Complete Simulation Results for On the Use of Information Criteriz for Subset Selection in Least Squares Regression

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- Orthogonal X, simulation setups are discussed in the Supplemental Material Section B.1.
  - The performance of selection rules for BS. The selection rules include C_p, AICc, BIC, GCV and 10-fold CV. For each selection rule except CV, there are two columns in the table indicating the degrees of freedoms to use in calculating the information criterion. The "edf" (effective degrees of freedom) is estimated using definition (3) by assuming the knowledge of μ and σ, and hence it is an infeasible rule. The "ndf/hdf/bdf" (naive degrees of freedom / heuristic degrees of freedom / degrees of freedom based on bootstrap) are feasible selection rules in practice.
    - Orth-Sparse-Ex1: tables S1-S2
    - Orth-Sparse-Ex2: tables S3-S4
    - Orth-Dense: tables S5-S6
- General X, simulation setups are discussed in Supplemental Material Section B.2.
  - The performance of BOSS compared to BS, FS and regularization methods ($n > p$). For BOSS, we consider three selection rules, that are AICc-hdf, C_p-hdf and CV. For lasso and gamma lasso, we consider AICc and CV. And for the remaining methods, we use CV. Note that for lasso, we use the number of non-zero coefficients $k(\lambda)$ in place of edf in the AICc formula (7). Zou et al. (2007) showed that $k(\lambda)$ is an unbiased estimator of edf for lasso. For gamma lasso, Taddy (2017) suggested a heuristic degrees of freedom to be plugged into (7) in order to use AICc as the selection rule.
    - Sparse-Ex1: tables S7-S12
    - Sparse-Ex2: tables S13-S18
    - Sparse-Ex3: tables S19-S24
    - Sparse-Ex4: tables S25-S30
    - Dense: tables S31-S36
  - The performance of BOSS compared to FS and regularization methods ($n < p$). For FS, we consider EBIC (Wang, 2009), HDBIC and HDHQ (Ing and Lai, 2011). We also consider the stopping rule, the trimming rule, and a combination of both introduced by Ing and Lai (2011) for FS.
    - Sparse-Ex1: tables S37-S39
    - Sparse-Ex2: tables S40-S42
    - Sparse-Ex3: tables S43-S45
    - Sparse-Ex4: tables S46-S48
    - Dense: tables S49-S51

Table S1: The performance of BS using different selection rules, Orth-Sparse-Ex1, n=200

		C_p		AICc		BIC		GCV		CV
		edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	
		% worse than the best possible BS								
hsnr	p=14	8	33/9/11	7	33/6/9	0	10/0/1	9	34/8/10	19
	p=30	4	84/5/7	2	83/2/5	0	28/0/0	4	86/4/7	24
	p=60	2	157/3/5	1	159/2/3	0	64/0/0	2	167/3/4	-
	p=180	1	338/30/32	0	392/1/2	0	206/0/0	0	431/2/3	-
msnr	p=14	8	33/14/12	7	33/11/10	0	10/2/1	9	34/14/12	19
	p=30	4	84/12/11	2	83/8/7	0	28/1/2	4	86/10/10	24
	p=60	2	157/13/10	1	159/8/7	0	64/2/2	2	167/11/9	-
	p=180	1	338/40/38	0	392/7/6	0	206/3/4	0	431/10/8	-
lsnr	p=14	18	16/23/23	18	17/24/24	93	43/97/93	18	16/23/23	26
	p=30	20	25/36/33	21	24/37/35	68	23/68/67	21	26/37/35	28
	p=60	18	44/28/27	21	45/31/30	43	17/43/43	20	48/30/29	-
	p=180	15	108/35/34	18	132/22/22	25	50/25/25	17	149/22/21	-
		Relative efficiency								
hsnr	p=14	0.93	0.75/0.92/0.9	0.94	0.75/0.94/0.92	1	0.91/1/0.99	0.92	0.75/0.92/0.91	0.84
	p=30	0.96	0.54/0.95/0.93	0.98	0.55/0.98/0.96	1	0.78/1/1	0.96	0.54/0.96/0.94	0.81
	p=60	0.98	0.39/0.97/0.95	0.99	0.39/0.98/0.97	1	0.61/1/1	0.98	0.38/0.97/0.96	-
	p=180	0.99	0.23/0.77/0.76	1	0.2/0.99/0.98	1	0.33/1/1	1	0.19/0.98/0.97	-
msnr	p=14	0.93	0.75/0.88/0.89	0.94	0.75/0.9/0.91	1	0.91/0.99/0.99	0.92	0.75/0.88/0.9	0.84
	p=30	0.96	0.54/0.89/0.9	0.98	0.55/0.92/0.93	1	0.78/0.99/0.98	0.96	0.54/0.91/0.91	0.81
	p=60	0.98	0.39/0.89/0.91	0.99	0.39/0.92/0.93	1	0.61/0.98/0.98	0.98	0.38/0.9/0.92	-
	p=180	0.99	0.23/0.71/0.72	1	0.2/0.93/0.94	1	0.33/0.97/0.96	1	0.19/0.91/0.92	-
lsnr	p=14	0.99	1/0.95/0.95	0.98	1/0.94/0.94	0.6	0.82/0.59/0.6	0.99	1/0.95/0.94	0.92
	p=30	1	0.97/0.89/0.9	1	0.97/0.88/0.89	0.72	0.98/0.72/0.72	0.99	0.96/0.88/0.89	0.94
	p=60	0.99	0.81/0.92/0.92	0.97	0.81/0.89/0.9	0.82	1/0.82/0.82	0.98	0.79/0.9/0.91	-
	p=180	1	0.55/0.86/0.86	0.97	0.5/0.95/0.95	0.93	0.77/0.93/0.93	0.98	0.46/0.95/0.95	-
		Sparsistency (number of extra variables)								
hsnr	p=14	6(0.2)	6(1.3)/6(0.4)/6(0.4)	6(0.2)	6(1.2)/6(0.2)/6(0.3)	6(0)	6(0.2)/6(0)/6(0)	6(0.3)	6(1.3)/6(0.4)/6(0.4)	6(0.7)
	p=30	6(0.1)	6(3.9)/6(0.2)/6(0.2)	6(0)	6(3.8)/6(0.1)/6(0.1)	6(0)	6(0.6)/6(0)/6(0)	6(0.1)	6(4.1)/6(0.1)/6(0.1)	6(0.7)
	p=60	6(0)	6(8.9)/6(0)/6(0.1)	6(0)	6(9.2)/6(0)/6(0)	6(0)	6(1.6)/6(0)/6(0)	6(0)	6(10.5)/6(0)/6(0.1)	-
	p=180	6(0)	6(32.2)/6(6.4)/6(6.3)	6(0)	6(48.9)/6(0)/6(0)	6(0)	6(9.5)/6(0)/6(0)	6(0)	6(74.6)/6(0)/6(0)	-
msnr	p=14	6(0.2)	6(1.3)/6(0.7)/6(0.4)	6(0.2)	6(1.2)/6(0.5)/6(0.3)	6(0)	6(0.2)/6(0)/6(0)	6(0.3)	6(1.3)/6(0.6)/6(0.4)	6(0.7)
	p=30	6(0.1)	6(3.9)/6(0.4)/6(0.3)	6(0)	6(3.8)/6(0.2)/6(0.1)	6(0)	6(0.6)/6(0)/6(0)	6(0.1)	6(4.1)/6(0.3)/6(0.2)	6(0.7)
	p=60	6(0)	6(8.9)/6(0.3)/6(0.2)	6(0)	6(9.2)/6(0.1)/6(0.1)	6(0)	6(1.6)/6(0)/6(0)	6(0)	6(10.5)/6(0.2)/6(0.1)	-
	p=180	6(0)	6(32.2)/6(6.6)/6(6.6)	6(0)	6(48.9)/6(0.1)/6(0.1)	6(0)	6(9.5)/6(0)/6(0)	6(0)	6(74.6)/6(0.1)/6(0.1)	-
lsnr	p=14	5.5(2.3)	5.2(1.3)/5.6(4.6)/5.4(3.6)	5.4(2.1)	5.2(1.2)/5.4(4.2)/5.3(3.2)	0.9(0.1)	3.6(0.2)/0.7(0.1)/0.9(0.1)	5.5(2.4)	5.3(1.3)/5.6(4.6)/5.4(3.5)	4.9(1.6)
	p=30	4.5(1.9)	5.3(3.9)/4.2(4.9)/4.2(4)	4.2(1.2)	5.2(3.8)/3.3(2.2)/3.4(1.8)	0.1(0)	3.7(0.6)/0.1(0)/0.2(0)	4.5(2)	5.3(4.1)/3.9(4.1)/3.9(3.3)	4(1.9)
	p=60	3.4(1.1)	5.2(8.9)/2.7(1.8)/2.8(1.6)	2.7(0.6)	5.3(9.2)/1.5(0.2)/1.7(0.3)	0(0)	3.8(1.4)/0.1(0)/0.1(0)	3.1(0.9)	5.4(10.4)/2(0.6)/2.1(0.7)	-
	p=180	1.9(0.5)	5.3(32.2)/1.8(10.9)/1.9(9.8)	1.1(0.1)	5.6(49)/0.5(0)/0.6(0)	0(0)	4.2(8.4)/0(0)/0(0)	1.4(0.2)	5.8(74.6)/0.7(0.1)/0.8(0.1)	-

Table S2: The performance of BS using different selection rules, Orth-Sparse-Ex1, n=2000

		C_p		AICc		BIC		GCV		CV
		edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	edf	ndf/hdf/bdf	
		% worse than the best possible BS								
hsnr	p=14	8	33/7/9	8	33/7/9	0	3/0/0	8	33/7/9	18
	p=30	3	85/3/6	3	85/3/6	0	9/0/0	3	86/3/6	23
	p=60	2	155/2/4	2	156/2/4	0	21/0/0	2	156/2/4	-
	p=180	0	334/1/3	1	337/1/3	0	60/0/0	1	340/1/3	-
msnr	p=14	8	33/7/9	8	33/7/9	0	3/0/0	8	33/7/9	18
	p=30	3	85/3/6	3	85/3/6	0	9/0/0	3	86/3/6	23
	p=60	2	155/2/4	2	156/2/4	0	21/0/0	2	156/2/4	-
	p=180	0	334/1/3	1	337/1/3	0	60/0/0	1	340/1/3	-
lsnr	p=14	8	33/9/9	8	33/9/9	0	3/0/0	8	33/9/9	18
	p=30	3	85/6/7	3	85/5/6	0	9/0/0	3	86/6/7	23
	p=60	2	155/5/5	2	156/5/5	0	21/0/0	2	156/5/5	-
	p=180	0	334/5/4	1	337/4/4	0	60/1/1	1	340/5/4	-
		Relative efficiency								
hsnr	p=14	0.93	0.75/0.94/0.92	0.93	0.75/0.94/0.92	1	0.97/1/1	0.92	0.75/0.94/0.92	0.84
	p=30	0.97	0.54/0.97/0.94	0.97	0.54/0.97/0.94	1	0.92/1/1	0.97	0.54/0.97/0.94	0.81
	p=60	0.98	0.39/0.98/0.96	0.98	0.39/0.98/0.96	1	0.83/1/1	0.98	0.39/0.98/0.96	-
	p=180	1	0.23/0.99/0.97	0.99	0.23/0.99/0.97	1	0.62/1/1	0.99	0.23/0.99/0.97	-
msnr	p=14	0.93	0.75/0.94/0.92	0.93	0.75/0.94/0.92	1	0.97/1/1	0.92	0.75/0.94/0.92	0.84
	p=30	0.97	0.54/0.97/0.94	0.97	0.54/0.97/0.94	1	0.92/1/1	0.97	0.54/0.97/0.94	0.81
	p=60	0.98	0.39/0.98/0.96	0.98	0.39/0.98/0.96	1	0.83/1/1	0.98	0.39/0.98/0.96	-
	p=180	1	0.23/0.99/0.97	0.99	0.23/0.99/0.97	1	0.62/1/1	0.99	0.23/0.99/0.97	-
lsnr	p=14	0.93	0.75/0.92/0.92	0.93	0.75/0.92/0.92	1	0.97/1/1	0.92	0.75/0.92/0.92	0.84
	p=30	0.97	0.54/0.95/0.94	0.97	0.54/0.95/0.94	1	0.92/1/1	0.97	0.54/0.95/0.94	0.81
	p=60	0.98	0.39/0.95/0.95	0.98	0.39/0.95/0.95	1	0.83/1/1	0.98	0.39/0.95/0.95	-
	p=180	1	0.23/0.96/0.96	0.99	0.23/0.96/0.96	1	0.62/0.99/0.99	0.99	0.23/0.96/0.96	-
		Sparsistency (number of extra variables)								
hsnr	p=14	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0)	6(0)/6(0)/6(0)	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0.6)
	p=30	6(0.1)	6(3.8)/6(0.1)/6(0.2)	6(0.1)	6(3.8)/6(0.1)/6(0.2)	6(0)	6(0.1)/6(0)/6(0)	6(0.1)	6(3.9)/6(0.1)/6(0.2)	6(0.6)
	p=60	6(0)	6(8.6)/6(0)/6(0)	6(0)	6(8.6)/6(0)/6(0)	6(0)	6(0.3)/6(0)/6(0)	6(0)	6(8.7)/6(0)/6(0)	-
	p=180	6(0)	6(27.5)/6(0)/6(0)	6(0)	6(28.2)/6(0)/6(0)	6(0)	6(1.1)/6(0)/6(0)	6(0)	6(28.9)/6(0)/6(0)	-
msnr	p=14	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0)	6(0)/6(0)/6(0)	6(0.3)	6(1.2)/6(0.3)/6(0.3)	6(0.6)
	p=30	6(0.1)	6(3.8)/6(0.1)/6(0.2)	6(0.1)	6(3.8)/6(0.1)/6(0.2)	6(0)	6(0.1)/6(0)/6(0)	6(0.1)	6(3.9)/6(0.1)/6(0.2)	6(0.6)
	p=60	6(0)	6(8.6)/6(0)/6(0)	6(0)	6(8.6)/6(0)/6(0)	6(0)	6(0.3)/6(0)/6(0)	6(0)	6(8.7)/6(0)/6(0)	-
	p=180	6(0)	6(27.5)/6(0)/6(0)	6(0)	6(28.2)/6(0)/6(0)	6(0)	6(1.1)/6(0)/6(0)	6(0)	6(28.9)/6(0)/6(0)	-
lsnr	p=14	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0)	6(0)/6(0)/6(0)	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0.6)
	p=30	6(0.1)	6(3.8)/6(0.2)/6(0.2)	6(0.1)	6(3.8)/6(0.2)/6(0.2)	6(0)	6(0.1)/6(0)/6(0)	6(0.1)	6(3.9)/6(0.2)/6(0.2)	6(0.6)
	p=60	6(0)	6(8.6)/6(0.1)/6(0.1)	6(0)	6(8.6)/6(0.1)/6(0.1)	6(0)	6(0.3)/6(0)/6(0)	6(0)	6(8.7)/6(0.1)/6(0.1)	-
	p=180	6(0)	6(27.5)/6(0.1)/6(0)	6(0)	6(28.2)/6(0.1)/6(0)	6(0)	6(1.1)/6(0)/6(0)	6(0)	6(28.9)/6(0.1)/6(0)	-

Table S3: The performance of BS using different selection rules, Orth-Sparse-Ex2, n=200

		C_p edf	C_p ndf/hdf/bdf	AICc edf	AICc ndf/hdf/bdf	BIC edf	BIC ndf/hdf/bdf	GCV edf	GCV ndf/hdf/bdf	CV
		% worse than the best possible BS								
hsnr	p=14	23	21/32/29	23	21/32/28	39	20/40/38	24	21/32/29	23
	p=30	21	48/27/26	21	47/25/23	27	20/27/26	21	49/27/25	24
	p=60	17	89/20/19	17	91/18/18	19	33/19/18	17	96/18/18	-
	p=180	12	200/32/33	11	236/11/11	11	112/11/11	11	262/11/12	-
msnr	p=14	13	33/23/20	11	33/21/17	3	14/12/11	14	34/23/19	21
	p=30	6	78/21/19	4	78/15/14	1	28/19/18	6	80/18/17	24
	p=60	3	146/20/17	3	148/14/13	1	59/34/28	3	155/17/15	-
	p=180	3	314/50/50	2	365/15/15	2	184/65/57	3	400/16/16	-
lsnr	p=14	25	26/34/32	25	26/34/33	52	24/91/77	26	27/34/33	27
	p=30	21	54/37/34	20	54/34/31	48	23/90/79	22	56/35/33	29
	p=60	19	95/34/32	17	97/33/31	49	35/84/76	18	102/33/32	-
	p=180	15	198/57/54	15	235/39/35	56	105/72/68	15	260/37/34	-
		Relative efficiency								
hsnr	p=14	0.97	0.99/0.91/0.93	0.97	0.99/0.91/0.93	0.86	1/0.86/0.87	0.97	0.99/0.91/0.93	0.97
	p=30	0.99	0.81/0.94/0.95	0.99	0.81/0.96/0.97	0.94	1/0.94/0.95	0.99	0.8/0.95/0.96	0.97
	p=60	0.99	0.62/0.97/0.98	1	0.61/0.99/0.99	0.98	0.87/0.98/0.98	1	0.59/0.98/0.99	-
	p=180	0.99	0.37/0.84/0.84	1	0.33/1/1	1	0.52/1/1	1	0.31/1/0.99	-
msnr	p=14	0.91	0.77/0.83/0.85	0.92	0.77/0.85/0.87	1	0.9/0.92/0.93	0.9	0.77/0.83/0.86	0.85
	p=30	0.95	0.57/0.84/0.85	0.97	0.57/0.88/0.89	1	0.79/0.85/0.86	0.95	0.56/0.86/0.86	0.81
	p=60	0.98	0.41/0.84/0.86	0.99	0.41/0.88/0.89	1	0.64/0.76/0.79	0.98	0.4/0.86/0.88	-
	p=180	0.99	0.25/0.68/0.68	1	0.22/0.89/0.89	1	0.36/0.62/0.65	1	0.2/0.88/0.88	-
lsnr	p=14	0.99	0.98/0.92/0.93	0.99	0.98/0.92/0.93	0.81	1/0.65/0.7	0.98	0.97/0.92/0.93	0.97
	p=30	0.99	0.78/0.88/0.9	1	0.78/0.9/0.92	0.82	0.98/0.63/0.67	0.98	0.77/0.89/0.9	0.93
	p=60	0.99	0.6/0.88/0.89	1	0.6/0.88/0.9	0.79	0.87/0.64/0.67	0.99	0.58/0.88/0.89	-
	p=180	1	0.39/0.73/0.75	1	0.34/0.83/0.85	0.74	0.56/0.67/0.69	1	0.32/0.84/0.86	-
		Sparsistency (number of extra variables)								
hsnr	p=14	5.3(0.9)	5.6(1.3)/5.1(1.7)/5.2(1.2)	5.2(0.7)	5.6(1.2)/5(1.3)/5.1(1)	4.1(0)	5.1(0.2)/4.1(0)/4.2(0)	5.3(0.9)	5.6(1.3)/5.1(1.6)/5.2(1.2)	5.3(1)
	p=30	4.8(0.4)	5.6(3.9)/4.6(0.8)/4.7(0.7)	4.7(0.2)	5.6(3.8)/4.5(0.2)/4.6(0.2)	4(0)	5.1(0.6)/4(0)/4.1(0)	4.8(0.4)	5.7(4.1)/4.5(0.6)/4.6(0.6)	5(0.9)
	p=60	4.5(0.2)	5.6(8.9)/4.3(0.3)/4.4(0.3)	4.4(0.1)	5.7(9.2)/4.2(0.1)/4.3(0.1)	4(0)	5.1(1.5)/4(0)/4(0)	4.4(0.1)	5.7(10.5)/4.3(0.1)/4.4(0.1)	-
	p=180	4.2(0)	5.7(32.2)/4.3(7.8)/4.4(7.4)	4.1(0)	5.8(48.9)/4.1(0)/4.1(0)	4(0)	5.3(9.1)/4(0)/4(0)	4.2(0)	5.9(74.6)/4.1(0)/4.2(0.1)	-
msnr	p=14	4.2(0.4)	4.8(1.3)/4.5(1.2)/4.4(0.9)	4.2(0.3)	4.8(1.2)/4.4(1)/4.3(0.6)	4(0)	4.3(0.2)/4(0)/4(0)	4.2(0.5)	4.8(1.3)/4.5(1.2)/4.4(0.8)	4.4(0.7)
	p=30	4.1(0.1)	4.8(3.9)/4.2(0.7)/4.2(0.6)	4(0.1)	4.8(3.8)/4.1(0.3)/4.1(0.2)	4(0)	4.3(0.6)/3.8(0)/3.9(0)	4.1(0.2)	4.8(4.1)/4.2(0.5)/4.1(0.5)	4.2(0.7)
	p=60	4(0)	4.8(8.9)/4.1(0.3)/4.1(0.3)	4(0)	4.8(9.3)/4(0.2)/4(0.1)	4(0)	4.3(1.5)/3.7(0)/3.8(0)	4(0)	4.9(10.5)/4.1(0.2)/4(0.2)	-
	p=180	4(0)	4.8(32.2)/4.1(7.3)/4.1(7)	4(0)	5.1(49.2)/3.9(0.1)/3.9(0.1)	4(0)	4.4(8.9)/3.4(0)/3.5(0)	4(0)	5.3(74.7)/3.9(0.1)/3.9(0.1)	-
lsnr	p=14	3.4(1)	3.8(1.3)/3.9(2.4)/3.7(1.8)	3.3(0.8)	3.8(1.2)/3.7(2.1)/3.5(1.5)	1.8(0)	2.8(0.2)/1.1(0)/1.4(0)	3.4(1.1)	3.8(1.3)/3.9(2.4)/3.6(1.7)	3.2(0.8)
	p=30	2.7(0.6)	3.8(3.9)/2.8(2)/2.7(1.5)	2.6(0.4)	3.8(3.8)/2.5(0.9)/2.4(0.7)	1.5(0)	2.8(0.6)/0.6(0)/0.9(0)	2.7(0.7)	3.9(4.1)/2.7(1.4)/2.6(1.2)	2.8(1)
	p=60	2.3(0.3)	3.8(8.9)/2.2(0.9)/2.2(0.8)	2.2(0.2)	3.9(9.3)/1.8(0.2)/1.9(0.3)	1.2(0)	2.8(1.5)/0.4(0)/0.6(0)	2.2(0.3)	4(10.5)/2(0.4)/2(0.4)	-
	p=180	1.9(0.3)	3.9(32.2)/1.8(9)/1.9(8.2)	1.8(0.2)	4.3(49.6)/1.1(0.1)/1.2(0.1)	0.6(0)	3.1(8.4)/0.2(0)/0.3(0)	1.9(0.2)	4.7(74.7)/1.2(0.1)/1.3(0.1)	-

Table S4: The performance of BS using different selection rules, Orth-Sparse-Ex2, n=2000

		C_p edf	C_p ndf/hdf/bdf	AICc edf	AICc ndf/hdf/bdf	BIC edf	BIC ndf/hdf/bdf	GCV edf	GCV ndf/hdf/bdf	CV
		% worse than the best possible BS								
hsnr	p=14	8	33/10/9	8	33/9/9	0	3/0/1	8	33/9/9	18
	p=30	3	85/6/7	3	85/5/6	0	9/0/0	3	86/6/7	23
	p=60	2	155/4/5	2	156/4/5	0	21/1/1	2	156/4/5	-
	p=180	0	334/4/4	1	337/4/3	0	60/4/3	1	340/4/4	-
msnr	p=14	13	27/31/28	13	27/32/27	47	26/95/80	13	27/32/28	22
	p=30	14	66/39/35	13	66/39/35	74	23/93/86	13	66/39/35	31
	p=60	15	111/39/35	14	111/38/35	80	22/82/78	15	112/39/35	-
	p=180	15	217/37/35	16	219/37/35	63	34/63/62	16	221/37/35	-
lsnr	p=14	15	29/19/17	15	29/19/17	4	8/8/7	15	29/19/17	20
	p=30	7	71/13/12	7	71/13/12	1	11/10/7	7	71/13/12	24
	p=60	3	131/10/9	3	131/10/9	1	19/16/11	3	132/10/9	-
	p=180	1	288/9/8	1	291/8/7	0	51/35/30	1	293/9/8	-
		Relative efficiency								
hsnr	p=14	0.93	0.75/0.91/0.91	0.93	0.75/0.92/0.92	1	0.97/1/0.99	0.92	0.75/0.91/0.91	0.84
	p=30	0.97	0.54/0.95/0.94	0.97	0.54/0.95/0.94	1	0.92/1/1	0.97	0.54/0.95/0.94	0.81
	p=60	0.98	0.39/0.96/0.95	0.98	0.39/0.96/0.96	1	0.83/0.99/0.99	0.98	0.39/0.96/0.95	-
	p=180	1	0.23/0.96/0.97	0.99	0.23/0.97/0.97	1	0.62/0.96/0.97	0.99	0.23/0.96/0.97	-
msnr	p=14	1	0.89/0.86/0.88	1	0.89/0.86/0.89	0.77	0.9/0.58/0.63	1	0.89/0.86/0.88	0.93
	p=30	1	0.68/0.82/0.84	1	0.68/0.82/0.84	0.65	0.92/0.59/0.61	1	0.68/0.82/0.84	0.87
	p=60	1	0.54/0.83/0.85	1	0.54/0.83/0.85	0.64	0.94/0.63/0.64	1	0.54/0.83/0.85	-
	p=180	1	0.36/0.84/0.85	1	0.36/0.84/0.85	0.71	0.86/0.71/0.71	1	0.36/0.84/0.85	-
lsnr	p=14	0.9	0.81/0.88/0.89	0.91	0.81/0.88/0.89	1	0.96/0.97/0.97	0.91	0.81/0.88/0.89	0.87
	p=30	0.95	0.59/0.9/0.9	0.95	0.59/0.9/0.91	1	0.92/0.92/0.95	0.95	0.59/0.9/0.91	0.82
	p=60	0.98	0.44/0.91/0.92	0.98	0.43/0.92/0.93	1	0.84/0.87/0.9	0.98	0.43/0.91/0.93	-
	p=180	0.99	0.26/0.92/0.93	0.99	0.26/0.93/0.93	1	0.66/0.74/0.77	0.99	0.26/0.92/0.93	-
		Sparsistency (number of extra variables)								
hsnr	p=14	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0)	6(0)/6(0)/6(0)	6(0.3)	6(1.2)/6(0.4)/6(0.3)	6(0.6)
	p=30	6(0.1)	6(3.8)/6(0.2)/6(0.2)	6(0.1)	6(3.8)/6(0.2)/6(0.2)	6(0)	6(0.1)/6(0)/6(0)	6(0.1)	6(3.9)/6(0.2)/6(0.2)	6(0.6)
	p=60	6(0)	6(8.6)/6(0.1)/6(0.1)	6(0)	6(8.6)/6(0.1)/6(0.1)	6(0)	6(0.3)/6(0)/6(0)	6(0)	6(8.7)/6(0.1)/6(0.1)	-
	p=180	6(0)	6(27.5)/6(0)/6(0)	6(0)	6(28.2)/6(0)/6(0)	6(0)	6(1.1)/6(0)/6(0)	6(0)	6(28.9)/6(0)/6(0)	-
msnr	p=14	5.9(0.6)	6(1.2)/5.8(1.7)/5.8(1.2)	5.9(0.6)	6(1.2)/5.8(1.7)/5.8(1.1)	5.1(0)	5.5(0)/4.4(0)/4.6(0)	5.9(0.6)	6(1.2)/5.8(1.7)/5.8(1.2)	5.9(0.8)
	p=30	5.8(0.4)	6(3.8)/5.5(1)/5.6(0.8)	5.8(0.4)	6(3.8)/5.5(0.9)/5.5(0.8)	4.5(0)	5.5(0.1)/4.2(0)/4.3(0)	5.8(0.4)	6(3.9)/5.5(1)/5.6(0.8)	5.7(1)
	p=60	5.6(0.4)	6(8.6)/5.2(0.3)/5.3(0.3)	5.6(0.4)	6(8.6)/5.2(0.3)/5.3(0.3)	4.1(0)	5.5(0.3)/4.1(0)/4.2(0)	5.6(0.4)	6(8.7)/5.2(0.3)/5.3(0.3)	-
	p=180	5.4(0.3)	6(27.5)/4.8(0.1)/4.8(0.1)	5.4(0.3)	6(28.2)/4.8(0.1)/4.8(0.1)	4(0)	5.5(1.1)/4(0)/4.1(0)	5.4(0.3)	6(28.9)/4.8(0.1)/4.8(0.1)	-
lsnr	p=14	4.3(0.5)	4.9(1.2)/4.5(1)/4.4(0.6)	4.3(0.5)	4.9(1.2)/4.5(0.9)/4.4(0.6)	4(0)	4.1(0)/4(0)/4(0)	4.3(0.5)	4.9(1.2)/4.5(1)/4.4(0.6)	4.5(0.7)
	p=30	4.1(0.2)	4.9(3.8)/4.2(0.4)/4.1(0.4)	4.1(0.2)	4.9(3.8)/4.2(0.4)/4.1(0.3)	4(0)	4.1(0.1)/3.9(0)/4(0)	4.1(0.2)	4.9(3.9)/4.2(0.4)/4.1(0.3)	4.3(0.8)
	p=60	4(0)	4.9(8.6)/4.1(0.2)/4.1(0.1)	4(0)	4.9(8.6)/4.1(0.1)/4.1(0.1)	4(0)	4.1(0.3)/3.9(0)/3.9(0)	4(0)	4.9(8.7)/4.1(0.2)/4.1(0.1)	-
	p=180	4(0)	4.9(27.5)/4(0.1)/4(0.1)	4(0)	4.9(28.2)/4(0.1)/4(0.1)	4(0)	4.1(1.1)/3.7(0)/3.8(0)	4(0)	4.9(28.9)/4(0.1)/4(0.1)	-

Table S5: The performance of BS using different selection rules, Orth-Dense, n=200

