**Vanderbilt University – College of Engineering**
**CS 2212, SPRING 2023 - Sections 01/02/03/04**
Homework Assignment #2 (maximum 50 points)

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**Section: 02/Arena**

**Honor Statement:**
By submitting this homework under your personal Gradescope account, you attest that you have neither given nor received unauthorized aid concerning this homework. You further acknowledge the instructors are the copyright owners of this HW. Any posting/uploading of HW questions for distribution (e.g., GitHub, Chegg) will be considered an honor code violation (even after finishing this class) and submitted to the honor council.

**HW Philosophy:**
The objective of HW in CS2212 is to expand on lecture topics and examine them in more detail and depth. For this reason, we typically give a couple of weeks to complete the HW. Keep in mind that this is not supposed to be an easy task. Expect HWs to take time, require thinking, and challenge you. The TAs and instructors are not here to provide you with an answer or tell you whether your answer is correct while you are working on the HW. We are here to help you think more critically and at a higher level than when you started this class.

**Clear, Correct, and Concise:**
One of the critical aspects of this course is learning to communicate your ideas effectively. This skill is not only necessary for this course (or CS in general) but also a skill that employers highly value. Answers on all HWs, exams, and other activities in this class are graded on correctness, clarity, and conciseness. Keeping this in mind, here are a couple of pieces of advice:

- **Know Your Audience** - For this class, you should imagine your audience are your fellow students. Keeping this in mind ensures you provide enough details to prove to the grader that you understand your solution while keeping your answer concise enough to avoid unnecessary information.
- **Reread, Rewrite and Refine** - As with any good piece of writing, a well-written proof or answer often goes through several revisions. Reread, rewrite, refine, and take the time to get it right.

**You must type your answers** (or you will lose 50% on the exercise). Each exercise should begin on a new page (you can put multiple parts to the same exercises on the same page). Save your file as a PDF and upload the PDF document in an electronic format to Gradescope (https://www.gradescope.com/).

**You can use different colors** to markup your solution but **avoid using a red font.** When submitting your work, designate the corresponding page(s) of your submission for the appropriate question (or risk losing 5 points). See the following video at the 0:46 mark: https://www.gradescope.com/get_started#student-submission

**Questions:**
Ask your instructor before/after class or your instructor or TAs during office hours.

**Due Date:**
Your completed homework, including all the questions and answers, must be uploaded as a PDF to Gradescope no later than **Wednesday, March 1st, @ 9 AM CT (Nashville Time).**

## Exercise 1: Logically Illogical [15 points – 4, 5, 6] – ARENA

**Skills:** logical reasoning, removing preconceived notions and bias, arriving at logical conclusions based solely on reasoning.

**There are (3) parts to this exercise (A, B, C).**

CS 2212 students often struggle with making assumptions because of preconceived notions that may (or may not) be present in the actual hypothesis or argument. Doing so often leads to reaching a conclusion that needs to be fully supported by the statements written or reaching an incorrect conclusion.

In this exercise, you will purposefully be given some nonsensical statements. Your job will be to decipher these nonsensical statements by looking at what is being said from a symbolic standpoint and, using all the statements provided, reach a logical (albeit possibly silly) conclusion.

Let's do one as an example, so you get the idea:

- All blocks are square.
- No square things can dance.

The crux of the "puzzle" is to chain together all the given statements so that each statement leads to the next. We do this until we reach the last statement when we can draw our conclusion. Let's start by assigning predicates and then representing the statements symbolically:

- B(x) = x is a block
- S(x) = x is square
- D(x) = x can dance

| # | Premise | Symbolic Representation |
|---|---|---|
| 1 | All blocks are square | $\forall x(B(x) \rightarrow S(x))$ |
| 2 | No square things can dance | $\forall x(S(x) \rightarrow \sim D(x))$ |

It's easy to see the connection between the 1st and 2nd premises without any fancy mechanics, but to do it formally will require using our inference rules for quantifiers and propositions:

| # | Statement | Justification |
|---|---|---|
| 1 | $\forall x(B(x) \rightarrow S(x))$ | Premise |
| 2 | $\forall x(S(x) \rightarrow \sim D(x))$ | Premise |
| 3 | $B(c) \rightarrow S(c)$ | 1, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 4 | $S(c) \rightarrow \sim D(c)$ | 2, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 5 | $B(c) \rightarrow \sim D(c)$ | 2, 3, Hypothetical Syllogism (HS) |
| 6 | $\forall x(B(x) \rightarrow \sim D(x))$ | 5, Universal Generalization (UG) |

With this completed, we have now successfully chained together the initial two premises leading us to the logical conclusion, $\forall x(B(x) \rightarrow \sim D(x))$ or, more precisely: "If you're a block, then you cannot dance."

