**Vanderbilt University – College of Engineering**
**CS 2212, SPRING 2023 - Sections 01/02/03/04**
Homework Assignment #4 (maximum 50 points)

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**Section:** **02/Arena**

**Honor Statement:**
By submitting this homework under your personal Gradescope account, you attest that you have neither given nor received unauthorized aid concerning this homework. You further acknowledge the instructors are the copyright owners of this HW. Any posting/uploading of HW questions for distribution (e.g., GitHub, Chegg) will be considered an honor code violation (even after finishing this class) and submitted to the honor council.

**HW Philosophy:**
The objective of HW in CS2212 is to expand on lecture topics and examine them in more detail and depth. For this reason, we typically give a couple of weeks to complete the HW. Keep in mind that this is not supposed to be an easy task. Expect HWs to take time, require thinking, and challenge you. The TAs and instructors are not here to provide you with an answer or tell you whether your answer is correct while you are working on the HW. We are here to help you think more critically and at a higher level than when you started this class.

**Clear, Correct, and Concise:**
One of the critical aspects of this course is learning to communicate your ideas effectively. This skill is not only necessary for this course (or CS in general) but also a skill that employers highly value. Answers on all HWs, exams, and other activities in this class are graded on correctness, clarity, and conciseness. Keeping this in mind, here are a couple of pieces of advice:

- **Know Your Audience** - For this class, you should imagine your audience are your fellow students. Keeping this in mind ensures you provide enough details to prove to the grader that you understand your solution while keeping your answer concise enough to avoid unnecessary information.
- **Reread, Rewrite and Refine** - As with any good piece of writing, a well-written proof or answer often goes through several revisions. Reread, rewrite, refine, and take the time to get it right.

**You must type your answers and show your work/reasoning** (or you will lose >=50% on the exercise). Each exercise should begin on a new page (you can put multiple parts to the same exercises on the same page). Save your file as a PDF and upload the PDF document in an electronic format to Gradescope (https://www.gradescope.com/).

**You can use different colors** to markup your solution but **avoid using a red font.** When submitting your work, designate the corresponding page(s) of your submission for the appropriate question (or risk losing 5 points). See the following video at the 0:46 mark: https://www.gradescope.com/get_started#student-submission

**Questions:**
Ask your instructor before/after class or your instructor or TAs during office hours.

**Due Date:**
Your completed homework, including all the questions and answers, must be uploaded as a PDF to Gradescope no later than **Wednesday, April 19th, @ 9 AM CT (Nashville Time).**

## Exercise 1: How High Can You Count? [20 points]

Skills: logical thinking, sequences, counting.

**A. [4 points] Identical Books.** How many ways can 13 identical books be placed on four different shelves?

Since order does **not** matter:

Bins = 4
Balls = 13

$$\binom{bins + balls - 1}{bins - 1} = \binom{4 + 13 - 1}{4 - 1} = \binom{16}{3} = C(16, 3) = \frac{16!}{3!\,13!} = \mathbf{560}$$

There are 560 ways that 13 idential books can be placed on four different shelves.

**B. [4 points] Unique Books.** Suppose there are 12 unique books and four shelves. You want to put at least three books on each shelf. How many ways can you accomplish this, assuming order matters?

If every shelf has at least three books, this means that from our 12 books, there is only three books per shelf for four shelves. Since order matters and each shelf has the same number of books, the number of ways we can accomplish this is $12! = 479{,}001{,}600$. There are **479,001,600 ways** to place 12 unique books on 4 shelves assuring that each shelf has at least 3 books.

**C. [6 points] Unlimited (no rules).** Suppose we needed to place 12 unique books on four shelves, but you can put any number of books on any shelf. How many ways can you accomplish this, assuming order matters?