		C_p edf	C_p ndf/hdf/bdf	AICc edf	AICc ndf/hdf/bdf	BIC edf	BIC ndf/hdf/bdf	GCV edf	GCV ndf/hdf/bdf	CV
		% worse than the best possible BS								
hsnr	p=14	0	0/0/0	0	0/0/0	0	1/0/0	0	0/0/0	0
	p=30	1	11/1/2	1	13/1/2	1	28/3/5	1	11/1/2	7
	p=60	8	7/9/9	9	7/11/11	20	8/32/33	8	8/10/10	-
	p=180	7	45/21/20	9	52/18/19	18	26/39/42	7	64/13/13	-
msnr	p=14	0	9/0/1	0	10/0/1	0	36/1/2	0	9/0/1	6
	p=30	3	10/3/4	3	11/4/5	21	27/19/25	3	10/4/4	11
	p=60	10	11/14/13	10	11/13/13	26	10/48/48	10	12/14/13	-
	p=180	8	52/23/23	10	62/18/19	21	25/61/56	8	74/14/14	-
lsnr	p=14	5	22/6/8	7	23/8/10	73	50/73/72	6	22/7/8	19
	p=30	15	10/16/16	20	10/21/20	27	16/27/27	17	10/18/18	16
	p=60	13	25/17/16	13	25/13/13	13	11/13/13	13	26/14/14	-
	p=180	8	86/22/22	7	102/7/7	7	39/7/7	7	116/7/7	-
		Relative efficiency								
hsnr	p=14	1	1/1/1	1	1/1/1	1	0.99/1/1	1	1/1/1	1
	p=30	1	0.91/1/1	1	0.9/1/0.99	1	0.79/0.98/0.96	1	0.91/1/1	0.95
	p=60	0.99	1/0.98/0.98	0.98	1/0.97/0.96	0.89	0.99/0.81/0.8	0.99	0.99/0.98/0.98	-
	p=180	1	0.74/0.89/0.89	0.99	0.71/0.91/0.9	0.91	0.85/0.77/0.76	1	0.65/0.95/0.95	-
msnr	p=14	1	0.92/1/0.99	1	0.91/1/0.99	1	0.74/1/0.99	1	0.92/1/0.99	0.95
	p=30	1	0.93/0.99/0.99	0.99	0.92/0.98/0.98	0.85	0.81/0.87/0.82	1	0.93/0.99/0.99	0.93
	p=60	1	0.99/0.96/0.97	1	0.99/0.97/0.97	0.87	1/0.74/0.74	1	0.98/0.97/0.97	-
	p=180	1	0.71/0.88/0.88	0.98	0.67/0.91/0.91	0.89	0.87/0.67/0.69	1	0.62/0.95/0.95	-
lsnr	p=14	0.98	0.85/0.97/0.96	0.97	0.84/0.96/0.94	0.6	0.69/0.6/0.6	0.98	0.85/0.97/0.95	0.86
	p=30	0.95	1/0.95/0.95	0.91	1/0.91/0.91	0.86	0.94/0.86/0.86	0.93	1/0.93/0.93	0.94
	p=60	0.98	0.89/0.95/0.96	0.99	0.89/0.99/0.99	0.98	1/0.98/0.98	0.98	0.88/0.97/0.98	-
	p=180	1	0.58/0.88/0.88	1	0.53/1/1	1	0.77/1/1	1	0.5/1/1	-
		Sparsistency (number of extra variables)								
hsnr	p=14	14	14/14/14	14	14/14/14	14	14/14/14	14	14/14/14	14
	p=30	30	24.7/29.5/29	30	24.2/29.4/28.8	30	20.9/28.8/27.5	30	24.7/29.5/29	26.6
	p=60	29.8	30.5/38.4/35.8	22.2	29.4/25.6/24.5	17.8	22.5/16.8/16.5	28.6	31.3/36.8/34	-
	p=180	20.5	53.3/37.4/35.5	18.3	62.3/16.3/16.3	16.1	35/13.7/13.5	19.4	89.8/17.8/17.8	-
msnr	p=14	14	13.2/14/13.9	14	13.2/14/13.9	14	11.8/13.9/13.8	14	13.2/14/13.9	13.4
	p=30	27.3	18.8/27.4/26.1	26.5	18.3/26.8/25.3	18	13.4/20.4/17.6	27.3	18.8/27.4/26.1	20.8
	p=60	19.4	24.1/29.6/27	13.9	23.4/15.6/15.2	9.3	14.5/7.5/7.4	18.3	25.2/26/24.1	-
	p=180	12.6	47.1/29.1/28.1	10.4	59/8.8/8.8	8.1	24.4/4.8/5	11.3	86.4/10/10	-
lsnr	p=14	13.6	7.7/12.7/11.7	13.4	7.6/12.3/11.3	0.7	3.6/0.7/0.8	13.5	7.8/12.6/11.6	8.8
	p=30	12.8	10.5/14.6/13	7.6	10.3/8.5/7.6	0	4/0/0	11.3	10.8/12.3/11.2	7.5
	p=60	3.4	15.7/6.5/6	1	15.8/0.8/1	0	4.9/0/0	2	17.3/2.4/2.4	-
	p=180	0.8	39/14.5/13.7	0.3	55.2/0.2/0.3	0	11.8/0/0	0.4	81.7/0.3/0.4	-

Table S6: The performance of BS using different selection rules, Orth-Dense, n=2000

		C_p edf	C_p ndf/hdf/bdf	AICc edf	AICc ndf/hdf/bdf	BIC edf	BIC ndf/hdf/bdf	GCV edf	GCV ndf/hdf/bdf	CV
		% worse than the best possible BS								
hsnr	p=14	0	0/0/0	0	0/0/0	0	0/0/0	0	0/0/0	0
	p=30	0	1/0/0	0	1/0/0	0	18/0/1	0	1/0/0	1
	p=60	5	5/5/5	5	5/5/5	25	17/34/37	5	5/5/5	-
	p=180	6	34/8/8	6	34/8/8	19	7/36/37	6	35/8/8	-
msnr	p=14	0	0/0/0	0	0/0/0	0	0/0/0	0	0/0/0	0
	p=30	1	9/1/1	1	9/1/1	1	40/2/5	1	9/1/1	5
	p=60	7	6/8/8	7	6/8/8	28	15/37/40	7	6/8/8	-
	p=180	6	39/9/9	6	39/8/9	21	7/38/40	6	40/8/9	-
lsnr	p=14	0	5/0/0	0	5/0/0	0	49/0/1	0	5/0/0	4
	p=30	2	11/3/3	2	11/3/3	44	41/36/45	2	11/3/3	10
	p=60	10	10/13/12	10	10/13/12	32	16/45/48	10	10/13/12	-
	p=180	8	48/10/10	8	48/10/10	24	8/45/47	8	49/10/10	-
		Relative efficiency								
hsnr	p=14	1	1/1/1	1	1/1/1	1	1/1/1	1	1/1/1	1
	p=30	1	0.99/1/1	1	0.99/1/1	1	0.85/1/0.99	1	0.99/1/1	0.99
	p=60	0.99	1/0.99/0.99	0.99	1/0.99/0.99	0.83	0.89/0.78/0.76	0.99	1/0.99/0.99	-
	p=180	1	0.79/0.98/0.98	1	0.79/0.98/0.98	0.89	1/0.78/0.78	1	0.79/0.98/0.98	-
msnr	p=14	1	1/1/1	1	1/1/1	1	1/1/1	1	1/1/1	1
	p=30	1	0.92/1/1	1	0.92/1/1	1	0.72/0.99/0.96	1	0.92/1/1	0.96
	p=60	0.99	1/0.98/0.98	0.99	1/0.98/0.98	0.83	0.92/0.77/0.76	0.99	1/0.98/0.98	-
	p=180	1	0.76/0.98/0.98	1	0.76/0.98/0.98	0.88	1/0.77/0.76	1	0.76/0.98/0.98	-
lsnr	p=14	1	0.95/1/1	1	0.95/1/1	1	0.67/1/0.99	1	0.95/1/1	0.96
	p=30	1	0.92/0.99/0.99	1	0.92/0.99/0.99	0.71	0.73/0.75/0.7	1	0.92/0.99/0.99	0.93
	p=60	1	1/0.97/0.98	1	1/0.97/0.98	0.83	0.94/0.75/0.74	1	1/0.97/0.98	-
	p=180	1	0.73/0.98/0.98	1	0.73/0.98/0.98	0.87	1/0.74/0.73	1	0.72/0.98/0.98	-
		Sparsistency (number of extra variables)								
hsnr	p=14	14	14/14/14	14	14/14/14	14	14/14/14	14	14/14/14	14
	p=30	30	29.8/30/29.9	30	29.8/30/29.9	30	28.6/30/29.9	30	29.8/30/29.9	29.8
	p=60	44.9	39.8/50.9/48.7	44.2	39.7/50.4/48.2	28.5	30.5/27.6/27.4	45.1	39.8/50.8/48.6	-
	p=180	32.1	58.9/32.4/32.3	31.8	58.9/31.6/31.6	27	31.3/25/24.9	32	59.9/32.1/32	-
msnr	p=14	14	14/14/14	14	14/14/14	14	14/14/14	14	14/14/14	14
	p=30	30	27.1/29.9/29.6	30	27.1/29.9/29.6	30	22.5/29.5/28.6	30	27.1/29.9/29.6	28.2
	p=60	34.8	33.3/42.8/40.1	33.9	33.2/41.9/39.2	20.4	22.9/19.4/19.2	34.6	33.3/42.8/40	-
	p=180	24.2	52.4/24.4/24.3	24	52.6/23.6/23.6	19.2	23.6/17.3/17.2	24.1	53.5/24.1/24.1	-
lsnr	p=14	14	13.6/14/13.9	14	13.6/14/13.9	14	11.7/14/13.9	14	13.6/14/13.9	13.7
	p=30	28.8	19.9/28.2/26.9	28.8	19.9/28.1/26.8	13.5	12.5/16.7/14.1	28.8	19.9/28.2/26.9	22.3
	p=60	21.6	24.8/30.6/27.9	20.9	24.7/29.2/26.9	10	12.7/8.7/8.5	21.8	24.9/30.4/27.7	-
	p=180	13.9	43.8/14/14	13.6	44.1/13.3/13.3	9.1	13.4/7/6.8	13.8	45/13.7/13.6	-

Table S7: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	8/6/18	19	19	42/41	16/20	13	15
	p=30	5/3/23	25	23	71/69	32/23	14	19
	p=60	4/2/21	-	23	87/85	51/24	16	19
	p=180	34/1/19	-	22	119/121	134/25	17	19
msnr	p=14	17/14/18	19	19	43/42	23/23	14	16
	p=30	15/11/23	25	23	71/69	49/28	16	20
	p=60	13/9/22	-	24	87/85	82/28	17	20
	p=180	44/7/20	-	22	119/121	222/30	17	20
lsnr	p=14	22/24/25	26	25	8/9	13/15	18	15
	p=30	32/34/26	26	26	2/2	14/9	10	7
	p=60	27/29/24	-	24	-1/-2	27/5	6	3
	p=180	32/22/18	-	19	-4/-2	83/2	1	1
		Relative efficiency						
hsnr	p=14	0.98/1/0.89	0.89	0.89	0.74/0.75	0.91/0.88	0.93	0.92
	p=30	0.98/1/0.84	0.82	0.83	0.6/0.61	0.78/0.83	0.9	0.87
	p=60	0.99/1/0.84	-	0.83	0.55/0.55	0.68/0.83	0.88	0.86
	p=180	0.75/1/0.85	-	0.83	0.46/0.46	0.43/0.81	0.87	0.85
msnr	p=14	0.98/1/0.96	0.96	0.96	0.8/0.81	0.93/0.93	1	0.98
	p=30	0.96/1/0.9	0.89	0.9	0.65/0.66	0.75/0.87	0.96	0.93
	p=60	0.97/1/0.9	-	0.88	0.58/0.59	0.6/0.85	0.94	0.91
	p=180	0.75/1/0.9	-	0.88	0.49/0.49	0.33/0.83	0.91	0.89
lsnr	p=14	0.89/0.88/0.87	0.86	0.86	1/1	0.95/0.94	0.92	0.95
	p=30	0.78/0.76/0.81	0.81	0.81	1/1	0.9/0.94	0.93	0.96
	p=60	0.77/0.76/0.79	-	0.79	1/1	0.78/0.94	0.93	0.96
	p=180	0.73/0.79/0.81	-	0.81	1/0.98	0.52/0.94	0.95	0.95
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.4)/6(0.2)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.6)	6(0.9)/6(1.4)	6(0.7)	6(0.6)
	p=30	6(0.1)/6(0)/6(0.6)	6(0.7)	6(0.6)	6(7.3)/6(8.4)	6(2.3)/6(1.7)	6(1)	6(0.7)
	p=60	6(0.1)/6(0)/6(0.5)	-	6(0.5)	6(10)/6(11.3)	6(4.6)/6(1.8)	6(1.4)	6(0.7)
	p=180	6(8.9)/6(0)/6(0.4)	-	6(0.4)	6(15.3)/6(20.3)	6(18.1)/6(2.1)	6(2.3)	6(0.7)
msnr	p=14	6(0.8)/6(0.6)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.6)	6(1.1)/6(1.4)	6(0.6)	6(0.6)
	p=30	6(0.4)/6(0.2)/6(0.6)	6(0.7)	6(0.6)	6(7.4)/6(8.4)	6(2.9)/6(1.6)	6(0.7)	6(0.7)
	p=60	6(0.2)/6(0.1)/6(0.5)	-	6(0.5)	6(10)/6(11.3)	6(6.2)/6(1.6)	6(1)	6(0.8)
	p=180	6(9.1)/6(0.1)/6(0.4)	-	6(0.4)	6(15.3)/6(20.3)	6(27.7)/6(1.7)	6(1.4)	6(0.8)
lsnr	p=14	5.4(4.5)/5.2(4.1)/4.6(1.7)	4.5(1.6)	4.6(1.7)	5.5(3.3)/5.5(3.9)	5(1.5)/5.1(2.9)	5(2.7)	5(2.1)
	p=30	4(4.7)/3.1(2.1)/3.7(2)	3.7(2)	3.7(2)	5.3(6.4)/5.3(7.1)	4.9(3.8)/4.8(4.9)	4.8(5.3)	4.6(3.6)
	p=60	2.8(1.9)/2(0.3)/2.9(1.4)	-	3(1.4)	5.1(8.5)/5.2(9.2)	4.8(8.1)/4.5(6.3)	4.5(7)	4.3(4.4)
	p=180	2.1(13.8)/1(0.1)/1.8(0.8)	-	1.8(0.8)	4.4(11.8)/4.5(15.4)	4.7(36.6)/3.8(10.1)	4.1(11.9)	3.7(7.5)

Table S8: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	6/6/17	18	17	40/40	12/14	11	13
	p=30	3/3/20	21	21	72/69	19/16	12	15
	p=60	2/2/23	-	22	100/96	30/19	15	17
	p=180	1/1/21	-	21	136/132	53/20	14	16
msnr	p=14	6/6/17	18	17	41/40	14/17	12	13
	p=30	3/3/20	21	21	72/69	26/18	12	15
	p=60	2/2/23	-	22	100/96	46/22	15	18
	p=180	1/1/21	-	21	136/132	97/22	13	17
lsnr	p=14	8/8/17	18	17	41/40	21/20	12	13
	p=30	5/4/20	21	21	72/69	44/23	12	15
	p=60	5/4/23	-	22	100/96	84/27	16	17
	p=180	5/5/21	-	21	136/132	192/27	13	17
		Relative efficiency						
hsnr	p=14	1/1/0.9	0.9	0.91	0.76/0.76	0.95/0.92	0.95	0.94
	p=30	1/1/0.86	0.85	0.85	0.6/0.61	0.87/0.88	0.92	0.89
	p=60	1/1/0.83	-	0.83	0.51/0.52	0.78/0.85	0.89	0.87
	p=180	1/1/0.84	-	0.84	0.43/0.44	0.66/0.85	0.89	0.87
msnr	p=14	1/1/0.9	0.9	0.91	0.75/0.76	0.93/0.91	0.95	0.94
	p=30	1/1/0.86	0.85	0.85	0.6/0.61	0.82/0.87	0.92	0.89
	p=60	1/1/0.83	-	0.83	0.51/0.52	0.7/0.84	0.89	0.86
	p=180	1/1/0.83	-	0.83	0.43/0.44	0.51/0.83	0.9	0.87
lsnr	p=14	1/1/0.93	0.92	0.93	0.77/0.77	0.9/0.9	0.97	0.96
	p=30	1/1/0.87	0.86	0.87	0.61/0.62	0.73/0.85	0.93	0.91
	p=60	1/1/0.85	-	0.85	0.52/0.53	0.57/0.82	0.9	0.89
	p=180	1/1/0.86	-	0.86	0.44/0.45	0.36/0.83	0.93	0.9
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.2)/6(0.2)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.4)	6(0.9)/6(1.2)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.5)	6(0.6)	6(0.6)	6(8.5)/6(8.8)	6(1.9)/6(1.7)	6(1)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13.4)/6(12.6)	6(4)/6(2.2)	6(1.6)	6(0.6)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.3)	6(23)/6(20.1)	6(10.4)/6(2.5)	6(2.1)	6(0.4)
msnr	p=14	6(0.2)/6(0.2)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.4)	6(1)/6(1.3)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.5)	6(0.6)	6(0.6)	6(8.5)/6(8.7)	6(2.2)/6(1.6)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13.3)/6(12.6)	6(5)/6(1.9)	6(1.5)	6(0.6)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.3)	6(22.8)/6(20)	6(15.7)/6(1.9)	6(1.8)	6(0.4)
lsnr	p=14	6(0.4)/6(0.4)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.4)	6(1.1)/6(1.3)	6(0.5)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.5)	6(0.6)	6(0.6)	6(8.5)/6(8.7)	6(2.9)/6(1.4)	6(0.7)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(13.4)/6(12.5)	6(6.7)/6(1.6)	6(1)	6(0.6)
	p=180	6(0.1)/6(0.1)/6(0.4)	-	6(0.3)	6(23)/6(20)	6(23.6)/6(1.1)	6(1)	6(0.4)

Table S9: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0.5, n=200

		BOSS	BS	FS	LASSO	Gamma LASSO	SparseNet	rLASSO
		C_p-hdf/AICc-hdf/CV	CV	CV	AICc/CV	AICc/CV	CV	CV
		% worse than the best possible BOSS						
hsnr	p=14	7/5/20	18	21	39/39	16/19	13	18
	p=30	4/2/20	22	21	66/65	34/21	15	18
	p=60	3/2/21	-	23	92/89	57/25	16	18
	p=180	65/1/19	-	22	139/134	136/25	16	16
msnr	p=14	19/17/20	15	20	34/34	21/20	12	21
	p=30	13/9/22	23	23	66/65	50/26	17	26
	p=60	12/8/22	-	25	90/88	85/30	17	29
	p=180	46/9/21	-	25	126/125	211/31	18	39
lsnr	p=14	19/22/23	23	21	-2/-2	12/6	7	4
	p=30	26/27/25	24	23	-4/-5	9/2	2	-1
	p=60	24/26/22	-	20	-4/-6	22/1	0	-3
	p=180	48/12/14	-	14	-2/-2	91/2	1	0
		Relative efficiency						
hsnr	p=14	0.98/1/0.87	0.89	0.87	0.75/0.75	0.9/0.88	0.93	0.89
	p=30	0.99/1/0.85	0.84	0.84	0.61/0.62	0.77/0.85	0.89	0.87
	p=60	0.99/1/0.85	-	0.83	0.53/0.54	0.65/0.82	0.88	0.87
	p=180	0.61/1/0.85	-	0.83	0.42/0.43	0.43/0.81	0.87	0.87
msnr	p=14	0.95/0.96/0.94	0.98	0.93	0.84/0.84	0.93/0.94	1	0.93
	p=30	0.97/1/0.9	0.89	0.89	0.66/0.66	0.73/0.87	0.93	0.87
	p=60	0.97/1/0.88	-	0.87	0.57/0.58	0.59/0.83	0.93	0.84
	p=180	0.75/1/0.9	-	0.88	0.48/0.49	0.35/0.83	0.93	0.79
lsnr	p=14	0.82/0.8/0.8	0.8	0.81	1/1	0.88/0.92	0.91	0.94
	p=30	0.75/0.75/0.76	0.77	0.77	0.99/1	0.87/0.93	0.93	0.96
	p=60	0.76/0.75/0.77	-	0.78	0.99/1	0.77/0.94	0.95	0.97
	p=180	0.66/0.87/0.86	-	0.86	1/1	0.51/0.96	0.97	0.98
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.2)/6(0.8)	6(0.6)	6(0.8)	6(3.8)/6(4.2)	6(0.9)/6(1.3)	6(0.6)	6(1)
	p=30	6(0.1)/6(0)/6(0.6)	6(0.7)	6(0.6)	6(6)/6(8.6)	6(2.5)/6(1.5)	6(1.1)	6(0.8)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.6)	6(8.8)/6(12.5)	6(4.9)/6(1.7)	6(1.4)	6(0.9)
	p=180	6(9.4)/6(0)/6(0.3)	-	6(0.4)	6(11.7)/6(21.3)	6(16.6)/6(1.7)	6(1.9)	6(0.6)
msnr	p=14	6(1.2)/6(1)/6(1.1)	6(0.7)	6(1.1)	6(3.8)/6(4.2)	6(1.1)/6(1.5)	6(0.7)	6(1.7)
	p=30	6(0.4)/6(0.2)/6(0.7)	6(0.7)	6(0.7)	6(6.1)/6(8.6)	6(3.1)/6(1.6)	6(1)	6(1.3)
	p=60	6(0.2)/6(0.2)/6(0.6)	-	6(0.6)	6(8.8)/6(12.4)	6(6.3)/6(1.8)	6(1.1)	6(1.6)
	p=180	6(10.2)/6(0.1)/6(0.4)	-	6(0.4)	6(16)/6(21.1)	6(26.5)/6(1.6)	6(1.3)	6(1.6)
lsnr	p=14	4.4(3.5)/4.1(3.2)/3.8(2.4)	3.7(2)	3.8(2.2)	5(3.3)/5.1(3.5)	4(1.6)/4.6(2.8)	4.6(2.7)	4.6(2.6)
	p=30	3.7(4.4)/2.9(2.1)/3.4(2.7)	3.3(2.3)	3.4(2.3)	4.8(4.8)/5.2(7.4)	4.5(3.9)/4.7(5.2)	4.8(5.7)	4.6(4.5)
	p=60	2.4(2)/1.6(0.5)/2.5(1.8)	-	2.5(1.6)	4.4(6.4)/4.8(10)	4.4(8.2)/4.2(7)	4.4(7.9)	4.1(5.7)
	p=180	1.3(14.1)/0.3(0.1)/1(0.8)	-	1.1(0.7)	2.5(4.9)/3.2(12)	4.2(35.8)/2.9(9.2)	3(9.7)	2.7(7.5)

Table S10: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0.5, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/6/17	19	17	38/39	12/14	12	14
	p=30	3/3/20	23	22	67/65	21/18	14	19
	p=60	2/2/21	-	21	91/89	29/19	15	20
	p=180	2/1/22	-	23	126/123	52/20	15	17
msnr	p=14	7/6/17	19	17	39/39	16/17	12	14
	p=30	3/3/20	23	22	67/66	29/20	14	19
	p=60	2/2/21	-	21	91/89	45/21	14	20
	p=180	2/1/22	-	23	126/123	95/22	15	17
lsnr	p=14	13/13/17	18	17	39/39	23/21	13	19
	p=30	6/6/20	23	22	67/65	49/25	15	21
	p=60	5/5/21	-	21	91/89	83/26	15	20
	p=180	4/4/22	-	23	126/123	192/28	15	17
		Relative efficiency						
hsnr	p=14	1/1/0.91	0.9	0.91	0.77/0.77	0.95/0.93	0.95	0.93
	p=30	1/1/0.86	0.84	0.84	0.62/0.62	0.85/0.87	0.9	0.87
	p=60	1/1/0.84	-	0.84	0.54/0.54	0.79/0.86	0.89	0.85
	p=180	1/1/0.83	-	0.83	0.45/0.45	0.67/0.85	0.88	0.87
msnr	p=14	1/1/0.91	0.9	0.91	0.77/0.76	0.92/0.91	0.95	0.93
	p=30	1/1/0.86	0.84	0.84	0.62/0.62	0.8/0.86	0.9	0.87
	p=60	1/1/0.84	-	0.84	0.54/0.54	0.71/0.84	0.89	0.85
	p=180	1/1/0.83	-	0.83	0.45/0.45	0.52/0.83	0.89	0.87
lsnr	p=14	1/1/0.96	0.95	0.96	0.81/0.81	0.92/0.93	1	0.95
	p=30	1/1/0.88	0.86	0.87	0.63/0.64	0.71/0.85	0.92	0.88
	p=60	1/1/0.87	-	0.87	0.55/0.56	0.57/0.84	0.91	0.87
	p=180	1/1/0.85	-	0.85	0.46/0.47	0.36/0.82	0.9	0.89
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.2)	6(0.9)/6(1)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.7)	6(0.6)	6(6.6)/6(8.3)	6(2.1)/6(1.7)	6(1)	6(0.8)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.4)	6(10.3)/6(12.1)	6(3.7)/6(2)	6(1.3)	6(0.7)
	p=180	6(0)/6(0)/6(0.5)	-	6(0.4)	6(13.8)/6(18.9)	6(9.3)/6(2.5)	6(2.3)	6(0.4)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.2)	6(1)/6(1.2)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.7)	6(0.6)	6(6.6)/6(8.3)	6(2.5)/6(1.6)	6(0.9)	6(0.8)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.4)	6(10.4)/6(12)	6(4.6)/6(1.7)	6(1.2)	6(0.7)
	p=180	6(0)/6(0)/6(0.5)	-	6(0.4)	6(14)/6(18.7)	6(14.2)/6(2)	6(2.1)	6(0.5)
lsnr	p=14	6(0.6)/6(0.6)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.3)	6(1.2)/6(1.3)	6(0.5)	6(1)
	p=30	6(0.2)/6(0.1)/6(0.6)	6(0.7)	6(0.6)	6(6.6)/6(8.4)	6(3.1)/6(1.5)	6(0.7)	6(0.8)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.4)	6(10.3)/6(12)	6(6.5)/6(1.4)	6(0.8)	6(0.8)
	p=180	6(0.1)/6(0)/6(0.5)	-	6(0.4)	6(13.8)/6(18.8)	6(23)/6(1.4)	6(1.4)	6(0.5)

Table S11: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0.9, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	20/21/19	16	18	6/5	10/6	12	7
	p=30	16/16/28	15	28	24/24	26/16	12	25
	p=60	15/15/34	-	34	58/59	66/38	28	23
	p=180	59/8/35	-	36	98/98	153/45	27	23
msnr	p=14	26/27/18	17	16	-8/-9	11/1	3	-3
	p=30	24/27/20	19	15	-11/-12	5/-5	-3	-7
	p=60	16/16/19	-	16	2/3	27/6	8	0
	p=180	26/7/15	-	16	31/31	111/17	15	12
lsnr	p=14	28/27/24	22	19	-9/-13	9/3	3	-1
	p=30	19/18/21	19	15	-18/-21	5/-7	-9	-10
	p=60	17/18/20	-	14	-20/-21	6/-12	-14	-18
	p=180	47/21/18	-	14	-13/-14	53/-9	-10	-12
		Relative efficiency						
hsnr	p=14	0.87/0.87/0.89	0.91	0.89	0.99/1	0.95/0.99	0.94	0.98
	p=30	0.97/0.97/0.87	0.98	0.88	0.91/0.91	0.89/0.96	1	0.89
	p=60	1/0.99/0.85	-	0.85	0.73/0.72	0.69/0.83	0.9	0.93
	p=180	0.68/1/0.8	-	0.79	0.55/0.54	0.43/0.74	0.85	0.88
msnr	p=14	0.72/0.71/0.77	0.77	0.78	0.98/1	0.82/0.9	0.88	0.93
	p=30	0.71/0.69/0.74	0.74	0.76	0.99/1	0.84/0.93	0.91	0.95
	p=60	0.86/0.85/0.83	-	0.86	0.98/0.97	0.78/0.94	0.92	1
	p=180	0.86/1/0.93	-	0.92	0.82/0.82	0.51/0.92	0.93	0.96
lsnr	p=14	0.68/0.68/0.7	0.71	0.73	0.95/1	0.8/0.85	0.85	0.88
	p=30	0.67/0.67/0.66	0.67	0.69	0.96/1	0.76/0.85	0.87	0.88
	p=60	0.68/0.67/0.66	-	0.69	0.98/1	0.74/0.89	0.92	0.96
	p=180	0.59/0.71/0.73	-	0.76	1/1	0.57/0.95	0.96	0.98
		Sparsistency (number of extra variables)						
hsnr	p=14	5.6(2.8)/5.5(2.6)/5.7(2.8)	5.6(1.9)	5.6(2.6)	5.9(4)/6(4)	5.6(1.4)/5.8(2)	5.6(2.2)	5.9(3.8)
	p=30	5.6(1.5)/5.6(1.2)/5.8(3.2)	5.8(1.5)	5.8(2.8)	6(7.5)/6(8.6)	5.8(3.4)/5.8(3.4)	5.8(2.3)	6(7.3)
	p=60	5.9(1)/5.8(0.9)/5.9(2.8)	-	5.9(1.9)	6(10.2)/6(12.1)	5.9(6.6)/5.9(4)	5.9(3)	6(3.1)
	p=180	6(9.9)/5.9(0.4)/6(2.4)	-	6(1.2)	6(13.6)/6(18)	6(21.6)/6(4.8)	6(4.3)	6(2.5)
msnr	p=14	2.9(2)/2.7(1.7)/3.7(2.9)	3.5(2.4)	3.6(2.7)	4.9(3.7)/5(3.7)	3.3(1.6)/4.2(2.6)	4(2.6)	4.6(3.4)
	p=30	3.3(2.9)/3(2.1)/3.9(4.9)	3.6(3.6)	3.8(4.1)	5.1(7.3)/5.3(8.2)	4.1(3.8)/4.7(5.7)	4.6(5.5)	4.9(7.4)
	p=60	4.3(2.8)/4.1(2.1)/4.4(4.3)	-	4.4(3.2)	5.6(10.1)/5.6(11.8)	4.7(7.4)/5(6.9)	5(6.8)	5.5(7.9)
	p=180	5.1(11.7)/5(1.2)/5(2.3)	-	5(1.7)	5.9(14.8)/5.9(17.5)	5.3(28.6)/5.4(6.8)	5.4(6.3)	5.6(5.5)
lsnr	p=14	1(1)/0.9(0.9)/1.5(1.7)	1.4(1.4)	1.5(1.4)	2.6(2.5)/2.9(2.7)	1.5(1.4)/2(1.8)	2(1.9)	2.1(2.1)
	p=30	0.9(1.8)/0.8(1.4)/1.2(2.9)	1.1(2.4)	1.2(2.2)	2.2(4.8)/2.6(6)	1.6(3.2)/2(4.2)	2.1(4.5)	1.9(4.5)
	p=60	0.9(2)/0.7(1.4)/1.1(3.2)	-	1.1(2.4)	2.6(7)/2.9(9)	2(6.5)/2.3(6.6)	2.5(7.2)	2.6(7.3)
	p=180	1.3(14.7)/0.5(0.6)/1(2.6)	-	1.2(1.9)	2.8(9.7)/3.1(13.9)	2.6(26.4)/2.6(10)	2.8(11.4)	2.7(9.4)