**A. [4 points – 1, 1, 2]** Using all the implications below, your goal is to chain them together to reach a logical conclusion, regardless of how nonsensical it may seem. Follow the template from above.

- Loyal commodore fans (L) are faithful Commodore fans (F).
- Those who cannot dance (¬D) are not faithful Commodore fans.

(i) **[1 point]** List your predicates:

L(x) = x is a loyal commodore fan
F(x) = x is a faithful commodore fan
D(x) = x can dance

(ii) **[1 point]** Write your premises symbolically:

| # | Premise | Symbolic Representation |
|---|---|---|
| 1 | Loyal commodore fans (L) are faithful Commodore fans (F) | $\forall x(L(x) \rightarrow F(x))$ |
| 2 | Those who cannot dance (¬D) are not faithful Commodore fans | $\forall x(\neg D(x) \rightarrow \neg F(x))$ |

(iii) **[2 points]** Construct your proof and state the conclusion. You may or may not need all rows.

| # | Statement | Justification |
|---|---|---|
| 1 | $\forall x(L(x) \rightarrow F(x))$ | Premise |
| 2 | $\forall x(\neg D(x) \rightarrow \neg F(x))$ | Premise |
| 3 | $L(s) \rightarrow F(s)$ | 1, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 4 | $\neg D(s) \rightarrow \neg F(s)$ | 2, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 5 | $F(s) \rightarrow D(s)$ | 4, Contrapositive |
| 6 | $L(s) \rightarrow D(s)$ | 2, 3, Hypothetical Syllogism (HS) |
| 7 | $\forall x(L(x) \rightarrow D(x))$ | 6, Universal Generalization (UG) |
| 8 | QED | |
| 9 | | |
| 10 | **Conclusion:** | $\forall x(L(x) \rightarrow D(x))$; If you are a loyal commodore fan, then you can dance |

**B. [5 points]** Using all the implications below, your goal is to chain them together to reach a logical conclusion, regardless of how nonsensical it may seem. Follow the initial template from above.

- No UT fans (U) can limbo (L).
- No Discrete Structures Professor (D) ever passes up the opportunity to limbo.
- All frogs (F) are UT Fans.

(i) **[1 point]** List your predicates:

U(x) = x is a UT fan
L(x) = x is someone who can limbo
D(x) = x is a discrete structure professor
F(x) = x is a frog

(ii) **[1 point]** Write your premises symbolically:

| # | Premise | Symbolic Representation |
|---|---|---|
| 1 | No UT fans (U) can limbo (L). | $\forall x(U(x) \rightarrow \neg L(x))$ |
| 2 | No Discrete Structures Professor (D) ever passes up the opportunity to limbo. | $\forall x(D(x) \rightarrow L(x))$ |
| 3 | All frogs (F) are UT Fans. | $\forall x(F(x) \rightarrow U(x))$ |

(iii) **[3 points]** Construct your proof and state the conclusion. You may or may not need all rows.

| # | Statement | Justification |
|---|---|---|
| 1 | $\forall x(U(x) \rightarrow \neg L(x))$ | Premise |
| 2 | $\forall x(D(x) \rightarrow L(x))$ | Premise |
| 3 | $\forall x(F(x) \rightarrow U(x))$ | Premise |
| 4 | $U(r) \rightarrow \neg L(r)$ | 1, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 5 | $D(r) \rightarrow L(r)$ | 2, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 6 | $F(r) \rightarrow U(r)$ | 3, Universal Instantiation (UI) aka Universal Elimination (UE) |
| 7 | $\neg L(r) \rightarrow \neg D(r)$ | 5, Contrapositive |
| 8 | $U(r) \rightarrow \neg D(r)$ | 4, 5, Hypothetical Syllogism (HS) |
| 9 | $F(r) \rightarrow \neg D(r)$ | 6, 8, Hypothetical Syllogism (HS) |
| 10 | $\forall x(F(x) \rightarrow \neg D(x))$ | 9, Universal Generalization (UG) |
| 11 | QED | |
| 12 | | |
| 13 | **Conclusion:** | $\forall x(F(x) \rightarrow \neg D(x)))$; If you are a frog, then you are not a discrete structures professor |

