0
0.5	0.5	0.5	0.5	0.75	1.0
0.75	0.75	0.75	0.75	0.75	1.0
1	1.0	1.0	1.0	1.0	1.0

back
menu

Fuzzy "or"

There are many ways to compute "**or**". The two most common are:

$$\sigma(x, y) = \max(x, y)$$

Zadeh OR

$$\max(\mu_A(x), \mu_B(x))$$

it simply computes the "or" by taking the **maximum** of the two (or more) membership values. This is the most common definition of the fuzzy "or".

$$\sigma(x, y) = x + y - xy$$

Product

$$(\mu_A(x) + \mu_B(x)) - (\mu_A(x) * \mu_B(x))$$

This technique uses the difference between the sum of the two (or more) membership values and the product of their membership values.

Similar to the fuzzy "and", both definitions of the fuzzy "or" also can be used to compute the Boolean "or".

▶ back

menu

Fuzzy “or”

Other ways to compute Fuzzy “**or**”:

$\sigma(x, y) = \min(1, x + y)$

Lukasiewicz Disjunction

min(1, *μA* (x) + *μB* (x))

$$\sigma(x, y) = \frac{x + y - 2xy}{1 - xy}$$

Hamacher Disjunction

$$\frac{\mu_A(x) + \mu_B(x) - 2\mu_A(x)\mu_B(x)}{1 - \mu_A(x)\mu_B(x)}$$

Fuzzy “or”

Other ways to compute Fuzzy “**or**”:

$$\sigma(x,y)=\frac{x+y}{1+xy}$$

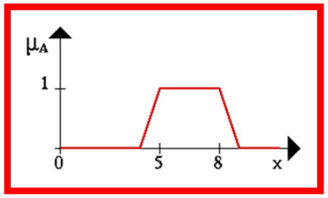
Einstein Disjunction

$$\frac{\mu_A(x)+\mu_B(x)}{1+\mu_A(x)\mu_B(x)}$$

Fuzzy “not”

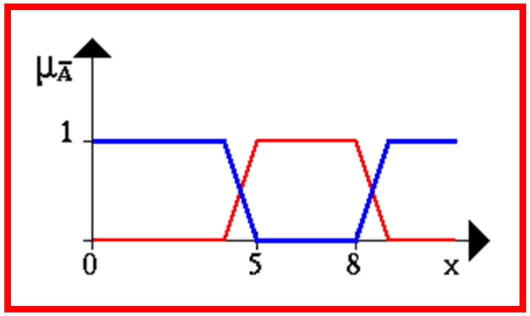
NOT (A) = 1 - A

A	NOT A
0	1
0.25	0.75
0.5	0.5
0.75	0.25
1	0



Fuzzy set **A**

Example:



Negation of **A**

▶ back

menu

Frequently used properties

Define three fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ on the universe X.

- *commutativity*

$$\underset{\sim}{A} \cup \underset{\sim}{B} = \underset{\sim}{B} \cup \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \underset{\sim}{B} = \underset{\sim}{B} \cap \underset{\sim}{A}$$

- *associativity*:

$$\underset{\sim}{A} \cup \left(\underset{\sim}{B} \cup \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cup \underset{\sim}{B}\right) \cup \underset{\sim}{C}$$

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$

- *distributivity*:

$$\underset{\sim}{A} \cup \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cup \underset{\sim}{B}\right) \cap \left(\underset{\sim}{A} \cup \underset{\sim}{C}\right)$$

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cup \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cup \left(\underset{\sim}{A} \cap \underset{\sim}{C}\right)$$

- *idempotency*:

$$\underset{\sim}{A} \cup \underset{\sim}{A} = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cap \underset{\sim}{A} = \underset{\sim}{A}$$

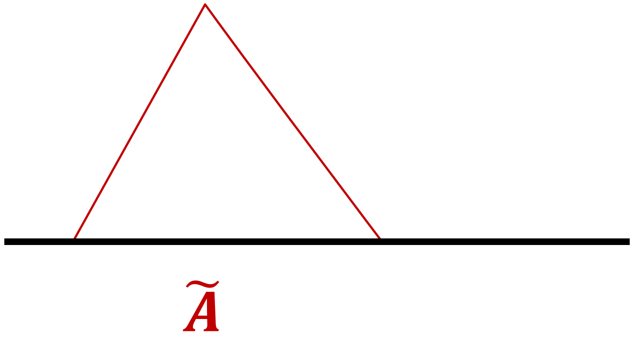
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



Fuzzy set A

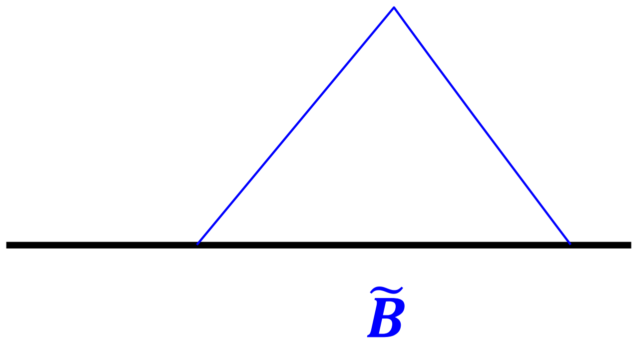
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



Fuzzy set B

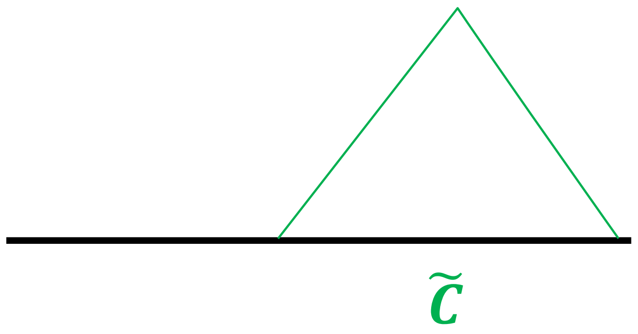
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



Fuzzy set C

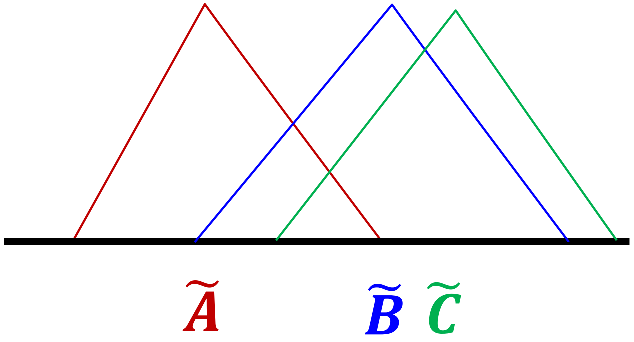
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$

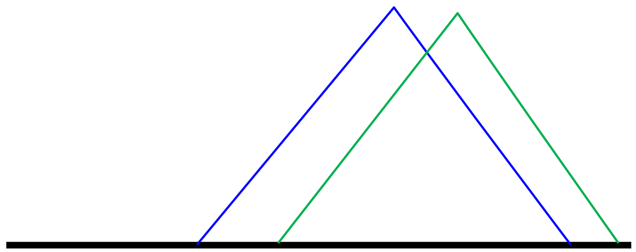


▶ back
menu

Some illustrations

- Associativity

$$\underbrace{\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right)}_{} = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$\tilde{B} \cap \tilde{C}$

B and C

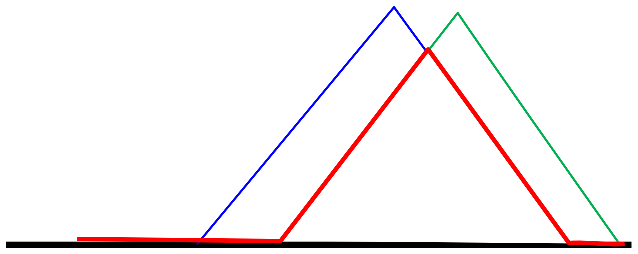
▶ back

menu

Some illustrations

- Associativity

$$\underbrace{\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right)} = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$\tilde{B} \cap \tilde{C}$

B and C

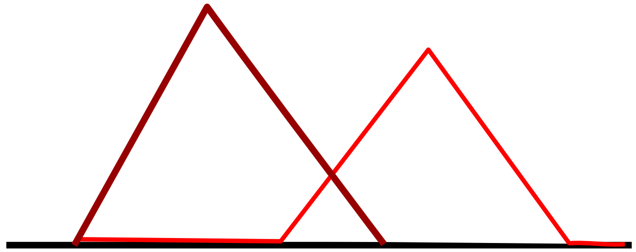
▶ back

menu

Some illustrations

- Associativity

$$\underbrace{\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right)} = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$\tilde{A} \cap (\tilde{B} \cap \tilde{C})$

A and (B and C)

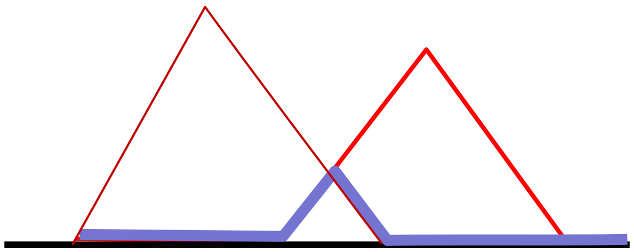
▶ back

menu

Some illustrations

- Associativity

$$\underbrace{\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right)} = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$\tilde{A} \cap (\tilde{B} \cap \tilde{C})$

A and (B and C)

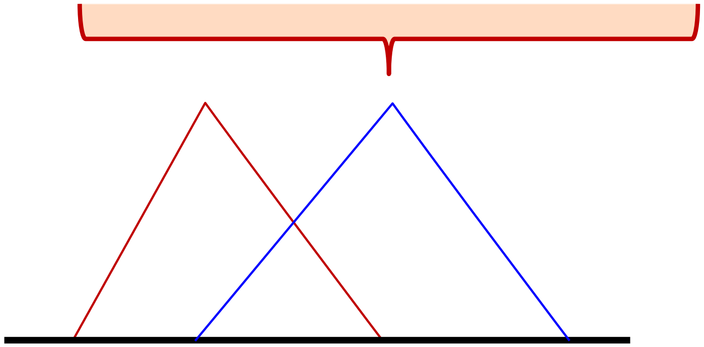
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$(\tilde{A} \cap \tilde{B})$

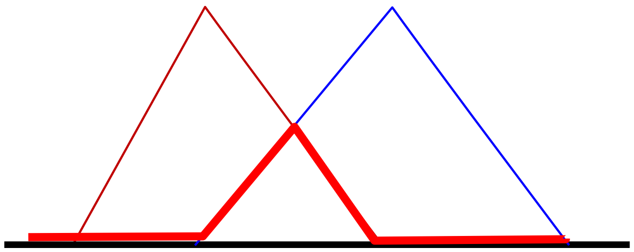
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \underbrace{\left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}}$$



$(\tilde{A} \cap \tilde{B})$

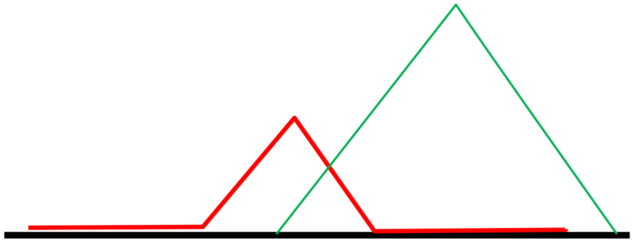
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \underbrace{\left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}}$$



$(\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

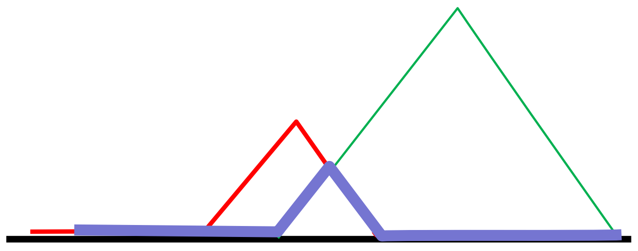
▶ back

menu

Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \underbrace{\left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}}$$



$(\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

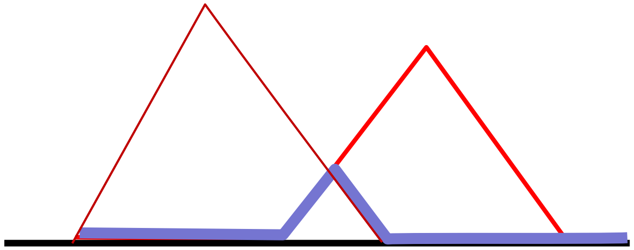
▶ back

menu

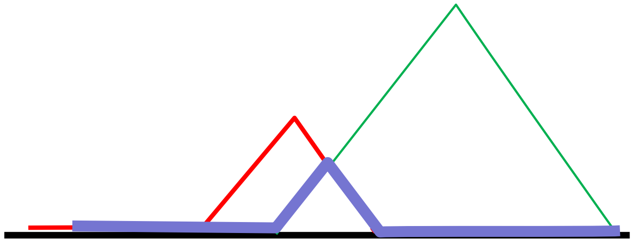
Some illustrations

- Associativity

$$\underset{\sim}{A} \cap \left(\underset{\sim}{B} \cap \underset{\sim}{C}\right) = \left(\underset{\sim}{A} \cap \underset{\sim}{B}\right) \cap \underset{\sim}{C}$$



$\tilde{A} \cap (\tilde{B} \cap \tilde{C})$



$(\tilde{A} \cap \tilde{B}) \cap \tilde{C}$

A and (B and C) ⟷ **(A and B) and C**

▶ back

menu

Frequently used properties

Define three fuzzy sets $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ on the universe X.