Table S12: The performance of BOSS compared to other methods, Sparse-Ex1, ρ=0.9, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/7/21	18	22	30/30	6/8	17	23
	p=30	4/3/22	25	23	58/57	14/12	21	33
	p=60	2/2/21	-	22	81/82	34/19	18	18
	p=180	1/1/18	-	22	114/113	68/18	15	17
msnr	p=14	15/15/21	14	21	22/20	14/12	14	23
	p=30	4/4/28	23	28	54/53	46/22	21	50
	p=60	2/2/22	-	24	81/82	76/24	20	24
	p=180	1/1/18	-	22	114/113	140/21	15	18
lsnr	p=14	27/28/17	17	15	-7/-9	10/2	4	-2
	p=30	22/22/20	16	17	-7/-7	10/-1	0	-4
	p=60	9/9/17	-	16	15/15	42/13	16	12
	p=180	3/3/13	-	15	59/58	146/21	33	18
		Relative efficiency						
hsnr	p=14	0.99/0.99/0.87	0.89	0.87	0.81/0.81	1/0.97	0.9	0.86
	p=30	1/1/0.85	0.83	0.84	0.65/0.66	0.9/0.92	0.86	0.77
	p=60	1/1/0.84	-	0.83	0.56/0.56	0.76/0.86	0.86	0.86
	p=180	1/1/0.85	-	0.83	0.47/0.47	0.6/0.85	0.88	0.86
msnr	p=14	0.98/0.98/0.93	0.99	0.93	0.92/0.93	0.99/1	0.98	0.91
	p=30	1/1/0.81	0.85	0.81	0.67/0.68	0.71/0.85	0.86	0.69
	p=60	1/1/0.83	-	0.82	0.56/0.56	0.58/0.82	0.85	0.82
	p=180	1/1/0.85	-	0.83	0.47/0.47	0.42/0.84	0.87	0.86
lsnr	p=14	0.72/0.72/0.78	0.78	0.79	0.98/1	0.83/0.9	0.88	0.93
	p=30	0.76/0.76/0.77	0.8	0.79	0.99/1	0.84/0.94	0.92	0.97
	p=60	1/1/0.93	-	0.93	0.94/0.94	0.77/0.96	0.94	0.97
	p=180	1/1/0.92	-	0.9	0.65/0.65	0.42/0.85	0.78	0.88
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.9)	6(0.7)	6(0.9)	6(3.4)/6(3.4)	6(0.2)/6(0.3)	6(0.6)	6(1.9)
	p=30	6(0.1)/6(0.1)/6(0.7)	6(0.7)	6(0.6)	6(7.6)/6(8)	6(0.9)/6(0.7)	6(1.2)	6(2.3)
	p=60	6(0)/6(0)/6(1)	-	6(0.5)	6(11.1)/6(12.1)	6(2.9)/6(1.4)	6(1.4)	6(0.8)
	p=180	6(0)/6(0)/6(1.3)	-	6(0.4)	6(15.6)/6(18.5)	6(9.5)/6(1.7)	6(1.9)	6(0.6)
msnr	p=14	5.9(1.2)/5.9(1.2)/6(1.7)	6(1)	6(1.6)	6(4)/6(3.9)	6(1)/6(1.2)	5.9(1)	6(3.3)
	p=30	6(0.2)/6(0.2)/6(1.2)	6(0.7)	6(1)	6(7.8)/6(8.2)	6(3.1)/6(1.4)	6(1.2)	6(4.8)
	p=60	6(0)/6(0)/6(1.1)	-	6(0.6)	6(11.1)/6(12.1)	6(6.4)/6(1.5)	6(1.2)	6(1.1)
	p=180	6(0)/6(0)/6(1.3)	-	6(0.4)	6(15.8)/6(18.5)	6(18.7)/6(1.4)	6(1.5)	6(0.6)
lsnr	p=14	3.5(2.1)/3.5(2.1)/4.3(3)	4(2.4)	4.2(2.8)	5.3(3.8)/5.4(3.8)	3.9(1.8)/4.6(2.7)	4.5(2.6)	5(3.5)
	p=30	4.1(2.7)/4.1(2.7)/4.6(4.5)	4.5(3.2)	4.5(3.7)	5.6(7.7)/5.7(8)	4.7(4)/5.1(5.3)	5.1(5)	5.5(7)
	p=60	5(1.4)/5(1.4)/5.1(3.1)	-	5.1(2.2)	5.9(11.1)/5.9(12)	5.3(8.2)/5.4(6)	5.3(5.7)	5.8(7.4)
	p=180	5.6(0.6)/5.6(0.6)/5.6(2)	-	5.6(0.9)	6(15.7)/6(18.4)	5.7(26.1)/5.6(3.6)	5.5(5.3)	5.9(3.1)

Table S13: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	8/6/20	21	20	41/41	17/20	15	16
	p=30	5/3/24	25	25	69/68	32/22	15	20
	p=60	4/2/21	-	23	95/94	53/23	16	19
	p=180	34/1/19	-	21	129/130	139/27	18	17
msnr	p=14	17/14/20	21	20	42/41	23/23	16	17
	p=30	17/13/24	25	25	69/68	48/27	16	21
	p=60	13/9/21	-	23	95/94	84/28	16	23
	p=180	49/10/20	-	22	129/130	224/32	18	29
lsnr	p=14	21/22/24	25	24	7/7	14/14	16	12
	p=30	29/31/26	25	26	1/1	13/8	8	6
	p=60	26/28/23	-	22	0/0	25/5	6	5
	p=180	31/18/15	-	16	-2/0	85/4	3	3
		Relative efficiency						
hsnr	p=14	0.98/1/0.89	0.88	0.89	0.75/0.75	0.91/0.89	0.92	0.91
	p=30	0.98/1/0.83	0.82	0.82	0.61/0.61	0.78/0.84	0.9	0.86
	p=60	0.99/1/0.85	-	0.83	0.52/0.53	0.67/0.83	0.88	0.86
	p=180	0.75/1/0.85	-	0.84	0.44/0.44	0.42/0.8	0.86	0.86
msnr	p=14	0.98/1/0.95	0.94	0.95	0.8/0.81	0.93/0.93	0.98	0.98
	p=30	0.96/1/0.91	0.9	0.9	0.67/0.67	0.76/0.89	0.97	0.93
	p=60	0.97/1/0.9	-	0.89	0.56/0.56	0.59/0.85	0.94	0.89
	p=180	0.74/1/0.92	-	0.9	0.48/0.48	0.34/0.83	0.93	0.85
lsnr	p=14	0.89/0.88/0.86	0.86	0.86	1/1	0.94/0.94	0.92	0.95
	p=30	0.78/0.77/0.8	0.8	0.8	1/1	0.89/0.93	0.93	0.95
	p=60	0.79/0.78/0.81	-	0.81	1/0.99	0.8/0.95	0.94	0.95
	p=180	0.75/0.83/0.85	-	0.85	1/0.98	0.53/0.94	0.95	0.95
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.2)/6(0.6)	6(0.7)	6(0.6)	6(3.6)/6(4.4)	6(1)/6(1.4)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0)/6(0.6)	6(0.7)	6(0.7)	6(7.5)/6(8.4)	6(2.4)/6(1.7)	6(1.1)	6(0.8)
	p=60	6(0.1)/6(0)/6(0.5)	-	6(0.5)	6(11.4)/6(13.2)	6(4.8)/6(1.6)	6(1.6)	6(0.7)
	p=180	6(8.8)/6(0)/6(0.4)	-	6(0.4)	6(15.9)/6(22.1)	6(18.5)/6(2.2)	6(2.6)	6(0.7)
msnr	p=14	6(0.8)/6(0.6)/6(0.6)	6(0.7)	6(0.6)	6(3.6)/6(4.4)	6(1.1)/6(1.4)	6(0.6)	6(0.6)
	p=30	6(0.5)/6(0.2)/6(0.6)	6(0.7)	6(0.7)	6(7.4)/6(8.5)	6(2.9)/6(1.6)	6(0.8)	6(0.9)
	p=60	6(0.2)/6(0.1)/6(0.5)	-	6(0.5)	6(11.3)/6(13.2)	6(6.4)/6(1.5)	6(1)	6(0.9)
	p=180	6(9.8)/6(0.1)/6(0.4)	-	6(0.4)	6(15.9)/6(22)	6(27.5)/6(2)	6(1.6)	6(1.3)
lsnr	p=14	5.5(4.5)/5.4(4.2)/4.8(1.8)	4.8(1.7)	4.8(1.8)	5.7(3.3)/5.7(4)	5.1(1.5)/5.3(2.9)	5.2(2.8)	5.2(2)
	p=30	3.9(4.3)/3.1(1.8)/3.6(2.2)	3.6(2)	3.6(2.1)	5.3(6.6)/5.3(7.1)	4.8(3.8)/4.7(4.9)	4.8(5.3)	4.5(3.6)
	p=60	2.3(1.8)/1.4(0.3)/2.7(1.4)	-	2.7(1.4)	4.7(8.8)/4.7(9.9)	4.9(8.6)/4.2(6.9)	4.3(7.5)	4(5.5)
	p=180	1.7(14.1)/0.5(0.1)/1.4(0.6)	-	1.4(0.7)	3.7(10.7)/3.8(13.5)	4.6(36.7)/3.2(9.4)	3.3(10.3)	3.1(8)

Table S14: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	6/6/17	17	17	40/41	12/14	11	13
	p=30	3/3/22	22	23	74/71	20/18	13	16
	p=60	2/2/24	-	23	97/93	29/19	15	18
	p=180	1/1/21	-	21	131/128	52/18	13	16
msnr	p=14	6/6/17	17	17	41/41	14/17	11	13
	p=30	3/3/22	22	23	74/71	27/20	13	16
	p=60	2/2/24	-	23	97/93	44/21	15	18
	p=180	1/1/21	-	21	131/127	97/22	13	16
lsnr	p=14	9/8/17	17	17	41/41	21/21	12	13
	p=30	5/5/22	22	23	74/71	46/25	13	16
	p=60	5/4/24	-	23	97/93	82/26	15	18
	p=180	5/5/21	-	21	131/128	192/26	12	17
		Relative efficiency						
hsnr	p=14	1/1/0.91	0.91	0.91	0.76/0.76	0.95/0.93	0.96	0.94
	p=30	1/1/0.84	0.84	0.84	0.59/0.6	0.86/0.87	0.91	0.89
	p=60	1/1/0.83	-	0.83	0.52/0.53	0.79/0.86	0.89	0.86
	p=180	1/1/0.83	-	0.84	0.44/0.44	0.66/0.86	0.89	0.87
msnr	p=14	1/1/0.91	0.91	0.91	0.75/0.75	0.93/0.91	0.96	0.94
	p=30	1/1/0.84	0.84	0.84	0.59/0.6	0.81/0.86	0.91	0.89
	p=60	1/1/0.83	-	0.83	0.52/0.53	0.71/0.84	0.89	0.86
	p=180	1/1/0.83	-	0.84	0.44/0.44	0.51/0.83	0.9	0.87
lsnr	p=14	1/1/0.93	0.93	0.93	0.77/0.77	0.9/0.9	0.97	0.96
	p=30	1/1/0.86	0.86	0.85	0.6/0.61	0.72/0.84	0.93	0.91
	p=60	1/1/0.84	-	0.85	0.53/0.54	0.57/0.83	0.91	0.89
	p=180	1/1/0.86	-	0.86	0.45/0.46	0.36/0.83	0.94	0.9
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.5)	6(0.9)/6(1.3)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.4)/6(8.7)	6(2)/6(1.8)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13.1)/6(12.2)	6(3.8)/6(2.1)	6(1.5)	6(0.6)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(21.6)/6(19.3)	6(10.3)/6(2.4)	6(2.1)	6(0.5)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.5)	6(0.9)/6(1.3)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.5)/6(8.7)	6(2.2)/6(1.7)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13.1)/6(12.2)	6(4.8)/6(1.8)	6(1.4)	6(0.6)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(21.8)/6(19.4)	6(15.6)/6(1.9)	6(1.9)	6(0.5)
lsnr	p=14	6(0.4)/6(0.4)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.5)	6(1.1)/6(1.4)	6(0.5)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.4)/6(8.7)	6(3)/6(1.6)	6(0.7)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(13.1)/6(12.2)	6(6.8)/6(1.4)	6(1)	6(0.6)
	p=180	6(0.1)/6(0.1)/6(0.3)	-	6(0.3)	6(21.7)/6(19.4)	6(23.9)/6(1.1)	6(0.9)	6(0.5)

Table S15: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0.5, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	8/6/17	18	17	46/44	15/20	13	15
	p=30	5/3/24	24	24	91/88	28/23	14	18
	p=60	4/2/22	-	23	123/119	43/24	16	17
	p=180	34/1/19	-	22	168/165	100/28	17	15
msnr	p=14	19/16/26	18	17	46/44	21/23	14	16
	p=30	22/18/29	23	25	89/87	42/28	14	41
	p=60	16/11/29	-	25	121/117	70/29	16	42
	p=180	48/9/29	-	30	160/157	179/32	14	52
lsnr	p=14	24/25/28	23	26	18/17	15/20	22	23
	p=30	33/34/26	20	30	17/17	15/19	18	22
	p=60	28/29/23	-	28	15/16	23/17	16	19
	p=180	28/14/14	-	17	8/9	71/11	10	12
		Relative efficiency						
hsnr	p=14	0.98/1/0.91	0.9	0.91	0.73/0.74	0.92/0.89	0.94	0.92
	p=30	0.98/1/0.83	0.83	0.83	0.54/0.55	0.8/0.84	0.9	0.87
	p=60	0.98/1/0.83	-	0.83	0.46/0.46	0.71/0.82	0.88	0.87
	p=180	0.75/1/0.85	-	0.83	0.38/0.38	0.5/0.79	0.86	0.88
msnr	p=14	0.96/0.98/0.9	0.97	0.97	0.78/0.79	0.94/0.92	1	0.98
	p=30	0.93/0.97/0.88	0.92	0.91	0.6/0.61	0.8/0.89	1	0.81
	p=60	0.96/1/0.86	-	0.89	0.5/0.51	0.66/0.87	0.96	0.78
	p=180	0.74/1/0.84	-	0.83	0.42/0.42	0.39/0.83	0.95	0.71
lsnr	p=14	0.92/0.92/0.9	0.94	0.91	0.97/0.98	1/0.96	0.94	0.94
	p=30	0.87/0.86/0.92	0.96	0.88	0.99/0.98	1/0.96	0.98	0.94
	p=60	0.89/0.89/0.93	-	0.89	1/0.99	0.93/0.98	0.99	0.96
	p=180	0.84/0.94/0.94	-	0.92	1/0.99	0.63/0.97	0.98	0.96
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.4)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(4.6)/6(5.5)	6(0.9)/6(1.5)	6(0.7)	6(0.6)
	p=30	6(0.1)/6(0)/6(0.7)	6(0.7)	6(0.7)	6(11.2)/6(12.8)	6(2.2)/6(1.5)	6(1.1)	6(0.7)
	p=60	6(0.1)/6(0)/6(0.5)	-	6(0.5)	6(16)/6(19.1)	6(4.3)/6(1.5)	6(1.7)	6(0.6)
	p=180	6(9.3)/6(0)/6(0.4)	-	6(0.4)	6(22.5)/6(31.5)	6(14.2)/6(2.1)	6(2.8)	6(0.6)
msnr	p=14	6(0.9)/6(0.8)/6(0.6)	6(0.6)	6(0.6)	6(4.6)/6(5.5)	6(1.1)/6(1.6)	6(0.7)	6(0.6)
	p=30	6(0.6)/6(0.4)/6(0.7)	6(0.7)	6(0.7)	6(11.2)/6(12.8)	6(2.7)/6(1.5)	6(0.9)	6(2)
	p=60	6(0.3)/6(0.2)/6(0.5)	-	6(0.6)	6(16)/6(19.1)	6(5.6)/6(1.3)	6(1.4)	6(1.9)
	p=180	6(10.1)/6(0.1)/6(0.5)	-	6(0.7)	6(22.3)/6(31.6)	6(23.9)/6(2)	6(2.1)	6(2.5)
lsnr	p=14	5.7(4.8)/5.6(4.5)/5.1(1.6)	5.1(1.4)	5.1(1.9)	5.7(4.3)/5.7(5.1)	5.4(1.6)/5.5(3.7)	5.3(2.7)	5.4(3.2)
	p=30	3.6(5.3)/2.4(2.2)/3.9(2.4)	3.9(1.9)	3.5(2.8)	4.5(8)/4.4(8.9)	5(4.7)/4.3(6.5)	4.2(5.3)	3.9(6.1)
	p=60	2.3(2.4)/1(0.3)/3.1(2.1)	-	2.5(1.8)	3.8(9.8)/3.7(10.9)	5(9)/3.7(8.1)	3.7(7)	3.3(7.8)
	p=180	1.7(14.9)/0.4(0.1)/1.4(1.1)	-	0.9(0.8)	2.4(9.8)/2.4(12.9)	4.5(35.8)/2.3(11)	2.3(9.3)	2.1(9.1)

Table S16: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0.5, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/7/18	19	18	48/47	14/18	13	15
	p=30	4/3/23	22	23	86/83	20/18	12	16
	p=60	2/2/23	-	23	124/120	28/20	14	15
	p=180	1/1/21	-	21	174/171	46/20	14	15
msnr	p=14	7/7/18	19	18	49/48	16/19	13	14
	p=30	4/3/23	22	23	85/83	25/20	12	16
	p=60	2/2/23	-	23	125/120	39/22	14	15
	p=180	1/1/21	-	21	174/171	77/23	13	15
lsnr	p=14	12/12/23	19	18	50/48	21/23	13	14
	p=30	9/8/27	22	23	85/83	40/25	13	16
	p=60	8/8/26	-	23	124/120	72/28	14	15
	p=180	10/9/24	-	21	175/170	163/29	12	16
		Relative efficiency						
hsnr	p=14	1/1/0.91	0.9	0.91	0.72/0.72	0.94/0.91	0.94	0.93
	p=30	1/1/0.84	0.85	0.84	0.56/0.56	0.86/0.87	0.92	0.89
	p=60	1/1/0.83	-	0.83	0.45/0.46	0.79/0.85	0.89	0.89
	p=180	1/1/0.84	-	0.84	0.37/0.37	0.69/0.84	0.89	0.88
msnr	p=14	1/1/0.91	0.9	0.91	0.72/0.72	0.92/0.89	0.94	0.93
	p=30	1/1/0.84	0.85	0.84	0.56/0.56	0.83/0.86	0.92	0.89
	p=60	1/1/0.83	-	0.83	0.45/0.46	0.73/0.83	0.89	0.88
	p=180	1/1/0.84	-	0.84	0.37/0.37	0.57/0.82	0.89	0.88
lsnr	p=14	1/1/0.91	0.94	0.95	0.75/0.76	0.92/0.91	0.99	0.98
	p=30	1/1/0.85	0.89	0.88	0.58/0.59	0.77/0.87	0.96	0.93
	p=60	1/1/0.85	-	0.88	0.48/0.49	0.63/0.84	0.95	0.93
	p=180	0.99/1/0.88	-	0.9	0.4/0.4	0.41/0.85	0.97	0.94
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.7)	6(0.6)	6(4.7)/6(5.5)	6(1)/6(1.4)	6(0.7)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(10.9)/6(11.3)	6(2.1)/6(1.9)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(18.1)/6(17.7)	6(3.8)/6(2.1)	6(1.4)	6(0.4)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(32.2)/6(29.7)	6(9.2)/6(2.4)	6(2.1)	6(0.3)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.7)	6(0.6)	6(4.8)/6(5.5)	6(1)/6(1.5)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(11)/6(11.3)	6(2.3)/6(1.7)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(18.1)/6(17.7)	6(4.6)/6(1.8)	6(1.4)	6(0.4)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(32.3)/6(29.7)	6(13.4)/6(1.8)	6(2)	6(0.3)
lsnr	p=14	6(0.6)/6(0.5)/6(0.6)	6(0.7)	6(0.6)	6(4.8)/6(5.5)	6(1.1)/6(1.5)	6(0.6)	6(0.5)
	p=30	6(0.2)/6(0.2)/6(0.6)	6(0.6)	6(0.6)	6(10.9)/6(11.3)	6(2.8)/6(1.5)	6(0.8)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(18.1)/6(17.8)	6(6.4)/6(1.4)	6(1.1)	6(0.4)
	p=180	6(0.1)/6(0.1)/6(0.3)	-	6(0.3)	6(32.5)/6(29.7)	6(21.7)/6(1.1)	6(1.2)	6(0.4)

Table S17: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0.9, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	18/15/25	17	23	53/51	20/27	14	18
	p=30	12/9/42	9	53	91/89	18/18	1	40
	p=60	9/6/49	-	69	131/126	16/12	-6	48
	p=180	22/5/75	-	111	138/120	-7/-16	-30	42
msnr	p=14	25/23/29	3	24	35/34	10/16	3	21
	p=30	23/20/31	-15	36	50/48	-1/2	-17	34
	p=60	21/18/30	-	48	65/61	-7/-7	-26	46
	p=180	23/22/20	-	64	48/35	-16/-1	-24	32
lsnr	p=14	40/41/31	32	50	41/40	38/39	39	43
	p=30	42/42/28	28	61	53/53	44/51	46	56
	p=60	36/35/25	-	50	45/45	52/47	45	48
	p=180	25/9/11	-	15	10/11	59/12	12	13
		Relative efficiency						
hsnr	p=14	0.97/0.99/0.91	0.97	0.93	0.75/0.75	0.95/0.89	1	0.97
	p=30	0.9/0.92/0.71	0.93	0.66	0.53/0.53	0.86/0.85	1	0.72
	p=60	0.87/0.89/0.63	-	0.56	0.41/0.42	0.81/0.84	1	0.64
	p=180	0.58/0.67/0.4	-	0.33	0.3/0.32	0.76/0.84	1	0.49
msnr	p=14	0.82/0.83/0.79	0.99	0.83	0.76/0.77	0.94/0.88	1	0.85
	p=30	0.68/0.69/0.63	0.98	0.61	0.55/0.56	0.84/0.82	1	0.62
	p=60	0.61/0.62/0.57	-	0.5	0.45/0.46	0.79/0.79	1	0.51
	p=180	0.62/0.62/0.63	-	0.46	0.51/0.56	0.9/0.76	1	0.57
lsnr	p=14	0.94/0.93/1	0.99	0.87	0.92/0.93	0.95/0.94	0.94	0.92
	p=30	0.9/0.9/1	1	0.8	0.83/0.84	0.89/0.85	0.88	0.82
	p=60	0.92/0.93/1	-	0.83	0.86/0.86	0.83/0.85	0.87	0.85
	p=180	0.87/1/0.98	-	0.95	0.98/0.98	0.68/0.97	0.97	0.96
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.9)/6(0.7)/6(0.7)	6(0.6)	6(1)	6(6.5)/6(7.2)	6(1.3)/6(2.2)	6(0.7)	6(0.8)
	p=30	6(1.1)/6(0.9)/6(2.5)	6(0.7)	6(3.7)	6(18.3)/6(19.9)	6(2.9)/6(2.5)	6(1.1)	6(5.7)
	p=60	6(1.7)/6(1.6)/6(3.3)	-	6(4.9)	6(31.8)/6(36.4)	6(4.2)/6(2.7)	6(1.8)	6(7.5)
	p=180	6(14.9)/6(6.8)/6(11.8)	-	5.9(19.7)	6(51.3)/6(76.1)	6(8.6)/6(4.1)	6(3.2)	6(17.5)
msnr	p=14	6(3)/6(2.7)/6(1.2)	6(0.6)	6(2.4)	6(6.5)/6(7.2)	6(1.5)/6(3)	6(1.3)	6(2.4)
	p=30	6(5.1)/6(4)/6(3.5)	6(0.7)	6(7)	6(18.2)/6(19.9)	6(3.6)/6(5.3)	6(3.4)	6(9.8)
	p=60	6(6.1)/6(4.7)/6(5.1)	-	6(10.3)	6(31.8)/6(36.3)	6(6.5)/6(7.7)	6(6)	6(16.8)
	p=180	5.9(38.2)/5.5(14.8)/5.8(18.2)	-	4.2(16.8)	5.8(49.6)/6(75.8)	6(23.5)/5.9(36.2)	5.9(31.7)	6(46.5)
lsnr	p=14	5.7(4.9)/5.6(4.6)/5.6(2.7)	5.2(1.2)	5.3(4)	5.8(6.3)/5.8(6.9)	5.4(3.7)/5.7(5.9)	5.4(4.7)	5.7(5.6)
	p=30	4(7.5)/2.9(4.4)/4.6(6.6)	4.1(1.7)	2.5(4.5)	2.7(8.3)/2.8(9.2)	4.4(9.7)/3.3(9.4)	3.4(8.7)	2.6(7.6)
	p=60	2.6(7.1)/1.4(2.5)/3.8(9.8)	-	0.8(1.7)	0.8(4.4)/0.9(5.7)	3.7(16)/1.3(7.2)	1.8(9.1)	0.9(4.9)
	p=180	1.2(20.1)/0.3(0.5)/0.7(2.7)	-	0.1(0.5)	0.3(4.4)/0.3(4.6)	2(35.8)/0.3(4.5)	0.3(4.9)	0.3(3.5)

Table S18: The performance of BOSS compared to other methods, Sparse-Ex2, ρ=0.9, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/7/18	19	18	56/55	22/28	14	15
	p=30	4/3/23	22	23	118/116	29/29	14	17
	p=60	2/2/23	-	23	186/182	37/33	15	15
	p=180	1/1/21	-	21	299/294	51/41	15	13
msnr	p=14	8/8/18	19	18	56/55	23/29	14	14
	p=30	4/4/24	22	23	118/115	31/30	13	17
	p=60	3/3/23	-	23	185/182	42/36	16	15
	p=180	3/3/22	-	21	298/293	61/43	14	13
lsnr	p=14	36/36/42	15	19	52/50	22/30	12	23
	p=30	37/36/43	12	32	100/97	28/26	7	40
	p=60	39/38/47	-	47	141/138	35/22	1	61
	p=180	38/38/37	-	72	178/175	45/16	-12	100
		Relative efficiency						
hsnr	p=14	1/1/0.91	0.9	0.91	0.68/0.69	0.88/0.84	0.94	0.93
	p=30	1/1/0.84	0.85	0.84	0.47/0.48	0.8/0.8	0.9	0.88
	p=60	1/1/0.83	-	0.83	0.35/0.36	0.74/0.76	0.88	0.88
	p=180	1/1/0.84	-	0.84	0.25/0.26	0.67/0.72	0.88	0.89
msnr	p=14	1/1/0.91	0.91	0.92	0.69/0.7	0.88/0.83	0.94	0.94
	p=30	1/1/0.84	0.85	0.84	0.48/0.48	0.79/0.8	0.91	0.89
	p=60	1/1/0.84	-	0.84	0.36/0.36	0.72/0.76	0.88	0.9
	p=180	1/1/0.84	-	0.85	0.26/0.26	0.64/0.72	0.9	0.91
lsnr	p=14	0.83/0.83/0.79	0.98	0.94	0.74/0.75	0.92/0.87	1	0.91
	p=30	0.79/0.79/0.75	0.96	0.81	0.54/0.54	0.84/0.85	1	0.76
	p=60	0.72/0.73/0.68	-	0.69	0.42/0.42	0.75/0.83	1	0.63
	p=180	0.64/0.64/0.64	-	0.51	0.32/0.32	0.61/0.76	1	0.44
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.7)	6(0.6)	6(6.6)/6(7.2)	6(1.5)/6(2.1)	6(0.7)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(17.8)/6(18.9)	6(2.7)/6(2.6)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(34.2)/6(35.3)	6(4.5)/6(2.9)	6(1.4)	6(0.4)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(72.8)/6(73.6)	6(8.7)/6(3.9)	6(2.2)	6(0.2)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.7)	6(0.6)	6(6.6)/6(7.2)	6(1.5)/6(2.2)	6(0.7)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(17.8)/6(18.8)	6(2.8)/6(2.3)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(34.2)/6(35.2)	6(4.8)/6(2.6)	6(1.4)	6(0.4)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(72.8)/6(73.6)	6(10.5)/6(3.3)	6(2.1)	6(0.2)
lsnr	p=14	6(2.7)/6(2.6)/6(0.6)	6(0.7)	6(0.9)	6(6.6)/6(7.2)	6(1.5)/6(2.5)	6(0.8)	6(1.3)
	p=30	6(2.1)/6(2)/6(0.8)	6(0.6)	6(1.5)	6(17.8)/6(18.8)	6(3.2)/6(2.7)	6(1.4)	6(2.5)
	p=60	6(1)/6(0.9)/6(0.9)	-	6(2.2)	6(34.2)/6(35.1)	6(6.3)/6(3.7)	6(2.6)	6(3.9)
	p=180	6(1.4)/6(1.4)/6(1.8)	-	6(4.6)	6(72.3)/6(73.5)	6(17.1)/6(8.4)	6(7.1)	6(10.4)

Table S19: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	8/6/20	21	20	44/43	17/20	15	16
	p=30	5/3/24	25	25	69/67	32/23	15	19
	p=60	4/2/21	-	23	97/96	52/24	16	19
	p=180	34/1/19	-	21	133/133	137/28	19	16
msnr	p=14	17/14/20	21	20	44/43	24/24	16	17
	p=30	18/13/24	25	25	69/67	48/27	16	21
	p=60	14/9/21	-	23	97/95	84/29	16	22
	p=180	50/11/20	-	22	132/133	224/33	19	29
lsnr	p=14	22/23/26	26	26	8/8	13/15	17	15
	p=30	29/32/26	26	25	1/1	14/8	8	6
	p=60	27/29/22	-	22	0/1	24/6	6	6
	p=180	30/16/14	-	14	-2/1	84/4	3	4
		Relative efficiency						
hsnr	p=14	0.98/1/0.89	0.88	0.89	0.74/0.75	0.91/0.88	0.93	0.91
	p=30	0.98/1/0.83	0.82	0.82	0.61/0.61	0.78/0.84	0.89	0.86
	p=60	0.99/1/0.85	-	0.83	0.52/0.52	0.67/0.83	0.88	0.86
	p=180	0.75/1/0.85	-	0.84	0.43/0.43	0.43/0.79	0.85	0.87
msnr	p=14	0.98/1/0.95	0.95	0.96	0.8/0.8	0.92/0.92	0.99	0.98
	p=30	0.96/1/0.92	0.91	0.91	0.67/0.68	0.76/0.89	0.98	0.94
	p=60	0.96/1/0.9	-	0.89	0.56/0.56	0.59/0.85	0.94	0.9
	p=180	0.74/1/0.92	-	0.91	0.48/0.47	0.34/0.83	0.93	0.86
lsnr	p=14	0.89/0.88/0.86	0.86	0.86	1/1	0.95/0.94	0.92	0.94
	p=30	0.78/0.76/0.8	0.8	0.81	1/1	0.88/0.93	0.93	0.95
	p=60	0.79/0.78/0.82	-	0.82	1/1	0.81/0.95	0.94	0.95
	p=180	0.76/0.85/0.86	-	0.86	1/0.98	0.53/0.94	0.95	0.95
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.2)/6(0.6)	6(0.7)	6(0.6)	6(3.7)/6(4.5)	6(1)/6(1.3)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0)/6(0.6)	6(0.7)	6(0.7)	6(7.4)/6(8.2)	6(2.4)/6(1.7)	6(1)	6(0.7)
	p=60	6(0.1)/6(0)/6(0.5)	-	6(0.5)	6(11.3)/6(13.1)	6(4.8)/6(1.7)	6(1.5)	6(0.7)
	p=180	6(8.8)/6(0)/6(0.4)	-	6(0.4)	6(16.5)/6(22.7)	6(18)/6(2.5)	6(2.7)	6(0.6)
msnr	p=14	6(0.8)/6(0.6)/6(0.6)	6(0.7)	6(0.6)	6(3.7)/6(4.5)	6(1.2)/6(1.4)	6(0.6)	6(0.6)
	p=30	6(0.5)/6(0.3)/6(0.6)	6(0.7)	6(0.7)	6(7.4)/6(8.2)	6(2.9)/6(1.6)	6(0.8)	6(0.8)
	p=60	6(0.3)/6(0.1)/6(0.5)	-	6(0.5)	6(11.4)/6(13.1)	6(6.4)/6(1.5)	6(1.1)	6(0.9)
	p=180	6(9.4)/6(0.1)/6(0.4)	-	6(0.4)	6(16.5)/6(22.8)	6(27.4)/6(2)	6(1.7)	6(1.1)
lsnr	p=14	5.4(4.4)/5.2(4.1)/4.7(1.8)	4.7(1.7)	4.7(1.7)	5.6(3.3)/5.6(4)	5.1(1.5)/5.3(3)	5.1(2.8)	5(2.1)
	p=30	4(4.4)/3.1(1.9)/3.7(2.1)	3.6(2.1)	3.7(2)	5.3(6.4)/5.4(7)	4.9(3.8)/4.8(4.9)	4.8(5.3)	4.6(3.6)
	p=60	2.2(1.8)/1.2(0.2)/2.6(1.4)	-	2.6(1.3)	4.6(8.6)/4.6(9.6)	4.9(8.5)/4.1(6.6)	4.2(7.2)	3.9(5.5)
	p=180	1.6(14.2)/0.5(0.1)/1.3(0.6)	-	1.3(0.6)	3.4(10.4)/3.5(13.1)	4.6(36.9)/3(9)	3.2(10.5)	2.9(7.7)