**C. [6 points]** Using all the implications below, your goal is to chain them together to reach a logical conclusion, regardless of how nonsensical it may seem. Follow the initial template from above.

- Every idea of mine that cannot be expressed as a haiku is completely ridiculous.
- None of my ideas about bird bagels are worth writing down.
- No idea of mine that fails to come true can be expressed as a haiku.
- I never have any ridiculous ideas that I do not immediately refer to as my exorcist baby.
- My dreams are all about bird bagels.
- I never refer to any idea of mine as my exorcist baby unless it is worth writing down.

(i) **[1 point]** List your predicates:

H(x) = x is expressed as a haiku
D(x) = x is a dream
R(x) = x is a ridiculous idea
F(x) = x is fails to come true
B(x) = x is a bird bagel
E(x) = x is an exorcist baby
W(x) = x is worth writing down

(ii) **[2 points]** Write your premises symbolically. Now that you understand the pattern, you do not have to list the quantifiers with every premise or the proof on this last exercise. For example, the symbolic representation of the statement "all birds can fly" you may use the form $B(x) \rightarrow F(x)$, where B(x) = x is a bird and F(x) = x can fly. In other words, you can overlook $\forall x$ to make the following proof easier. You may or may not need all rows.

| # | Premise | Symbolic Representation |
|---|---|---|
| 1 | Every idea of mine that cannot be expressed as a haiku is completely ridiculous | $\neg H(x) \rightarrow R(x)$ |
| 2 | None of my ideas about bird bagels are worth writing down. | $B(x) \rightarrow \neg W(x)$ |
| 3 | No idea of mine that fails to come true can be expressed as a haiku. | $F(x) \rightarrow \neg H(x)$ |
| 4 | I never have any ridiculous ideas that I do not immediately refer to as my exorcist baby. | $R(x) \rightarrow E(x)$ |
| 5 | My dreams are all about bird bagels. | $D(x) \rightarrow B(x)$ |
| 6 | I never refer to any idea of mine as my exorcist baby unless it is worth writing down | $E(x) \rightarrow W(x)$ |

(iii) **[4 points]** Construct your proof and state the conclusion. Now that you understand the pattern, you do not have to list the quantifiers with every premise or the proof on this last exercise. However, if you would like to do so, you can:

| # | Statement | Justification |
|---|---|---|
| 1 | $\neg H(x) \to R(x)$ | Premise |
| 2 | $B(x) \to \neg W(x)$ | Premise |
| 3 | $F(x) \to \neg H(x)$ | Premise |
| 4 | $R(x) \to E(x)$ | Premise |
| 5 | $D(x) \to B(x)$ | Premise |
| 6 | $E(x) \to W(x)$ | Premise |
| 7 | $F(x) \to R(x)$ | 1, 3 Hypothetical Syllogism |
| 8 | $R(x) \to W(x)$ | 4, 6 Hypothetical Syllogism |
| 9 | $D(x) \to \neg W(x)$ | 2, 5 Hypothetical Syllogism |
| 10 | $F(x) \to W(x)$ | 7, 8 Hypothetical Syllogism |
| 11 | $\neg W(x) \to \neg F(x)$ | 10, Contrapositive |
| 12 | $D(x) \to \neg F(x)$ | 9, 11 Hypothetical Syllogism |
| 13 | QED | |
| 14 | | |
| 15 | **Conclusion:** | $D(x) \to \neg F(x)$; If I have dreams, then they will come true |

## Exercise 2: Prove It [20 points – 5, 7, 8] – ARENA

**Skills:** Critical thinking and proof techniques.

**There are (3) parts to this problem A, B, C.**

**A. [5 points].** Write a well-structured proof demonstrating n(n - 1) is even for all n >= 1, n ∈ N+ (the set of positive natural numbers)

**Proof**: Using a **direct proof**, we will prove if n is greater than or equal to 1, n(n-1) is even.

There are two cases we must consider, when n is odd or when n is even.