- *identity*

$$\underset{\sim}{A} \cup \emptyset = \underset{\sim}{A} \quad \text{and} \quad \underset{\sim}{A} \cap X = \underset{\sim}{A}$$

$$\underset{\sim}{A} \cap \emptyset = \emptyset \quad \text{and} \quad \underset{\sim}{A} \cup X = X$$

- *transitivity*:

If $\underset{\sim}{A} \subseteq \underset{\sim}{B}$ and $\underset{\sim}{B} \subseteq \underset{\sim}{C}$, then $\underset{\sim}{A} \subseteq \underset{\sim}{C}$

$A \subset B$	$\Rightarrow$	A is fully contained in B (if $x \in A$, then $x \in B$)
$A \subseteq B$	$\Rightarrow$	A is contained in or is equivalent to B
$(A \leftrightarrow B)$	$\Rightarrow$	$A \subseteq B$ and $B \subseteq A$ (A is equivalent to B)

- *involution*:

$$\overline{\overline{\underset{\sim}{A}}} = \underset{\sim}{A}$$

an involutory function is a function f that is its own inverse.

e.g. f(f(x) = x

e.g. NOT(NOT(A) = A

menu

Fuzzy Set operations

Fuzzy logic is a superset of conventional (Boolean) logic

All other operations on classical sets also hold for fuzzy sets, except for the **excluded middle laws**.

$$A \cup \bar{A} \neq X$$
$$A \cap \bar{A} \neq 0$$

for any proposition, either that proposition is true or its negation is true

The **whole set X** is the set of all elements in the universe

▶ back

menu

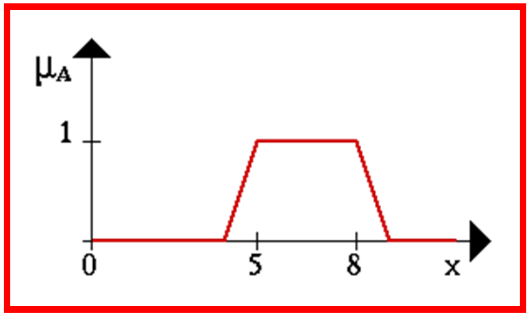
Fuzzy Set operations

All other operations on classical sets also hold for fuzzy sets, except for the **excluded middle laws**.

$A \cup \bar{A} \neq X$

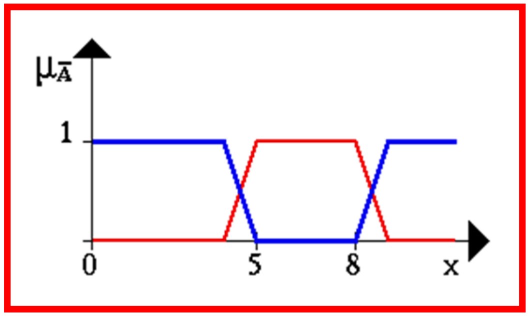
for any proposition, either that proposition is true or its negation is true

The **whole set X** is the set of all elements in the universe



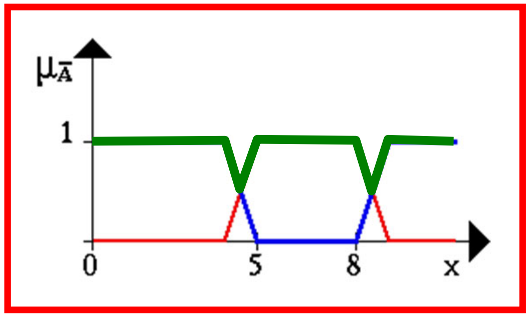
Fuzzy set **A**

(red curve)



NOT(**A**)

(blue curve)



$A \cup \bar{A} \neq X$

(green curve)

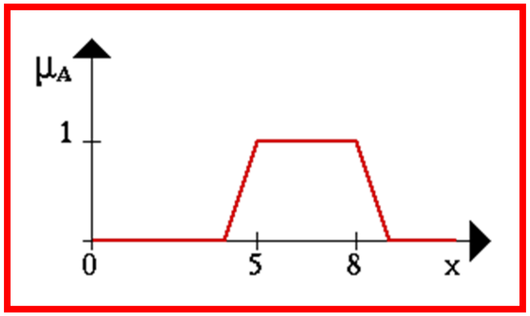
▶ back
menu

Fuzzy Set operations

All other operations on classical sets also hold for fuzzy sets, except for the **excluded middle laws**.

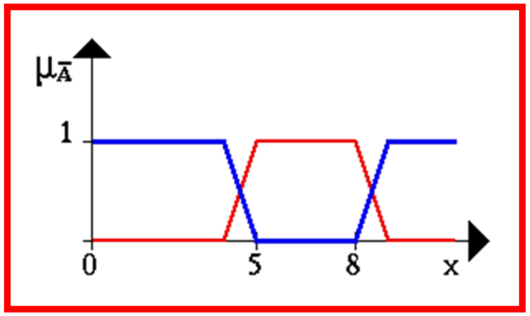
$$A \cap \bar{A} \neq 0$$

The **whole set X** is the set of all elements in the universe



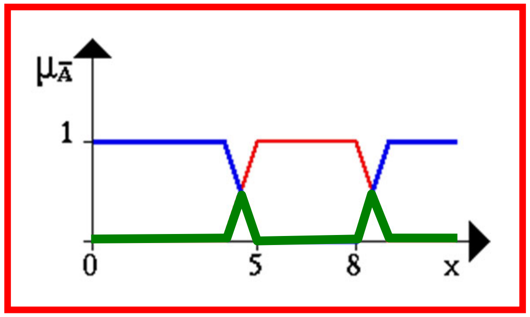
Fuzzy set **A**

(red curve)



NOT(**A**)

(blue curve)



$A \cap \bar{A} \neq 0$

(green curve)

▶ back

menu

More details…

- Fuzzy Inference Systems (FIS)
- Fuzzy Sets
- Fuzzy Combination Operators
- Membership functions
- Implication and Aggregation Operators
- Defuzzification Methods

▶ back

menu

Fuzzy Inference

Fuzzy inference is the process of formulating the mapping from a given input to an output using fuzzy logic. The mapping then provides a basis from which decisions can be made, or patterns discerned. The process of fuzzy inference involves all of the pieces that were described in the previous sections: Membership Functions, Logical Operations, and If-Then Rules.

http://www.mathworks.com/help/fuzzy/fuzzy-inference-process.html

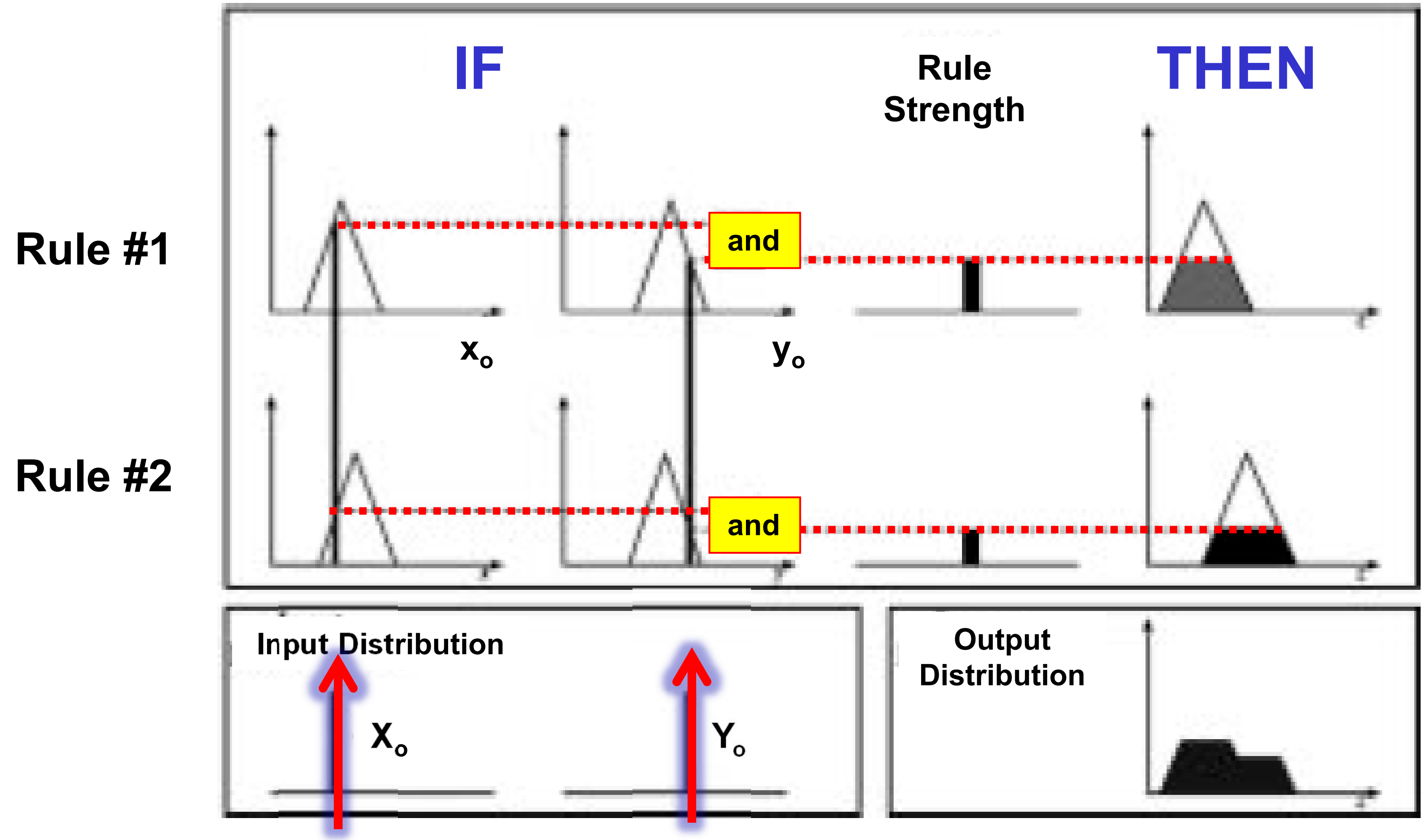
back

menu

Mamdani Fuzzy Inference System

Mamdani Inference System

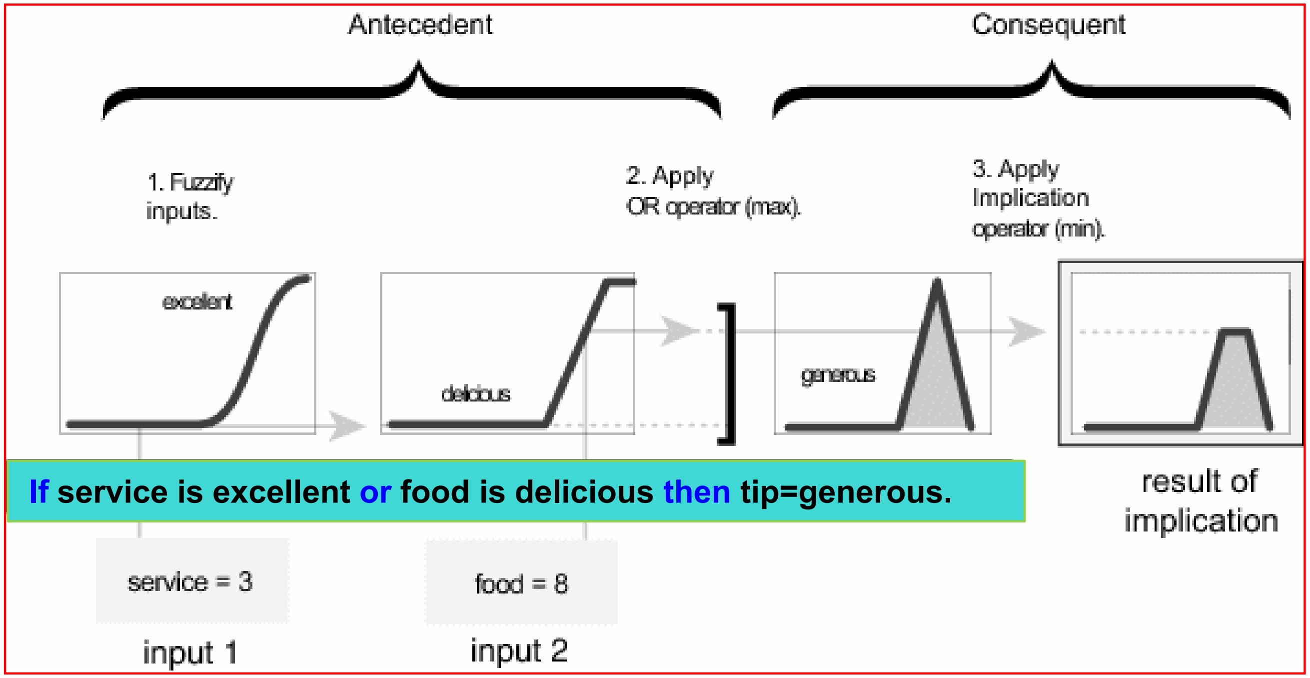
Two input, two rule Mamdani FIS with crisp inputs



Fuzzy rules are a collection of linguistic statements that describe how the FIS should make a decision regarding classifying an input or controlling an output.