You can order the number of books 12! ways. To determine the number of ways how many books can be placed on each shelf is solved via the bins and balls method:

Bins = 4
Balls = 12

$$\binom{bins + balls - 1}{bins - 1} = \binom{4 + 12 - 1}{4 - 1} = \binom{15}{3} = C(15, 3) = \frac{15!}{3!\,12!} = \mathbf{455}$$

Multiply 455 by 12! to get the total number of ways to place 12 unique books on four shelves. This gives us $455 * 12! = \mathbf{217{,}945{,}728{,}000}$ ways to place 12 unique books on four shelves.

**D. [6 points – 3, 3] Skolar++.** Nadha's older brother, Bedha Skolar has just finished designing a new programming language aptly named "Skolar++." The syntax of Skolar++ requires all variable names to adhere to certain restrictions. Based on these restrictions, Bedha is now determining how many different variable names are possible. All variables must be named according to the following conventions:

- Zero or more letters from the set {b, e, d}

followed by

- Zero or more digits from the set {0, 1, 2, 3}.

In other words, the empty string, b, 12, bed100, and bedb2012 are all legal variable names. However, 1b and be2d are not valid variable names in Skolar++. Following this logic, we can see:

- There is exactly 1 legal variable name of length 0, the empty string.
- There are exactly 7 legal variable names of length 1: b, e, d, 0, 1, 2, and 3.
- There are exactly 37 legal variable names of length 2: bb, be, bd, b0, b1, b2, b3, eb, ee, ed, e0, e1, e2, e3, db, de, dd, d0, d1, d2, d3, 00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, and 33.

(i) **[3 points]** By illustrating the formula, indicate how you would determine the the number of possible legal variables names of length n.

$$\textbf{\# of possible variable names} = \sum_{k=0}^{n}(4^{n-k} * 3^k)$$

***k***: the number of times a letter from the set {b, e, d}, which is length 3, occupies a position in the string
***n* – *k*:** the number of times a number from the set {0, 1, 2, 3}, which is length 4, occupies a position in the string. It has to be *n-k* because if a letter is in the string, it has to be before the number.

The summation is from 0 times a letter from the set {b, e, d} occupies a position in the string to *n* times a letter from the set {b, e, d} occupies a position in the string.

(ii) **[3 points]** Now, using your formula, find a closed-form expression for the number of possible variable names of length n in the Skolar++ language.

We are using the summation formula $\sum_{k=0}^{n} a^k = \frac{a^{n+1}-1}{a-1}$ from step 4 to 5.

$$\textbf{Simplify Summation: Step 1)} \sum_{k=0}^{n}(4^{n-k} * 3^k) \rightarrow \textbf{Step 2)} \sum_{k=0}^{n}\left(\frac{4^n}{4^k}\right) * 3^k \rightarrow \textbf{Step 3)} \sum_{k=0}^{n}(4^n) * \left(\frac{3^k}{4^k}\right) \rightarrow$$

$$\textbf{Simplify Summation: Step 4)}\ (4^n)\sum_{k=0}^{n}(\frac{3}{4})^k \rightarrow \textbf{Step 5)}\ (4^n)(\frac{\left(\frac{3}{4}\right)^{n+1} - 1}{\left(\frac{3}{4}\right) - 1}) \rightarrow \textbf{Step 6)}\ \frac{4^n\left(\frac{3}{4}\right)^{n+1} - 4^n}{\left(\frac{3}{4}\right) - 1}$$

$$\textbf{Simplify Summation: Step 7)}\ \frac{4^n(\frac{3}{4})^{n+1}}{(-\frac{1}{4})} - \frac{4^n}{(-\frac{1}{4})} \rightarrow \textbf{Step 8)} \left(-4 * 4^n\left(\frac{3}{4}\right)^{n+1}\right) + (4 * 4^n)$$

$$\textbf{Simplify Summation: Step 9)} \left(-4 * 4^n\left(\frac{3}{4}\right)^{n+1}\right) + (4^{n+1}) \rightarrow \textbf{Step 10)} - (4^{n+1}) * \frac{3^{n+1}}{4^{n+1}} + (4^{n+1}) \rightarrow \textbf{Step 11)} -3^{n+1} + 4^{n+1}$$