Table S20: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	6/6/17	17	17	41/41	12/15	11	13
	p=30	3/3/22	22	23	72/69	20/18	13	16
	p=60	2/2/24	-	23	97/93	29/19	14	17
	p=180	1/1/21	-	21	132/129	53/19	13	17
msnr	p=14	6/6/17	17	17	41/41	14/17	11	13
	p=30	3/3/22	22	23	72/69	26/20	13	17
	p=60	2/2/24	-	23	97/93	43/21	14	18
	p=180	1/1/21	-	21	132/129	97/22	13	17
lsnr	p=14	9/9/17	17	17	42/41	21/20	12	13
	p=30	5/5/22	22	23	72/69	45/25	13	16
	p=60	5/4/24	-	23	97/93	82/26	15	18
	p=180	5/4/21	-	21	132/129	192/26	13	17
		Relative efficiency						
hsnr	p=14	1/1/0.91	0.91	0.91	0.76/0.76	0.95/0.93	0.96	0.94
	p=30	1/1/0.84	0.84	0.84	0.6/0.61	0.86/0.87	0.91	0.88
	p=60	1/1/0.83	-	0.83	0.52/0.53	0.79/0.86	0.89	0.87
	p=180	1/1/0.83	-	0.84	0.44/0.44	0.66/0.85	0.89	0.86
msnr	p=14	1/1/0.91	0.91	0.91	0.75/0.75	0.93/0.91	0.96	0.94
	p=30	1/1/0.84	0.84	0.84	0.6/0.61	0.81/0.86	0.91	0.88
	p=60	1/1/0.83	-	0.83	0.52/0.53	0.71/0.84	0.89	0.87
	p=180	1/1/0.83	-	0.84	0.44/0.44	0.51/0.83	0.89	0.86
lsnr	p=14	1/1/0.93	0.93	0.93	0.77/0.77	0.9/0.9	0.97	0.96
	p=30	0.99/1/0.86	0.86	0.85	0.61/0.62	0.72/0.84	0.93	0.9
	p=60	1/1/0.84	-	0.85	0.53/0.54	0.57/0.83	0.91	0.89
	p=180	0.99/1/0.86	-	0.86	0.45/0.46	0.36/0.82	0.92	0.89
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.5)	6(0.9)/6(1.3)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.3)/6(8.5)	6(2)/6(1.8)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13)/6(12.2)	6(3.9)/6(2.1)	6(1.5)	6(0.5)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(21.7)/6(19.1)	6(10.3)/6(2.4)	6(2.1)	6(0.4)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.9)/6(4.5)	6(1)/6(1.3)	6(0.6)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.3)/6(8.6)	6(2.2)/6(1.8)	6(0.9)	6(0.5)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13)/6(12.2)	6(4.8)/6(1.8)	6(1.4)	6(0.6)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(21.6)/6(19.1)	6(15.6)/6(1.8)	6(1.8)	6(0.4)
lsnr	p=14	6(0.4)/6(0.4)/6(0.6)	6(0.6)	6(0.6)	6(3.9)/6(4.5)	6(1.1)/6(1.4)	6(0.5)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.3)/6(8.5)	6(3)/6(1.6)	6(0.7)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(13.2)/6(12.2)	6(6.8)/6(1.5)	6(1)	6(0.6)
	p=180	6(0.1)/6(0)/6(0.3)	-	6(0.3)	6(21.9)/6(19.1)	6(23.7)/6(1)	6(1)	6(0.5)

Table S21: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0.5, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/6/18	19	18	40/39	15/18	14	15
	p=30	5/2/22	24	22	70/68	30/21	14	19
	p=60	3/1/22	-	23	93/92	51/22	16	19
	p=180	35/1/18	-	21	135/135	134/26	17	20
msnr	p=14	15/13/18	19	18	40/39	20/21	15	17
	p=30	14/10/22	25	23	69/68	45/25	15	22
	p=60	13/9/23	-	24	91/89	78/26	15	27
	p=180	48/11/20	-	23	132/133	218/31	18	43
lsnr	p=14	19/21/24	24	24	5/4	11/11	13	10
	p=30	28/30/25	25	25	0/0	12/6	7	6
	p=60	23/25/21	-	21	-3/-3	22/3	3	2
	p=180	27/11/13	-	13	-3/-1	83/3	3	3
		Relative efficiency						
hsnr	p=14	0.98/1/0.9	0.89	0.9	0.75/0.76	0.92/0.9	0.92	0.92
	p=30	0.98/1/0.84	0.82	0.84	0.6/0.61	0.79/0.85	0.9	0.86
	p=60	0.98/1/0.83	-	0.83	0.53/0.53	0.67/0.83	0.88	0.85
	p=180	0.75/1/0.85	-	0.84	0.43/0.43	0.43/0.8	0.86	0.84
msnr	p=14	0.98/1/0.95	0.95	0.95	0.8/0.81	0.93/0.93	0.98	0.96
	p=30	0.97/1/0.9	0.88	0.89	0.65/0.66	0.76/0.88	0.96	0.9
	p=60	0.97/1/0.88	-	0.88	0.57/0.57	0.61/0.86	0.94	0.86
	p=180	0.75/1/0.92	-	0.9	0.48/0.47	0.35/0.84	0.94	0.77
lsnr	p=14	0.88/0.86/0.84	0.84	0.84	0.99/1	0.94/0.94	0.93	0.95
	p=30	0.78/0.77/0.8	0.8	0.8	1/1	0.9/0.94	0.93	0.95
	p=60	0.79/0.78/0.8	-	0.8	1/1	0.79/0.94	0.94	0.95
	p=180	0.76/0.87/0.86	-	0.86	1/0.98	0.53/0.94	0.94	0.94
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.4)/6(0.2)/6(0.6)	6(0.7)	6(0.6)	6(3.7)/6(4.5)	6(0.9)/6(1.3)	6(0.7)	6(0.6)
	p=30	6(0.2)/6(0)/6(0.6)	6(0.7)	6(0.6)	6(7.9)/6(9)	6(2.4)/6(1.5)	6(1.1)	6(0.9)
	p=60	6(0.1)/6(0)/6(0.5)	-	6(0.5)	6(11.5)/6(13.3)	6(4.9)/6(1.6)	6(1.6)	6(0.8)
	p=180	6(9.6)/6(0)/6(0.3)	-	6(0.4)	6(16.6)/6(23.8)	6(18.1)/6(2.1)	6(2.4)	6(0.8)
msnr	p=14	6(0.7)/6(0.6)/6(0.6)	6(0.7)	6(0.6)	6(3.7)/6(4.5)	6(1)/6(1.3)	6(0.6)	6(0.8)
	p=30	6(0.4)/6(0.2)/6(0.7)	6(0.8)	6(0.7)	6(8)/6(9)	6(2.9)/6(1.5)	6(0.8)	6(1)
	p=60	6(0.3)/6(0.2)/6(0.6)	-	6(0.6)	6(11.5)/6(13.3)	6(6.1)/6(1.5)	6(1.1)	6(1.3)
	p=180	6(10)/6(0.1)/6(0.4)	-	6(0.5)	6(17)/6(23.8)	6(27.9)/6(1.8)	6(1.7)	6(1.8)
lsnr	p=14	5.3(4.3)/5.2(4)/4.6(2.1)	4.6(2)	4.6(2)	5.5(3.4)/5.6(4)	4.9(1.4)/5.2(3)	5.2(2.6)	5.1(2.4)
	p=30	3.8(4.3)/2.9(2)/3.6(2.4)	3.4(2.1)	3.5(2.3)	5.1(6.9)/5.2(7.5)	4.6(3.9)/4.6(5.3)	4.7(5.3)	4.5(4.2)
	p=60	2.1(1.7)/1.3(0.4)/2.3(1.5)	-	2.3(1.4)	4.4(8.6)/4.5(9.7)	4.4(8.3)/3.9(6.7)	4(7.5)	3.8(5.8)
	p=180	1.4(15.1)/0.3(0.1)/1(0.7)	-	1(0.7)	3(9.7)/3(12.5)	4.3(37.2)/2.5(8.3)	2.6(9.3)	2.4(7.7)

Table S22: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0.5, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/6/18	19	19	43/42	13/16	13	15
	p=30	3/3/22	22	21	73/71	21/18	14	17
	p=60	1/1/21	-	20	96/92	29/18	13	17
	p=180	1/1/22	-	22	130/126	53/20	14	18
msnr	p=14	7/6/18	19	19	44/43	16/18	13	16
	p=30	3/3/22	22	21	73/71	28/20	14	17
	p=60	1/1/21	-	20	96/92	43/21	13	17
	p=180	1/1/22	-	22	130/126	98/22	13	18
lsnr	p=14	9/9/18	19	19	44/43	22/22	13	16
	p=30	5/5/22	21	21	73/71	47/25	14	17
	p=60	4/4/21	-	20	96/92	80/26	14	17
	p=180	5/5/22	-	22	129/126	192/27	13	19
		Relative efficiency						
hsnr	p=14	1/1/0.9	0.9	0.9	0.75/0.75	0.94/0.92	0.95	0.92
	p=30	1/1/0.85	0.85	0.85	0.59/0.6	0.85/0.87	0.91	0.88
	p=60	1/1/0.84	-	0.84	0.52/0.53	0.79/0.86	0.89	0.86
	p=180	1/1/0.83	-	0.83	0.44/0.45	0.66/0.84	0.89	0.86
msnr	p=14	1/1/0.9	0.9	0.9	0.74/0.75	0.92/0.9	0.94	0.92
	p=30	1/1/0.85	0.85	0.85	0.59/0.6	0.8/0.85	0.9	0.88
	p=60	1/1/0.84	-	0.84	0.52/0.53	0.71/0.84	0.9	0.87
	p=180	1/1/0.83	-	0.83	0.44/0.45	0.51/0.83	0.89	0.86
lsnr	p=14	1/1/0.92	0.92	0.92	0.75/0.76	0.89/0.89	0.96	0.94
	p=30	1/1/0.86	0.86	0.86	0.61/0.61	0.71/0.84	0.92	0.9
	p=60	1/1/0.86	-	0.86	0.53/0.54	0.58/0.83	0.91	0.88
	p=180	1/1/0.86	-	0.86	0.46/0.46	0.36/0.82	0.93	0.88
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.9)/6(4.4)	6(0.9)/6(1.3)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.4)/6(8.6)	6(2.1)/6(1.7)	6(1)	6(0.6)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13.2)/6(12.3)	6(3.9)/6(2.1)	6(1.6)	6(0.6)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.4)	6(21.5)/6(18.7)	6(10.4)/6(2.6)	6(2.3)	6(0.6)
msnr	p=14	6(0.3)/6(0.3)/6(0.6)	6(0.6)	6(0.6)	6(3.9)/6(4.5)	6(1)/6(1.4)	6(0.6)	6(0.6)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.3)/6(8.6)	6(2.4)/6(1.6)	6(0.9)	6(0.6)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(13)/6(12.3)	6(4.7)/6(1.8)	6(1.4)	6(0.6)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.4)	6(21.5)/6(18.7)	6(15.7)/6(1.9)	6(2.1)	6(0.5)
lsnr	p=14	6(0.4)/6(0.4)/6(0.6)	6(0.6)	6(0.6)	6(3.9)/6(4.5)	6(1.1)/6(1.4)	6(0.5)	6(0.6)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.3)/6(8.6)	6(3)/6(1.4)	6(0.7)	6(0.6)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(13.1)/6(12.2)	6(6.5)/6(1.5)	6(1)	6(0.6)
	p=180	6(0.1)/6(0.1)/6(0.4)	-	6(0.4)	6(21.2)/6(18.7)	6(23.5)/6(1.2)	6(0.9)	6(0.5)

Table S23: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0.9, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	7/6/24	13	24	33/33	14/16	12	16
	p=30	7/5/41	17	41	66/64	26/29	12	23
	p=60	6/4/43	-	43	84/83	44/38	13	19
	p=180	35/3/27	-	29	126/126	132/35	16	18
msnr	p=14	14/13/19	16	19	18/17	17/17	6	9
	p=30	15/13/24	8	25	30/29	29/24	0	8
	p=60	12/10/20	-	20	43/42	50/25	-6	6
	p=180	35/7/16	-	18	87/87	164/23	2	13
lsnr	p=14	17/20/22	22	21	-2/-3	5/3	5	4
	p=30	26/28/24	23	23	-2/-2	8/6	5	3
	p=60	23/26/22	-	22	-3/-3	20/3	4	2
	p=180	29/16/16	-	15	-5/-3	79/1	1	0
		Relative efficiency						
hsnr	p=14	0.99/1/0.85	0.93	0.85	0.79/0.79	0.92/0.91	0.95	0.91
	p=30	0.98/1/0.74	0.9	0.74	0.63/0.64	0.83/0.81	0.93	0.86
	p=60	0.98/1/0.73	-	0.73	0.57/0.57	0.73/0.75	0.92	0.87
	p=180	0.76/1/0.8	-	0.79	0.45/0.45	0.44/0.76	0.88	0.87
msnr	p=14	0.93/0.94/0.9	0.92	0.9	0.9/0.91	0.91/0.91	1	0.97
	p=30	0.87/0.88/0.8	0.93	0.8	0.77/0.77	0.77/0.8	1	0.92
	p=60	0.84/0.85/0.78	-	0.78	0.66/0.66	0.63/0.75	1	0.89
	p=180	0.75/0.95/0.88	-	0.87	0.55/0.55	0.39/0.83	1	0.9
lsnr	p=14	0.83/0.81/0.8	0.8	0.81	0.99/1	0.93/0.94	0.93	0.94
	p=30	0.78/0.76/0.79	0.8	0.8	1/1	0.91/0.93	0.93	0.95
	p=60	0.78/0.77/0.79	-	0.8	1/1	0.8/0.94	0.93	0.95
	p=180	0.74/0.82/0.82	-	0.83	1/0.98	0.53/0.94	0.95	0.95
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.6)/6(0.5)/6(1.4)	6(0.6)	6(1.4)	6(3.9)/6(4.5)	6(1)/6(1.4)	6(0.7)	6(1.2)
	p=30	6(0.7)/6(0.6)/6(2.1)	6(0.8)	6(2.1)	6(9.2)/6(10.3)	6(2.5)/6(3.4)	6(1.6)	6(2.3)
	p=60	6(0.8)/6(0.7)/6(2)	-	6(1.9)	6(12.4)/6(14.2)	6(5.1)/6(4.7)	6(2.5)	6(2.1)
	p=180	6(9.2)/6(0.1)/6(0.6)	-	6(0.6)	6(16.2)/6(22.2)	6(18.1)/6(3.4)	6(2.4)	6(0.8)
msnr	p=14	5.6(1.2)/5.6(1.1)/5.8(2.1)	5.8(1.8)	5.8(2.1)	6(4)/6(4.5)	5.7(1.5)/5.8(2.6)	5.8(1)	5.9(1.9)
	p=30	5.1(1.5)/5.1(1.2)/5.3(2.6)	5.5(1.4)	5.3(2.6)	6(9.1)/6(10.2)	5.2(4.1)/5.4(5.8)	5.7(1.8)	5.7(3.2)
	p=60	5.2(1.2)/5.2(1)/5.2(1.7)	-	5.2(1.6)	5.9(12.4)/6(14.1)	5.2(7.5)/5.3(5.2)	5.8(1.7)	5.7(2.8)
	p=180	5.6(10)/5.6(0.5)/5.6(0.8)	-	5.6(0.8)	6(16.2)/6(22.2)	5.6(27.5)/5.6(3.1)	5.9(1.7)	5.9(1.9)
lsnr	p=14	4.4(4)/4.2(3.6)/3.7(2.5)	3.6(2.4)	3.7(2.3)	4.8(3.6)/4.9(4.1)	3.9(2.1)/4.4(3.2)	4.5(2.5)	4.5(3)
	p=30	2.6(4.4)/1.9(2.3)/2.4(3)	2.5(2.8)	2.4(2.9)	3.9(7.5)/4(8.2)	3.3(4.9)/3.3(6.3)	3.7(6.1)	3.4(5.3)
	p=60	1.7(2)/1.1(0.8)/1.8(2)	-	1.8(1.9)	3.7(9.4)/3.8(10.4)	3.4(8.9)/3.2(7.6)	3.4(7.7)	3.2(6.5)
	p=180	1.4(14.4)/0.5(0.2)/1.1(1.1)	-	1.1(1.1)	3.2(11.1)/3.3(14.7)	3.5(36.7)/2.8(10.2)	3(10.8)	2.8(9.1)

Table S24: The performance of BOSS compared to other methods, Sparse-Ex3, ρ=0.9, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	6/6/19	19	19	40/39	12/14	13	15
	p=30	2/2/21	21	22	74/72	20/18	13	17
	p=60	1/1/22	-	22	101/97	29/19	14	19
	p=180	1/1/21	-	22	135/131	52/18	14	21
msnr	p=14	6/6/19	19	19	42/40	18/18	14	15
	p=30	2/2/21	21	22	74/72	27/20	14	17
	p=60	1/1/22	-	22	101/97	43/21	14	19
	p=180	1/1/21	-	22	135/132	96/21	13	21
lsnr	p=14	9/9/19	19	19	27/26	17/18	6	14
	p=30	5/5/21	20	21	53/51	34/23	3	14
	p=60	4/4/20	-	20	75/71	62/22	1	12
	p=180	4/4/17	-	17	92/89	145/23	-5	11
		Relative efficiency						
hsnr	p=14	1/1/0.89	0.89	0.89	0.76/0.76	0.95/0.93	0.94	0.92
	p=30	1/1/0.84	0.85	0.84	0.59/0.6	0.85/0.87	0.9	0.87
	p=60	1/1/0.83	-	0.83	0.5/0.51	0.79/0.85	0.89	0.85
	p=180	1/1/0.84	-	0.83	0.43/0.44	0.67/0.86	0.89	0.84
msnr	p=14	1/1/0.89	0.89	0.89	0.75/0.76	0.9/0.9	0.93	0.92
	p=30	1/1/0.84	0.85	0.84	0.59/0.6	0.81/0.85	0.9	0.87
	p=60	1/1/0.83	-	0.83	0.5/0.51	0.71/0.84	0.89	0.85
	p=180	1/1/0.84	-	0.83	0.43/0.44	0.52/0.83	0.9	0.84
lsnr	p=14	0.97/0.97/0.89	0.89	0.89	0.83/0.84	0.91/0.89	1	0.93
	p=30	0.98/0.98/0.85	0.86	0.85	0.67/0.68	0.77/0.83	1	0.9
	p=60	0.97/0.97/0.84	-	0.84	0.58/0.59	0.62/0.82	1	0.9
	p=180	0.91/0.91/0.81	-	0.81	0.49/0.5	0.39/0.77	1	0.85
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.3)/6(0.3)/6(0.7)	6(0.6)	6(0.6)	6(4.1)/6(4.6)	6(0.8)/6(1)	6(0.7)	6(0.6)
	p=30	6(0)/6(0)/6(0.6)	6(0.6)	6(0.6)	6(9.2)/6(9.5)	6(2)/6(1.8)	6(1)	6(0.7)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(14)/6(13.3)	6(3.7)/6(2.1)	6(1.5)	6(0.8)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.4)	6(23.2)/6(20.6)	6(10.1)/6(2.4)	6(2.2)	6(0.7)
msnr	p=14	6(0.3)/6(0.3)/6(0.7)	6(0.6)	6(0.6)	6(4.2)/6(4.6)	6(1.1)/6(1.3)	6(0.7)	6(0.6)
	p=30	6(0)/6(0)/6(0.6)	6(0.6)	6(0.6)	6(9.2)/6(9.5)	6(2.3)/6(1.7)	6(1)	6(0.7)
	p=60	6(0)/6(0)/6(0.5)	-	6(0.5)	6(14.1)/6(13.3)	6(4.7)/6(1.8)	6(1.4)	6(0.7)
	p=180	6(0)/6(0)/6(0.4)	-	6(0.4)	6(23.2)/6(20.5)	6(15.3)/6(1.9)	6(1.9)	6(0.7)
lsnr	p=14	5.8(0.4)/5.8(0.4)/5.9(1.4)	5.9(1.5)	5.9(1.5)	6(4.2)/6(4.6)	5.9(1.2)/5.9(2.1)	6(0.8)	6(1.6)
	p=30	5.8(0.3)/5.8(0.3)/5.8(1.1)	5.8(1.1)	5.8(1.1)	6(9.2)/6(9.5)	5.8(2.8)/5.9(3)	6(0.8)	6(1.6)
	p=60	5.8(0.3)/5.8(0.3)/5.8(0.8)	-	5.8(0.8)	6(14)/6(13.3)	5.8(6.3)/5.8(2.3)	6(1.1)	6(1.4)
	p=180	5.7(0.3)/5.7(0.3)/5.7(0.7)	-	5.7(0.7)	6(23)/6(20.6)	5.7(23.2)/5.7(2.6)	6(1)	5.9(1.7)

Table S25: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	30/30/22	22	22	19/19	16/18	20	20
	p=30	24/21/24	25	25	29/28	32/19	16	20
	p=60	16/15/20	-	21	37/36	57/17	13	21
	p=180	29/5/14	-	15	50/51	168/14	11	16
msnr	p=14	25/22/22	23	22	31/31	26/21	18	18
	p=30	26/20/27	28	28	52/51	43/24	21	26
	p=60	18/14/24	-	25	69/68	72/25	21	27
	p=180	55/16/26	-	27	97/99	237/29	27	36
lsnr	p=14	34/34/29	30	29	16/16	19/25	25	19
	p=30	37/34/31	31	31	20/18	33/25	24	22
	p=60	32/33/29	-	29	20/20	55/24	22	22
	p=180	49/29/27	-	26	18/20	142/23	21	17
		Relative efficiency						
hsnr	p=14	0.89/0.89/0.95	0.95	0.95	0.98/0.98	1/0.98	0.97	0.97
	p=30	0.94/0.96/0.94	0.93	0.93	0.9/0.91	0.88/0.98	1	0.97
	p=60	0.97/0.99/0.94	-	0.94	0.83/0.83	0.72/0.97	1	0.93
	p=180	0.82/1/0.92	-	0.92	0.7/0.7	0.39/0.92	0.94	0.91
msnr	p=14	0.95/0.96/0.97	0.96	0.97	0.9/0.9	0.94/0.98	1	1
	p=30	0.96/1/0.95	0.94	0.94	0.79/0.8	0.84/0.97	0.99	0.95
	p=60	0.96/1/0.92	-	0.92	0.68/0.68	0.66/0.91	0.95	0.9
	p=180	0.74/1/0.92	-	0.91	0.59/0.58	0.34/0.9	0.91	0.85
lsnr	p=14	0.86/0.87/0.9	0.89	0.9	1/1	0.97/0.93	0.93	0.97
	p=30	0.87/0.89/0.9	0.91	0.9	0.99/1	0.89/0.95	0.96	0.97
	p=60	0.91/0.9/0.93	-	0.93	1/1	0.77/0.97	0.98	0.98
	p=180	0.78/0.91/0.92	-	0.92	0.99/0.97	0.48/0.95	0.96	1
		Sparsistency (number of extra variables)						
hsnr	p=14	5(1.7)/4.9(1.3)/5.2(0.9)	5.2(0.9)	5.2(0.9)	5.8(3.4)/5.9(4.2)	5.5(1.3)/5.3(1.5)	5.3(1.3)	5.2(1.1)
	p=30	4.6(0.7)/4.4(0.2)/4.9(0.9)	4.9(1.1)	4.9(1)	5.7(7)/5.8(8)	5.4(3.3)/5(1.9)	5(2)	5(1.4)
	p=60	4.3(0.2)/4.2(0)/4.6(0.7)	-	4.6(0.7)	5.5(10.4)/5.6(12.1)	5.4(7.5)/4.8(2.2)	4.9(2.8)	4.6(1.6)
	p=180	4.3(10.2)/4.1(0)/4.2(0.4)	-	4.3(0.5)	5.2(14)/5.3(19)	5.4(35.4)/4.4(2.3)	4.5(3.9)	4.3(1.5)
msnr	p=14	4.5(1.3)/4.4(1)/4.4(0.6)	4.4(0.7)	4.4(0.7)	5.2(3.1)/5.2(3.6)	4.6(1.1)/4.5(1)	4.4(0.8)	4.4(0.7)
	p=30	4.2(0.8)/4.1(0.4)/4.2(0.8)	4.2(0.8)	4.2(0.8)	5(6.5)/5(7.1)	4.6(2.9)/4.3(1.3)	4.3(1.3)	4.3(1.2)
	p=60	4(0.3)/4(0.2)/4.1(0.6)	-	4.1(0.6)	4.7(9.5)/4.7(10.7)	4.5(6.5)/4.1(1.3)	4.2(1.7)	4.1(1.2)
	p=180	4.1(10.4)/3.9(0.1)/4(0.5)	-	4(0.5)	4.4(13.1)/4.5(17.1)	4.6(31.6)/4(1.6)	4.1(2.6)	4(1.7)
lsnr	p=14	3.9(2.4)/3.7(2)/3.3(0.9)	3.3(1.1)	3.2(0.9)	4.4(2.7)/4.5(3)	3.7(1.3)/3.6(1.5)	3.6(1.6)	3.6(1.2)
	p=30	2.6(1.8)/2.2(0.7)/2.5(1.1)	2.5(1.1)	2.5(1.1)	3.8(5.3)/3.9(5.5)	3.4(3.5)/2.9(2.5)	3(2.9)	3(2.4)
	p=60	2(0.8)/1.7(0.2)/2.2(0.8)	-	2.2(0.8)	3.4(7.3)/3.5(8.1)	3.5(7.8)/2.7(3.3)	2.8(3.9)	2.7(3.1)
	p=180	1.8(12)/1.3(0.1)/1.6(0.6)	-	1.7(0.6)	2.9(9.7)/2.9(11.2)	3.7(34)/2.2(5)	2.4(6.2)	2.2(3.8)

Table S26: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	9/9/17	17	17	41/41	19/15	12	13
	p=30	5/5/22	22	23	74/71	45/22	13	16
	p=60	4/4/24	-	23	97/93	83/23	16	17
	p=180	4/4/21	-	21	131/127	200/23	13	17
msnr	p=14	31/31/21	21	22	34/33	20/20	22	22
	p=30	40/40/29	31	29	52/50	42/29	28	34
	p=60	40/40/32	-	32	64/61	76/29	29	33
	p=180	39/40/31	-	32	67/64	157/27	27	30
lsnr	p=14	19/19/19	19	19	28/28	24/18	16	16
	p=30	11/11/21	22	21	49/47	57/20	14	17
	p=60	10/9/20	-	20	64/61	102/19	13	17
	p=180	9/8/20	-	20	91/88	235/21	14	18
		Relative efficiency						
hsnr	p=14	1/1/0.93	0.93	0.93	0.77/0.77	0.91/0.95	0.97	0.96
	p=30	1/1/0.86	0.86	0.86	0.6/0.61	0.72/0.86	0.93	0.91
	p=60	1/1/0.84	-	0.84	0.53/0.54	0.57/0.85	0.9	0.89
	p=180	1/1/0.86	-	0.86	0.45/0.46	0.35/0.85	0.92	0.89
msnr	p=14	0.91/0.91/0.99	0.99	0.98	0.9/0.9	1/0.99	0.98	0.98
	p=30	0.91/0.91/0.99	0.97	0.99	0.84/0.85	0.9/0.99	1	0.95
	p=60	0.92/0.92/0.97	-	0.98	0.78/0.8	0.73/1	1	0.97
	p=180	0.91/0.91/0.96	-	0.96	0.76/0.77	0.49/1	1	0.97
lsnr	p=14	0.97/0.97/0.97	0.97	0.97	0.9/0.91	0.93/0.98	1	1
	p=30	1/1/0.92	0.91	0.92	0.74/0.76	0.71/0.93	0.98	0.95
	p=60	1/1/0.91	-	0.91	0.67/0.68	0.54/0.92	0.97	0.93
	p=180	1/1/0.9	-	0.9	0.57/0.58	0.32/0.9	0.95	0.92
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.4)/6(0.4)/6(0.6)	6(0.6)	6(0.6)	6(3.8)/6(4.5)	6(1.1)/6(0.8)	6(0.5)	6(0.5)
	p=30	6(0.1)/6(0.1)/6(0.6)	6(0.6)	6(0.6)	6(8.4)/6(8.7)	6(2.9)/6(1.3)	6(0.7)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(13)/6(12.2)	6(6.8)/6(1.5)	6(1.1)	6(0.6)
	p=180	6(0)/6(0)/6(0.3)	-	6(0.3)	6(21.7)/6(19.4)	6(24.4)/6(1.5)	6(1.2)	6(0.5)
msnr	p=14	5.8(1.9)/5.8(1.8)/5.8(0.8)	5.8(0.8)	5.8(0.8)	6(3.8)/6(4.4)	5.9(1.2)/5.9(1.5)	5.8(1.1)	5.8(0.9)
	p=30	5.4(1)/5.4(0.9)/5.6(0.9)	5.6(0.9)	5.6(0.9)	6(8.5)/6(8.6)	5.9(3.4)/5.7(2.2)	5.7(1.9)	5.6(1.5)
	p=60	5.2(0.3)/5.2(0.3)/5.6(0.8)	-	5.6(0.8)	6(13)/6(12.1)	5.9(7.8)/5.7(2.6)	5.7(2.7)	5.6(1.4)
	p=180	4.7(0.1)/4.7(0.1)/5.2(0.6)	-	5.2(0.6)	5.9(21.7)/5.9(19)	5.9(26.9)/5.4(2.9)	5.4(4.5)	5.2(1.4)
lsnr	p=14	4.5(1)/4.5(1)/4.5(0.6)	4.5(0.6)	4.5(0.6)	5.3(3.4)/5.4(3.8)	4.8(1.3)/4.6(1)	4.5(0.7)	4.5(0.6)
	p=30	4.1(0.3)/4.1(0.3)/4.3(0.6)	4.3(0.6)	4.3(0.6)	5.1(7.2)/5.1(7.3)	4.8(3.6)/4.3(1.2)	4.3(1)	4.2(0.9)
	p=60	4.1(0.2)/4.1(0.2)/4.2(0.5)	-	4.2(0.5)	5(11.2)/4.9(10.1)	4.8(8)/4.2(1.2)	4.2(1.2)	4.2(0.8)
	p=180	4(0.1)/4(0.1)/4.1(0.4)	-	4.1(0.4)	4.7(18.7)/4.6(15.6)	4.8(27)/4.1(1)	4.1(1.4)	4.1(0.8)