Mamdani FIS

Mamdani-type inference, expects the output membership functions to be fuzzy sets. After the aggregation process, there is a fuzzy set for each output variable that needs defuzzification.

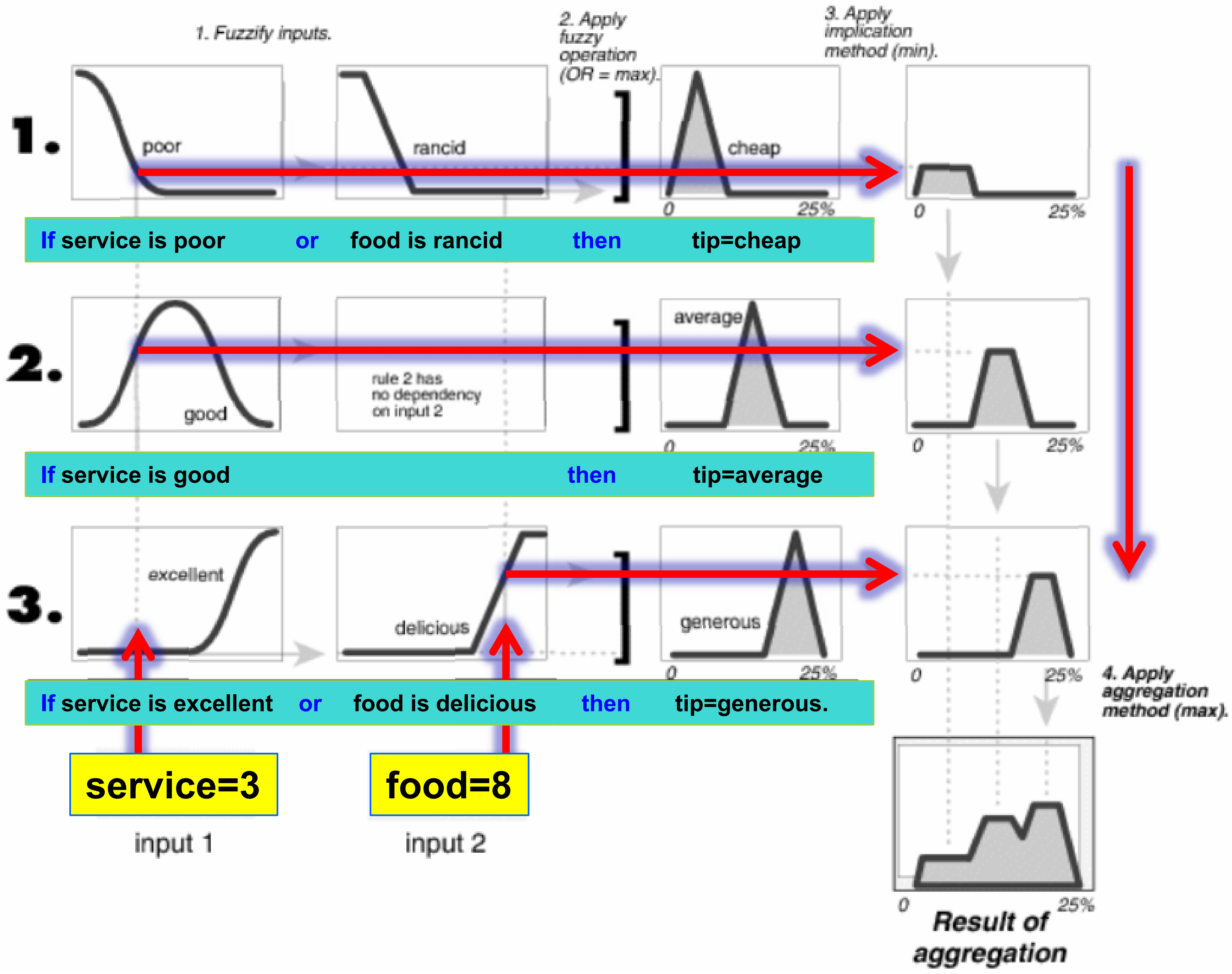


If service is excellent or food is delicious then tip=generous.

http://www.mathworks.com/access/helpdesk/help/toolbox/fuzzy/index.html?/access/helpdesk/help/toolbox/fuzzy/bp78l6_-1.html&http://www.mathworks.com/cgi-bin/texis/webinator/search/?db=MSS&prox=page&rorder=750&rprox=750&rdfreq=500&rwfreq=500&rlead=250&sufs=0&order=r&is_summary_on=1&ResultCount=10&query=fuzzy+logic&submitButtonName=Search

back
menu

Mamdani FIS



If service is poor **or** food is rancid **then** tip=cheap

If service is good **then** tip=average

If service is excellent **or** food is delicious **then** tip=generous.

service=3

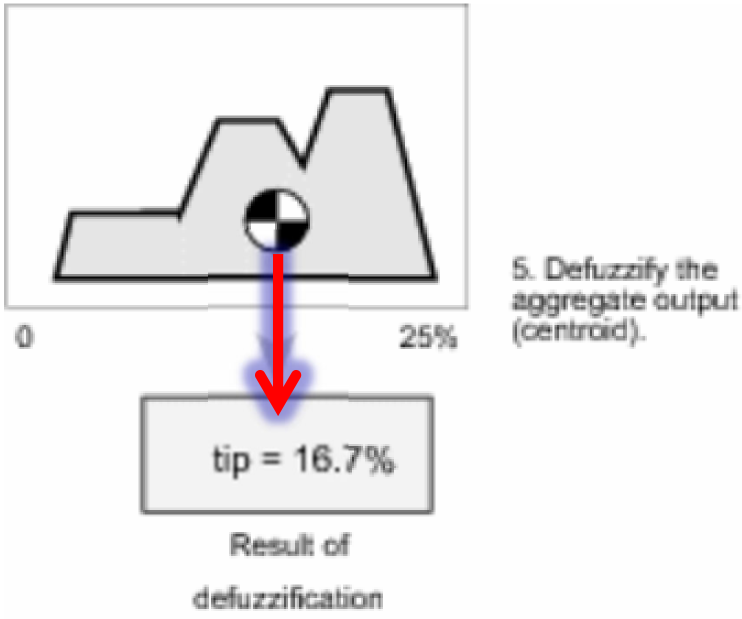
input 1

food=8

input 2

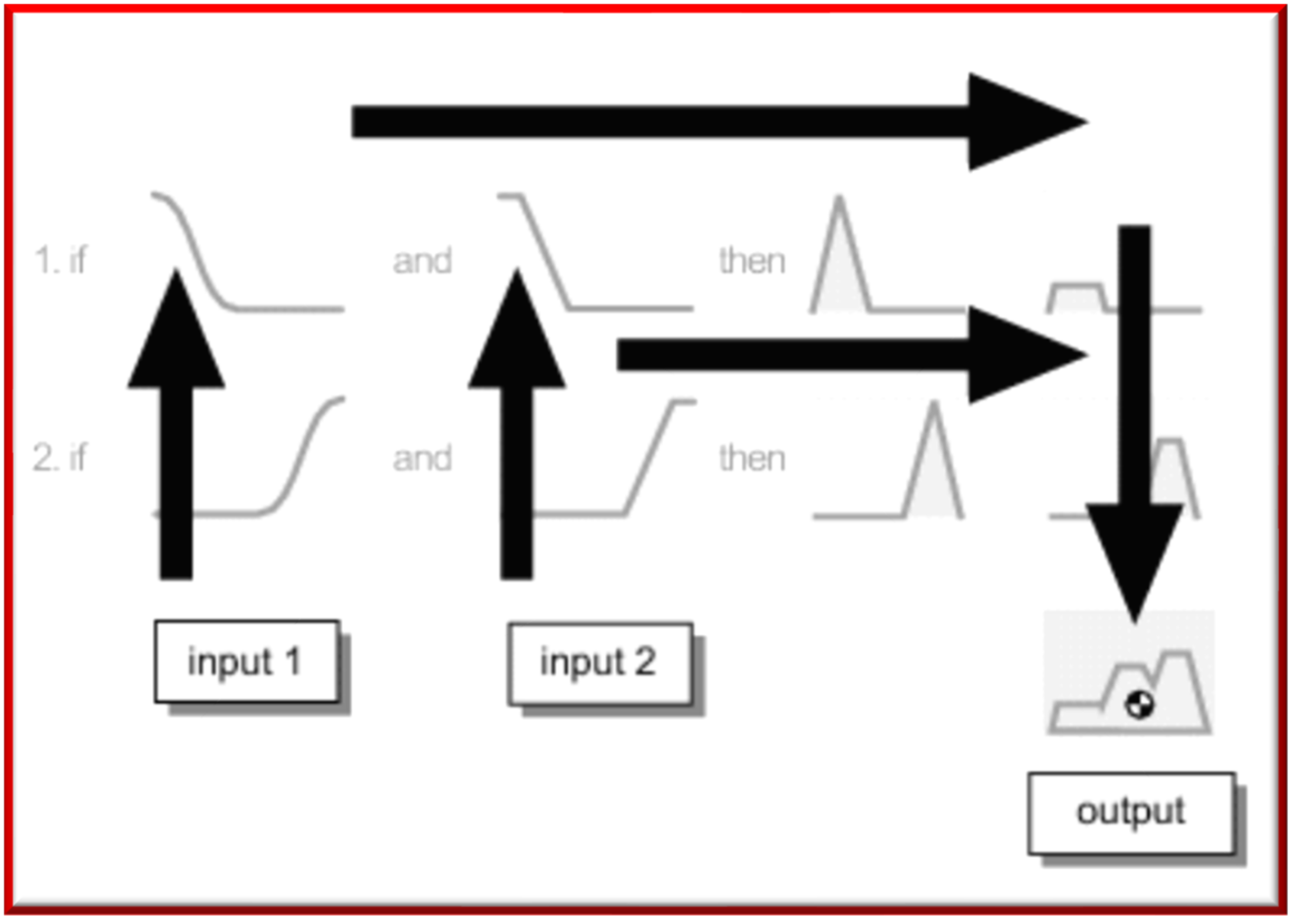
menu

Mamdani FIS



5. Defuzzify the aggregate output (centroid).

Flow of Fuzzy Inference



In this figure, the flow proceeds up from the inputs in the lower left, then across each row, or rule, and then down the rule outputs to finish in the lower right. This compact flow shows everything at once, from linguistic variable fuzzification all the way through defuzzification of the aggregate output.

Mamdani FIS

OUTPUT MEMBERSHIP FUNCTION

- It is possible, and in many cases much more efficient, to use a **single spike** as the **output membership function** rather than a distributed fuzzy set.
- This type of output is sometimes known as a **singleton output membership function**, and it can be thought of as a **pre-defuzzified fuzzy set**.
- It enhances the efficiency of the defuzzification process because it **greatly simplifies the computation** required by the more general Mamdani method, which finds the **centroid of a 2-D function**.
- Rather than integrating across the two-dimensional function to find the centroid, you use the **weighted average of a few data points**.

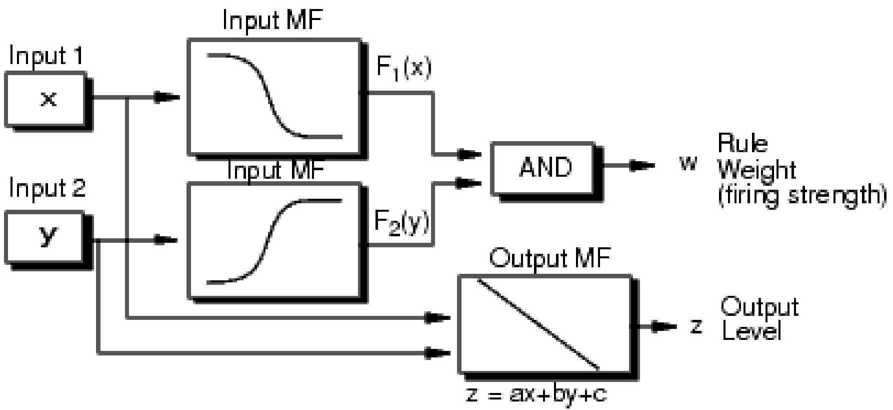
▶ back

menu

Sugeno Fuzzy Inference System

Sugeno FIS

Sugeno FIS is similar to the Mamdani method in many respects. The first two parts of the fuzzy inference process, fuzzifying the inputs and applying the fuzzy operator, are exactly the same. The main difference between Mamdani and Sugeno is that the Sugeno output membership functions are either linear or constant.



