PRINCIPLES OF INTEGRAL EVALUATION

Integration by Parts:

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

GUIDELINES FOR INTEGRATION BY PARTS:

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, **I**nverse trigonometric, **A**lgebraic, **T**rigonometric, **E**xponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works often enough to be useful.

Note, for example, that the integrand in Example 1 consists of the product of the *algebraic* function x and the *trigonometric* function $\cos x$. Thus, the LIATE method suggests that we should let $u = x$ and $dv = \cos x\,dx$, which proved to be a successful choice.

The LIATE method is discussed in the article "A Technique for Integration by Parts," *American Mathematical Monthly*, Vol. 90, 1983, pp. 210–211, by Herbert Kasube.

▶ **Example 1** Use integration by parts to evaluate $\int x\cos x\,dx$.

Solution. We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u\,dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

(Other possibilities will be considered later.) The second step is to compute du from u and v from dv. This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x\,dx = \sin x$$

The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$
$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C \blacktriangleleft$$

▶ **Example 2** Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C \blacktriangleleft$$

▶ Example 3 Evaluate $\int \ln x\, dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\, dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\, dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\, dx = \int u\, dv = uv - \int v\, du = x\ln x - \int dx = x\ln x - x + C \blacktriangleleft$$

REPEATED INTEGRATION BY PARTS:

▶ Example 4 Evaluate $\int x^2 e^{-x}\, dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\, dx, \quad du = 2x\, dx, \quad v = \int e^{-x}\, dx = -e^{-x}$$

so that from (3)

$$\begin{aligned} \int x^2 e^{-x}\, dx &= \int u\, dv = uv - \int v\, du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\, dx \\ &= -x^2 e^{-x} + 2\int x e^{-x}\, dx \end{aligned} \tag{4}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int x e^{-x}\, dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\, dx, \quad du = dx, \quad v = \int e^{-x}\, dx = -e^{-x}$$

so that

$$\int x e^{-x}\, dx = x(-e^{-x}) - \int -e^{-x}\, dx = -xe^{-x} + \int e^{-x}\, dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned} \int x^2 e^{-x}\, dx &= -x^2 e^{-x} + 2\int x e^{-x}\, dx = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C \blacktriangleleft \end{aligned}$$

▶ Example 5 Evaluate $\int e^x \cos x \, dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x \, dx, \quad du = -\sin x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,

$$\int e^x \cos x \, dx = \int u \, dv = uv - \int v \, du = e^x \cos x + \int e^x \sin x \, dx \tag{5}$$

Since the integral $\int e^x \sin x \, dx$ is similar in form to the original integral $\int e^x \cos x \, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x \, dx, \quad du = \cos x \, dx, \quad v = \int e^x \, dx = e^x$$

Thus,

$$\int e^x \sin x \, dx = \int u \, dv = uv - \int v \, du = e^x \sin x - \int e^x \cos x \, dx$$

Together with Equation (5) this yields

$$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx \tag{6}$$

which is an equation we can solve for the unknown integral. We obtain

$$2 \int e^x \cos x \, dx = e^x \cos x + e^x \sin x$$

and hence

$$\int e^x \cos x \, dx = \tfrac{1}{2} e^x \cos x + \tfrac{1}{2} e^x \sin x + C \quad ◀$$

A TABULAR METHOD FOR REPEATED INTEGRATION BY PARTS:

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

► Example 6 In Example 11 of Section 5.3 we evaluated $\int x^2\sqrt{x-1}\,dx$ using u-substitution. Evaluate this integral using tabular integration by parts.

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

Thus, it follows that

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C \blacktriangleleft$$

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS:

► **Example 7** Evaluate $\int_0^1 \tan^{-1} x\,dx$.

Solution. Let

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\,dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1} x\,dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$

$$= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

But

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$ ◄

REDUCTION FORMULAS:

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Example 8 Evaluate $\int \cos^4 x\,dx$.

Solution. From (10) with $n = 4$

$$\int \cos^4 x\,dx = \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\int \cos^2 x\,dx$$

Now apply (10) with $n = 2$.