Table S27: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0.5, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	34/33/25	22	26	29/28	17/25	25	33
	p=30	23/19/24	23	28	49/47	34/26	21	32
	p=60	18/17/21	-	24	56/54	52/21	17	28
	p=180	29/4/15	-	16	82/81	170/16	13	15
msnr	p=14	27/25/29	20	21	38/37	25/20	17	24
	p=30	24/19/34	24	30	74/72	52/26	16	52
	p=60	17/13/33	-	28	90/89	78/26	14	62
	p=180	52/14/37	-	29	137/137	263/29	17	91
lsnr	p=14	37/35/31	24	30	30/29	23/29	26	33
	p=30	37/35/32	23	34	35/35	30/31	23	38
	p=60	37/38/32	-	36	38/40	53/34	24	42
	p=180	49/31/26	-	34	33/35	120/31	20	38
		Relative efficiency						
hsnr	p=14	0.88/0.88/0.94	0.97	0.93	0.91/0.92	1/0.94	0.94	0.89
	p=30	0.97/1/0.96	0.97	0.93	0.8/0.81	0.89/0.95	0.99	0.9
	p=60	0.99/1/0.96	-	0.94	0.75/0.76	0.77/0.96	1	0.91
	p=180	0.81/1/0.91	-	0.9	0.57/0.58	0.39/0.9	0.92	0.91
msnr	p=14	0.92/0.94/0.91	0.97	0.96	0.84/0.85	0.93/0.97	1	0.94
	p=30	0.93/0.98/0.87	0.93	0.89	0.66/0.67	0.76/0.92	1	0.76
	p=60	0.97/1/0.85	-	0.88	0.6/0.6	0.64/0.9	1	0.7
	p=180	0.74/1/0.83	-	0.88	0.48/0.48	0.31/0.88	0.97	0.59
lsnr	p=14	0.9/0.91/0.94	0.99	0.95	0.95/0.95	1/0.95	0.98	0.92
	p=30	0.9/0.91/0.93	1	0.92	0.91/0.91	0.94/0.93	1	0.89
	p=60	0.91/0.9/0.94	-	0.91	0.9/0.89	0.81/0.93	1	0.87
	p=180	0.81/0.92/0.96	-	0.9	0.91/0.89	0.55/0.92	1	0.87
		Sparsistency (number of extra variables)						
hsnr	p=14	5.2(1.9)/5.1(1.4)/5.5(0.9)	5.5(0.9)	5.4(1)	5.9(4.6)/6(5.4)	5.7(1.3)/5.4(1.9)	5.4(1.4)	5.2(1.4)
	p=30	4.5(0.9)/4.4(0.2)/5(1)	5(1)	4.8(1.1)	5.7(10.4)/5.7(12)	5.4(3.6)/4.8(2.3)	4.8(2.1)	4.5(1.8)
	p=60	4.4(0.3)/4.2(0.1)/4.7(0.8)	-	4.6(0.8)	5.6(15)/5.7(17.8)	5.5(7.4)/4.7(2.3)	4.8(2.8)	4.4(1.6)
	p=180	4.3(11.4)/4(0)/4.2(0.5)	-	4.1(0.5)	5.1(20.2)/5.3(27.9)	5.2(35.3)/4.2(2.1)	4.2(3.5)	4.1(0.8)
msnr	p=14	4.7(1.6)/4.6(1.3)/4.4(0.6)	4.5(0.7)	4.5(0.7)	5.4(4.2)/5.5(5)	4.7(1.1)/4.6(1.2)	4.5(0.8)	4.6(1.1)
	p=30	4.2(1)/4.1(0.5)/4.2(0.7)	4.2(0.7)	4.2(1)	5(9.3)/5.1(10.7)	4.5(3.1)/4.2(1.4)	4.2(1.2)	4.3(2.9)
	p=60	4.1(0.4)/4(0.2)/4.1(0.6)	-	4.1(0.8)	4.8(13.6)/4.9(15.9)	4.5(6.8)/4.1(1.6)	4.1(1.8)	4.1(3.8)
	p=180	4.1(10.7)/4(0.1)/4(0.4)	-	4(0.6)	4.5(19.3)/4.6(26.2)	4.5(35.4)/4(1.7)	4.1(2.6)	4(5.2)
lsnr	p=14	4(2.7)/3.8(2.2)/3.3(0.8)	3.3(0.8)	3.3(1)	4.5(3.6)/4.7(4.1)	3.7(1.4)/3.5(1.7)	3.4(1.4)	3.7(2)
	p=30	2.7(2.3)/2.3(1)/2.7(1.3)	2.6(1)	2.6(1.5)	3.6(6.9)/3.6(7.5)	3.4(3.7)/2.9(3.3)	2.8(2.9)	2.9(4)
	p=60	1.9(1.1)/1.5(0.3)/2.3(1.2)	-	2(1.1)	2.9(8.5)/2.9(9.5)	3.4(7.8)/2.4(4)	2.5(3.9)	2.4(5.2)
	p=180	1.7(12.7)/1(0.2)/1.6(0.9)	-	1.3(0.8)	2.2(10.7)/2.1(13.3)	3.5(32.4)/1.9(6.4)	2.1(6.5)	1.9(8)

Table S28: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0.5, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	12/11/25	19	18	48/47	19/16	13	14
	p=30	8/7/29	21	23	86/83	39/20	12	16
	p=60	7/7/28	-	23	125/120	73/22	14	15
	p=180	7/6/28	-	21	174/171	171/23	12	18
msnr	p=14	33/33/26	18	20	44/42	20/23	18	34
	p=30	40/39/33	21	31	71/69	43/35	26	54
	p=60	42/43/35	-	42	93/90	76/44	31	69
	p=180	41/42/33	-	46	105/102	153/48	37	67
lsnr	p=14	22/22/30	21	21	36/35	24/20	17	19
	p=30	14/13/30	22	23	61/59	55/20	15	19
	p=60	11/10/28	-	21	89/85	104/21	13	20
	p=180	8/8/27	-	20	125/121	242/19	10	26
		Relative efficiency						
hsnr	p=14	1/1/0.89	0.94	0.95	0.75/0.76	0.94/0.96	0.99	0.98
	p=30	1/1/0.83	0.89	0.87	0.58/0.59	0.77/0.89	0.96	0.93
	p=60	1/1/0.83	-	0.87	0.47/0.48	0.62/0.87	0.93	0.92
	p=180	1/1/0.83	-	0.88	0.39/0.39	0.39/0.87	0.95	0.9
msnr	p=14	0.88/0.88/0.93	1	0.98	0.82/0.83	0.99/0.96	1	0.88
	p=30	0.87/0.87/0.91	1	0.93	0.71/0.72	0.85/0.9	0.97	0.79
	p=60	0.92/0.92/0.98	-	0.92	0.68/0.69	0.75/0.91	1	0.77
	p=180	0.94/0.94/1	-	0.91	0.65/0.66	0.53/0.9	0.97	0.8
lsnr	p=14	0.96/0.96/0.9	0.97	0.96	0.86/0.87	0.94/0.98	1	0.98
	p=30	0.99/1/0.87	0.93	0.92	0.7/0.71	0.73/0.95	0.99	0.95
	p=60	1/1/0.86	-	0.91	0.58/0.59	0.54/0.91	0.98	0.92
	p=180	0.99/1/0.85	-	0.9	0.48/0.49	0.32/0.91	0.98	0.86
		Sparsistency (number of extra variables)						
hsnr	p=14	6(0.6)/6(0.5)/6(0.6)	6(0.7)	6(0.6)	6(4.7)/6(5.5)	6(1)/6(1)	6(0.6)	6(0.5)
	p=30	6(0.2)/6(0.2)/6(0.6)	6(0.6)	6(0.6)	6(10.9)/6(11.3)	6(2.7)/6(1.4)	6(0.8)	6(0.5)
	p=60	6(0.1)/6(0.1)/6(0.5)	-	6(0.5)	6(18)/6(17.7)	6(6.4)/6(1.5)	6(1.1)	6(0.4)
	p=180	6(0.1)/6(0.1)/6(0.3)	-	6(0.3)	6(32.2)/6(29.8)	6(22.3)/6(1.3)	6(1.3)	6(0.4)
msnr	p=14	5.9(2.1)/5.9(2)/6(0.7)	5.9(0.7)	5.9(0.9)	6(4.8)/6(5.5)	6(1.2)/5.9(1.9)	5.9(1.1)	5.9(1.7)
	p=30	5.6(1.1)/5.6(1)/5.9(0.8)	5.9(0.7)	5.8(1)	6(10.9)/6(11.3)	6(3.3)/5.8(2.8)	5.8(2)	5.6(2.2)
	p=60	5.4(0.4)/5.3(0.3)/5.8(0.8)	-	5.6(1.1)	6(18)/6(17.6)	6(7.7)/5.6(3.7)	5.7(3.1)	5.3(2.3)
	p=180	4.8(0.2)/4.8(0.2)/5.5(1)	-	5.1(0.8)	5.9(31.7)/5.9(29)	6(26.5)/5.1(4.5)	5.2(4.7)	4.7(2)
lsnr	p=14	4.6(1.4)/4.6(1.3)/4.6(0.7)	4.6(0.7)	4.6(0.8)	5.5(4.4)/5.5(5)	4.9(1.3)/4.6(1.2)	4.5(0.8)	4.4(0.8)
	p=30	4.2(0.5)/4.2(0.4)/4.3(0.7)	4.3(0.6)	4.2(0.7)	5.2(9.6)/5.3(9.9)	4.8(3.6)/4.3(1.1)	4.3(1)	4.2(0.8)
	p=60	4.1(0.2)/4.1(0.2)/4.2(0.5)	-	4.1(0.5)	5(15.5)/5(14.7)	4.8(8.2)/4.1(1.2)	4.1(1.2)	4.1(0.9)
	p=180	4(0.1)/4(0.1)/4(0.4)	-	4(0.4)	4.6(26.1)/4.6(23.6)	4.8(27.9)/4(1)	4.1(1.3)	4(0.9)

Table S29: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0.9, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	34/33/29	24	46	45/44	33/43	40	53
	p=30	25/21/33	21	56	73/71	40/36	28	58
	p=60	23/22/37	-	57	95/91	50/31	22	64
	p=180	27/7/34	-	68	123/110	91/2	-10	62
msnr	p=14	29/27/28	15	33	41/39	23/28	16	39
	p=30	25/19/30	-4	52	61/60	29/23	3	56
	p=60	22/14/24	-	83	90/86	57/38	16	91
	p=180	31/17/26	-	66	78/66	78/26	6	70
lsnr	p=14	41/39/31	27	54	51/50	46/47	38	55
	p=30	39/32/27	18	78	71/72	58/66	44	75
	p=60	29/28/23	-	81	80/80	79/79	57	84
	p=180	37/17/18	-	36	33/33	110/37	35	36
		Relative efficiency						
hsnr	p=14	0.93/0.94/0.96	1	0.85	0.86/0.86	0.94/0.87	0.89	0.81
	p=30	0.97/1/0.91	1	0.78	0.7/0.71	0.87/0.9	0.95	0.77
	p=60	0.99/1/0.89	-	0.77	0.62/0.64	0.81/0.93	0.99	0.74
	p=180	0.71/0.84/0.68	-	0.54	0.41/0.43	0.47/0.88	1	0.56
msnr	p=14	0.89/0.91/0.9	1	0.87	0.82/0.83	0.93/0.9	0.99	0.83
	p=30	0.77/0.81/0.74	1	0.63	0.6/0.6	0.75/0.78	0.93	0.62
	p=60	0.93/1/0.92	-	0.62	0.6/0.61	0.73/0.83	0.99	0.6
	p=180	0.81/0.9/0.84	-	0.64	0.59/0.64	0.59/0.84	1	0.62
lsnr	p=14	0.9/0.91/0.97	1	0.83	0.84/0.84	0.87/0.86	0.92	0.82
	p=30	0.85/0.89/0.93	1	0.66	0.69/0.69	0.75/0.71	0.82	0.67
	p=60	0.95/0.97/1	-	0.68	0.69/0.68	0.69/0.69	0.78	0.67
	p=180	0.86/1/0.99	-	0.86	0.88/0.88	0.56/0.86	0.87	0.86
		Sparsistency (number of extra variables)						
hsnr	p=14	5.8(2.9)/5.7(2.5)/5.6(1.6)	5.6(0.8)	5.3(2)	6(6.5)/6(7.2)	5.5(2.3)/5.4(3.4)	5.3(2.4)	5.4(3.2)
	p=30	5.2(3.7)/5.1(2.8)/5.3(3.8)	5(1)	4.8(4.1)	5.8(17.8)/5.8(19.5)	4.9(4.5)/4.5(3.4)	4.4(2.7)	4.8(6.8)
	p=60	5(4.2)/4.8(3)/4.9(4.7)	-	4.5(4.3)	5.6(30.2)/5.7(35.1)	4.9(8.8)/4.3(3.6)	4.3(3.3)	4.3(8.7)
	p=180	4.4(13.2)/4.2(2.4)/4.3(4.3)	-	4.3(8)	4.6(44.2)/4.8(64.9)	4.4(29.2)/4.1(4)	4.1(3.1)	4.1(15.4)
msnr	p=14	5(2.5)/4.9(2.2)/4.7(1.4)	4.6(0.7)	4.9(2.4)	5.6(6.3)/5.7(7)	4.6(1.8)/4.9(3.3)	4.6(2.4)	5(3.9)
	p=30	4.5(4.5)/4.4(3.6)/4.5(3.7)	4.2(0.7)	4.6(6.5)	5.2(16.9)/5.3(18.5)	4.4(4.7)/4.5(7)	4.4(5.8)	4.8(11.2)
	p=60	4.3(5.9)/4.2(5.1)/4.3(5.7)	-	3.7(6.3)	4.8(28.3)/5(32.8)	4.4(10.4)/4.2(11.8)	4.3(11.7)	4.1(17.8)
	p=180	4.1(24.5)/3.6(7.6)/3.7(9.2)	-	3(8.8)	3.8(37.7)/4.3(59.6)	4.3(37.5)/3.6(20.4)	3.8(22.8)	3.2(26)
lsnr	p=14	4.2(3)/4(2.7)/3.8(1.9)	3.3(0.7)	3.9(2.9)	4.9(5.9)/5.1(6.5)	3.8(2.6)/4.1(4.1)	3.7(3.2)	4.4(4.7)
	p=30	3.1(5.5)/2.7(3.9)/3.2(5)	2.7(0.9)	2.2(4.4)	3(10.1)/3(11)	3.3(7.6)/2.8(8)	2.9(8)	2.6(8.5)
	p=60	2.2(5.3)/1.9(3.9)/2.6(6.8)	-	0.9(2)	1.2(7.4)/1.2(8.4)	2.8(12.8)/1.3(6.6)	2(10)	1.1(6.4)
	p=180	1.4(18.2)/0.7(1.7)/1.1(5.5)	-	0.2(0.6)	0.3(4.8)/0.3(5.4)	2.3(40.2)/0.3(4.4)	0.6(9)	0.3(4.4)

Table S30: The performance of BOSS compared to other methods, Sparse-Ex4, ρ=0.9, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	33/33/27	16	20	53/51	17/20	15	27
	p=30	33/32/33	16	33	108/106	26/22	12	46
	p=60	27/26/30	-	48	157/154	37/22	9	73
	p=180	15/15/25	-	90	226/223	68/33	10	140
msnr	p=14	28/28/22	18	45	53/51	26/39	30	57
	p=30	22/22/26	23	89	108/105	56/75	55	109
	p=60	21/21/27	-	114	166/162	100/107	92	124
	p=180	25/25/30	-	107	253/250	190/110	105	106
lsnr	p=14	33/33/32	20	29	45/44	24/28	19	37
	p=30	28/27/34	16	40	85/82	44/30	12	57
	p=60	19/19/30	-	59	122/119	75/32	9	89
	p=180	15/14/27	-	104	179/176	167/48	20	159
		Relative efficiency						
hsnr	p=14	0.86/0.86/0.9	0.99	0.96	0.75/0.76	0.98/0.95	1	0.91
	p=30	0.84/0.84/0.84	0.96	0.84	0.54/0.54	0.89/0.92	1	0.77
	p=60	0.86/0.86/0.84	-	0.74	0.42/0.43	0.8/0.89	1	0.63
	p=180	0.95/0.96/0.88	-	0.58	0.34/0.34	0.65/0.82	1	0.46
msnr	p=14	0.92/0.92/0.97	1	0.81	0.77/0.78	0.94/0.85	0.91	0.76
	p=30	0.99/1/0.96	0.99	0.64	0.58/0.59	0.78/0.7	0.78	0.58
	p=60	1/1/0.95	-	0.56	0.45/0.46	0.6/0.58	0.63	0.54
	p=180	1/1/0.96	-	0.6	0.35/0.36	0.43/0.6	0.61	0.61
lsnr	p=14	0.9/0.9/0.9	1	0.92	0.82/0.83	0.96/0.93	1	0.87
	p=30	0.87/0.88/0.83	0.97	0.8	0.61/0.61	0.78/0.86	1	0.71
	p=60	0.92/0.92/0.84	-	0.69	0.49/0.5	0.62/0.83	1	0.58
	p=180	1/1/0.9	-	0.56	0.41/0.41	0.43/0.77	0.95	0.44
		Sparsistency (number of extra variables)						
hsnr	p=14	6(1.4)/6(1.4)/6(0.6)	6(0.7)	6(0.9)	6(6.6)/6(7.2)	6(1.2)/6(1.8)	6(0.9)	6(1.4)
	p=30	6(0.5)/6(0.4)/6(0.7)	6(0.6)	6(1.3)	6(17.8)/6(18.9)	6(2.6)/6(2.5)	6(1.6)	6(2.4)
	p=60	6(0.4)/6(0.4)/6(0.8)	-	6(2)	6(34.2)/6(35.3)	6(5.8)/6(3.9)	6(3.1)	6(4.1)
	p=180	6(1.4)/6(1.4)/6(1.7)	-	5.9(3.8)	6(72.7)/6(73.8)	6(17.2)/6(9.4)	6(8.7)	5.9(9.4)
msnr	p=14	6(2)/6(2)/6(1.3)	6(0.7)	5.9(2.4)	6(6.6)/6(7.2)	6(2.4)/5.9(4.1)	5.9(2.9)	5.8(3.9)
	p=30	5.9(2.4)/5.9(2.4)/5.9(2.6)	5.9(0.7)	5.4(3.4)	6(17.8)/6(18.8)	5.9(5.9)/5.6(7.2)	5.6(6.2)	5.1(5.6)
	p=60	5.8(4.8)/5.8(4.8)/5.9(5.6)	-	4.7(1.9)	6(34)/6(35.1)	5.8(12.4)/4.9(7.7)	5(8.3)	4.3(2.6)
	p=180	5.5(12.9)/5.5(12.7)/5.7(16.2)	-	4.2(0.6)	5.7(68.6)/5.7(69.3)	5.4(31.9)/4.2(3.4)	4.3(5)	4(0.3)
lsnr	p=14	5(2.5)/5(2.4)/4.7(1.1)	4.6(0.7)	4.6(1.5)	5.7(6.4)/5.7(7)	4.6(1.5)/4.7(2.4)	4.5(1.5)	4.7(2.6)
	p=30	4.5(2.1)/4.4(1.9)/4.4(1.7)	4.3(0.6)	4.3(2.2)	5.4(16.6)/5.4(17.6)	4.5(3.7)/4.3(3.5)	4.2(2.5)	4.4(4.6)
	p=60	4.2(1.9)/4.2(1.8)/4.3(2.3)	-	4.2(3.2)	5.1(30.6)/5.1(31.3)	4.3(8.1)/4.2(5.8)	4.2(5)	4.3(8.8)
	p=180	4.1(3.7)/4.1(3.6)/4.1(4.4)	-	3.7(3.7)	4.6(60.3)/4.6(60.5)	4.3(26.8)/4.1(14.8)	4.2(14.2)	3.6(13.7)

Table S31: The performance of BOSS compared to other methods, Dense, ρ=0, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/0	0	0	0/0	0/1	0	0
	p=30	2/2/7	8	7	1/1	8/2	4	4
	p=60	9/11/11	-	11	0/-2	-1/-1	1	2
	p=180	18/13/11	-	12	18/12	5/5	3	13
msnr	p=14	0/0/6	6	6	1/1	7/1	3	3
	p=30	4/5/10	10	10	0/-1	8/2	4	3
	p=60	11/12/12	-	12	-3/-5	-1/-2	-1	-1
	p=180	19/14/13	-	12	3/0	13/1	2	5
lsnr	p=14	7/9/20	19	20	3/3	17/8	10	10
	p=30	15/19/16	16	16	-8/-8	5/-1	-3	-3
	p=60	16/14/14	-	14	-9/-8	11/-4	-4	-4
	p=180	22/7/8	-	9	-5/-3	65/0	-1	0
		Relative efficiency						
hsnr	p=14	1/1/1	1	1	1/1	1/0.99	1	1
	p=30	0.99/0.99/0.95	0.94	0.94	1/1	0.93/0.99	0.97	0.98
	p=60	0.9/0.88/0.89	-	0.89	0.98/1	1/0.99	0.98	0.96
	p=180	0.88/0.91/0.93	-	0.93	0.88/0.93	0.99/0.99	1	0.92
msnr	p=14	1/1/0.95	0.95	0.95	1/1	0.94/0.99	0.97	0.98
	p=30	0.96/0.95/0.91	0.9	0.91	0.99/1	0.92/0.98	0.96	0.96
	p=60	0.86/0.85/0.85	-	0.85	0.98/1	0.96/0.97	0.96	0.96
	p=180	0.84/0.88/0.89	-	0.89	0.97/1	0.88/0.98	0.98	0.95
lsnr	p=14	0.96/0.94/0.86	0.86	0.85	0.99/1	0.88/0.95	0.93	0.93
	p=30	0.8/0.77/0.79	0.79	0.79	1/1	0.88/0.93	0.94	0.94
	p=60	0.79/0.8/0.8	-	0.8	1/1	0.83/0.95	0.96	0.96
	p=180	0.78/0.89/0.88	-	0.88	1/0.98	0.58/0.95	0.96	0.95
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	29.2/29/26	25.7	25.9	28.6/29.2	25.3/28.6	27.2	26.8
	p=60	35.8/24.3/28	-	27.6	40.5/44.9	29.8/38.4	36.8	32.8
	p=180	36.9/17/19.3	-	19.2	47.5/67.6	38.7/36.4	32.4	36.5
msnr	p=14	14/14/13.4	13.4	13.4	13.9/13.9	13.4/13.9	13.6	13.7
	p=30	26.9/26/21.1	20.7	20.9	25.1/26.3	18.8/24.9	23.7	22.6
	p=60	26/14.9/18.4	-	18.3	32.2/36.4	22.7/29.2	29.6	25
	p=180	30.7/7.9/9.9	-	9.9	33.8/49.1	40.6/30.1	27.9	25.8
lsnr	p=14	12.2/11.8/8.3	8.4	8.2	10.6/11.2	7.6/10.4	10	9.1
	p=30	13/7.8/8	7.5	7.8	14/15.1	10.3/12.3	12.7	10.6
	p=60	5.4/1/4.2	-	4.1	15.2/16.6	15.3/13.7	14.6	12.2
	p=180	16.6/0.3/1.4	-	1.5	12.4/15.8	41.6/12.2	13.5	11.3

Table S32: The performance of BOSS compared to other methods, Dense, ρ=0, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/0	0	0	2/2	1/4	0	0
	p=30	0/0/1	1	1	1/1	3/6	1	1
	p=60	5/5/7	-	7	1/0	0/0	3	4
	p=180	7/7/9	-	9	15/14	3/5	2	9
msnr	p=14	0/0/0	0	0	0/0	0/2	0	0
	p=30	1/1/5	5	5	1/1	4/2	3	3
	p=60	8/8/9	-	9	1/0	-1/0	3	4
	p=180	8/8/10	-	10	13/12	3/4	3	7
lsnr	p=14	0/0/4	4	4	0/0	3/1	3	3
	p=30	3/3/10	10	9	1/0	7/2	5	5
	p=60	13/13/12	-	12	0/-1	0/1	3	3
	p=180	9/9/12	-	13	9/8	14/5	4	7
		Relative efficiency						
hsnr	p=14	1/1/1	1	1	0.98/0.98	0.99/0.96	1	1
	p=30	1/1/0.99	0.99	0.99	0.99/0.99	0.97/0.94	0.99	0.99
	p=60	0.95/0.95/0.93	-	0.93	0.99/0.99	1/1	0.97	0.96
	p=180	0.95/0.96/0.94	-	0.94	0.89/0.9	0.99/0.98	1	0.94
msnr	p=14	1/1/1	1	1	1/1	1/0.99	1	1
	p=30	1/1/0.96	0.96	0.96	1/1	0.97/0.99	0.97	0.97
	p=60	0.92/0.92/0.91	-	0.91	0.98/0.99	1/0.99	0.96	0.96
	p=180	0.95/0.95/0.93	-	0.93	0.91/0.92	1/0.99	1	0.96
lsnr	p=14	1/1/0.96	0.96	0.96	1/1	0.98/0.99	0.98	0.97
	p=30	0.97/0.97/0.91	0.91	0.92	1/1	0.94/0.98	0.96	0.96
	p=60	0.88/0.88/0.88	-	0.88	0.99/1	0.99/0.99	0.96	0.96
	p=180	0.95/0.96/0.93	-	0.93	0.96/0.97	0.91/1	1	0.97
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	30/30/29.8	29.8	29.8	30/30	30/30	29.9	29.9
	p=60	50.1/49.6/39.8	-	39.7	53.1/54.3	46.4/50.2	44.8	42.6
	p=180	32.5/31.6/32.3	-	32.1	88.1/88.8	62.2/60.9	46.4	38.7
msnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	29.8/29.8/28.1	28.2	28.1	29.6/29.8	28.9/29.7	28.7	28.7
	p=60	42/41.1/31.3	-	31.1	48.1/49.1	37.1/43.5	37.5	34.2
	p=180	23.9/23.2/24.1	-	24.1	75.1/74.9	51.3/44.6	38.9	29.6
lsnr	p=14	14/14/13.7	13.7	13.7	14/14	13.8/14	13.8	13.8
	p=30	27.8/27.8/22.1	21.9	22.1	26.8/27.5	21/26.2	24.6	23.4
	p=60	30/28.5/19.9	-	19.5	38.5/38.9	25.4/31.4	29.1	25.3
	p=180	14.1/13.5/14.1	-	14	55.3/53.9	42/31.8	29.7	23.4

Table S33: The performance of BOSS compared to other methods, Dense, ρ=0.5, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/4	0	0	0/0	1/2	0	-1
	p=30	1/1/8	9	8	2/1	6/2	5	1
	p=60	13/13/12	-	12	10/7	5/5	7	4
	p=180	37/14/13	-	16	47/27	10/13	8	7
msnr	p=14	0/0/5	5	4	0/0	2/1	2	1
	p=30	4/5/10	11	10	4/3	8/4	8	3
	p=60	16/15/14	-	14	11/6	4/6	7	4
	p=180	43/14/14	-	17	37/21	18/18	11	14
lsnr	p=14	5/7/17	19	17	9/7	16/8	14	12
	p=30	18/18/17	16	18	10/7	10/10	11	7
	p=60	17/13/13	-	14	6/5	19/8	8	5
	p=180	47/4/8	-	9	2/4	79/6	6	4
		Relative efficiency						
hsnr	p=14	0.99/0.99/0.95	0.99	0.99	0.99/0.99	0.99/0.97	0.99	1
	p=30	0.98/0.98/0.93	0.91	0.93	0.98/0.98	0.94/0.98	0.94	0.98
	p=60	0.92/0.92/0.93	-	0.93	0.95/0.98	0.99/0.99	0.97	1
	p=180	0.78/0.94/0.95	-	0.92	0.73/0.84	0.97/0.95	0.99	1
msnr	p=14	1/1/0.95	0.95	0.96	1/1	0.98/0.99	0.98	0.99
	p=30	0.98/0.98/0.93	0.92	0.93	0.99/1	0.95/0.98	0.95	0.99
	p=60	0.89/0.9/0.91	-	0.91	0.93/0.98	1/0.98	0.97	1
	p=180	0.77/0.97/0.97	-	0.95	0.81/0.91	0.94/0.94	1	0.97
lsnr	p=14	0.97/0.96/0.87	0.86	0.87	0.93/0.96	0.88/0.94	0.9	0.91
	p=30	0.9/0.9/0.92	0.92	0.91	0.97/1	0.98/0.97	0.96	1
	p=60	0.9/0.93/0.93	-	0.93	0.99/1	0.88/0.98	0.97	1
	p=180	0.69/0.97/0.94	-	0.94	1/0.98	0.57/0.96	0.96	0.98
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	29.7/29.6/26.1	25.1	26	29.1/29.7	27/29.2	27	28
	p=60	42/29.1/29.4	-	28.6	44.7/50.7	33.7/43.1	35	42
	p=180	43.6/17/20.2	-	19.6	52.2/87.1	40.3/42.5	32.4	63.5
msnr	p=14	14/14/13.7	13.6	13.7	14/14	13.8/14	13.8	13.9
	p=30	28.3/27.8/21.1	18.8	20.9	26.7/28.2	20.3/26.2	22.6	25.1
	p=60	34.4/17.9/19.9	-	19.6	34.7/43.5	24.7/33.7	27.1	35.8
	p=180	36.7/8.3/11	-	9.7	24.5/61.4	39.6/28.9	22.1	47.6
lsnr	p=14	13.1/12.7/9.4	8.7	9.3	10.8/12	8.7/11.4	10.3	10.6
	p=30	13.4/5.7/7.5	6.6	7.1	5.3/12.9	11.8/11.8	10.3	11.4
	p=60	4.8/0.6/3.5	-	3	3.7/11.2	15.7/9.4	8.9	8.7
	p=180	19/0.2/1.1	-	0.9	2.7/8.6	38.9/7.2	6.9	5.9

Table S34: The performance of BOSS compared to other methods, Dense, ρ=0.5, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/0	0	0	2/2	2/5	0	0
	p=30	0/0/3	0	0	1/1	9/10	0	-1
	p=60	6/6/7	-	7	4/3	4/5	5	6
	p=180	8/8/9	-	12	37/36	20/20	8	34
msnr	p=14	0/0/2	0	0	0/0	1/4	0	0
	p=30	0/0/5	6	5	1/0	3/5	3	1
	p=60	10/10/10	-	10	7/6	4/5	7	9
	p=180	9/9/11	-	14	36/34	16/18	9	33
lsnr	p=14	0/0/5	2	2	0/0	1/1	1	1
	p=30	2/2/9	10	10	2/1	7/2	7	3
	p=60	17/16/13	-	14	11/10	6/8	8	13
	p=180	11/11/13	-	16	32/30	19/18	10	30
		Relative efficiency						
hsnr	p=14	1/1/1	1	1	0.98/0.98	0.98/0.95	1	1
	p=30	0.98/0.98/0.96	0.98	0.98	0.97/0.97	0.9/0.89	0.98	0.99
	p=60	0.98/0.98/0.96	-	0.96	1/1	0.99/0.98	0.98	0.98
	p=180	1/1/0.99	-	0.96	0.79/0.79	0.9/0.89	1	0.81
msnr	p=14	1/1/0.98	1	1	1/1	0.99/0.96	1	1
	p=30	0.98/0.98/0.94	0.93	0.94	0.98/0.98	0.95/0.94	0.96	0.97
	p=60	0.95/0.95/0.95	-	0.95	0.98/0.98	1/1	0.98	0.96
	p=180	0.99/1/0.98	-	0.95	0.8/0.81	0.93/0.92	1	0.81
lsnr	p=14	1/1/0.95	0.98	0.98	1/1	0.99/0.99	0.99	0.99
	p=30	0.98/0.98/0.91	0.9	0.91	0.98/0.98	0.93/0.97	0.93	0.96
	p=60	0.91/0.91/0.93	-	0.93	0.95/0.96	1/0.98	0.98	0.94
	p=180	0.99/1/0.97	-	0.95	0.83/0.85	0.93/0.93	1	0.85
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	30/30/30	30	30	30/30	30/30	30	30
	p=60	53.6/53.3/40.3	-	40	55.4/56.9	48.4/49.4	43.8	49
	p=180	36/34.5/35.1	-	32.6	106.5/113.5	76.5/77.1	43	66.3
msnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	30/30/28.6	28.3	28.5	29.8/29.9	29.4/29.7	29.1	29.2
	p=60	47.5/46.6/31.6	-	31.2	51.5/53.4	42.1/48.1	36.3	43.6
	p=180	27.3/26.2/27.1	-	24.1	90.5/98.9	60/54.4	35.2	54.9
lsnr	p=14	14/14/13.9	13.9	13.9	14/14	14/14	13.9	13.9
	p=30	29.1/29/22.7	21.2	22.1	28/28.9	23/27.6	24.1	26.2
	p=60	36.1/34.7/21.1	-	19.8	42.2/45.5	28.3/34.9	26.5	35.2
	p=180	17/16/17	-	14.3	61.8/72	43.7/33.1	25.3	46.9