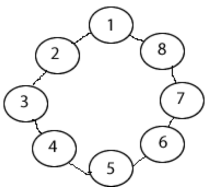
**Closed Form:** $4^{n+1} - 3^{n+1}$

## Exercise 2: Put A Ring On It [12 points]

**Skills:** logical thinking, proofs, graphs

**There are (3) parts to this exercise (A, B, C)**

Let's define a **ring graph** as an undirected graph G with a cycle utilizing all vertices and edges that can be visually represented in the shape of a ring. An example below is an 8-node ring graph. Let's further define the distance from vertex u to another vertex v in the graph ring as the **minimum** number of edges that must be traveled beginning from u to reach v. The distance from vertex 1 to vertex 8 in the ring graph below is 1.




An 8-vertex ring graph

**A.** **[2 points]** Determine the average distance from vertex 1 to all other vertices in the above graph. For full credit, be sure to show your work.

Summing up all the distances – 4, 3, 2, 1, 1, 2, 3 – gives us 16. Divide 16 by 7, and this gives us the average distance of 2.29 from vertex 1 to all other vertices.

**B.** **[4 points]** Let's generalize your formula from part A. Given a **ring cycle** graph G with n vertices, determine the average distance from a vertex u to all other vertices. You may assume n is even.

$$Average\ Distance = \frac{\frac{n}{2} + 2\left(\sum_{i=0}^{\left(\frac{n}{2}-1\right)} i\right)}{n-1} = \frac{\frac{n}{2} + 2\left(\frac{\left(\frac{n}{2}-1\right) * \left(\left(\frac{n}{2}-1\right)+1\right)}{2}\right)}{n-1}$$

$$Average\ Distance\ (cont.) = \frac{\frac{n}{2} + \left(\left(\frac{n}{2}-1\right) * \left(\frac{n}{2}\right)\right)}{n-1} = \boldsymbol{\frac{n^2}{4 * (n-1)}}$$

The formula above adds the distance between vertex u and the furthest vertex with the value that is twice the summation of the distances from vertex u to one less than the average vertex – this is the sum of all the distances. This number is then divided by the number of vertices in the ring not including vertex u.

**C.** **[6 points]** Now, let's now suppose you are given some random undirected graph G=(V, E) with V vertices and no self-loops (i.e., no vertex has an edge directly to itself). You want to delete all vertices in G, one at a time. As you are about to begin deleting vertices, a troll appears, preventing you from doing anything. Instead, the troll challenges you to a game. At every step, you are to pick some vertex v in graph G that has a degree at most 1 for removal. If you can find such a vertex, you may delete it and continue playing. The troll wins if you cannot find such a vertex. If you can manage to delete all vertices in G, you win.

**Prove/Disprove** the following claim: You can only defeat the troll if graph G is acyclic. You may include a diagram supporting your proof (but it is not required).

**Proof**: Using a **direct proof**, we will prove that the only way to defeat the troll is if graph G is acyclic.

To be able to delete all vertices in G, you need to find a degree of at most 1, delete it, and then continue this process until all vertices are deleted. Once you remove all vertices of degree at most 1, you either win the game, or are in situation where all the remaining vertices are in a cycle with each vertex being of degree 2. The reason why each remaining vertex in a cycle is degree 2 is because in a cycle the number of vertices is equal to the number of edges. Since each vertex has to be connected to two other vertexes in a cycle, you can never win the game if graph G is cyclic. Graph G must be acyclic to win the game. QED.

## Exercise 3: Hammy The Hamster [18 points]

**Skills:** GCD, Congruence, Mod, and Pigeonhole Principle

A. **[4 points]** Hammy The Hamster from lecture has been training on his hamster wheel. He can now travel 23 minutes per day in a clockwise fashion. Is it possible for Hammy to reach 9 minutes before 12 o'clock? If so, how many days will it take? If not, why not?