$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x \sin x + \tfrac{1}{2}\int dx\right)$$

$$= \tfrac{1}{4}\cos^3 x \sin x + \tfrac{3}{8}\cos x \sin x + \tfrac{3}{8}x + C$$

EXERCISE SET 7.2

1–38 Evaluate the integral.

1. $\int xe^{-2x}\,dx$

2. $\int xe^{3x}\,dx$

3. $\int x^2 e^x\,dx$

4. $\int x^2 e^{-2x}\,dx$

5. $\int x\sin 3x\,dx$

6. $\int x\cos 2x\,dx$

7. $\int x^2\cos x\,dx$

8. $\int x^2\sin x\,dx$

9. $\int x\ln x\,dx$

10. $\int \sqrt{x}\ln x\,dx$

11. $\int (\ln x)^2\,dx$

12. $\int \frac{\ln x}{\sqrt{x}}\,dx$

13. $\int \ln(3x-2)\,dx$

14. $\int \ln(x^2+4)\,dx$

15. $\int \sin^{-1} x\,dx$

16. $\int \cos^{-1}(2x)\,dx$

17. $\int \tan^{-1}(3x)\,dx$

18. $\int x\tan^{-1} x\,dx$

19. $\int e^x\sin x\,dx$

20. $\int e^{3x}\cos 2x\,dx$

21. $\int \sin(\ln x)\,dx$

22. $\int \cos(\ln x)\,dx$

23. $\int x\sec^2 x\,dx$

24. $\int x\tan^2 x\,dx$

25. $\int x^3 e^{x^2}\,dx$

26. $\int \frac{xe^x}{(x+1)^2}\,dx$

27. $\int_0^2 xe^{2x}\,dx$

28. $\int_0^1 xe^{-5x}\,dx$

29. $\int_1^e x^2\ln x\,dx$

30. $\int_{\sqrt{e}}^{e} \frac{\ln x}{x^2}\,dx$

69. Use reduction formula (9) to evaluate

(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

70. Use reduction formula (10) to evaluate

(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

71. Derive reduction formula (9).

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{9}$$

SOLUTION SET

1. $u = x$, $dv = e^{-2x}dx$, $du = dx$, $v = -\frac{1}{2}e^{-2x}$; $\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$.

3. $u = x^2$, $dv = e^x dx$, $du = 2x\,dx$, $v = e^x$; $\int x^2e^x dx = x^2e^x - 2\int xe^x dx$. For $\int xe^x dx$ use $u = x$, $dv = e^x dx$, $du = dx$, $v = e^x$ to get $\int xe^x dx = xe^x - e^x + C_1$ so $\int x^2e^x dx = x^2e^x - 2xe^x + 2e^x + C$.

5. $u = x$, $dv = \sin 3x\,dx$, $du = dx$, $v = -\frac{1}{3}\cos 3x$; $\int x\sin 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx = -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$.

7. $u = x^2$, $dv = \cos x\,dx$, $du = 2x\,dx$, $v = \sin x$; $\int x^2\cos x\,dx = x^2\sin x - 2\int x\sin x\,dx$. For $\int x\sin x\,dx$ use $u = x$, $dv = \sin x\,dx$ to get $\int x\sin x\,dx = -x\cos x + \sin x + C_1$ so $\int x^2\cos x\,dx = x^2\sin x + 2x\cos x - 2\sin x + C$.

9. $u = \ln x$, $dv = x\,dx$, $du = \frac{1}{x}dx$, $v = \frac{1}{2}x^2$; $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$.

11. $u = (\ln x)^2$, $dv = dx$, $du = 2\frac{\ln x}{x}dx$, $v = x$; $\int (\ln x)^2dx = x(\ln x)^2 - 2\int \ln x\,dx$. Use $u = \ln x$, $dv = dx$ to get $\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C_1$ so $\int (\ln x)^2dx = x(\ln x)^2 - 2x\ln x + 2x + C$.