Table S35: The performance of BOSS compared to other methods, Dense, ρ=0.9, n=200

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/5	0	0	0/0	6/7	1	1
	p=30	2/2/9	10	8	2/2	14/15	8	3
	p=60	15/19/14	-	12	12/11	12/13	14	6
	p=180	46/15/12	-	12	71/43	23/24	20	23
msnr	p=14	1/1/7	8	6	2/1	8/8	5	5
	p=30	4/5/12	12	10	4/3	8/5	9	7
	p=60	20/23/16	-	13	30/17	12/15	15	18
	p=180	58/26/15	-	17	61/46	31/37	26	44
lsnr	p=14	11/13/18	19	18	11/10	13/12	18	16
	p=30	19/15/14	15	13	10/11	19/12	12	15
	p=60	14/11/12	-	12	9/12	34/13	12	4
	p=180	59/12/10	-	10	15/17	106/12	9	10
		Relative efficiency						
hsnr	p=14	0.99/0.99/0.95	0.99	0.99	0.99/0.99	0.94/0.93	0.99	0.99
	p=30	0.98/0.98/0.91	0.9	0.92	0.98/0.98	0.87/0.87	0.93	0.97
	p=60	0.87/0.84/0.87	-	0.89	0.89/0.9	0.89/0.88	0.88	0.94
	p=180	0.77/0.97/1	-	1	0.65/0.78	0.91/0.9	0.93	0.91
msnr	p=14	0.99/0.99/0.94	0.93	0.94	0.98/0.99	0.93/0.93	0.95	0.96
	p=30	0.99/0.98/0.92	0.92	0.93	0.99/1	0.95/0.98	0.95	0.96
	p=60	0.93/0.92/0.97	-	0.99	0.86/0.96	1/0.97	0.98	0.95
	p=180	0.73/0.91/1	-	0.98	0.71/0.79	0.88/0.84	0.91	0.8
lsnr	p=14	0.93/0.91/0.87	0.87	0.88	0.94/0.94	0.91/0.93	0.88	0.89
	p=30	0.92/0.96/0.97	0.96	0.97	1/1	0.92/0.98	0.98	0.96
	p=60	0.91/0.94/0.93	-	0.93	0.96/0.93	0.78/0.93	0.93	1
	p=180	0.69/0.97/0.99	-	1	0.95/0.93	0.53/0.98	1	1
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	29.5/29.3/25.2	23	24.6	28.9/29.5	24.4/24.8	26.2	27.6
	p=60	44.3/26.1/27.4	-	23.6	46.5/49.8	31.6/34.4	32.9	41.2
	p=180	46.7/15.6/21.3	-	17.2	54.4/93.4	44.8/46.9	37.7	67.6
msnr	p=14	14/14/13.5	13.2	13.4	13.9/14	13.4/13.7	13.5	13.6
	p=30	28.6/28/20.2	16.6	19	26.4/27.9	19.3/24.4	22.3	24.5
	p=60	33/13/18.4	-	14.9	30.3/42.2	24.7/32.3	25.8	34.3
	p=180	41.6/5.8/14.1	-	9.3	4.8/42.5	46.7/30.9	26	29.2
lsnr	p=14	10.6/9.8/7.1	6.4	6.6	8.9/9.3	7.5/8.6	7	7.6
	p=30	8.6/4.1/5.2	4.3	4.5	8.4/9.2	10.3/7.6	5.9	6.8
	p=60	3.7/1.4/4	-	2.7	2.7/8.1	14.5/6.2	5	4.2
	p=180	18/1/2.1	-	1.6	3.9/6.8	37.7/4	3.3	2.7

Table S36: The performance of BOSS compared to other methods, Dense, ρ=0.9, n=2000

		BOSS C_p-hdf/AICc-hdf/CV	BS CV	FS CV	LASSO AICc/CV	Gamma LASSO AICc/CV	SparseNet CV	rLASSO CV
		% worse than the best possible BOSS						
hsnr	p=14	0/0/0	0	0	3/4	27/27	2	0
	p=30	0/0/3	1	1	2/2	106/106	1	3
	p=60	7/8/9	-	6	6/6	74/74	8	7
	p=180	10/10/10	-	9	39/38	56/56	17	35
msnr	p=14	0/0/3	0	0	1/1	8/8	1	0
	p=30	0/0/5	7	5	1/1	25/25	3	6
	p=60	12/12/10	-	8	9/8	11/11	9	10
	p=180	10/11/12	-	9	41/39	22/23	16	38
lsnr	p=14	1/1/7	5	4	1/1	5/8	3	3
	p=30	3/3/11	11	9	3/2	7/6	8	8
	p=60	19/18/14	-	10	14/13	8/11	12	15
	p=180	12/12/13	-	11	40/38	27/26	18	37
		Relative efficiency						
hsnr	p=14	1/1/1	1	1	0.97/0.97	0.79/0.79	0.98	1
	p=30	1/1/0.97	0.99	0.99	0.98/0.98	0.49/0.49	0.99	0.98
	p=60	0.98/0.98/0.97	-	1	1/1	0.61/0.61	0.97	0.99
	p=180	0.99/0.99/0.99	-	1	0.78/0.79	0.7/0.7	0.93	0.8
msnr	p=14	1/1/0.97	1	1	0.99/0.99	0.92/0.92	0.99	1
	p=30	1/1/0.96	0.94	0.95	1/1	0.8/0.8	0.97	0.95
	p=60	0.96/0.96/0.98	-	1	0.99/0.99	0.97/0.97	0.99	0.98
	p=180	0.98/0.98/0.98	-	1	0.77/0.78	0.89/0.88	0.94	0.79
lsnr	p=14	1/1/0.94	0.95	0.97	1/1	0.96/0.93	0.98	0.98
	p=30	0.99/0.99/0.92	0.92	0.94	1/1	0.96/0.96	0.94	0.95
	p=60	0.91/0.92/0.95	-	0.99	0.95/0.96	1/0.98	0.97	0.95
	p=180	1/0.99/0.99	-	1	0.8/0.81	0.88/0.88	0.94	0.81
		Sparsistency (number of extra variables)						
hsnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	30/30/29.9	29.9	29.9	30/30	26.1/26.1	30	29.9
	p=60	53.2/52.7/39.2	-	36.3	55.5/57.2	29.4/29.4	46	49.5
	p=180	37/35/38.6	-	30.2	109.6/118.6	43.1/43.1	52.4	74.3
msnr	p=14	14/14/14	14	14	14/14	14/14	14	14
	p=30	29.9/29.9/28.2	27.3	27.9	29.7/29.9	26.3/26.3	28.8	28.7
	p=60	47.5/46.6/31.3	-	27.4	51.1/53.4	35.5/35.6	37.2	44
	p=180	30.8/28.4/32	-	23	95.1/104.1	62.7/59	44.6	71.3
lsnr	p=14	14/14/13.8	13.6	13.7	14/14	13.8/13.9	13.8	13.8
	p=30	28.8/28.8/21.2	18.5	20.2	27.6/28.4	21.7/24.8	23.5	24.8
	p=60	35.3/33.6/20.6	-	16.3	41/43.9	28.5/33.1	26.4	35.4
	p=180	18.5/16.6/21.4	-	11.8	65.3/76.1	49.6/41.2	32.3	60.8

Table S37: The performance of BOSS for high dimensional data, Sparse-Ex1, ρ=0, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	6/1/17	502/502/502/18	1/0/1	502	0	141/142	183/142	17	16
	p=500	26/3/18	511/511/511/19	2/0/1	511	0	163/165	368/165	24	18
	p=1000	86/13/21	512/512/512/22	2/0/0	512	0	184/192	424/192	27	17
	p=1500	122/44/19	505/505/505/21	2/0/0	505	0	206/209	433/209	26	15
msnr	p=250	23/12/18	501/501/501/19	4/269/6	501	269	140/141	313/141	18	28
	p=500	61/24/19	508/508/508/21	6/415/18	508	416	161/164	459/164	25	28
	p=1000	135/56/23	511/511/511/24	7/571/18	511	571	184/191	478/191	27	34
	p=1500	190/128/23	499/499/499/25	10/604/53	499	604	202/206	472/206	26	63
lsnr	p=250	16/15/13	166/166/166/13	15/16/15	166	16	-3/0	122/0	3	3
	p=500	22/12/12	149/149/149/12	12/12/12	149	12	-3/0	143/0	3	3
	p=1000	33/16/9	136/136/136/9	9/9/9	136	9	-3/2	133/2	3	4
	p=1500	34/20/6	138/138/138/6	3/3/3	138	3	-3/1	135/1	3	5
		Relative efficiency								
hsnr	p=250	0.95/0.99/0.86	0.17/0.17/0.17/0.85	0.99/1/0.99	0.17	1	0.42/0.41	0.35/0.41	0.85	0.87
	p=500	0.79/0.98/0.85	0.16/0.16/0.16/0.84	0.98/1/0.99	0.16	1	0.38/0.38	0.21/0.38	0.81	0.85
	p=1000	0.54/0.89/0.83	0.16/0.16/0.16/0.82	0.98/1/1	0.16	1	0.35/0.34	0.19/0.34	0.79	0.86
	p=1500	0.45/0.7/0.84	0.17/0.17/0.17/0.82	0.98/1/1	0.17	1	0.33/0.32	0.19/0.32	0.79	0.87
msnr	p=250	0.84/0.93/0.88	0.17/0.17/0.17/0.87	1/0.28/0.98	0.17	0.28	0.43/0.43	0.25/0.43	0.88	0.81
	p=500	0.66/0.85/0.89	0.17/0.17/0.17/0.88	1/0.21/0.9	0.17	0.21	0.41/0.4	0.19/0.4	0.85	0.83
	p=1000	0.46/0.69/0.87	0.18/0.18/0.18/0.86	1/0.16/0.91	0.18	0.16	0.38/0.37	0.18/0.37	0.84	0.8
	p=1500	0.38/0.48/0.89	0.18/0.18/0.18/0.88	1/0.16/0.72	0.18	0.16	0.36/0.36	0.19/0.36	0.87	0.68
lsnr	p=250	0.84/0.85/0.86	0.37/0.37/0.37/0.86	0.85/0.84/0.84	0.37	0.84	1/0.97	0.44/0.97	0.94	0.94
	p=500	0.79/0.87/0.87	0.39/0.39/0.39/0.87	0.87/0.86/0.86	0.39	0.86	1/0.97	0.4/0.97	0.94	0.94
	p=1000	0.73/0.84/0.89	0.41/0.41/0.41/0.89	0.89/0.89/0.89	0.41	0.89	1/0.95	0.42/0.95	0.94	0.93
	p=1500	0.73/0.81/0.92	0.41/0.41/0.41/0.92	0.94/0.95/0.95	0.41	0.95	1/0.96	0.41/0.96	0.94	0.93
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/3.6/6	6	3.6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/2.4/5.9	6	2.4	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/1.1/5.9	6	1.1	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	5.9/0.1/5.6	6	0.1	6/6	6/6	6	5.9
lsnr	p=250	0.8/0.4/1.1	5.6/5.6/5.6/1.1	0.3/0/0.1	5.6	0	3.3/3.3	4.6/3.3	2.9	2.6
	p=500	1.1/0.4/0.8	4.3/4.3/4.3/0.8	0.2/0/0.1	4.3	0	3/3	3.9/3	2.6	2.4
	p=1000	1.3/0.7/0.6	3.2/3.2/3.2/0.6	0.1/0/0	3.2	0	2.5/2.6	3.2/2.6	2.2	2.1
	p=1500	0.8/0.5/0.3	2.3/2.3/2.3/0.3	0.1/0/0	2.3	0	1.5/1.6	2.4/1.6	1.4	1.3
		Number of extra variables								
hsnr	p=250	0.2/0/0.3	193/193/193/0.3	0/0/0	193	0	16.8/24.2	27.7/24.2	2.4	0.5
	p=500	1.4/0/0.2	193/193/193/0.3	0/0/0	193	0	18.1/29.6	73.7/29.6	3.7	0.8
	p=1000	4.8/0.5/0.3	193/193/193/0.3	0/0/0	193	0	18.9/37.9	87/37.9	4.7	0.6
	p=1500	6.4/2.1/0.2	193/193/193/0.3	0/0/0	192.9	0	21.4/43.8	89.7/43.8	5.1	0.5
msnr	p=250	0.5/0.2/0.3	193/193/193/0.3	0/0/0	193	0	16.9/24.2	49.6/24.2	1.4	1
	p=500	2/0.3/0.3	193/193/193/0.3	0/0/0	193	0	18.2/29.6	97.9/29.6	2.3	1.2
	p=1000	5.7/1.1/0.3	193/193/193/0.3	0/0/0	193	0	19.2/37.9	99.9/37.9	2.5	1.3
	p=1500	8.1/4/0.3	193/193/193/0.3	0/0/0	192.9	0	21.3/44.1	99/44.1	2.6	2.5
lsnr	p=250	0.7/0.1/0.5	193.4/193.4/193.4/0.5	0/0/0	193.4	0	10.1/13.1	70/13.1	9.6	7.2
	p=500	2.8/0.2/0.5	194.7/194.7/194.7/0.4	0/0/0	194.7	0	11.3/15.9	113.7/15.9	12.4	8.9
	p=1000	5.8/1.5/0.4	195.8/195.8/195.8/0.3	0/0/0	195.8	0	10.7/20	113/20	14.1	11.5
	p=1500	6.1/3.2/0.3	196.7/196.7/196.7/0.3	0/0/0	196.6	0	9.5/15.6	111.9/15.6	12.8	10.7

Table S38: The performance of BOSS for high dimensional data, Sparse-Ex1, ρ=0.5, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	4/1/17	495/495/495/20	1/0/1	495	0	139/138	177/138	17	16
	p=500	21/2/18	507/507/507/18	3/0/1	507	0	161/161	359/161	20	19
	p=1000	70/9/20	513/513/513/21	3/0/0	513	0	184/186	425/186	24	20
	p=1500	113/30/16	502/502/502/19	3/0/0	502	0	208/209	428/209	23	14
msnr	p=250	22/13/19	490/490/490/22	4/303/8	490	304	137/136	293/136	18	37
	p=500	53/24/20	498/498/498/20	8/412/20	498	414	157/158	450/158	21	35
	p=1000	116/52/23	506/506/506/25	8/570/26	506	570	180/184	475/184	24	39
	p=1500	182/113/22	491/491/491/24	11/584/61	491	584	203/204	465/204	26	73
lsnr	p=250	13/12/11	165/165/165/12	12/13/13	165	13	-4/-2	117/-2	1	1
	p=500	20/12/11	150/150/150/11	11/12/12	150	12	-3/-1	144/-1	2	2
	p=1000	30/14/9	136/136/136/9	8/8/8	136	8	-4/0	133/0	2	3
	p=1500	31/17/6	141/141/141/6	3/2/2	141	2	-3/0	138/0	3	4
		Relative efficiency								
hsnr	p=250	0.96/0.99/0.85	0.17/0.17/0.17/0.83	0.99/1/0.99	0.17	1	0.42/0.42	0.36/0.42	0.85	0.86
	p=500	0.83/0.98/0.84	0.16/0.16/0.16/0.85	0.97/1/0.99	0.16	1	0.38/0.38	0.22/0.38	0.83	0.84
	p=1000	0.59/0.92/0.84	0.16/0.16/0.16/0.82	0.97/1/1	0.16	1	0.35/0.35	0.19/0.35	0.81	0.83
	p=1500	0.47/0.77/0.86	0.17/0.17/0.17/0.84	0.98/1/1	0.17	1	0.32/0.32	0.19/0.32	0.81	0.88
msnr	p=250	0.85/0.92/0.87	0.18/0.18/0.18/0.85	1/0.26/0.96	0.18	0.26	0.44/0.44	0.26/0.44	0.88	0.76
	p=500	0.7/0.87/0.89	0.18/0.18/0.18/0.9	1/0.21/0.9	0.18	0.21	0.42/0.42	0.2/0.42	0.89	0.8
	p=1000	0.5/0.71/0.88	0.18/0.18/0.18/0.87	1/0.16/0.86	0.18	0.16	0.39/0.38	0.19/0.38	0.87	0.78
	p=1500	0.4/0.52/0.91	0.19/0.19/0.19/0.9	1/0.16/0.69	0.19	0.16	0.37/0.37	0.2/0.37	0.89	0.64
lsnr	p=250	0.85/0.86/0.86	0.36/0.36/0.36/0.86	0.86/0.85/0.85	0.36	0.85	1/0.98	0.44/0.98	0.95	0.95
	p=500	0.8/0.86/0.87	0.39/0.39/0.39/0.87	0.87/0.86/0.86	0.39	0.86	1/0.97	0.4/0.97	0.94	0.94
	p=1000	0.74/0.84/0.88	0.41/0.41/0.41/0.88	0.89/0.88/0.88	0.41	0.88	1/0.96	0.41/0.96	0.94	0.93
	p=1500	0.74/0.83/0.92	0.4/0.4/0.4/0.92	0.95/0.95/0.95	0.4	0.95	1/0.97	0.41/0.97	0.94	0.94
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/3.1/5.9	6	3.1	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/2.3/5.9	6	2.3	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/1/5.8	6	1	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	5.9/0/5.5	6	0	6/6	6/6	6	5.9
lsnr	p=250	0.7/0.3/1	5.5/5.5/5.5/1	0.2/0/0.1	5.5	0	3/3.1	4.3/3.1	2.7	2.5
	p=500	1/0.4/0.8	4.2/4.2/4.2/0.8	0.2/0/0.1	4.2	0	2.9/2.9	3.7/2.9	2.5	2.4
	p=1000	1.2/0.6/0.6	3.1/3.1/3.1/0.6	0.1/0/0	3.1	0	2.5/2.5	3/2.5	2.2	2
	p=1500	0.7/0.4/0.2	2.1/2.1/2.1/0.2	0/0/0	2.1	0	1.3/1.3	2.1/1.3	1.2	1.1
		Number of extra variables								
hsnr	p=250	0.2/0/0.3	193/193/193/0.3	0/0/0	193	0	17/23.7	26.6/23.7	2	0.5
	p=500	1/0/0.3	193/193/193/0.3	0/0/0	193	0	18.3/28.3	71.3/28.3	3	0.8
	p=1000	3.6/0.3/0.2	193/193/193/0.3	0/0/0	193	0	18.7/35.2	87.8/35.2	3.6	0.6
	p=1500	5.9/1.3/0.2	193/193/193/0.2	0/0/0	192.9	0	22.5/44.6	89.1/44.6	3.9	0.5
msnr	p=250	0.5/0.2/0.3	193/193/193/0.4	0/0/0	193	0	16.7/23.7	46.2/23.7	1.2	1.2
	p=500	1.7/0.3/0.3	193/193/193/0.3	0/0/0	193	0	18.4/28.3	98.4/28.3	1.8	1.3
	p=1000	4.6/1/0.3	193/193/193/0.3	0/0/0	193	0	18.9/35.2	101/35.2	2	1.4
	p=1500	7.6/3.2/0.3	193/193/193/0.3	0/0/0	192.9	0	22.4/44.6	100.2/44.6	2.4	2.9
lsnr	p=250	0.4/0.1/0.5	193.5/193.5/193.5/0.6	0/0/0	193.5	0	9.4/12.1	67.6/12.1	8.7	6.7
	p=500	2.4/0.2/0.5	194.8/194.8/194.8/0.5	0/0/0	194.8	0	10.8/15.6	114.8/15.6	12.2	9.3
	p=1000	5.2/1.3/0.4	195.9/195.9/195.9/0.4	0/0/0	195.8	0	10.8/18	115.1/18	12.8	10.6
	p=1500	5.2/2.3/0.3	196.9/196.9/196.9/0.3	0/0/0	196.8	0	9/13.3	113.5/13.3	11.4	9.6

Table S39: The performance of BOSS for high dimensional data, Sparse-Ex1, ρ=0.9, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	5/4/39	447/447/447/39	3/25/4	447	22	108/107	201/107	22	25
	p=500	7/5/40	473/473/473/42	4/17/6	473	16	135/134	341/134	31	26
	p=1000	18/7/50	447/447/447/51	3/46/6	447	43	143/141	360/141	30	22
	p=1500	37/9/48	459/459/459/49	5/27/7	459	26	176/173	389/173	54	30
msnr	p=250	10/9/19	272/272/272/19	14/199/18	272	199	41/40	170/40	25	20
	p=500	12/8/13	262/262/262/14	7/232/14	262	232	48/47	239/47	27	11
	p=1000	21/12/13	242/242/242/15	7/291/18	242	291	52/50	226/50	26	5
	p=1500	41/21/14	237/237/237/15	11/273/58	237	273	66/64	223/64	26	31
lsnr	p=250	11/11/11	164/164/164/10	11/11/11	164	11	-13/-12	97/-12	-8	-8
	p=500	14/12/12	155/155/155/12	13/13/13	155	13	-9/-9	149/-9	-5	-5
	p=1000	18/11/10	139/139/139/10	9/9/9	139	9	-9/-8	136/-8	-5	-4
	p=1500	19/7/6	151/151/151/6	2/2/2	151	2	-5/-3	147/-3	-1	0
		Relative efficiency								
hsnr	p=250	0.98/0.99/0.74	0.19/0.19/0.19/0.74	1/0.82/0.99	0.19	0.85	0.49/0.5	0.34/0.5	0.84	0.83
	p=500	0.97/0.99/0.75	0.18/0.18/0.18/0.74	1/0.89/0.99	0.18	0.9	0.44/0.45	0.24/0.45	0.8	0.83
	p=1000	0.87/0.96/0.69	0.19/0.19/0.19/0.68	1/0.71/0.97	0.19	0.72	0.42/0.43	0.22/0.43	0.8	0.84
	p=1500	0.77/0.97/0.71	0.19/0.19/0.19/0.71	1/0.83/0.99	0.19	0.83	0.38/0.38	0.21/0.38	0.68	0.81
msnr	p=250	0.99/1/0.91	0.29/0.29/0.29/0.91	0.96/0.36/0.92	0.29	0.36	0.77/0.78	0.4/0.78	0.87	0.9
	p=500	0.96/0.99/0.95	0.3/0.3/0.3/0.94	1/0.32/0.94	0.3	0.32	0.72/0.73	0.32/0.73	0.85	0.97
	p=1000	0.87/0.93/0.93	0.31/0.31/0.31/0.91	0.98/0.27/0.89	0.31	0.27	0.69/0.7	0.32/0.7	0.83	1
	p=1500	0.79/0.92/0.98	0.33/0.33/0.33/0.97	1/0.3/0.71	0.33	0.3	0.67/0.68	0.34/0.68	0.88	0.85
lsnr	p=250	0.78/0.78/0.78	0.33/0.33/0.33/0.79	0.78/0.78/0.78	0.33	0.78	1/0.99	0.44/0.99	0.95	0.95
	p=500	0.79/0.81/0.81	0.36/0.36/0.36/0.81	0.81/0.8/0.8	0.36	0.8	1/0.99	0.36/0.99	0.95	0.95
	p=1000	0.77/0.82/0.83	0.38/0.38/0.38/0.82	0.83/0.83/0.83	0.38	0.83	1/0.98	0.39/0.98	0.95	0.95
	p=1500	0.8/0.89/0.9	0.38/0.38/0.38/0.9	0.93/0.94/0.94	0.38	0.94	1/0.98	0.38/0.98	0.96	0.95
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/5.8/6	6	5.8	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/5.9/6	6	5.9	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/5.7/5.9	6	5.7	6/6	5.9/6	6	6
	p=1500	6/6/6	6/6/6/6	6/5.8/5.9	6	5.8	6/6	5.9/6	5.9	6
msnr	p=250	5/5/5	5.9/5.9/5.9/5.1	4.9/2.1/4.8	5.9	2.1	5.9/5.9	5.2/5.9	5	5.6
	p=500	5/5/5	5.6/5.6/5.6/5	4.9/1.8/4.8	5.6	1.7	5.9/5.9	4.9/5.9	4.8	5.7
	p=1000	4.9/4.9/4.9	5.4/5.4/5.4/4.9	4.8/0.6/4.7	5.4	0.6	5.9/5.9	4.9/5.9	4.7	5.7
	p=1500	4.9/4.9/4.8	5.2/5.2/5.2/4.8	4.7/0/3.9	5.2	0	5.8/5.8	4.9/5.8	4.6	5.4
lsnr	p=250	0.4/0.2/0.6	5.3/5.3/5.3/0.7	0.1/0/0.1	5.3	0	2.4/2.5	2.9/2.5	2.2	2.1
	p=500	0.6/0.3/0.6	3.6/3.6/3.6/0.6	0.1/0/0.1	3.6	0	2.2/2.4	2.6/2.4	2	1.9
	p=1000	0.7/0.3/0.4	2.6/2.6/2.6/0.4	0.1/0/0	2.6	0	2/2.2	2.2/2.2	1.8	1.7
	p=1500	0.4/0.1/0.2	1.9/1.9/1.9/0.2	0/0/0	1.9	0	1/1	1.6/1	0.9	0.8
		Number of extra variables								
hsnr	p=250	0.5/0.5/2.8	193/193/193/1.3	0.5/0.5/0.5	193	0.2	17.9/21.4	35.7/21.4	3.9	2.7
	p=500	0.3/0.2/1.4	193/193/193/0.9	0.2/0.2/0.2	193	0.1	19.5/25.2	72.1/25.2	5.1	1.6
	p=1000	1.2/0.6/1.6	193/193/193/1.3	0.6/0.5/0.5	193	0.3	21.2/29.9	85.1/29.9	7	2.3
	p=1500	2/0.3/1.1	193/193/193/1	0.2/0.2/0.2	193	0.2	25.2/40.1	91.9/40.1	10	1.9
msnr	p=250	1.6/1.4/2.9	193.1/193.1/193.1/2	1.2/0.5/1.2	193.1	0.5	17.9/21.4	57/21.4	6.9	6.8
	p=500	1.7/1.3/1.7	193.4/193.4/193.4/1.6	1.1/0.3/1	193.4	0.3	19.4/25	103.4/25	7.3	5.3
	p=1000	2.5/1.7/1.7	193.6/193.6/193.6/1.7	1.2/0.2/1.2	193.6	0.2	21.1/29.8	107/29.8	7.8	4.7
	p=1500	4.6/2.2/1.7	193.7/193.7/193.7/1.7	1.2/0/0.9	193.7	0	25.1/39.7	107.2/39.7	7.5	8.5
lsnr	p=250	0.9/0.4/1.6	193.7/193.7/193.7/1.5	0.2/0/0.1	193.7	0	12.2/13.8	62.3/13.8	11.5	9.6
	p=500	1.7/0.5/1.5	195.4/195.4/195.4/1.2	0.1/0/0	195.4	0	12.7/16	119.1/16	12.9	10.2
	p=1000	3.3/0.9/1	196.4/196.4/196.4/0.9	0.1/0/0	196.4	0	13.8/18	120.8/18	14	10.6
	p=1500	3.3/0.8/0.5	197.1/197.1/197.1/0.5	0/0/0	197	0	10.8/14.5	119.7/14.5	12.4	9.2

Table S40: The performance of BOSS for high dimensional data, Sparse-Ex2, ρ=0, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	5/1/18	501/501/501/18	2/0/1	501	0	152/152	189/152	18	16
	p=500	30/3/18	501/501/501/19	2/0/1	501	0	178/179	366/179	23	17
	p=1000	86/11/20	501/501/501/20	2/0/0	501	0	203/204	420/204	25	14
	p=1500	132/40/19	501/501/501/21	2/0/0	501	0	218/218	432/218	26	15
msnr	p=250	29/16/19	498/498/498/20	5/419/13	498	419	151/151	311/151	18	60
	p=500	83/38/22	495/495/495/23	10/514/39	495	514	175/176	448/176	23	87
	p=1000	162/88/26	491/491/491/27	18/538/89	491	538	197/200	462/200	27	117
	p=1500	214/151/27	485/485/485/29	26/536/142	485	536	210/211	460/211	26	140
lsnr	p=250	10/8/9	169/169/169/9	8/7/8	169	7	-3/0	125/0	3	3
	p=500	16/6/8	162/162/162/8	5/4/5	162	4	-2/1	156/1	4	4
	p=1000	27/9/6	158/158/158/6	3/3/3	158	3	-1/3	155/3	5	5
	p=1500	36/20/6	156/156/156/6	2/2/2	156	2	-2/2	153/2	4	6
		Relative efficiency								
hsnr	p=250	0.95/0.99/0.85	0.17/0.17/0.17/0.85	0.98/1/0.99	0.17	1	0.4/0.4	0.35/0.4	0.84	0.86
	p=500	0.77/0.97/0.85	0.17/0.17/0.17/0.84	0.98/1/0.99	0.17	1	0.36/0.36	0.21/0.36	0.81	0.86
	p=1000	0.54/0.9/0.84	0.17/0.17/0.17/0.83	0.98/1/1	0.17	1	0.33/0.33	0.19/0.33	0.8	0.87
	p=1500	0.43/0.72/0.84	0.17/0.17/0.17/0.83	0.98/1/1	0.17	1	0.31/0.31	0.19/0.31	0.79	0.87
msnr	p=250	0.81/0.9/0.88	0.18/0.18/0.18/0.88	1/0.2/0.93	0.18	0.2	0.42/0.42	0.26/0.42	0.89	0.66
	p=500	0.6/0.8/0.9	0.18/0.18/0.18/0.89	1/0.18/0.79	0.18	0.18	0.4/0.4	0.2/0.4	0.89	0.59
	p=1000	0.45/0.63/0.94	0.2/0.2/0.2/0.93	1/0.19/0.63	0.2	0.19	0.4/0.39	0.21/0.39	0.93	0.54
	p=1500	0.4/0.5/0.99	0.22/0.22/0.22/0.98	1/0.2/0.52	0.22	0.2	0.41/0.4	0.22/0.4	1	0.52
lsnr	p=250	0.88/0.9/0.89	0.36/0.36/0.36/0.89	0.9/0.91/0.9	0.36	0.91	1/0.98	0.43/0.98	0.95	0.95
	p=500	0.85/0.93/0.91	0.38/0.38/0.38/0.91	0.94/0.94/0.94	0.38	0.94	1/0.97	0.38/0.97	0.94	0.94
	p=1000	0.78/0.91/0.93	0.38/0.38/0.38/0.93	0.96/0.96/0.96	0.38	0.96	1/0.96	0.39/0.96	0.94	0.94
	p=1500	0.73/0.82/0.93	0.38/0.38/0.38/0.93	0.96/0.97/0.96	0.38	0.97	1/0.96	0.39/0.96	0.95	0.93
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/1.7/5.9	6	1.7	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	5.9/0.5/5.7	6	0.5	6/6	6/6	6	5.9
	p=1000	6/6/6	6/6/6/6	5.8/0.1/5.2	6	0.1	6/6	6/6	6	5.8
	p=1500	6/6/6	6/6/6/6	5.8/0/4.6	6	0	6/6	6/6	6	5.7
lsnr	p=250	0.4/0.2/0.7	5.5/5.5/5.5/0.7	0.1/0/0.1	5.5	0	2.4/2.3	4.3/2.3	2.1	1.9
	p=500	0.5/0.2/0.4	4/4/4/0.4	0.1/0/0	4	0	1.7/1.8	3.5/1.8	1.6	1.5
	p=1000	0.5/0.2/0.3	2.7/2.7/2.7/0.3	0.1/0/0	2.7	0	1.3/1.3	2.6/1.3	1.2	1.2
	p=1500	0.6/0.3/0.2	2.1/2.1/2.1/0.2	0/0/0	2.1	0	1.1/1.1	2.1/1.1	1	0.9
		Number of extra variables								
hsnr	p=250	0.2/0/0.3	193/193/193/0.3	0/0/0	193	0	18.2/26.6	29.1/26.6	2.5	0.6
	p=500	1.7/0/0.3	193/193/193/0.3	0/0/0	193	0	20.5/35	75/35	3.5	0.5
	p=1000	4.7/0.3/0.3	193/193/193/0.3	0/0/0	193	0	22.3/44.2	88.9/44.2	4.2	0.4
	p=1500	7.1/1.9/0.2	193/193/193/0.3	0/0/0	192.9	0	23/48.8	90.4/48.8	4.4	0.4
msnr	p=250	0.5/0.2/0.3	193/193/193/0.3	0/0/0	193	0	18.2/26.6	49.3/26.6	1.6	2.3
	p=500	3/0.6/0.3	193/193/193/0.4	0/0/0	193	0	20.7/34.9	98.2/34.9	2.1	3.7
	p=1000	6.6/1.9/0.4	193/193/193/0.4	0/0/0	193	0	22.4/44.3	100.9/44.3	2.6	4.6
	p=1500	9.3/4.8/0.4	193/193/193/0.4	0/0/0	192.9	0	23/48.6	100/48.6	2.5	6.2
lsnr	p=250	0.5/0/0.4	193.5/193.5/193.5/0.4	0/0/0	193.5	0	8.9/11	69.1/11	8.1	6.5
	p=500	2.1/0.1/0.3	195/195/195/0.3	0/0/0	195	0	8.5/11.9	114.5/11.9	9.6	7.6
	p=1000	3.9/0.7/0.3	196.3/196.3/196.3/0.3	0/0/0	196.3	0	8.5/13.5	114/13.5	10.6	8.7
	p=1500	5.6/2.6/0.2	196.9/196.9/196.9/0.3	0/0/0	196.9	0	8.2/12.9	112.5/12.9	10.4	8.9