**N is Odd**

| | | |
|---|---|---|
| 1 | Expand $n(n-1)$ $to$ $n^2 - n$ | Math (distrib) |
| 2 | Assume n (any integer) is odd | Premise |
| 3 | n = 2k + 1 for some integer k | Defn. of Odd |
| 4 | $n^2 - n = (2k+1)^2 - (2k+1)$ | 3, Math (square) |
| 5 | $n^2 - n = 4k^2 + 2k$ | 4, Math (simp.) |
| 6 | $n^2 - n = 2(2k^2 + k)$ | 5, Math (distrib) |
| 7 | $m = 2k^2 + k$; m is an integer | Math substitution |
| 8 | $n^2 - n = 2m$ for some integer m | 6, 7 Math Sub |
| 9 | $n^2 - n$ must be even by definition | Defn. of even |
| 10 | QED | 1-9, CP |

**N is Even**

| | | |
|---|---|---|
| 1 | Expand $n(n-1)$ $to$ $n^2 - n$ | Math (distrib) |
| 2 | Assume n (any integer) is even | Premise |
| 3 | n = 2k for some integer k | Defn. of even |
| 4 | $n^2 - n = (2k)^2 - (2k)$ | 3, Math (square) |
| 5 | $n^2 - n = 4k^2 - 2k$ | 4, Math (simp.) |
| 6 | $n^2 - n = 2(2k^2 - k)$ | 5, Math (distrib) |
| 7 | $m = 2k^2 - k$; m is an integer | Math substitution |
| 8 | $n^2 - n = 2m$ for some integer m | 6, 7 Math Sub |
| 9 | $n^2 - n$ must be even by definition | Defn. of even |
| 10 | QED | 1-9, CP |

Whether n is odd or even, n(n-1) is even. This statement is proven true. QED.

**B.** **[7 points]** Use your result from part A to show that $n^2 - 1$ is a multiple of 8 whenever n is an odd integer greater than or equal to 1, $n \in N+$. Tip: You know for any odd integer x, x = 2k+1 for some integer k. You may find it helpful to use the additional fact that when x is odd, x also equals 2k – 1 for some integer k.

**Proof**: Using a **direct proof**, we will prove if n is an odd integer, $n^2 - 1$ is a multiple of 8.

Since we are assuming that n is odd, we can use the definition of odd (x = 2k – 1) and plug that into our expression: $n^2 - 1$. After some simplification steps, we conclude that $n^2 - 1$ is equal to 4k (k − 1). Based on part A, we have proved n (n – 1) is even. Using this result, we can conclude that k (k – 1) is an even number. Furthermore, we know that from our definition of even, an even number has the form x = 2m. Plugging this back into our original expression, $n^2 - 1$ is equal to 8m. $n^2 - 1$ is a multiple of 8. QED.

**C.** **[8 points]** Write a well-structured written proof given integers a, b, c, d, e that when a + 2b + 3c + 4d + 5e >= 70, then one or more of the following must be true:

- a >= 2
- b >= 3
- c >= 4
- d >= 5
- e >= 6

**Proof**: Using a proof by way of contradiction (BWOC), we will prove that when a + 2b + 3c + 4d + 5e >= 70, then a >= 2, b >=3, c >= 4, d >=5, e >= 6.

To solve this problem, let's use propositional logic. Let's say p is "a + 2b + 3c + 4d + 5e >= 70" and q is "one or more of the following must be true a >= 2, b >=3, c >= 4, d >=5, e >= 6". Then the statement, p → q means if a + 2b + 3c + 4d + 5e >= 70 then one or more of the following must be true a >= 2, b >=3, c >= 4, d >=5, e >= 6. The contrapositive of this statement ($\neg q \rightarrow \neg p$) is "If a is not greater than 2, b is not greater than 3, c is not greater than 4, d is not greater than 5, and e is not greater than 6, then a + 2b + 3c + 4d + 5e is not greater than or equal to 70." To prove that the contrapositive is true, plug in the max value from the conditional statement. This means plugging in 1 for a, 2 for b, 3 for c, 4 for d, and 5 for e. Plug this into the equation: 1+ 2(2) + 3(3)+ 4(4) + 5(5), which gives us 55. By proving the contrapositive true, we are proving the original statement true as well (Defn. of a contrapositive). For a + 2b + 3c + 4d + 5e >= 70, then one or more of the following must be true a >= 2, b >=3, c >= 4, d >=5, e >= 6. QED.