A typical rule in a Sugeno fuzzy model has the form:

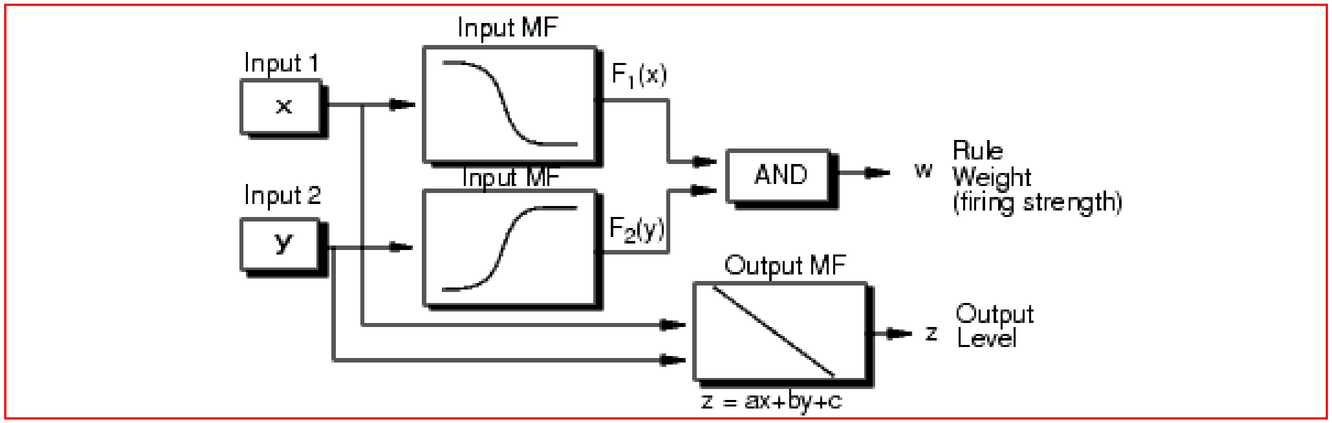
If (Input 1 = x and Input 2 = y) **then** Output is **z = ax + by + c**

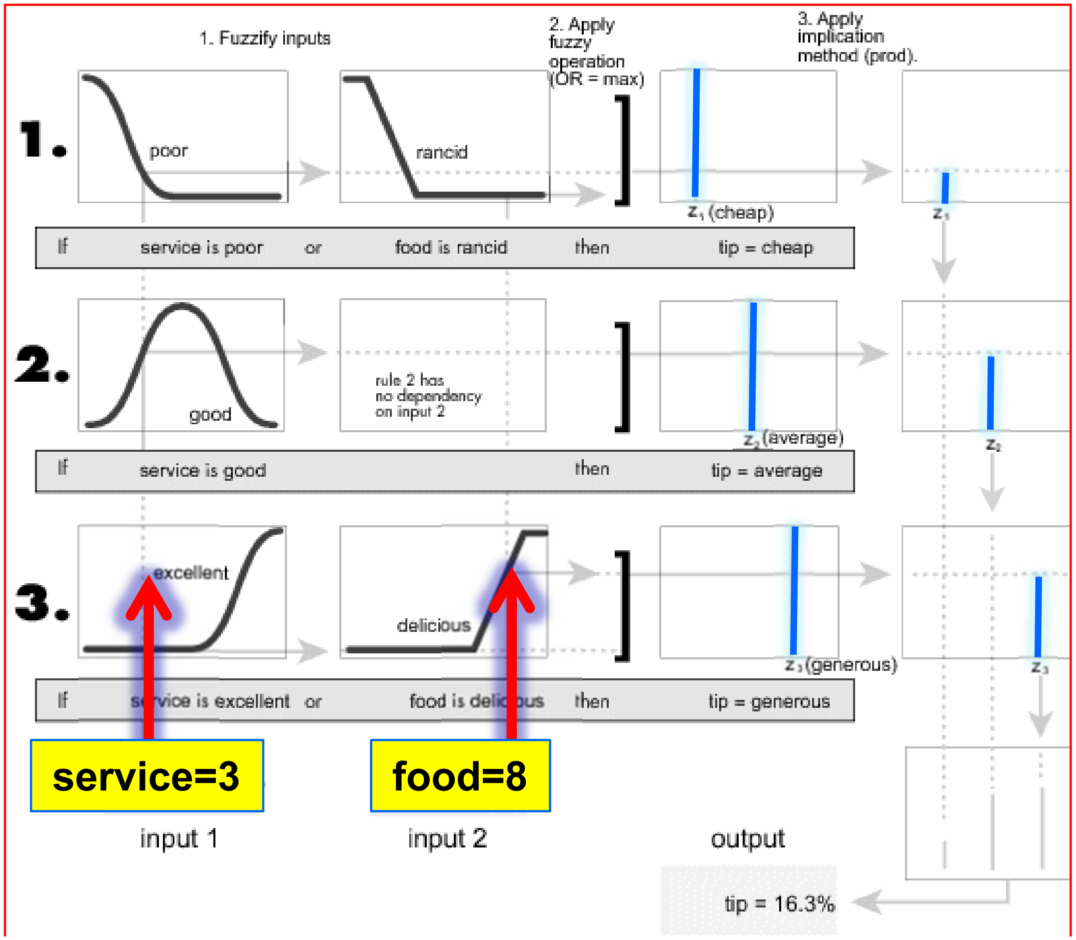
For a **zero-order Sugeno model**, the output **z** is a **constant** (where a=b=0).

back

menu

Sugeno FIS





$$\text{Final Output} = \frac{\sum_{i=1}^{N} w_i z_i}{\sum_{i=1}^{N} w_i}$$

▶ back

menu

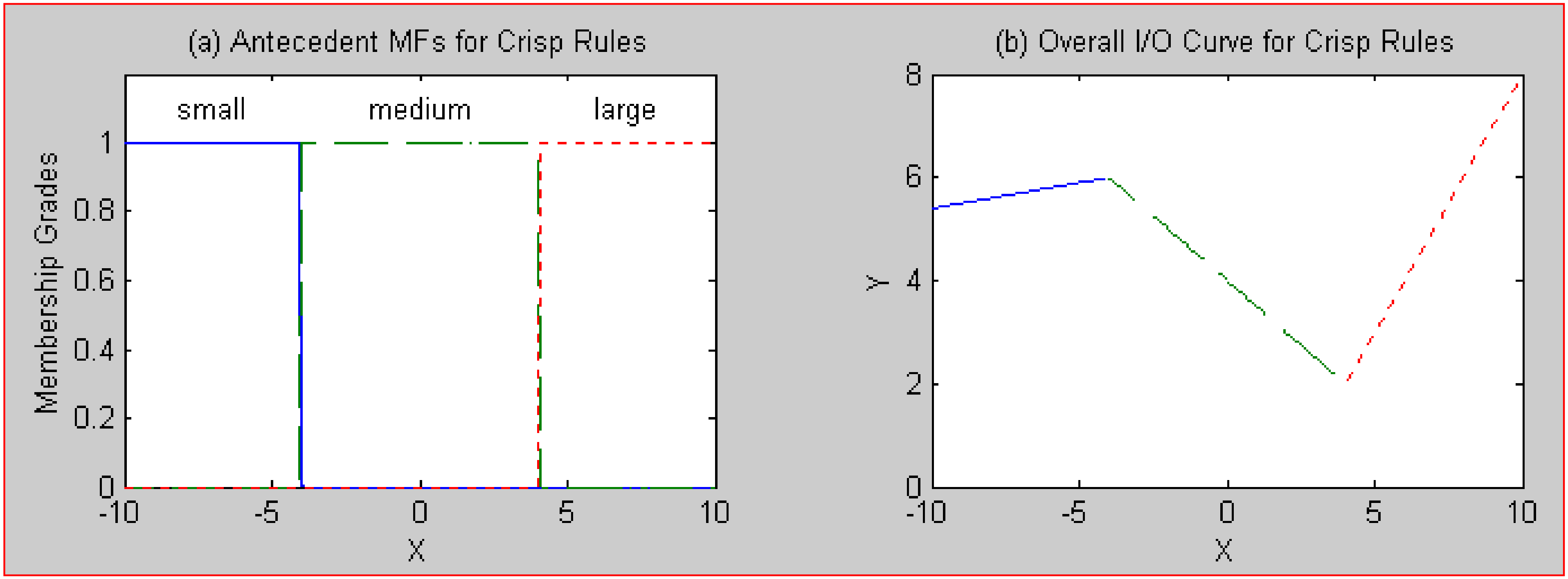
Sugeno FIS Example

X = input $\in [-10, 10]$

R1: If X is small then $Y = 0.1X + 6.4$

R2: If X is medium then $Y = -0.5X + 4$

R3: If X is large then $Y = X - 2$

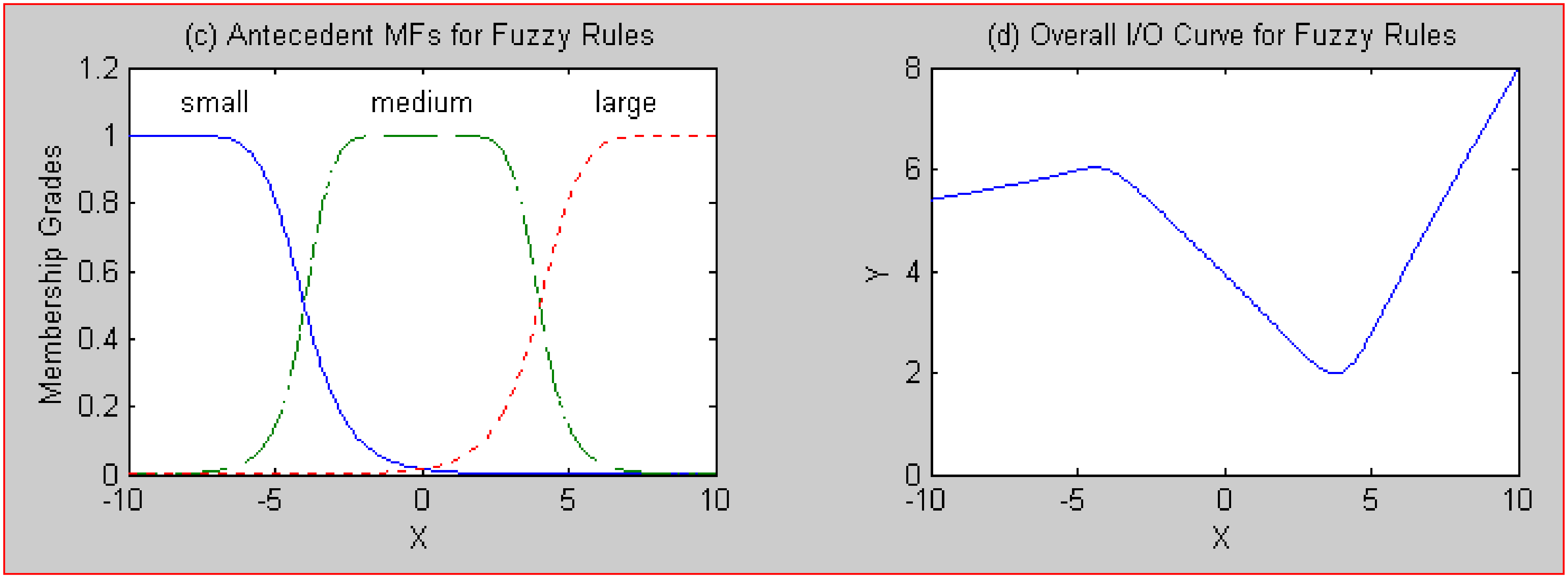


Using **jagged** membership functions, the overall input-output curve produced contains sharp edges.

Sugeno FIS Example

X = input $\in [-10, 10]$

R1: If X **is small** then $Y = 0.1X + 6.4$
R2: If X **is medium** then $Y = -0.5X + 4$
R3: If X **is large** then $Y = X - 2$



Using **smooth** membership functions, the overall input-output curve produced becomes smoother.