Table S41: The performance of BOSS for high dimensional data, Sparse-Ex2, ρ=0.5, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	6/1/18	501/501/501/18	2/0/1	501	0	214/206	125/206	18	27
	p=500	40/3/18	501/501/501/19	2/0/1	501	0	259/245	269/245	24	37
	p=1000	110/16/20	501/501/501/22	2/0/0	501	0	304/285	373/285	26	55
	p=1500	154/49/21	501/501/501/25	2/0/0	501	0	337/306	391/306	27	75
msnr	p=250	30/13/34	457/457/457/43	3/382/6	457	382	191/184	232/184	12	140
	p=500	74/26/41	412/412/412/62	8/437/27	412	437	206/195	347/195	9	177
	p=1000	113/57/44	331/331/331/66	20/368/73	331	368	195/179	298/179	3	180
	p=1500	104/80/42	263/263/263/67	29/296/96	263	296	172/153	240/153	-2	157
lsnr	p=250	7/5/8	162/162/162/9	5/5/5	162	5	3/4	113/4	7	7
	p=500	9/3/6	155/155/155/6	2/2/2	155	2	1/3	147/3	5	5
	p=1000	16/5/5	153/153/153/5	1/1/1	153	1	1/3	148/3	4	5
	p=1500	26/13/4	152/152/152/5	1/0/0	152	0	0/2	148/2	3	5
		Relative efficiency								
hsnr	p=250	0.94/0.99/0.85	0.17/0.17/0.17/0.85	0.98/1/0.99	0.17	1	0.32/0.33	0.44/0.33	0.85	0.79
	p=500	0.72/0.97/0.85	0.17/0.17/0.17/0.84	0.98/1/0.99	0.17	1	0.28/0.29	0.27/0.29	0.8	0.73
	p=1000	0.48/0.86/0.83	0.17/0.17/0.17/0.82	0.98/1/1	0.17	1	0.25/0.26	0.21/0.26	0.79	0.65
	p=1500	0.39/0.67/0.83	0.17/0.17/0.17/0.8	0.98/1/1	0.17	1	0.23/0.25	0.2/0.25	0.79	0.57
msnr	p=250	0.8/0.91/0.77	0.19/0.19/0.19/0.72	1/0.21/0.97	0.19	0.21	0.36/0.36	0.31/0.36	0.92	0.43
	p=500	0.62/0.86/0.76	0.21/0.21/0.21/0.67	1/0.2/0.85	0.21	0.2	0.35/0.37	0.24/0.37	0.99	0.39
	p=1000	0.48/0.66/0.72	0.24/0.24/0.24/0.62	0.86/0.22/0.6	0.24	0.22	0.35/0.37	0.26/0.37	1	0.37
	p=1500	0.48/0.54/0.69	0.27/0.27/0.27/0.59	0.76/0.25/0.5	0.27	0.25	0.36/0.39	0.29/0.39	1	0.38
lsnr	p=250	0.97/0.98/0.96	0.39/0.39/0.39/0.95	0.98/0.99/0.98	0.39	0.99	1/0.99	0.49/0.99	0.97	0.97
	p=500	0.93/0.99/0.96	0.4/0.4/0.4/0.95	0.99/0.99/0.99	0.4	0.99	1/0.99	0.41/0.99	0.96	0.96
	p=1000	0.87/0.96/0.96	0.4/0.4/0.4/0.96	1/1/1	0.4	1	1/0.98	0.41/0.98	0.96	0.96
	p=1500	0.8/0.89/0.96	0.4/0.4/0.4/0.96	0.99/1/1	0.4	1	1/0.98	0.4/0.98	0.97	0.95
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/1.5/5.9	6	1.5	6/6	6/6	6	5.9
	p=500	6/6/6	6/6/6/5.9	5.9/0.3/5.6	6	0.3	5.9/6	6/6	6	5.8
	p=1000	5.9/5.9/5.9	6/6/6/5.7	5.6/0/4.7	6	0	5.7/5.9	6/5.9	6	5.6
	p=1500	5.8/5.8/5.7	5.9/5.9/5.9/5.3	5.1/0/3.8	5.9	0	5.3/5.6	5.9/5.6	6	5.2
lsnr	p=250	0.2/0.1/0.5	5.5/5.5/5.5/0.3	0/0/0	5.5	0	1/1	4/1	1	0.9
	p=500	0.2/0.1/0.2	3.6/3.6/3.6/0.2	0/0/0	3.6	0	0.7/0.7	2.8/0.7	0.7	0.6
	p=1000	0.2/0.1/0.1	2.1/2.1/2.1/0.1	0/0/0	2.1	0	0.5/0.5	1.8/0.5	0.5	0.4
	p=1500	0.2/0.1/0.1	1.5/1.5/1.5/0.1	0/0/0	1.5	0	0.4/0.4	1.3/0.4	0.4	0.3
		Number of extra variables								
hsnr	p=250	0.2/0/0.3	193/193/193/0.3	0/0/0	193	0	27.9/43.8	19.2/43.8	2.7	1
	p=500	2.5/0/0.3	193/193/193/0.3	0/0/0	193	0	31.8/58.1	53.8/58.1	3.9	1.3
	p=1000	6.9/0.7/0.3	193/193/193/0.3	0/0/0	193	0	35.4/76.2	80.3/76.2	4.5	2
	p=1500	8.6/2.4/0.3	193/193/193/0.3	0/0/0	192.9	0	36.9/86	83.5/86	5	3.2
msnr	p=250	0.9/0.3/0.6	193/193/193/1	0.1/0/0.1	193	0	27.7/43.8	39.5/43.8	2.4	10.9
	p=500	4.3/0.8/0.9	193/193/193/1.6	0.2/0/0.1	193	0	31.4/57.4	89.5/57.4	4.1	20.8
	p=1000	9.2/3/1.3	193/193/193/2.1	0.3/0/0.1	193	0	32/71.9	97.2/71.9	7.6	35.4
	p=1500	10.9/6.7/1.7	193.1/193.1/193.1/2.2	0.3/0/0.1	193	0	29.5/74.2	97.6/74.2	11.5	42.2
lsnr	p=250	0.3/0/0.6	193.5/193.5/193.5/0.4	0/0/0	193.5	0	6.1/7.2	65.5/7.2	6.3	5.1
	p=500	1.2/0.1/0.3	195.4/195.4/195.4/0.3	0/0/0	195.4	0	6.6/7.7	111.7/7.7	7.1	5.6
	p=1000	2.5/0.5/0.3	196.9/196.9/196.9/0.3	0/0/0	196.9	0	6.5/9.2	112.7/9.2	7.9	5.9
	p=1500	4/1.7/0.2	197.5/197.5/197.5/0.3	0/0/0	197.4	0	6.8/9.4	111/9.4	7.4	6.5

Table S42: The performance of BOSS for high dimensional data, Sparse-Ex2, ρ=0.9, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	18/13/68	225/225/225/128	37/137/49	225	116	186/147	-7/147	-36	116
	p=500	106/69/105	70/70/70/141	96/180/111	70	174	275/234	-31/234	-30	221
	p=1000	97/68/108	22/22/22/129	91/159/106	22	158	177/178	21/178	158	176
	p=1500	91/84/116	7/7/7/132	111/147/122	7	147	137/133	8/133	134	131
msnr	p=250	23/21/18	68/68/68/61	57/82/64	68	82	75/70	-4/70	-9	70
	p=500	14/14/15	21/21/21/26	24/32/27	21	32	27/27	6/27	27	27
	p=1000	11/11/13	11/11/11/19	18/22/20	11	22	18/18	2/18	19	18
	p=1500	10/10/12	7/7/7/15	15/17/16	7	17	12/12	-2/12	12	12
lsnr	p=250	3/3/6	157/157/157/7	3/3/3	157	3	3/4	83/4	5	6
	p=500	5/1/5	153/153/153/5	1/1/1	153	1	2/2	132/2	4	5
	p=1000	12/2/4	151/151/151/5	0/0/0	151	0	1/2	138/2	4	5
	p=1500	21/9/4	151/151/151/5	0/0/0	151	0	1/2	140/2	3	5
		Relative efficiency								
hsnr	p=250	0.54/0.56/0.38	0.2/0.2/0.2/0.28	0.47/0.27/0.43	0.2	0.3	0.22/0.26	0.69/0.26	1	0.3
	p=500	0.34/0.41/0.34	0.41/0.41/0.41/0.29	0.35/0.25/0.33	0.41	0.25	0.18/0.21	1/0.21	0.98	0.21
	p=1000	0.61/0.72/0.58	0.99/0.99/0.99/0.53	0.63/0.47/0.59	0.99	0.47	0.44/0.43	1/0.43	0.47	0.44
	p=1500	0.56/0.58/0.49	1/1/1/0.46	0.51/0.43/0.48	1	0.43	0.45/0.46	0.98/0.46	0.45	0.46
msnr	p=250	0.74/0.76/0.77	0.54/0.54/0.54/0.57	0.58/0.5/0.56	0.54	0.5	0.52/0.54	0.95/0.54	1	0.54
	p=500	0.93/0.93/0.92	0.88/0.88/0.88/0.84	0.85/0.8/0.83	0.88	0.8	0.83/0.83	1/0.83	0.84	0.83
	p=1000	0.92/0.91/0.9	0.91/0.91/0.91/0.86	0.86/0.83/0.85	0.91	0.83	0.86/0.86	1/0.86	0.86	0.86
	p=1500	0.89/0.89/0.87	0.92/0.92/0.92/0.85	0.85/0.84/0.84	0.92	0.84	0.87/0.87	1/0.87	0.87	0.87
lsnr	p=250	0.99/1/0.97	0.4/0.4/0.4/0.96	1/1/1	0.4	1	0.99/0.99	0.56/0.99	0.97	0.97
	p=500	0.96/1/0.96	0.4/0.4/0.4/0.96	1/1/1	0.4	1	0.99/0.99	0.43/0.99	0.97	0.96
	p=1000	0.9/0.98/0.96	0.4/0.4/0.4/0.96	1/1/1	0.4	1	0.99/0.98	0.42/0.98	0.97	0.96
	p=1500	0.83/0.92/0.96	0.4/0.4/0.4/0.96	1/1/1	0.4	1	1/0.98	0.42/0.98	0.97	0.95
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/5.8	5.6/4.7/5.5	6	4.7	6/6	6/6	6	6
	p=500	4.7/4.7/4.7	5.4/5.4/5.4/3.4	3.9/2.2/3.6	5.4	2.2	1.4/2.2	6/2.2	6	2.3
	p=1000	3.7/3.7/3.5	4.3/4.3/4.3/1.9	2.9/0.8/2.5	4.3	0.8	0.8/0.7	3.1/0.7	1.2	0.8
	p=1500	2.4/2.4/2.1	3/3/3/0.8	1.5/0.1/1.1	3	0.1	0.8/0.8	1.9/0.8	0.8	0.8
msnr	p=250	4.1/3.9/4.5	5.8/5.8/5.8/1.6	1.5/0.1/1.1	5.8	0.1	1/1.5	5.8/1.5	5.7	1.5
	p=500	1.9/1.8/1.9	4/4/4/0.6	0.7/0/0.4	4	0	0.7/0.6	2.5/0.6	0.7	0.5
	p=1000	1.3/1.2/1.1	2.6/2.6/2.6/0.4	0.4/0/0.2	2.6	0	0.4/0.4	1.7/0.4	0.4	0.4
	p=1500	0.8/0.8/0.6	1.8/1.8/1.8/0.2	0.2/0/0.1	1.8	0	0.4/0.4	1.3/0.4	0.3	0.3
lsnr	p=250	0.1/0.1/0.2	4.9/4.9/4.9/0.1	0/0/0	4.9	0	0.2/0.2	1.4/0.2	0.2	0.2
	p=500	0.1/0/0	2.5/2.5/2.5/0	0/0/0	2.5	0	0.2/0.1	1.3/0.1	0.2	0.1
	p=1000	0.1/0/0	1.3/1.3/1.3/0	0/0/0	1.3	0	0.1/0.1	0.8/0.1	0.1	0.1
	p=1500	0/0/0	0.9/0.9/0.9/0	0/0/0	0.9	0	0.1/0.1	0.5/0.1	0.1	0.1
		Number of extra variables								
hsnr	p=250	9.4/6.3/11.3	193/193/193/23.8	4/2/3.6	193	0	68.2/108.6	9.5/108.6	4.2	49.7
	p=500	12.7/9/11.2	193.6/193.6/193.6/10.4	3.4/1/2.6	193.6	0	9.6/35.3	33.7/35.3	56.3	36.3
	p=1000	6.3/6.8/5.8	194.7/194.7/194.7/3.5	2.3/0.3/1.5	194.6	0	6.1/6.3	89/6.3	16.4	4.9
	p=1500	6.4/5.7/3.7	196/196/196/1.1	1.1/0/0.6	195.9	0	8.6/13.4	94/13.4	11.3	9.9
msnr	p=250	13.6/10.4/16.5	193.2/193.2/193.2/2.9	0.5/0/0.2	193.2	0	6.8/19.3	52.7/19.3	65.4	16.7
	p=500	3.8/3.2/4.4	195/195/195/0.9	0.2/0/0.1	195	0	5.9/7.3	82.8/7.3	7.7	5.1
	p=1000	2.9/2.1/1.8	196.4/196.3/196.4/0.4	0.1/0/0	196.3	0	5.5/6.2	95.5/6.2	5.2	4.8
	p=1500	3.9/2.7/1	197.2/197.2/197.2/0.4	0/0/0	197.1	0	6.8/8.8	97.8/8.8	7.3	6.1
lsnr	p=250	0.2/0.1/0.6	194.1/194.1/194.1/0.3	0/0/0	194.1	0	4.5/4.4	51.1/4.4	4.2	3.3
	p=500	0.7/0.1/0.3	196.5/196.5/196.5/0.3	0/0/0	196.5	0	5.5/5.6	102.5/5.6	5.4	4.5
	p=1000	1.9/0.3/0.3	197.7/197.7/197.7/0.3	0/0/0	197.7	0	5.6/7.5	105.8/7.5	6	5.2
	p=1500	3.4/1.3/0.2	198.1/198.1/198.1/0.3	0/0/0	198.1	0	6.1/7.7	105.5/7.7	6.3	6

Table S43: The performance of BOSS for high dimensional data, Sparse-Ex3, ρ=0, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	6/1/18	501/501/501/18	2/0/1	501	0	131/134	189/134	18	15
	p=500	24/2/18	501/501/501/19	2/0/1	501	0	151/155	364/155	23	17
	p=1000	74/12/20	501/501/501/20	2/0/0	501	0	173/179	417/179	24	16
	p=1500	128/40/19	501/501/501/21	2/0/0	501	0	185/193	430/193	27	15
msnr	p=250	21/10/19	498/498/498/19	6/221/10	498	223	130/133	313/133	20	20
	p=500	57/24/20	496/496/496/21	9/383/21	496	385	149/153	448/153	25	26
	p=1000	134/63/24	494/494/494/24	13/509/45	494	510	170/177	464/177	29	36
	p=1500	194/121/24	493/493/493/26	17/553/64	493	553	181/190	467/190	29	42
lsnr	p=250	18/18/15	168/168/168/15	19/24/21	168	24	-4/0	124/0	3	2
	p=500	24/14/13	154/154/154/12	14/18/16	154	18	-3/0	148/0	4	3
	p=1000	37/18/10	144/144/144/10	11/13/12	144	13	-3/2	140/2	4	4
	p=1500	46/29/9	139/139/139/10	9/11/10	139	11	-3/3	137/3	5	5
		Relative efficiency								
hsnr	p=250	0.95/0.99/0.85	0.17/0.17/0.17/0.85	0.98/1/0.99	0.17	1	0.43/0.43	0.35/0.43	0.85	0.87
	p=500	0.81/0.98/0.85	0.17/0.17/0.17/0.84	0.98/1/0.99	0.17	1	0.4/0.39	0.22/0.39	0.81	0.86
	p=1000	0.58/0.9/0.84	0.17/0.17/0.17/0.83	0.98/1/1	0.17	1	0.37/0.36	0.19/0.36	0.8	0.86
	p=1500	0.44/0.72/0.84	0.17/0.17/0.17/0.83	0.98/1/1	0.17	1	0.35/0.34	0.19/0.34	0.79	0.87
msnr	p=250	0.88/0.96/0.89	0.18/0.18/0.18/0.89	1/0.33/0.96	0.18	0.33	0.46/0.46	0.26/0.46	0.88	0.88
	p=500	0.69/0.88/0.91	0.18/0.18/0.18/0.91	1/0.23/0.91	0.18	0.23	0.44/0.43	0.2/0.43	0.88	0.87
	p=1000	0.48/0.69/0.91	0.19/0.19/0.19/0.91	1/0.19/0.78	0.19	0.19	0.42/0.41	0.2/0.41	0.88	0.83
	p=1500	0.4/0.53/0.94	0.2/0.2/0.2/0.93	1/0.18/0.71	0.2	0.18	0.42/0.4	0.21/0.4	0.91	0.83
lsnr	p=250	0.82/0.82/0.84	0.36/0.36/0.36/0.84	0.81/0.78/0.8	0.36	0.78	1/0.97	0.43/0.97	0.94	0.94
	p=500	0.78/0.84/0.86	0.38/0.38/0.38/0.86	0.84/0.82/0.83	0.38	0.82	1/0.97	0.39/0.97	0.93	0.94
	p=1000	0.71/0.82/0.88	0.4/0.4/0.4/0.88	0.88/0.86/0.87	0.4	0.86	1/0.95	0.4/0.95	0.94	0.94
	p=1500	0.67/0.75/0.89	0.41/0.41/0.41/0.89	0.89/0.88/0.88	0.41	0.88	1/0.95	0.41/0.95	0.93	0.93
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/4.2/5.9	6	4.2	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	5.9/2.7/5.9	6	2.7	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	5.9/1.4/5.7	6	1.4	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	5.9/0.9/5.5	6	0.9	6/6	6/6	6	6
lsnr	p=250	1.2/0.7/1.4	5.6/5.6/5.6/1.4	0.5/0/0.3	5.6	0	3.9/3.9	4.5/3.9	3.3	3.1
	p=500	1.3/0.7/1	4.4/4.4/4.4/1	0.3/0/0.1	4.4	0	3.3/3.4	4/3.4	2.8	2.6
	p=1000	1.3/0.8/0.7	3.2/3.2/3.2/0.7	0.2/0/0.1	3.2	0	2.6/2.8	3.1/2.8	2.3	2.2
	p=1500	1.3/0.9/0.6	2.6/2.6/2.6/0.6	0.2/0/0.1	2.6	0	2.4/2.4	2.7/2.4	2	2
		Number of extra variables								
hsnr	p=250	0.4/0/0.3	193/193/193/0.3	0/0/0	193	0	15.1/22.3	29.1/22.3	2.5	0.5
	p=500	1.4/0/0.3	193/193/193/0.3	0/0/0	193	0	17.1/27.7	74.4/27.7	3.6	0.8
	p=1000	3.9/0.4/0.3	193/193/193/0.3	0/0/0	193	0	19.2/35.4	87.2/35.4	4.2	0.8
	p=1500	6.8/1.9/0.2	193/193/193/0.3	0/0/0	192.9	0	19.3/40.1	89.7/40.1	4.7	0.8
msnr	p=250	0.6/0.1/0.3	193/193/193/0.3	0/0/0	193	0	15.1/22.3	49.9/22.3	1.7	0.6
	p=500	2/0.3/0.3	193/193/193/0.3	0/0/0	193	0	17.2/27.7	97.2/27.7	2.4	1.2
	p=1000	5.4/1.4/0.3	193/193/193/0.3	0/0/0	193	0	18.9/35.8	100.1/35.8	2.7	1.3
	p=1500	8.3/3.8/0.3	193/193/193/0.3	0/0/0	192.9	0	19.4/40.4	99.6/40.4	2.6	1.8
lsnr	p=250	1/0.1/0.5	193.4/193.4/193.4/0.5	0/0/0	193.4	0	10.8/14.6	69.8/14.6	10.3	7
	p=500	2.6/0.2/0.5	194.6/194.6/194.6/0.4	0/0/0	194.6	0	11.3/15.7	113.8/15.7	11	8.2
	p=1000	6.1/1.5/0.3	195.8/195.8/195.8/0.3	0/0/0	195.8	0	11.2/18.9	113.7/18.9	13.5	10.2
	p=1500	8/4/0.3	196.3/196.3/196.3/0.3	0/0/0	196.3	0	11.1/20	111.8/20	14.5	11.4

Table S44: The performance of BOSS for high dimensional data, Sparse-Ex3, ρ=0.5, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	6/1/16	495/495/495/19	2/0/1	495	0	135/137	182/137	19	16
	p=500	24/2/18	495/495/495/20	2/0/1	495	0	156/159	356/159	24	16
	p=1000	79/11/19	495/495/495/20	2/0/0	495	0	179/185	412/185	25	15
	p=1500	124/44/18	495/495/495/20	2/0/0	495	0	191/199	425/199	26	14
msnr	p=250	22/11/18	492/492/492/20	4/234/8	492	236	134/136	304/136	19	24
	p=500	61/26/19	490/490/490/22	7/408/19	490	409	153/157	441/157	24	33
	p=1000	133/62/23	488/488/488/24	11/515/41	488	515	176/182	458/182	27	46
	p=1500	190/122/22	486/486/486/24	15/549/63	486	549	187/195	460/195	27	56
lsnr	p=250	17/17/14	166/166/166/14	17/21/19	166	21	-3/0	122/0	4	3
	p=500	23/13/12	153/153/153/12	13/15/14	153	15	-3/1	148/1	4	3
	p=1000	35/17/10	144/144/144/10	9/11/10	144	11	-3/2	141/2	4	4
	p=1500	44/28/9	140/140/140/9	8/9/9	140	9	-2/3	138/3	5	6
		Relative efficiency								
hsnr	p=250	0.95/0.99/0.86	0.17/0.17/0.17/0.84	0.98/1/0.99	0.17	1	0.43/0.42	0.35/0.42	0.84	0.86
	p=500	0.8/0.98/0.85	0.17/0.17/0.17/0.83	0.98/1/0.99	0.17	1	0.39/0.39	0.22/0.39	0.81	0.86
	p=1000	0.56/0.9/0.84	0.17/0.17/0.17/0.84	0.98/1/1	0.17	1	0.36/0.35	0.2/0.35	0.8	0.87
	p=1500	0.45/0.7/0.85	0.17/0.17/0.17/0.84	0.98/1/1	0.17	1	0.34/0.33	0.19/0.33	0.79	0.88
msnr	p=250	0.86/0.94/0.89	0.18/0.18/0.18/0.87	1/0.31/0.96	0.18	0.31	0.45/0.44	0.26/0.44	0.87	0.84
	p=500	0.67/0.85/0.9	0.18/0.18/0.18/0.88	1/0.21/0.9	0.18	0.21	0.42/0.42	0.2/0.42	0.87	0.81
	p=1000	0.48/0.68/0.9	0.19/0.19/0.19/0.9	1/0.18/0.79	0.19	0.18	0.4/0.39	0.2/0.39	0.88	0.76
	p=1500	0.39/0.52/0.94	0.2/0.2/0.2/0.92	1/0.18/0.7	0.2	0.18	0.4/0.39	0.2/0.39	0.9	0.73
lsnr	p=250	0.83/0.83/0.85	0.36/0.36/0.36/0.85	0.83/0.8/0.82	0.36	0.8	1/0.97	0.44/0.97	0.94	0.95
	p=500	0.79/0.86/0.87	0.38/0.38/0.38/0.87	0.86/0.84/0.85	0.38	0.84	1/0.96	0.39/0.96	0.94	0.94
	p=1000	0.72/0.83/0.89	0.4/0.4/0.4/0.89	0.89/0.88/0.88	0.4	0.88	1/0.96	0.41/0.96	0.94	0.93
	p=1500	0.68/0.76/0.9	0.41/0.41/0.41/0.9	0.9/0.89/0.9	0.41	0.89	1/0.95	0.41/0.95	0.93	0.92
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	6/6/6	6	6	6/6	6/6	6	6
msnr	p=250	6/6/6	6/6/6/6	6/4/5.9	6	4	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/2.3/5.9	6	2.3	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	5.9/1.2/5.7	6	1.2	6/6	6/6	6	6
	p=1500	6/6/6	6/6/6/6	5.9/0.8/5.5	6	0.8	6/6	6/6	6	5.9
lsnr	p=250	1.1/0.6/1.2	5.5/5.5/5.5/1.3	0.4/0/0.2	5.5	0	3.6/3.6	4.3/3.6	3	2.9
	p=500	1.2/0.5/0.9	4.2/4.2/4.2/0.9	0.3/0/0.1	4.2	0	3/3	3.8/3	2.6	2.4
	p=1000	1.1/0.7/0.6	3/3/3/0.6	0.2/0/0.1	3	0	2.3/2.4	2.9/2.4	2.1	2
	p=1500	1.2/0.8/0.5	2.5/2.5/2.5/0.5	0.2/0/0	2.5	0	2.1/2.2	2.5/2.2	1.9	1.7
		Number of extra variables								
hsnr	p=250	0.3/0/0.3	193/193/193/0.3	0/0/0	193	0	15.8/23.3	28.2/23.3	2.6	0.5
	p=500	1.4/0/0.3	193/193/193/0.3	0/0/0	193	0	17.7/29.6	73.4/29.6	3.7	0.6
	p=1000	4.4/0.4/0.3	193/193/193/0.3	0/0/0	193	0	19.6/37.8	87.6/37.8	4.1	0.6
	p=1500	6.7/2.2/0.2	193/193/193/0.3	0/0/0	192.9	0	20.4/42.7	89.5/42.7	4.6	0.5
msnr	p=250	0.5/0.1/0.3	193/193/193/0.3	0/0/0	193	0	15.9/23.3	48.6/23.3	1.6	0.8
	p=500	2.3/0.4/0.3	193/193/193/0.3	0/0/0	193	0	18/29.7	96.8/29.7	2.2	1.7
	p=1000	5.5/1.4/0.3	193/193/193/0.3	0/0/0	193	0	19.6/37.9	100.4/37.9	2.5	1.8
	p=1500	8.4/3.9/0.3	193/193/193/0.3	0/0/0	192.9	0	20.6/43	99.7/43	2.5	2.4
lsnr	p=250	0.8/0.1/0.5	193.5/193.5/193.5/0.5	0/0/0	193.5	0	10.3/14.4	69.3/14.4	10.3	7.1
	p=500	2.9/0.2/0.4	194.8/194.8/194.8/0.4	0/0/0	194.8	0	11.2/15.8	114.6/15.8	11.2	8.6
	p=1000	5.7/1.5/0.4	196/196/196/0.4	0/0/0	196	0	10.9/18.1	113.7/18.1	13.8	10.1
	p=1500	7.6/3.8/0.3	196.5/196.5/196.5/0.3	0/0/0	196.4	0	10.7/18.8	112.5/18.8	14.1	12

