New Zealand Mathematical Olympiad Committee

Maths Workshop June 2024

mathsolympiad.org.nz/workshops/

Problems

1. X, Y and Z each have 3 digits and between them contain all the digits from 1 to 9.
   - $X + Y = Z$.
   - Z is a power of a prime.
   - Each digit of X is less than the corresponding digit of Y.

   What are X, Y and Z?

2. We are given a large cube. By connecting the centres of the faces of the large cube with lines we form an octahedron. By connecting the centres of each face of the octohedron with lines we get a small cube. What is the ratio between the side length of the two cubes?

3. Suppose a 5×9 rectangle is partitioned into 10 rectangles with integer dimensions. Is it necessarily true that some two of these smaller rectangles are congruent?

4. For all real numbers a, b, c, d with $a^2 + b^2 + c^2 + d^2 = 4$ prove that

   $$(a + 2)(b + 2) \geqslant cd$$

   and determine when equality holds.

5. Let $\mathbb{Z}^+$ denote the set of positive integers. A function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by: $f(1) = 1$ and $f(3) = 3$ and
   - $f(2n) = f(n)$,
   - $f(4n + 1) = 2f(2n + 1) - f(n)$, and
   - $f(4n + 3) = 3f(2n + 1) - 2f(n)$.

   for all positive integers n. Determine the number of positive integers $n \leqslant 2024$ such that $f(n) = n$.

6. Find the smallest positive integer such that when its last digit is moved to the start of the number (for example: 1234 becomes 4123) the resulting number is larger than and is a multiple of the original number. Numbers are written in standard decimal notation, with no leading zeroes.