Sugeno FIS Example

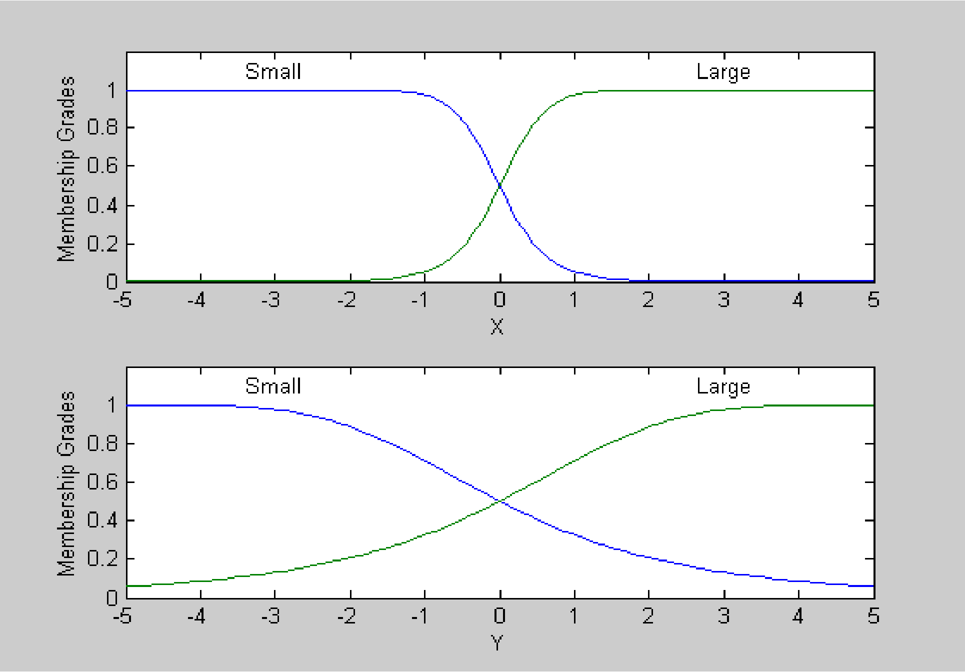
R1: if X is small and Y is small then $z = -x + y + 1$

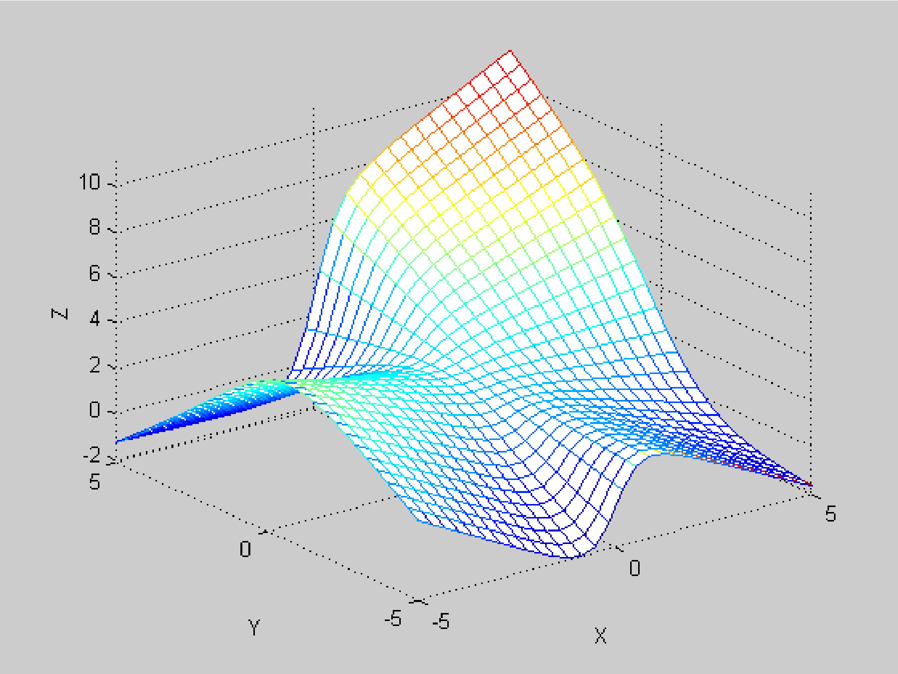
R2: if X is small and Y is large then $z = -y + 3$

R3: if X is large and Y is small then $z = -x + 3$

R4: if X is large and Y is large then $z = x + y + 2$

$$X, Y \in [-5, 5]$$





Highly non-linear problems could be solved.

http://aimm02.cse.ttu.edu.tw/

▶ back
menu

FIS: Sugeno vs. Mamdani

Advantages of the Sugeno Method

It is computationally efficient.
It can be used to model any inference system in which the output membership functions are either linear or constant.
It works well with linear techniques (e.g., PID control).
It works well with optimization and adaptive techniques.
It has guaranteed continuity of the output surface.
It is well suited to mathematical analysis.

Advantages of the Mamdani Method

It is intuitive.
It has widespread acceptance.
It is well suited to human input.

▶ back
menu

Variations in computing for the rule consequence

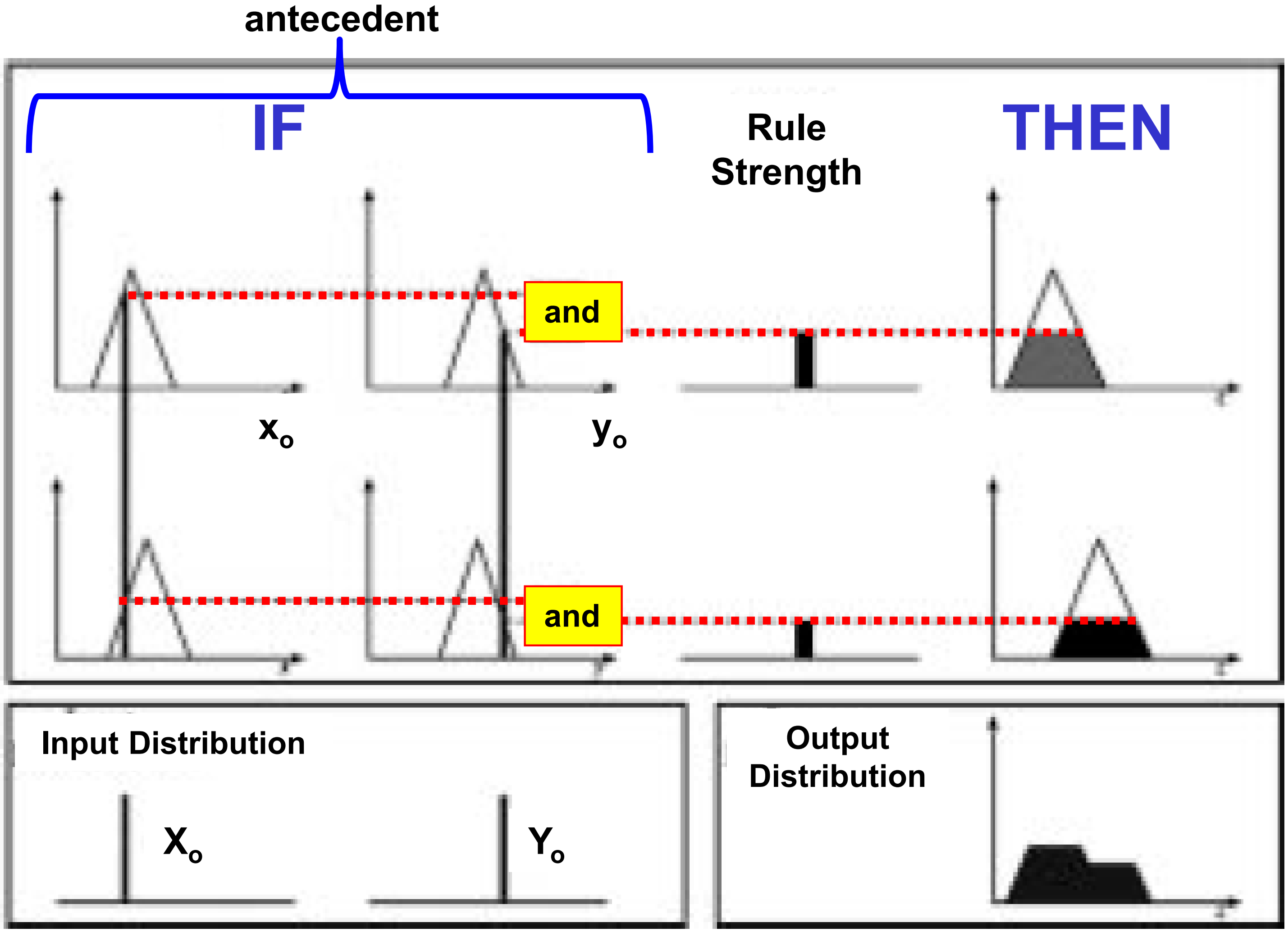
▶ back

menu

Consequence

The consequence of a fuzzy rule is computed using two steps:

1 Computing the **rule strength** by combining the **fuzzified** inputs using the **fuzzy combination** process



In this example, the fuzzy “and” is used to combine the membership functions to compute the rule strength.

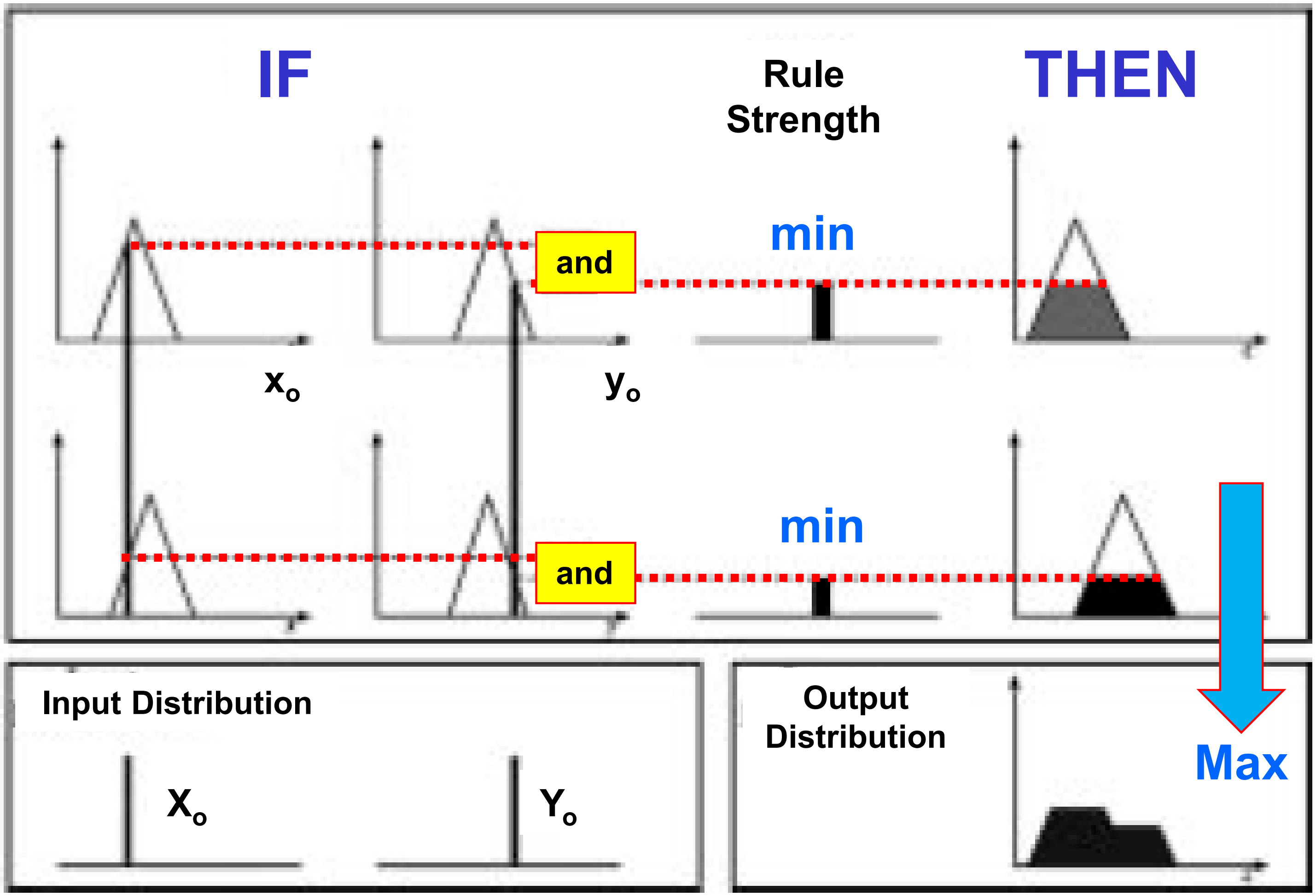
back

menu

Consequence

2a Max-Min Composition

Clipping the output membership function at the rule strength.