13. $u = \ln(3x-2)$, $dv = dx$, $du = \frac{3}{3x-2}dx$, $v = x$; $\int \ln(3x-2)dx = x\ln(3x-2) - \int \frac{3x}{3x-2}dx$, but $\int \frac{3x}{3x-2}dx = \int \left(1 + \frac{2}{3x-2}\right)dx = x + \frac{2}{3}\ln(3x-2) + C_1$ so $\int \ln(3x-2)dx = x\ln(3x-2) - x - \frac{2}{3}\ln(3x-2) + C$.

15. $u = \sin^{-1}x$, $dv = dx$, $du = 1/\sqrt{1-x^2}dx$, $v = x$; $\int \sin^{-1}x\,dx = x\sin^{-1}x - \int x/\sqrt{1-x^2}dx = x\sin^{-1}x + \sqrt{1-x^2} + C$.

17. $u = \tan^{-1}(3x)$, $dv = dx$, $du = \frac{3}{1+9x^2}dx$, $v = x$; $\int \tan^{-1}(3x)dx = x\tan^{-1}(3x) - \int \frac{3x}{1+9x^2}dx = x\tan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$.

19. $u = e^x$, $dv = \sin x\,dx$, $du = e^x dx$, $v = -\cos x$; $\int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx$. For $\int e^x\cos x\,dx$ use $u = e^x$, $dv = \cos x\,dx$ to get $\int e^x\cos x = e^x\sin x - \int e^x\sin x\,dx$, so $\int e^x\sin x\,dx = -e^x\cos x + e^x\sin x - \int e^x\sin x\,dx$, $2\int e^x\sin x\,dx = e^x(\sin x - \cos x) + C_1$, $\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.

21. $u = \sin(\ln x)$, $dv = dx$, $du = \frac{\cos(\ln x)}{x}dx$, $v = x$; $\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$. Use $u = \cos(\ln x)$, $dv = dx$ to get $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$ so $\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$, $\int \sin(\ln x)dx = \frac{1}{2}x[\sin(\ln x) - \cos(\ln x)] + C$.

23. $u = x$, $dv = \sec^2 x\,dx$, $du = dx$, $v = \tan x$; $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x - \int \frac{\sin x}{\cos x}dx = x\tan x + \ln|\cos x| + C$.

25. $u = x^2$, $dv = xe^{x^2}dx$, $du = 2x\,dx$, $v = \frac{1}{2}e^{x^2}$; $\int x^3e^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$.

27. $u = x$, $dv = e^{2x}dx$, $du = dx$, $v = \frac{1}{2}e^{2x}$; $\int_0^2 xe^{2x}dx = \frac{1}{2}xe^{2x}\Big]_0^2 - \frac{1}{2}\int_0^2 e^{2x}dx = e^4 - \frac{1}{4}e^{2x}\Big]_0^2 = e^4 - \frac{1}{4}(e^4 - 1) = (3e^4 + 1)/4$.

29. $u = \ln x$, $dv = x^2dx$, $du = \frac{1}{x}dx$, $v = \frac{1}{3}x^3$; $\int_1^e x^2\ln x\,dx = \frac{1}{3}x^3\ln x\Big]_1^e - \frac{1}{3}\int_1^e x^2dx = \frac{1}{3}e^3 - \frac{1}{9}x^3\Big]_1^e = \frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1) = (2e^3 + 1)/9$.

69. **(a)** $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(-\frac{1}{2}\sin x\cos x + \frac{1}{2}x\right) + C = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C$.

(b) $\int_0^{\pi/2} \sin^5 x\,dx = -\frac{1}{5}\sin^4 x\cos x\Big]_0^{\pi/2} + \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x\Big]_0^{\pi/2} + \frac{2}{3}\int_0^{\pi/2} \sin x\,dx\right)$

$= -\frac{8}{15}\cos x\Big]_0^{\pi/2} = \frac{8}{15}$.

71. $u = \sin^{n-1} x$, $dv = \sin x\,dx$, $du = (n-1)\sin^{n-2} x\cos x\,dx$, $v = -\cos x$; $\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\cos^2 x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,(1 - \sin^2 x)dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx$, so $n\int \sin^n x\,dx = -\sin^{n-1} x\cos x + (n-1)\int \sin^{n-2} x\,dx$, and $\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$.