## Exercise 3: Set of Sensors and Functions [15 points] - HASAN

**Skills:** understanding sets, functions, logical reasoning

**There are two (2) parts to this exercise (A and B).**

### A. [3 points] Power Set

Given a set S = {◆, ▣}, list all elements in the powerset of the powerset of S, i.e., P(P(S)). List one element per line. You may add or remove lines as needed.

P(S) = {{}, {◆}, {▣}, {◆, ▣}}

P(P(S)):

{}
{{}}
{{◆}}
{{▣}}
{{◆, ▣}}
{{}, {◆}}
{{}, {▣}}
{{}, {◆, ▣}}
{{◆}, {▣}}
{{◆}, {{◆}, {▣}}}
{{▣}, {{◆}, {▣}}}
{{}, {◆}, {▣}}
{{}, {◆}, {◆, ▣}}
{{}, {▣}, {◆, ▣}}
{{◆}, {▣}, {◆, ▣}}
{{}, {◆}, {▣}, {◆, ▣}}

### B. [10 points]: Sensor and Function

Consider a set of ten target locations **T = {$t_1$, $t_2$, … , $t_{10}$}** and three sensors **$S_1$, $S_2$**, and **$S_3$**. Each sensor has a sensing range (circular region) whereby a sensor detects an event if it occurs at a target location lying within the sensor's sensing range. For example, **$S_3$** detects an event if it occurs at any of the target locations **{$t_4$, $t_5$, … , $t_8$}**. The output of each sensor is either **0** or **1**. If an event occurs at a target location that is within the sensor's sensing range, then the output of that sensor is **1**, otherwise it is **0**.

Based on sensors outputs, we can assign a 3-bit label **[$s_1(t_i)$ $s_2(t_i)$ $s_3(t_i)$ ]** to each target location **$t_i$** as follows: if sensor **$S_1$** monitors target **$t_i$**, then **$s_1(t_i)$ = 1**, otherwise **$s_1(t_i)$ = 0**. Similarly, we define **$s_2(t_i)$** and **$s_3(t_i)$** corresponding to Sensors **$S_2$**, and **$S_3$**, respectively. As an example, the label corresponding to the target location **$t_1$** is **[1 0 0]** because sensor $S_1$ monitors **$t_1$** (thus, **$s_1(t_1)$ = 1**), whereas sensors **$S_2$** and **$S_3$** do not monitor **$t_1$** (thus, **$s_2(t_1)$ = 0** and **$s_3(t_1)$ = 1**).

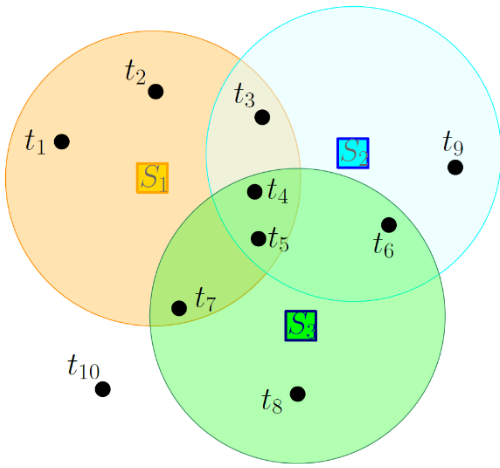
Let **L** be the set of labels corresponding to all target locations, that is,

**L = { [ $s_1(t_1)$ $s_2(t_1)$ $s_3(t_1)$ ], [ $s_1(t_2)$ $s_2(t_2)$ $s_3(t_2)$ ], …. , [ $s_1(t_{10})$ $s_2(t_{10})$ $s_3(t_{10})$ ] }.**

Now, consider the following **function** (map)

$$f: \quad \mathbf{T} \rightarrow \mathbf{L}$$

In case of an event at some target location, we have access only to sensors' outputs, that is, the label of the target location at which event occurs. For instance, if event occurs at $\mathbf{t_1}$, we only get the label of $\mathbf{t_1}$, which is **[ 1 0 0]**.