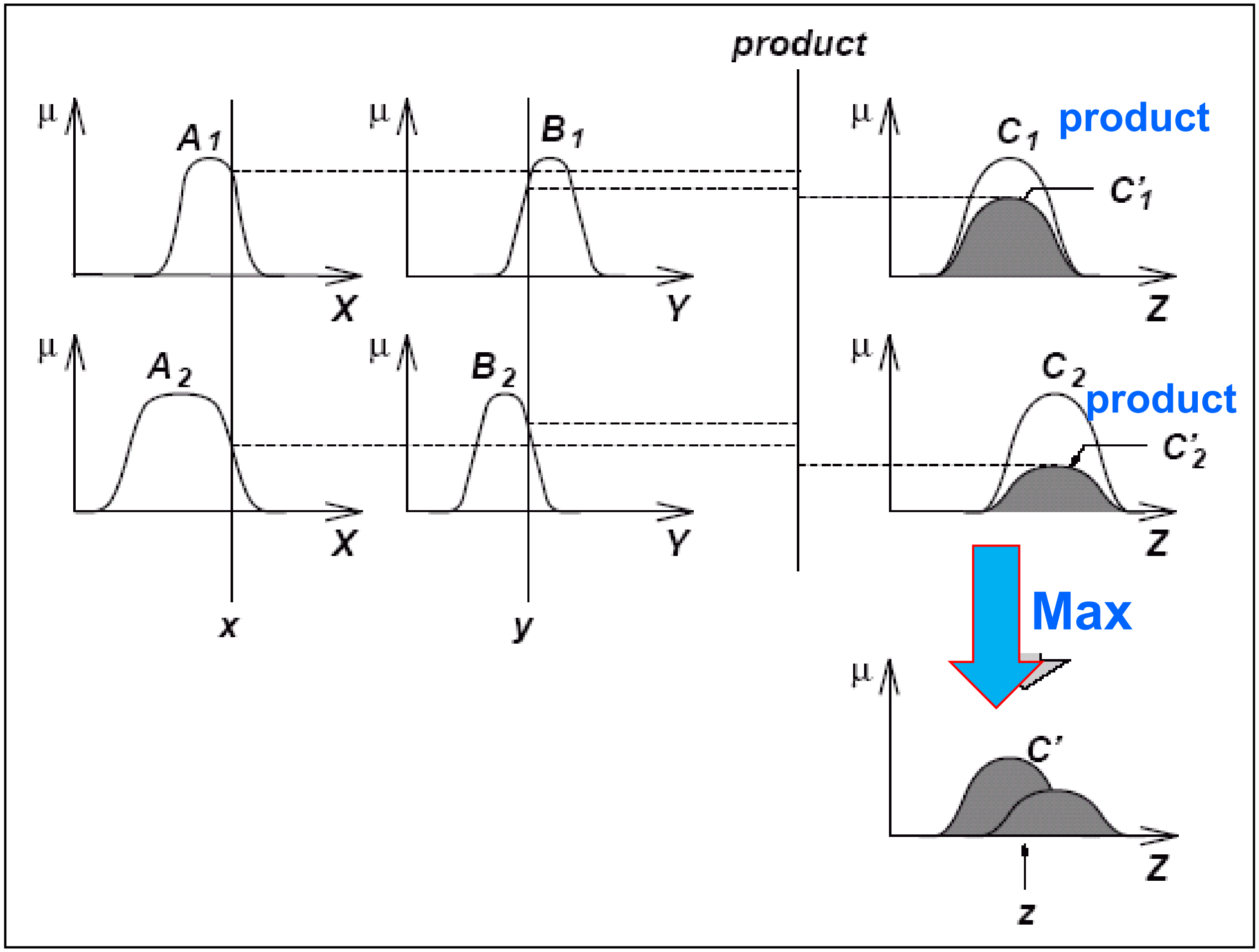
In this example, the fuzzy “and” is used to combine the membership functions to compute the rule strength.

▶ back
menu

Consequence

2b **(Alternatively) Max-Product Composition**

Multiplying the output membership function by the rule strength.

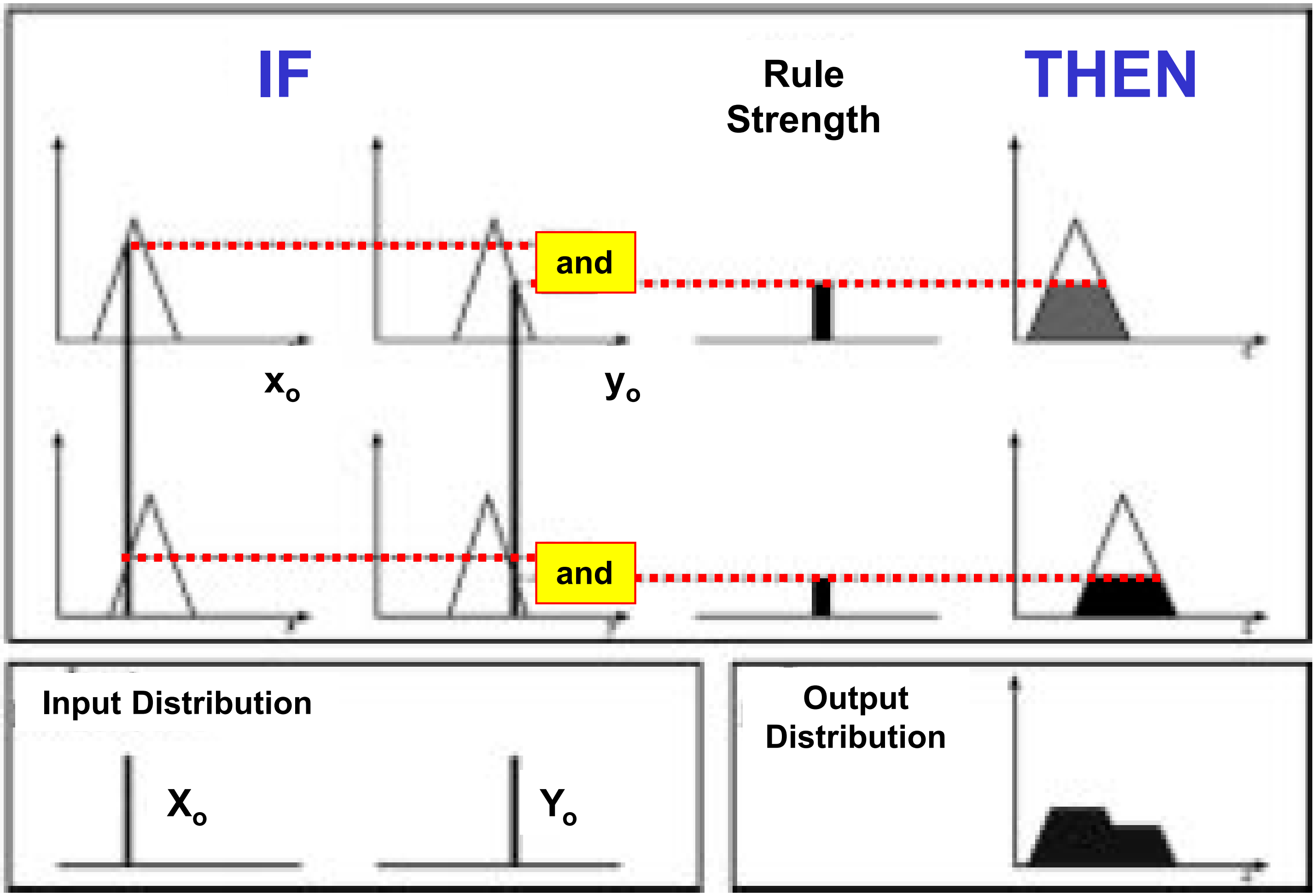


In this example, the fuzzy "and" is used to combine the membership functions to compute the rule strength.

back
menu

Consequence

The outputs of all of the fuzzy rules must now be combined to obtain one **fuzzy output distribution**. This is usually, but not always, done by using the fuzzy “or”. The figure below shows an example of this.



Aggregation Method

The output membership functions on the right hand side of the figure are combined using the fuzzy “**or**” to obtain the output distribution shown on the lower right corner of the figure.

▶ back
menu

Defuzzification techniques

▶ back

menu

3

Defuzzification of Output Distribution

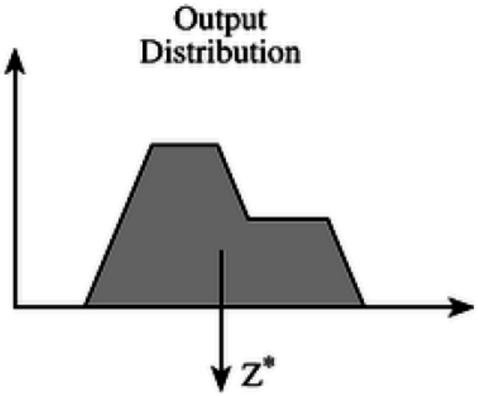
In many instances, it is desired to come up with a single crisp output from a FIS. For example, if one was trying to classify a letter drawn by hand on a drawing tablet, ultimately the FIS would have to come up with a crisp number to tell the computer which letter was drawn. This crisp number is obtained in a process known as defuzzification.

There are two common techniques for **defuzzifying**:

a) **Center of mass** - This technique takes the output distribution found in the previous slide and finds its center of mass to come up with one crisp number. This is computed as follows:

$$z = \frac{\sum_{j=1}^{q} Z_{\text{out}j} u_c(Z_j)}{\sum_{j=1}^{q} u_c(Z_j)}$$

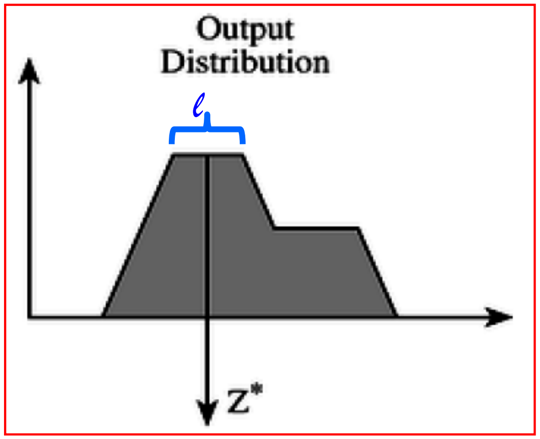
where z is the center of mass and μ_c is the membership in class c at value Z_j. An example outcome of this computation is shown in the figure at the right.



Defuzzification of Output Distribution

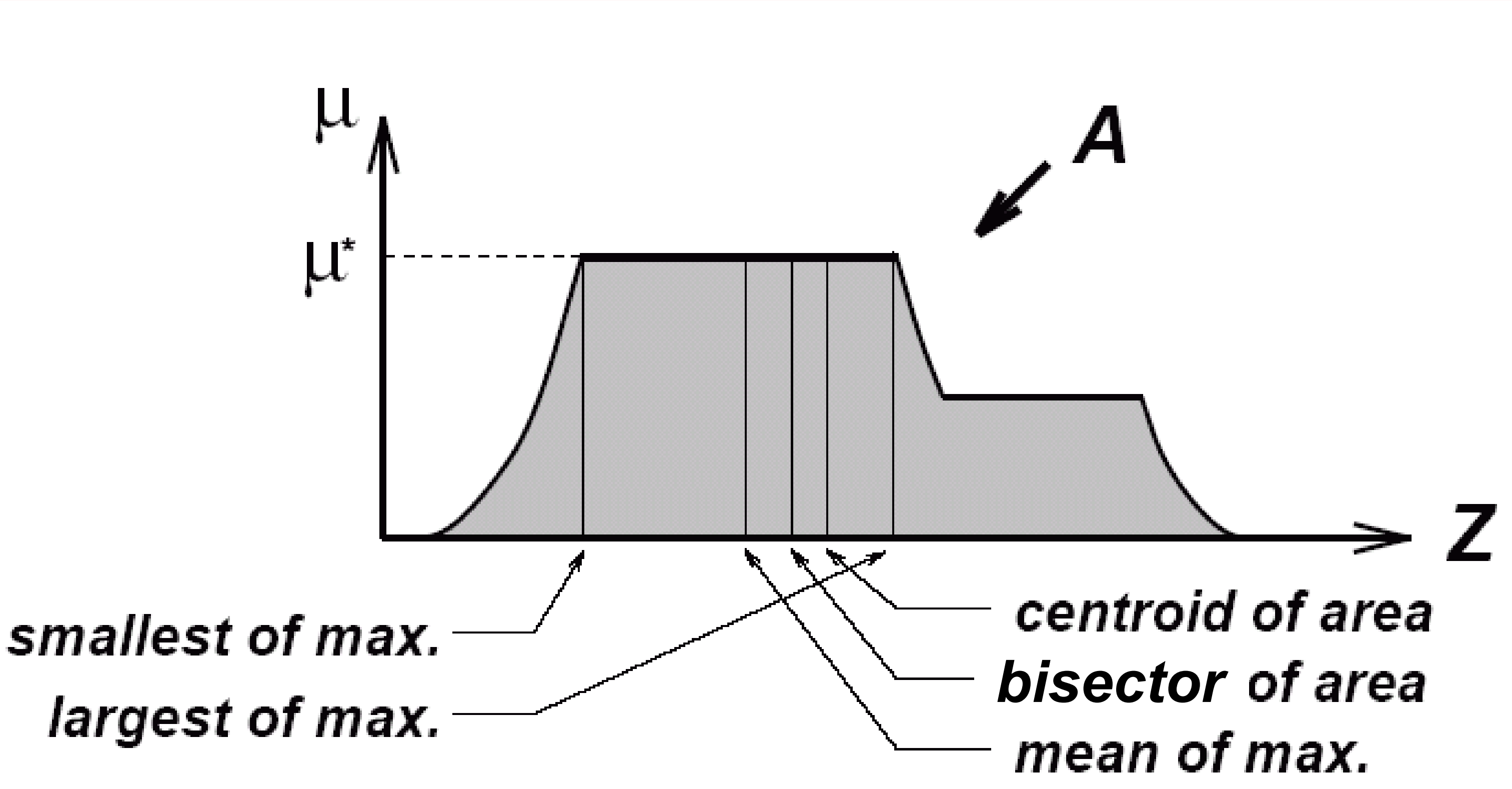
b) Mean of maximum - This technique takes the output distribution found in the previous section and finds its mean of maxima to come up with one crisp number. This is computed as follows:

$$z = \sum_{j=1}^{l} \frac{z_j}{l}$$



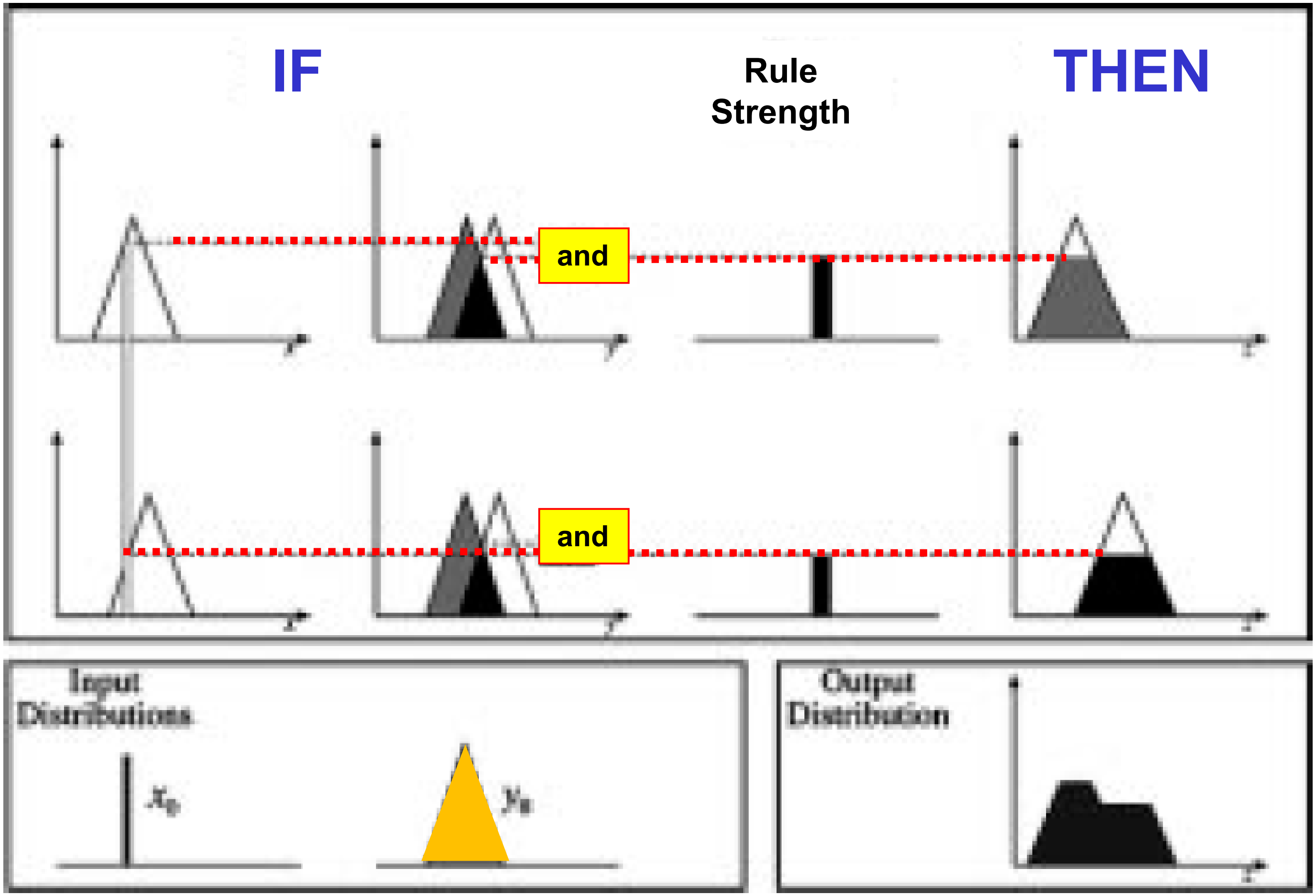
where **z** is the mean of maximum, Z_j is the point at which the membership function is maximum, and ℓ is the number of times the output distribution reaches the maximum level.
An example outcome of this computation is shown on the figure at the right.

Other Defuzzification Methods



Mamdani FIS with a Fuzzy Input

A two Input, two rule Mamdani FIS with a fuzzy input



shows a modification of the Mamdani FIS where the input **y0** is **fuzzy**, not crisp.

This can be used to model inaccuracies in the measurement.

For example, we may be measuring the output of a pressure sensor. Even with the exact same pressure applied, the sensor is measured to have slightly different voltages.

The fuzzy input membership function models this uncertainty.

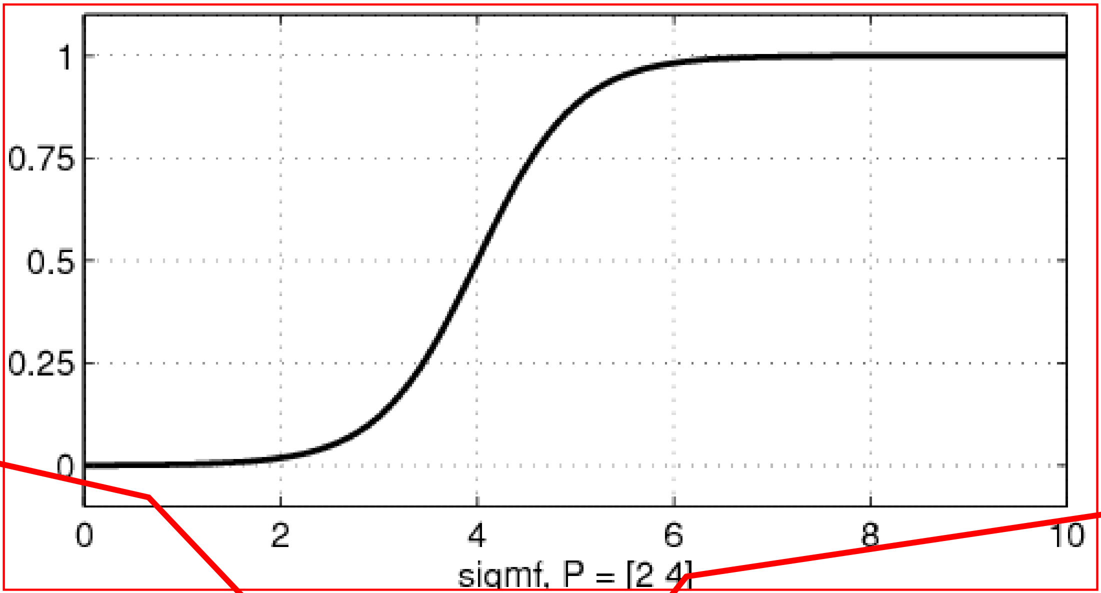
The input fuzzy function is combined with the rule input membership function by using the fuzzy “**and**”

▶ back
menu

Membership Functions

The Sigmoidal function

sigmf(x,[**a** **c**]), as given in the following equation by f(x,a,c) is a mapping on a vector x, and depends on two parameters **a** and **c**.



Slope

crossover point

$$f(x,a,c) = \frac{1}{1+e^{-a(x-c)}}$$

back
menu

Membership Functions

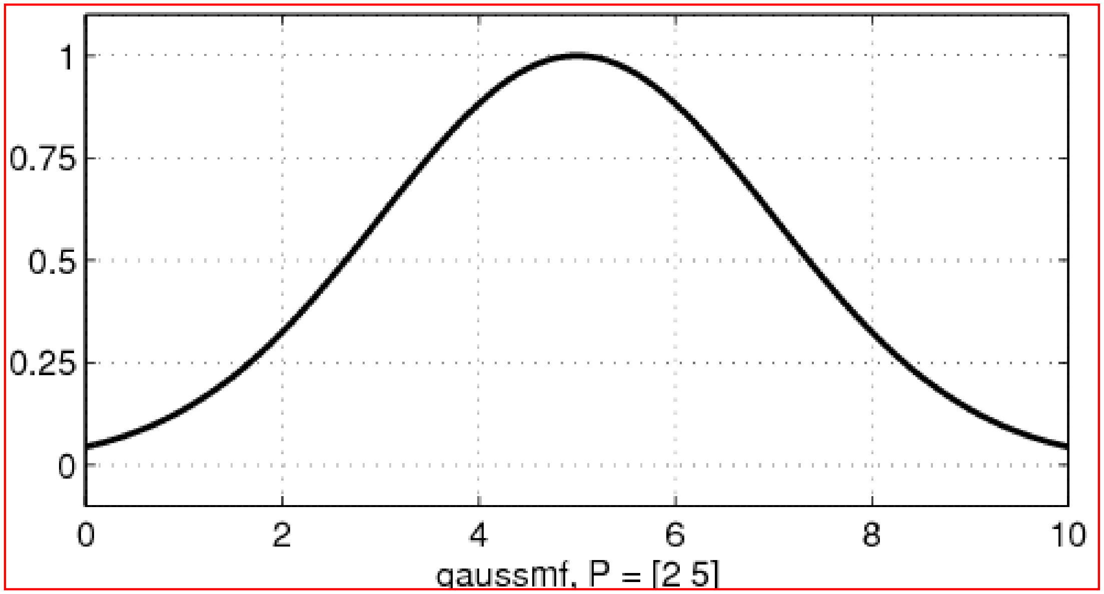
The Gaussian function

The symmetric Gaussian function depends on two parameters σ and c as given by

centre

$$f(x;\sigma,c) = e^{\frac{-(x-c)^2}{2\sigma^2}}$$

width



back

menu

Membership Functions

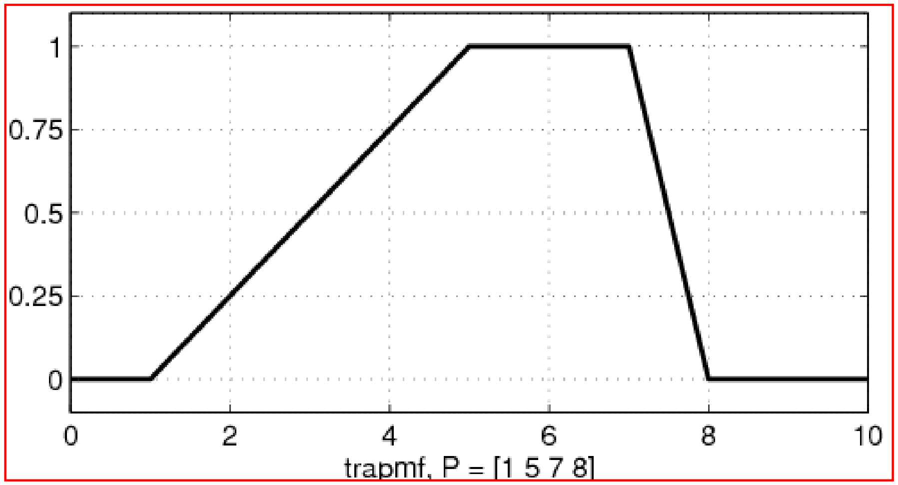
The Trapezoidal function

The trapezoidal curve is a function of a vector, x, and depends on four scalar parameters a, b, c, and d, as given by

$$f(x;a,b,c,d)=\left\{\begin{array}{ll} 0, & x\le a \\ \frac{x-a}{b-a}, & a\le x\le b \\ 1, & b\le x\le c \\ \frac{d-x}{d-c}, & c\le x\le d \\ 0, & d\le x \end{array}\right\}$$