$23x \equiv 51 \text{ mod } 60$, where x is number of days

The equation above illustrates that we are trying to determine the number of days – x – that it will take the hamster traveling at 23 minutes per day to reach the 51 minute mark (9 minutes before 12 o' clock). We are using 51 mod 60 since there are 60 minutes. To determine if there are possible solutions to this statement, find the gcd(60, 23), which in this case is 1, and 1 divides 51. Now time to solve the mod.

1. $23x \equiv 51 \text{ mod } 60 \equiv y \text{ mod } 60$, where y is divisible by 23
2. y = -69 based on mod properties
3. $23x \equiv -69 \text{ mod } 60 \rightarrow x \equiv -3 \text{ mod } 60$
4. Since -3 days is not possible, we must find another congruence
5. $x \equiv -3 \text{ mod } 60 \equiv 57 \text{ mod } 60$

It will take Hammy the Hamster 57 days to reach 9 minutes before 12 o' clock.

B. **[4 points]** Suppose Hammy gets turned around and runs counterclockwise? Is it possible for Hammy to reach 9 minutes before 12 o'clock going counterclockwise? If so, how many days will it take? If not, why not?

$23x \equiv 9 \text{ mod } 60$, where x is number of days

The equation above illustrates that we are trying to determine the number of days – x – that it will take the hamster traveling at 23 minutes per day to reach the 51 minute mark (9 minutes before 12 o' clock). We are using 9 mod 60 since there are 60 minutes. Since we are traveling in a counter-clockwise direction, instead of a clockwise direction, it will be 9 mod 60 instead of 51 mod 60. To determine if there are possible solutions to this statement, find the gcd(60, 23), which in this case is 1, and 1 divides 9. Now time to solve the mod.

1. $23x \equiv 9 \text{ mod } 60 \equiv y \text{ mod } 60$, where y is divisible by 23
2. y = 69 based on mod properties
3. $23x \equiv 69 \text{ mod } 60 \rightarrow x \equiv 3 \text{ mod } 60$
4. $x \equiv 3 \text{ mod } 60$

It will take Hammy the Hamster 3 days to reach 9 minutes before 12 o' clock.

C. **[4 points]** Hammy The Hamster accidentally drinks a few sips of some spilled Red Bull by his cage. Now hyped up on caffeine, Hammy is traveling 2 full hours per day in a clockwise fashion. Is it possible for Hammy to reach 3 o'clock traveling clockwise? If so, how many days will it take? If not, why not?

$2x \equiv 3 \bmod 12$, where x is # of days

The equation above illustrates that we are trying to determine the number of days – x – that it will take the hamster traveling at 2 hours per day to reach 3 o'clock (since there are 12 hours, we are doing 3 mod 12). To determine that there are possible solutions to this statement, find the gcd(12, 2), which in this case is 2, and 2 does not divide 3. Therefore, it is impossible for Hammy to reach 3 o'clock traveling in a clockwise direction.

D. [**6 points]** At a ladies-night party, Nadha's girlfriend lost her precious amulet and burst into tears. Seeing her situation, all n attendees started whispering. It should be noted that if lady x whispers to lady y, it implies that lady y whispers to lady x. Note further that it is not possible for an attendee to whisper to herself.

**Prove:** There must be two ladies who have whispered with the same number of others present.

**Proof**: We will prove by **direct proof** via the Pigeonhole Principle that there must be two ladies who have whispered with the same number of others present. There are two cases to consider:

**Case 1**: One of the *n* attendees whispers to no one. The number of ladies this specific lady whispers to is 0 to n-2. There are *n* attendees and *n-1* number of ladies that each specific lady whispers to. By the pigeonhole principle, at least one of the ladies whispered to two others. QED.

**Case 2**: All attendees have whispered to someone else. In this scenario, the number of ladies that each specific lady whispers to is 1 to n-1. There are *n* attendees and *n-1* number of ladies that each specific lady whispers to. Because of the pigeonhole principle, at least one of the ladies whispered to two others. QED.