We are interested in the **identification problem**, that is, *from the labels, identify exactly the target location at which the event occurred.*

Answer parts B(i), B(ii), B(iii), and B(iv) below in detail and with complete explanations.

**B(i) [2 points]** For the sensor placement in the above figure, write down the sets **T**, **L**, and compute the values of ***f*** **($t_i$)** for each element **$t_i \in$T**. Does ***f*** solve the identification problem? You can use the following table to show the mapping between T and L.

**Note:** In the Table 1, the first row shows the mapping for $t_1$ is [1 0 0 ] because s1($t_1$) = 1, $s_2(t_1)$ = 0, and $s_3(t_1)$ = 0.

Table 1: Mapping between T and L.

| T | L |
|---|---|
| $t_1$ | [1 0 0] |
| $t_2$ | [1 0 0] |
| $t_3$ | [1 1 0] |
| $t_4$ | [1 1 1] |
| $t_5$ | [1 1 1] |
| $t_6$ | [0 1 1] |
| $t_7$ | [1 0 1] |
| $t_8$ | [0 0 1] |
| $t_9$ | [0 1 0] |
| $t_{10}$ | [0 0 0] |

No, ***f*** **($t_i$)** does not solve the identification problem since there are multiple targets with the same mapping. For example, for $t_1$ and $t_2$ they have the same labels. Therefore, from the labels, one cannot identify exactly the target location at which the event occurred.

**B(ii) [2 points]** In general, assuming that the sensor placement solves the *identification* problem, then explain if the corresponding ***f*** is surjective, injective, or bijective?

Since the sensor placement solves the identification problem then this means that ***f*** **($t_i$)** is injective – each target has a unique label. Is ***f*** **($t_i$)** bijective? The definition of bijective (onto) is if in the co-domain, everyone maps to at least one element in the domain. Based on the codomain of L, for the elements - [1 0 0], [1 1 0], [1 1 1], [0 1 1], [1 0 1], [0 0 1], and [0 1 0] – they map to at least one element in the domain (targets). Since ***f*** **($t_i$)** is both injective and surjective, then ***f*** **($t_i$)** is bijective.

**B(iii) [2 points]** If there are ***n*** target locations, then what is the minimum number of sensors that are needed to solve the **identification** problem (i.e., all labels are associated with a target)? Don't worry about the intersections and particular sensing model (i.e., whether it is, circular or not). Just consider the number of labels that can be generated by a k-bit label and map it to the given problem.

$$\text{\# of sensors} \geq \text{ ceil } (\log_2(\text{n}))$$

We get this formula from the fact that $2^x = \text{ n}$, where *x* stands for number of sensors and *n* stands for number of targets. Take the log of both sides to get the formula above.

**B(iv) It has two parts: a and b [4 points, 2 each]**

Let **$M(S_k,t_i)$** mean *"Sensor $S_k$ monitors target $t_i$"*.

**B(iv)-(a) (2 points)** What does the following expression mean?
$\forall t_i \, \forall t_j \, \exists \, S_k \, ( (t_i \neq t_j) \rightarrow (M(S_k,t_i) \oplus M(S_k,t_j)))$

$\oplus$ is the exclusive or = or and not both

For **all** targets, there exists **some** sensor that monitors exactly one target (not both).

**B(iv)-(b) (2 points)** Prove or disprove the statement below:

**If *f* is injective, then** $\forall t_i \, \forall t_j \, \exists \, S_k \, ( (t_i \neq t_j) \rightarrow (M(S_k,t_i) \oplus M(S_k,t_j)) )$

**Proof**: Using a proof by way of contradiction (BWOC), we will prove that when f is injective, then for all targets, there exists some sensor that monitors exactly one target (not both).

Assume that for $t_i$ and $t_j$ (two different targets), there is not a sensor that produces different labels and that the hypothesis "if f is injective" is true. This means that every sensor has the same labels as $t_i$ and $t_j$. By having the same output, this means that f is not injective. However, this is a contradiction since we stated that f is injective. Therefore, the whole statement above is true. QED.