or

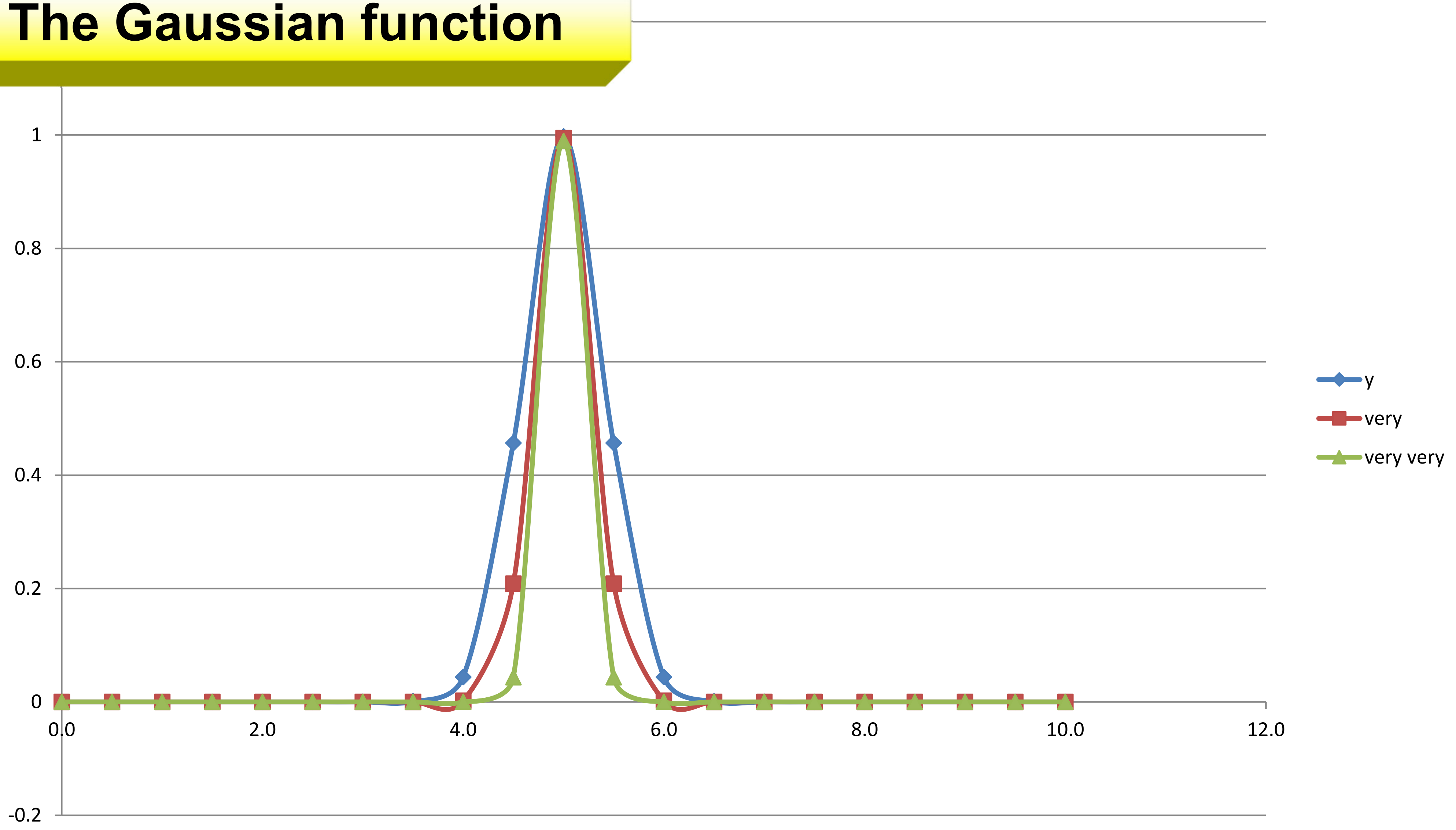
$$f(x;a,b,c,d)=\max\left(\min\left(\frac{x-a}{b-a},1,\frac{d-x}{d-c}\right),o\right)$$



The parameters a and d locate the "feet" of the trapezoid and the parameters b and c locate the "shoulders."

Fuzzy Hedges

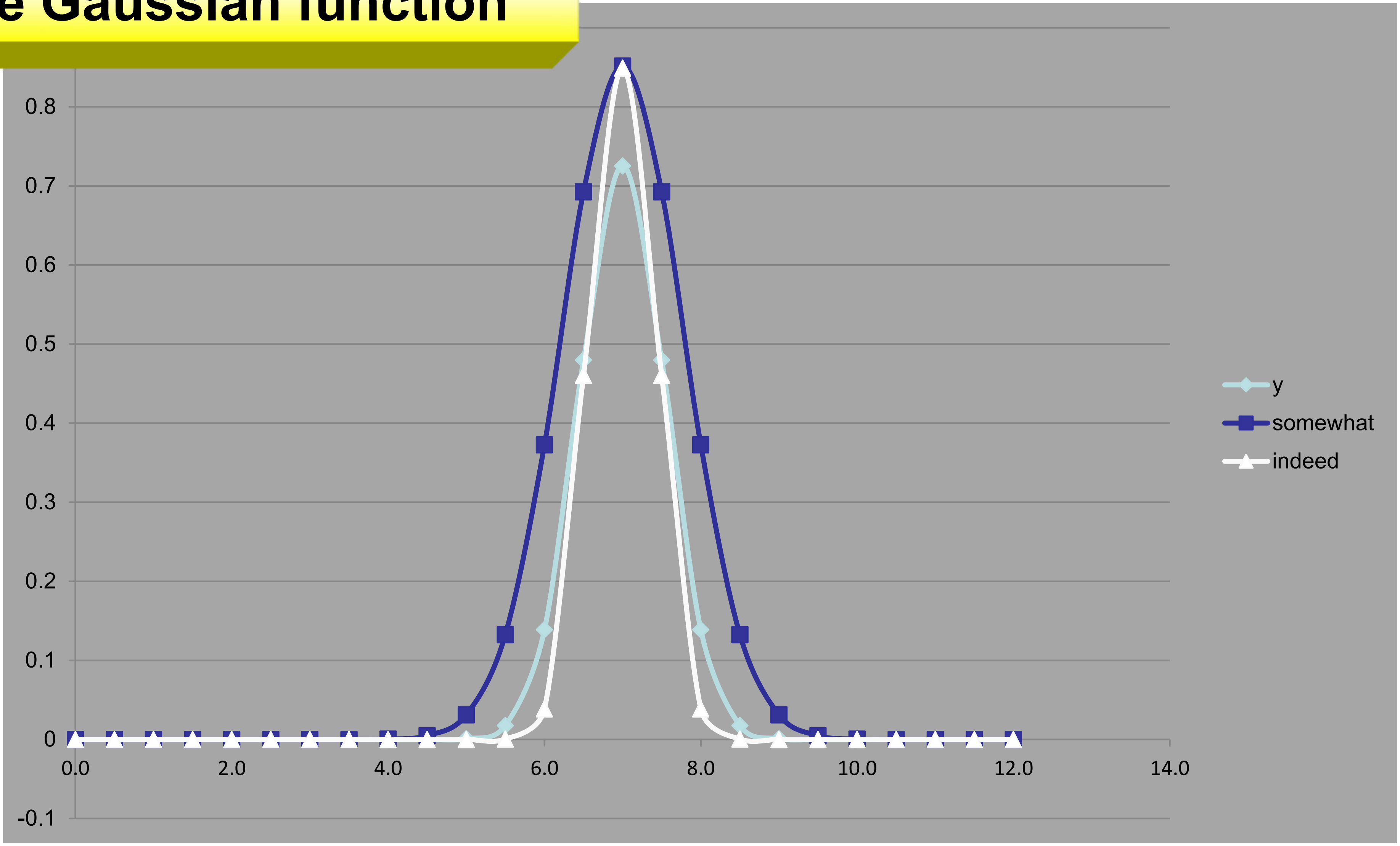
The Gaussian function



sigma	0.4
c	5

Fuzzy Hedges

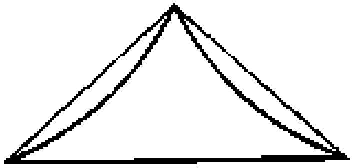
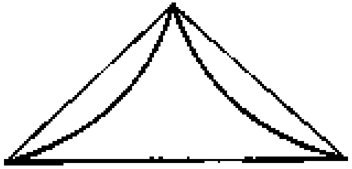
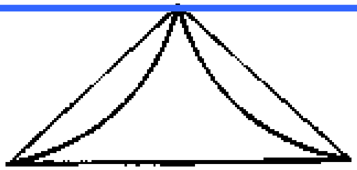
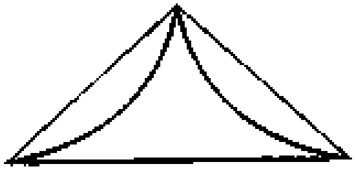
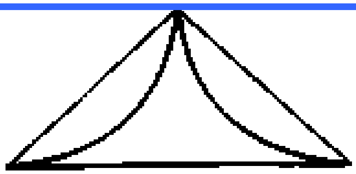
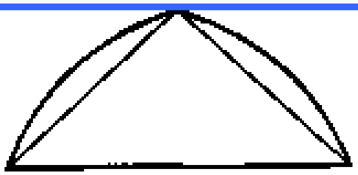
The Gaussian function



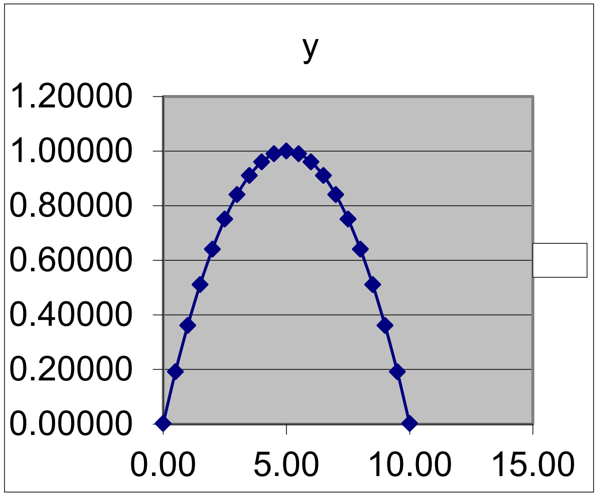
sigma	0.55
c	7

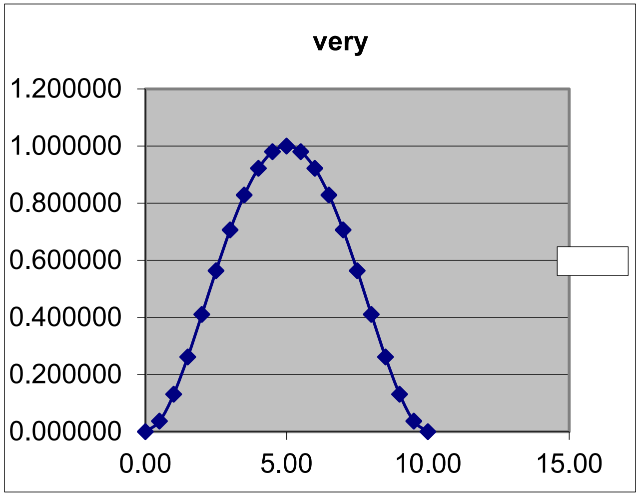
Fuzzy Hedges

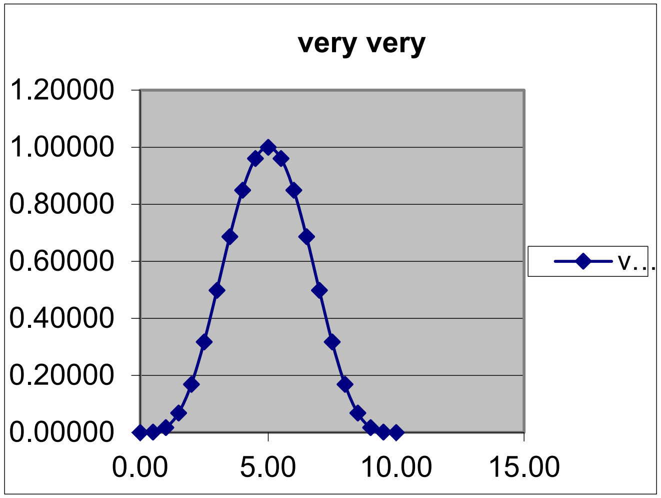
http://blog.peltarion.com/2006/10/25/fuzzy-math-part-1-the-theory/

Hedge	Operator	Effect
A little	$\mu_A(x)^{1.3}$	
Slightly	$\mu_A(x)^{1.7}$	
Very	$\mu_A(x)^2$	
Extremely	$\mu_A(x)^3$	
Very very	$\mu_A(x)^4$	
Somewhat	$\mu_A(x)^{\frac{1}{2}}$	
Indeed	$\begin{array}{rl} 2\mu_A(x)^2 & \text{if } 0 \le \mu_A(x) \le 0.5 \\ 1-2(1-\mu_A(x))^2 & \text{if } 0.5 < \mu_A(x) \le 1 \end{array}$	

More Examples







Sigma	3.54
c	5

References

- Genetic fuzzy systems by Oscar Cordón, Francisco Herrera, Frank Hoffmann
- Neural Network and Fuzzy Logic Applications in C/C++ (Wiley Professional Computing) by Stephen Welstead
- Fuzzy Logic with Engineering Applications by Timothy Ross
- Fuzzy Sets and Pattern Recognition by Benjamin Knapp

▶ back

menu

The End.

▶ back

menu