Table S45: The performance of BOSS for high dimensional data, Sparse-Ex3, ρ=0.9, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	8/3/30	477/477/477/33	3/13/4	477	13	135/137	167/137	17	16
	p=500	28/6/33	476/476/476/34	4/13/7	476	13	156/159	334/159	22	17
	p=1000	85/17/34	475/475/475/35	6/13/9	475	13	179/183	393/183	22	17
	p=1500	123/52/32	474/474/474/34	6/13/10	474	13	191/196	405/196	24	18
msnr	p=250	15/7/12	367/367/367/13	2/172/4	367	172	91/92	217/92	3	13
	p=500	44/17/13	366/366/366/13	4/315/11	366	315	107/110	327/110	5	21
	p=1000	103/44/15	365/365/365/17	7/401/27	365	401	126/130	342/130	10	37
	p=1500	140/92/16	364/364/364/17	9/426/42	364	426	135/139	343/139	11	45
lsnr	p=250	16/15/14	164/164/164/14	16/18/17	164	18	-3/0	118/0	3	3
	p=500	21/12/11	151/151/151/11	11/12/12	151	12	-3/0	145/0	2	3
	p=1000	33/15/9	142/142/142/9	8/9/8	142	9	-3/2	139/2	4	4
	p=1500	41/27/8	139/139/139/8	7/7/7	139	7	-3/2	136/2	4	6
		Relative efficiency								
hsnr	p=250	0.96/1/0.79	0.18/0.18/0.18/0.77	1/0.91/0.99	0.18	0.91	0.44/0.43	0.39/0.43	0.88	0.89
	p=500	0.81/0.99/0.79	0.18/0.18/0.18/0.78	1/0.92/0.98	0.18	0.92	0.41/0.4	0.24/0.4	0.86	0.89
	p=1000	0.57/0.91/0.79	0.18/0.18/0.18/0.78	1/0.94/0.97	0.18	0.94	0.38/0.37	0.21/0.37	0.87	0.91
	p=1500	0.47/0.7/0.8	0.18/0.18/0.18/0.79	1/0.94/0.96	0.18	0.94	0.36/0.36	0.21/0.36	0.85	0.9
msnr	p=250	0.89/0.96/0.91	0.22/0.22/0.22/0.9	1/0.38/0.98	0.22	0.38	0.54/0.53	0.32/0.53	0.99	0.91
	p=500	0.72/0.88/0.92	0.22/0.22/0.22/0.91	1/0.25/0.93	0.22	0.25	0.5/0.49	0.24/0.49	0.99	0.86
	p=1000	0.53/0.74/0.93	0.23/0.23/0.23/0.92	1/0.21/0.84	0.23	0.21	0.47/0.47	0.24/0.47	0.98	0.78
	p=1500	0.45/0.57/0.94	0.23/0.23/0.23/0.93	1/0.21/0.77	0.23	0.21	0.46/0.46	0.25/0.46	0.98	0.75
lsnr	p=250	0.83/0.84/0.85	0.37/0.37/0.37/0.85	0.84/0.82/0.83	0.37	0.82	1/0.97	0.45/0.97	0.94	0.94
	p=500	0.8/0.87/0.87	0.38/0.38/0.38/0.87	0.87/0.86/0.86	0.38	0.86	1/0.97	0.39/0.97	0.94	0.94
	p=1000	0.73/0.84/0.89	0.4/0.4/0.4/0.89	0.9/0.89/0.89	0.4	0.89	1/0.95	0.41/0.95	0.93	0.93
	p=1500	0.69/0.76/0.9	0.41/0.41/0.41/0.9	0.91/0.91/0.91	0.41	0.91	1/0.95	0.41/0.95	0.93	0.92
		Sparsistency								
hsnr	p=250	6/6/6	6/6/6/6	6/5.9/6	6	5.9	6/6	6/6	6	6
	p=500	6/6/6	6/6/6/6	6/5.9/6	6	5.9	6/6	6/6	6	6
	p=1000	6/6/6	6/6/6/6	6/5.9/5.9	6	5.9	6/6	5.9/6	6	6
	p=1500	6/6/6	6/6/6/6	6/5.9/5.9	6	5.9	6/6	5.9/6	6	6
msnr	p=250	5.5/5.5/5.5	5.9/5.9/5.9/5.5	5.5/3.7/5.5	5.9	3.7	6/6	5.6/6	5.8	5.9
	p=500	5.5/5.5/5.5	5.8/5.8/5.8/5.5	5.5/2/5.4	5.8	2	6/6	5.6/6	5.8	5.9
	p=1000	5.5/5.5/5.5	5.6/5.6/5.6/5.5	5.5/0.9/5.3	5.6	0.9	6/6	5.5/6	5.8	5.8
	p=1500	5.5/5.5/5.5	5.6/5.6/5.6/5.5	5.5/0.6/5.1	5.6	0.6	5.9/6	5.5/6	5.8	5.8
lsnr	p=250	0.8/0.5/1	5.2/5.2/5.2/1	0.3/0/0.2	5.2	0	3/3	3.5/3	2.6	2.4
	p=500	0.9/0.4/0.7	3.7/3.7/3.7/0.7	0.2/0/0.1	3.7	0	2.4/2.4	3.1/2.4	2.2	2
	p=1000	0.9/0.5/0.5	2.6/2.6/2.6/0.5	0.1/0/0	2.6	0	1.9/1.9	2.3/1.9	1.8	1.6
	p=1500	0.9/0.7/0.4	2.1/2.1/2.1/0.4	0.1/0/0	2.1	0	1.6/1.7	2/1.7	1.5	1.4
		Number of extra variables								
hsnr	p=250	0.3/0.1/0.6	193/193/193/0.7	0.1/0.1/0.1	193	0.1	17.2/25.2	26.8/25.2	2.8	0.8
	p=500	1.6/0.1/0.7	193/193/193/0.6	0.1/0.1/0.1	193	0.1	19.1/32.2	71.2/32.2	4	0.8
	p=1000	5/0.7/0.6	193/193/193/0.6	0.1/0.1/0.1	193	0.1	21.2/40.2	87.3/40.2	4.6	0.8
	p=1500	7.3/2.8/0.5	193/193/193/0.5	0.1/0.1/0.1	192.9	0.1	22/45	89.8/45	5.4	1
msnr	p=250	1/0.6/0.8	193.1/193.1/193.1/0.8	0.5/0.3/0.5	193.1	0.3	17.2/25.1	47.5/25.1	2	2
	p=500	2.7/0.8/0.8	193.2/193.2/193.2/0.8	0.5/0.1/0.5	193.2	0.1	19.1/32.2	97.3/32.2	2.2	2.6
	p=1000	6.6/1.9/0.8	193.4/193.3/193.4/0.8	0.5/0/0.4	193.3	0	21.1/40.6	100.4/40.6	3	3.4
	p=1500	9.1/4.8/0.8	193.4/193.4/193.4/0.8	0.5/0/0.4	193.3	0	21.8/44.9	100.5/44.9	3.3	4
lsnr	p=250	1/0.2/0.8	193.8/193.8/193.8/0.9	0.1/0/0	193.8	0	11.2/14.5	68.1/14.5	10.9	8.1
	p=500	2.9/0.3/0.6	195.3/195.3/195.3/0.6	0/0/0	195.3	0	11.3/15.9	114.2/15.9	11.5	8.8
	p=1000	5.9/1.4/0.5	196.4/196.4/196.4/0.5	0/0/0	196.4	0	11.2/18.2	114.5/18.2	14	11
	p=1500	7.6/4.2/0.4	196.9/196.9/196.9/0.4	0/0/0	196.8	0	10.9/19.3	112.4/19.3	14.3	12.6

Table S46: The performance of BOSS for high dimensional data, Sparse-Ex4, ρ=0, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	10/6/13	302/302/302/13	6/5/6	302	5	63/63	234/63	11	16
	p=500	23/6/11	294/294/294/12	4/3/4	294	3	73/75	281/75	12	16
	p=1000	56/13/12	289/289/289/12	3/2/2	289	2	84/87	280/87	13	12
	p=1500	78/31/11	286/286/286/11	3/1/2	286	1	90/93	278/93	13	12
msnr	p=250	33/30/29	452/452/452/29	39/153/58	452	153	108/109	363/109	29	74
	p=500	65/36/31	427/427/427/31	46/155/75	427	155	115/119	421/119	35	86
	p=1000	116/62/36	399/399/399/37	51/150/89	399	150	119/127	396/127	38	95
	p=1500	153/109/38	386/386/386/37	55/146/95	386	146	122/131	383/131	41	100
lsnr	p=250	35/35/28	275/275/275/28	38/47/42	275	47	19/22	201/22	21	23
	p=500	39/27/23	244/244/244/23	29/35/32	244	35	14/19	242/19	19	21
	p=1000	51/27/19	217/217/217/19	21/24/23	217	24	10/16	217/16	16	19
	p=1500	63/40/17	207/207/207/18	18/20/20	207	20	9/15	206/15	14	19
		Relative efficiency								
hsnr	p=250	0.96/0.99/0.93	0.26/0.26/0.26/0.93	0.99/1/1	0.26	1	0.65/0.65	0.32/0.65	0.95	0.91
	p=500	0.84/0.98/0.93	0.26/0.26/0.26/0.92	0.99/1/1	0.26	1	0.6/0.59	0.27/0.59	0.92	0.89
	p=1000	0.65/0.9/0.91	0.26/0.26/0.26/0.91	0.99/1/1	0.26	1	0.55/0.54	0.27/0.54	0.9	0.91
	p=1500	0.57/0.78/0.91	0.26/0.26/0.26/0.91	0.99/1/1	0.26	1	0.53/0.52	0.27/0.52	0.89	0.91
msnr	p=250	0.97/0.99/1	0.23/0.23/0.23/1	0.93/0.51/0.81	0.23	0.51	0.62/0.62	0.28/0.62	0.99	0.74
	p=500	0.79/0.96/1	0.25/0.25/0.25/1	0.89/0.51/0.75	0.25	0.51	0.61/0.59	0.25/0.59	0.97	0.7
	p=1000	0.63/0.84/1	0.27/0.27/0.27/0.99	0.9/0.54/0.72	0.27	0.54	0.62/0.6	0.27/0.6	0.99	0.69
	p=1500	0.54/0.66/0.99	0.28/0.28/0.28/1	0.88/0.56/0.7	0.28	0.56	0.62/0.59	0.28/0.59	0.97	0.69
lsnr	p=250	0.88/0.88/0.93	0.32/0.32/0.32/0.93	0.86/0.81/0.84	0.32	0.81	1/0.97	0.39/0.97	0.98	0.96
	p=500	0.82/0.9/0.93	0.33/0.33/0.33/0.93	0.89/0.85/0.86	0.33	0.85	1/0.96	0.33/0.96	0.96	0.95
	p=1000	0.73/0.87/0.93	0.35/0.35/0.35/0.93	0.91/0.89/0.89	0.35	0.89	1/0.95	0.35/0.95	0.95	0.93
	p=1500	0.67/0.78/0.93	0.35/0.35/0.35/0.92	0.92/0.9/0.91	0.35	0.9	1/0.94	0.35/0.94	0.95	0.91
		Sparsistency								
hsnr	p=250	4.1/4.1/4.3	5.8/5.8/5.8/4.2	4.1/4/4	5.8	4	5/5.2	5.4/5.2	4.5	4.2
	p=500	4.2/4.1/4.2	5.3/5.3/5.3/4.2	4/4/4	5.3	4	4.8/4.9	5.1/4.9	4.4	4.1
	p=1000	4.2/4.1/4.1	4.9/4.9/4.9/4.1	4/4/4	4.9	4	4.6/4.8	4.8/4.8	4.3	4
	p=1500	4.2/4.1/4.1	4.7/4.7/4.7/4.1	4/4/4	4.7	4	4.5/4.6	4.6/4.6	4.3	4
msnr	p=250	3.8/3.7/3.9	5.6/5.6/5.6/3.9	3.6/2.3/3.4	5.6	2.3	4.3/4.3	4.8/4.3	4	3.8
	p=500	3.9/3.7/3.8	4.9/4.9/4.9/3.8	3.4/2.2/3.1	4.9	2.2	4.1/4.2	4.6/4.2	3.9	3.6
	p=1000	3.9/3.7/3.7	4.5/4.5/4.5/3.7	3.3/2/2.8	4.5	2	4/4.1	4.3/4.1	3.9	3.4
	p=1500	3.9/3.8/3.6	4.3/4.3/4.3/3.6	3.2/2/2.7	4.3	2	3.8/4	4.1/4	3.8	3.3
lsnr	p=250	0.8/0.5/1.2	5.3/5.3/5.3/1.2	0.4/0/0.2	5.3	0	2.2/2.2	3.7/2.2	2	1.8
	p=500	0.9/0.5/0.9	3.8/3.8/3.8/0.9	0.3/0/0.1	3.8	0	1.9/1.8	3.3/1.8	1.7	1.6
	p=1000	0.9/0.6/0.6	2.7/2.7/2.7/0.6	0.2/0/0.1	2.7	0	1.6/1.5	2.5/1.5	1.4	1.3
	p=1500	1/0.7/0.5	2.3/2.3/2.3/0.5	0.2/0/0	2.3	0	1.4/1.4	2.1/1.4	1.3	1.2
		Number of extra variables								
hsnr	p=250	0.2/0/0.4	193.2/193.2/193.2/0.3	0/0/0	193.2	0	16.1/22.7	67.5/22.7	3.2	1.1
	p=500	2/0.1/0.3	193.7/193.7/193.7/0.3	0/0/0	193.7	0	17.9/29.1	101.9/29.1	4.2	1.6
	p=1000	5.1/0.7/0.3	194.1/194.1/194.1/0.3	0/0/0	194.1	0	19.2/36.7	96.1/36.7	5.1	0.9
	p=1500	7.3/2.4/0.3	194.3/194.3/194.3/0.3	0/0/0	194.2	0	20.2/40.2	92.2/40.2	5.7	1
msnr	p=250	0.5/0.1/0.4	193.4/193.4/193.4/0.4	0/0/0	193.4	0	15/20.1	73/20.1	2.7	3.1
	p=500	2.4/0.3/0.4	194.1/194.1/194.1/0.4	0/0/0	194.1	0	16.8/26.3	117/26.3	4.2	4.1
	p=1000	5.9/1.5/0.4	194.5/194.5/194.5/0.4	0/0/0	194.5	0	18.1/32.9	111.4/32.9	5	5.3
	p=1500	8.2/4.3/0.4	194.7/194.7/194.7/0.4	0/0/0	194.7	0	18.2/35.6	106.8/35.6	6.1	5.6
lsnr	p=250	0.4/0.1/0.5	193.7/193.7/193.7/0.5	0/0/0	193.7	0	9.1/11	67/11	6.8	5.6
	p=500	1.9/0.1/0.4	195.2/195.2/195.2/0.4	0/0/0	195.2	0	10.2/12.8	128.2/12.8	8.4	6.8
	p=1000	4.6/1/0.3	196.2/196.2/196.2/0.3	0/0/0	196.2	0	10.3/15.7	126.3/15.7	10.6	9.1
	p=1500	6.6/2.9/0.3	196.7/196.7/196.7/0.3	0/0/0	196.7	0	10/16.5	121.8/16.5	11.5	10.8

Table S47: The performance of BOSS for high dimensional data, Sparse-Ex4, ρ=0.5, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	9/5/12	292/292/292/13	5/5/5	292	5	95/93	215/93	13	19
	p=500	26/4/11	283/283/283/12	3/2/2	283	2	112/107	268/107	12	21
	p=1000	59/13/11	278/278/278/12	2/1/1	278	1	132/125	268/125	12	30
	p=1500	84/37/11	277/277/277/12	1/0/1	277	0	145/135	268/135	12	39
msnr	p=250	26/22/31	374/374/374/44	29/117/44	374	118	123/121	306/121	15	115
	p=500	48/22/28	314/314/314/47	30/100/49	314	100	117/115	308/115	20	102
	p=1000	72/36/27	267/267/267/45	30/83/47	267	84	112/110	264/110	24	87
	p=1500	78/56/26	235/235/235/39	26/72/42	235	73	102/101	232/101	23	72
lsnr	p=250	25/24/21	251/251/251/33	32/37/35	251	37	30/32	180/32	25	35
	p=500	26/17/17	216/216/216/24	21/24/23	216	24	20/23	214/23	21	26
	p=1000	30/14/12	187/187/187/15	11/12/12	187	12	11/14	186/14	14	17
	p=1500	36/22/11	177/177/177/12	8/9/8	177	9	7/11	177/11	12	15
		Relative efficiency								
hsnr	p=250	0.96/0.99/0.93	0.27/0.27/0.27/0.92	0.99/1/1	0.27	1	0.53/0.54	0.33/0.54	0.93	0.88
	p=500	0.81/0.98/0.92	0.27/0.27/0.27/0.91	0.99/1/1	0.27	1	0.48/0.49	0.28/0.49	0.91	0.84
	p=1000	0.63/0.89/0.91	0.27/0.27/0.27/0.9	0.99/1/1	0.27	1	0.43/0.45	0.27/0.45	0.9	0.77
	p=1500	0.55/0.73/0.91	0.27/0.27/0.27/0.9	0.99/1/1	0.27	1	0.41/0.43	0.27/0.43	0.9	0.72
msnr	p=250	0.91/0.94/0.88	0.24/0.24/0.24/0.8	0.89/0.53/0.8	0.24	0.53	0.52/0.52	0.28/0.52	1	0.54
	p=500	0.81/0.98/0.94	0.29/0.29/0.29/0.82	0.93/0.6/0.81	0.29	0.6	0.55/0.56	0.29/0.56	1	0.6
	p=1000	0.72/0.91/0.97	0.34/0.34/0.34/0.85	0.95/0.67/0.84	0.34	0.67	0.58/0.59	0.34/0.59	1	0.66
	p=1500	0.69/0.79/0.98	0.37/0.37/0.37/0.88	0.98/0.71/0.87	0.37	0.71	0.61/0.61	0.37/0.61	1	0.71
lsnr	p=250	0.97/0.98/1	0.35/0.35/0.35/0.91	0.92/0.88/0.9	0.35	0.88	0.94/0.92	0.43/0.92	0.97	0.9
	p=500	0.93/1/1	0.37/0.37/0.37/0.94	0.97/0.95/0.95	0.37	0.95	0.97/0.95	0.37/0.95	0.97	0.92
	p=1000	0.85/0.97/0.98	0.39/0.39/0.39/0.96	1/0.99/0.99	0.39	0.99	1/0.97	0.39/0.97	0.97	0.94
	p=1500	0.79/0.88/0.97	0.39/0.39/0.39/0.96	1/0.99/0.99	0.39	0.99	1/0.96	0.39/0.96	0.96	0.93
		Sparsistency								
hsnr	p=250	4.1/4.1/4.2	5.8/5.8/5.8/4.1	4/4/4	5.8	4	4.8/5	5.3/5	4.3	4
	p=500	4.1/4.1/4.1	5.2/5.2/5.2/4.1	4/4/4	5.2	4	4.5/4.7	5/4.7	4.2	4
	p=1000	4.1/4.1/4	4.7/4.7/4.7/4	4/4/4	4.7	4	4.3/4.5	4.6/4.5	4.1	4
	p=1500	4.1/4.1/4	4.5/4.5/4.5/4	4/4/4	4.5	4	4.2/4.4	4.4/4.4	4.1	4
msnr	p=250	3.9/3.9/3.9	5.6/5.6/5.6/3.7	3.5/2.3/3.3	5.6	2.3	4.1/4.3	4.8/4.3	4	3.3
	p=500	3.8/3.7/3.7	4.8/4.8/4.8/3.3	3.2/2.2/2.9	4.8	2.1	3.8/4	4.4/4	3.8	2.9
	p=1000	3.7/3.6/3.5	4.2/4.2/4.2/3.1	3/2.1/2.7	4.2	2	3.4/3.7	4.1/3.7	3.6	2.6
	p=1500	3.5/3.5/3.3	4/4/4/2.9	2.9/2/2.5	4	2	3.2/3.4	3.8/3.4	3.4	2.5
lsnr	p=250	0.8/0.6/1.2	5.3/5.3/5.3/0.7	0.3/0/0.1	5.3	0	1.4/1.2	3.4/1.2	1.4	1.1
	p=500	0.7/0.5/0.7	3.6/3.6/3.6/0.4	0.1/0/0.1	3.6	0	0.9/0.9	2.8/0.9	1	0.7
	p=1000	0.5/0.3/0.3	2.3/2.3/2.3/0.2	0.1/0/0	2.3	0	0.6/0.6	1.7/0.6	0.6	0.5
	p=1500	0.4/0.4/0.2	1.7/1.7/1.7/0.2	0.1/0/0	1.7	0	0.5/0.5	1.3/0.5	0.5	0.4
		Number of extra variables								
hsnr	p=250	0.4/0/0.4	193.2/193.2/193.2/0.3	0/0/0	193.2	0	23.7/36.4	63.8/36.4	3.4	1.1
	p=500	2.7/0.1/0.3	193.8/193.8/193.8/0.3	0/0/0	193.8	0	26.1/46.1	104.6/46.1	4.1	1.4
	p=1000	6.2/0.8/0.3	194.3/194.3/194.3/0.3	0/0/0	194.2	0	28.5/59.2	103.1/59.2	5.1	2.3
	p=1500	8.3/3.1/0.3	194.5/194.5/194.5/0.3	0/0/0	194.4	0	29.8/66	100.1/66	5.3	3.7
msnr	p=250	0.9/0.4/0.8	193.4/193.4/193.4/0.9	0.1/0/0	193.4	0	21.7/32.8	74.5/32.8	4.6	6.5
	p=500	3.4/0.7/0.9	194.2/194.2/194.2/0.9	0.1/0/0	194.2	0	22.8/39.7	116.2/39.7	7.4	5.5
	p=1000	6.6/1.9/1	194.8/194.7/194.8/0.8	0.1/0/0	194.7	0	22.9/46.5	114.4/46.5	9.3	5.1
	p=1500	8.2/4.6/0.9	195/195/195/0.7	0.1/0/0	194.9	0	22.5/47.9	111.5/47.9	10.8	4.2
lsnr	p=250	0.6/0.2/1.1	193.7/193.7/193.7/0.6	0/0/0	193.7	0	8.6/10.2	68.8/10.2	6.8	7
	p=500	1.7/0.3/0.7	195.4/195.4/195.4/0.4	0/0/0	195.4	0	7.9/9.8	126.8/9.8	8.1	6.8
	p=1000	3.3/0.8/0.4	196.7/196.7/196.7/0.3	0/0/0	196.7	0	7.6/11.5	126.4/11.5	8.5	7.6
	p=1500	4.7/2.3/0.3	197.3/197.3/197.3/0.3	0/0/0	197.3	0	7.5/12.2	122.8/12.2	9.4	9.2

Table S48: The performance of BOSS for high dimensional data, Sparse-Ex4, ρ=0.9, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	13/5/35	249/249/249/75	6/10/6	249	3	182/149	140/149	4	151
	p=500	26/8/48	196/196/196/90	8/32/12	196	22	254/169	170/169	10	196
	p=1000	15/60/119	51/51/51/147	105/146/115	51	144	105/58	47/58	4	50
	p=1500	26/48/123	23/23/23/146	114/141/124	23	140	97/39	21/39	3	36
msnr	p=250	19/21/23	144/144/144/79	51/102/60	144	101	86/72	100/72	20	67
	p=500	16/14/17	66/66/66/53	34/60/40	66	60	60/50	62/50	-11	48
	p=1000	12/12/15	22/22/22/25	19/27/22	22	27	24/25	21/25	15	25
	p=1500	10/10/13	13/13/13/18	15/19/16	13	19	15/15	12/15	14	15
lsnr	p=250	11/10/14	197/197/197/20	16/16/16	197	16	17/17	160/17	20	20
	p=500	8/5/9	173/173/173/11	7/7/7	173	7	8/9	170/9	10	11
	p=1000	15/4/6	162/162/162/7	2/2/2	162	2	3/5	160/5	6	8
	p=1500	23/10/5	158/158/158/6	1/1/1	158	1	2/3	157/3	5	7
		Relative efficiency								
hsnr	p=250	0.92/0.99/0.77	0.3/0.3/0.3/0.59	0.98/0.94/0.98	0.3	1	0.37/0.42	0.43/0.42	1	0.41
	p=500	0.85/1/0.73	0.36/0.36/0.36/0.57	1/0.82/0.96	0.36	0.88	0.3/0.4	0.4/0.4	0.98	0.36
	p=1000	0.91/0.65/0.48	0.69/0.69/0.69/0.42	0.51/0.42/0.49	0.69	0.43	0.51/0.66	0.71/0.66	1	0.7
	p=1500	0.81/0.69/0.46	0.83/0.83/0.83/0.42	0.48/0.43/0.46	0.83	0.43	0.52/0.74	0.85/0.74	1	0.76
msnr	p=250	1/0.99/0.97	0.49/0.49/0.49/0.67	0.79/0.59/0.75	0.49	0.6	0.64/0.69	0.6/0.69	1	0.72
	p=500	0.76/0.78/0.76	0.53/0.53/0.53/0.58	0.66/0.55/0.63	0.53	0.55	0.55/0.59	0.55/0.59	1	0.6
	p=1000	1/1/0.97	0.92/0.92/0.92/0.89	0.94/0.88/0.92	0.92	0.88	0.9/0.9	0.93/0.9	0.97	0.9
	p=1500	1/0.99/0.97	0.97/0.97/0.97/0.93	0.95/0.92/0.94	0.97	0.92	0.95/0.95	0.98/0.95	0.96	0.95
lsnr	p=250	0.99/1/0.97	0.37/0.37/0.37/0.91	0.95/0.95/0.95	0.37	0.95	0.94/0.94	0.42/0.94	0.92	0.92
	p=500	0.98/1/0.96	0.39/0.39/0.39/0.95	0.98/0.98/0.99	0.39	0.98	0.97/0.97	0.39/0.97	0.95	0.95
	p=1000	0.89/0.98/0.97	0.39/0.39/0.39/0.96	1/1/1	0.39	1	0.99/0.97	0.39/0.97	0.96	0.95
	p=1500	0.82/0.92/0.96	0.39/0.39/0.39/0.95	1/1/1	0.39	1	0.99/0.98	0.39/0.98	0.96	0.95
		Sparsistency								
hsnr	p=250	4.1/4.1/4.2	5.6/5.6/5.6/4.1	4/3.9/4	5.6	3.9	4.3/4.6	4.5/4.6	4.2	4.1
	p=500	4/4/4	4.8/4.8/4.8/4	3.9/3.7/3.8	4.8	3.7	3.2/4	4.3/4	4.2	3.1
	p=1000	3.1/3/2.8	3.6/3.6/3.6/1.9	2/1.4/1.8	3.6	1.4	3/3	2.5/3	3.9	3
	p=1500	2.2/2.1/1.7	2.6/2.6/2.6/0.8	1.2/0.7/1	2.6	0.7	2.5/2.8	1.9/2.8	3.3	2.8
msnr	p=250	3.5/3.3/3.5	5.4/5.4/5.4/2.3	2.2/1/1.9	5.4	1	2.9/3.2	3.9/3.2	2.9	2.9
	p=500	2.6/2.4/2.5	4/4/4/1	1.3/0.4/1	4	0.4	1.3/1.6	2.8/1.6	2.6	1.7
	p=1000	1.1/1/1	2.4/2.4/2.4/0.4	0.5/0.1/0.4	2.4	0.1	0.6/0.6	1.5/0.6	1	0.5
	p=1500	0.8/0.7/0.5	1.6/1.6/1.6/0.2	0.3/0.1/0.2	1.6	0.1	0.5/0.5	1/0.5	0.5	0.4
lsnr	p=250	0.3/0.3/0.4	4.7/4.7/4.7/0.1	0.1/0/0	4.7	0	0.2/0.2	1.9/0.2	0.2	0.2
	p=500	0.2/0.1/0.1	2.5/2.5/2.5/0	0/0/0	2.5	0	0.2/0.1	1.3/0.1	0.1	0.1
	p=1000	0.1/0.1/0	1.3/1.3/1.3/0	0/0/0	1.3	0	0.1/0.1	0.7/0.1	0.1	0.1
	p=1500	0.1/0/0	0.8/0.8/0.8/0	0/0/0	0.8	0	0.1/0.1	0.5/0.1	0.1	0.1
		Number of extra variables								
hsnr	p=250	2.9/1.2/3.6	193.4/193.4/193.4/7.3	0.8/0.7/0.8	193.4	0	55.7/90.9	49.1/90.9	9.3	39.6
	p=500	7.8/2.5/4.9	194.2/194.2/194.2/11.5	1.8/1.2/1.6	194.2	0	44.1/104.6	96/104.6	33.9	18.9
	p=1000	24.7/10.1/6.6	195.4/195.4/195.4/5.9	2.5/0.9/2	195.4	0	50.1/93.8	108.1/93.8	87.7	40.7
	p=1500	28.5/18.4/4.5	196.4/196.4/196.4/2.1	1.7/0.5/1.1	196.3	0	43/101.4	109.6/101.4	84.5	61.6
msnr	p=250	9.9/7.2/9.2	193.6/193.6/193.6/6.6	1.6/0.2/1.2	193.6	0	33.4/55.5	68.7/55.5	24.4	27.8
	p=500	11.9/6.4/7.4	195/195/195/2	1.1/0.1/0.6	195	0	15.2/36.7	112.9/36.7	41	24.6
	p=1000	4.7/2.9/2.7	196.6/196.6/196.6/0.8	0.4/0/0.2	196.5	0	8.2/11	115.1/11	25.6	7.5
	p=1500	5.4/3.3/1.4	197.4/197.4/197.4/0.6	0.2/0/0.1	197.3	0	8.6/11.9	113.4/11.9	10.1	7.8
lsnr	p=250	0.7/0.6/1.5	194.3/194.3/194.3/0.3	0/0/0	194.3	0	4.2/3.8	79.1/3.8	4.2	2.8
	p=500	0.6/0.2/0.5	196.5/196.5/196.5/0.3	0/0/0	196.5	0	4.7/5.2	123.5/5.2	4.7	3.9
	p=1000	2.1/0.3/0.3	197.7/197.7/197.7/0.3	0/0/0	197.7	0	5.9/8.1	121.3/8.1	6.3	5.5
	p=1500	3.6/1.3/0.3	198.2/198.2/198.2/0.3	0/0/0	198.1	0	6.5/8.5	119.2/8.5	7	6.6

Table S49: The performance of BOSS for high dimensional data, Dense, ρ=0, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	11/14/14	68/68/68/13	35/166/50	68	166	18/9	5/9	3	16
	p=500	15/9/12	55/55/55/12	33/147/74	55	147	25/12	23/12	3	19
	p=1000	18/9/10	45/45/45/10	38/132/101	45	132	32/13	27/13	1	20
	p=1500	20/13/10	41/41/41/10	46/126/109	41	126	36/15	26/15	1	18
msnr	p=250	13/19/16	76/76/76/16	32/69/35	76	69	5/1	27/1	3	7
	p=500	15/11/13	60/60/60/13	21/68/26	60	68	5/1	47/1	3	7
	p=1000	17/10/9	49/49/49/9	14/66/21	49	66	5/1	41/1	2	7
	p=1500	18/13/8	44/44/44/8	12/64/22	44	64	5/1	38/1	1	6
lsnr	p=250	11/9/9	156/156/156/9	8/9/9	156	9	-3/0	112/0	2	3
	p=500	16/7/8	150/150/150/8	6/6/6	150	6	-2/1	143/1	3	3
	p=1000	26/11/7	145/145/145/7	4/4/4	145	4	-2/2	141/2	4	5
	p=1500	35/21/6	143/143/143/6	4/3/3	143	3	-2/3	140/3	4	6
		Relative efficiency								
hsnr	p=250	0.93/0.91/0.91	0.62/0.62/0.62/0.91	0.76/0.39/0.69	0.62	0.39	0.88/0.95	0.99/0.95	1	0.89
	p=500	0.89/0.94/0.91	0.66/0.66/0.66/0.92	0.77/0.42/0.59	0.66	0.42	0.82/0.92	0.84/0.92	1	0.87
	p=1000	0.86/0.93/0.92	0.7/0.7/0.7/0.92	0.74/0.44/0.5	0.7	0.44	0.76/0.89	0.8/0.89	1	0.85
	p=1500	0.84/0.89/0.92	0.72/0.72/0.72/0.92	0.69/0.45/0.48	0.72	0.45	0.74/0.88	0.8/0.88	1	0.85
msnr	p=250	0.9/0.85/0.87	0.57/0.57/0.57/0.87	0.77/0.6/0.75	0.57	0.6	0.97/1	0.8/1	0.98	0.95
	p=500	0.87/0.9/0.89	0.63/0.63/0.63/0.89	0.83/0.6/0.8	0.63	0.6	0.96/1	0.68/1	0.98	0.94
	p=1000	0.86/0.92/0.92	0.68/0.68/0.68/0.92	0.88/0.61/0.83	0.68	0.61	0.96/1	0.72/1	0.99	0.95
	p=1500	0.86/0.89/0.94	0.7/0.7/0.7/0.94	0.9/0.62/0.83	0.7	0.62	0.96/1	0.73/1	1	0.96
lsnr	p=250	0.88/0.89/0.89	0.38/0.38/0.38/0.89	0.9/0.89/0.89	0.38	0.89	1/0.97	0.46/0.97	0.95	0.95
	p=500	0.85/0.91/0.91	0.39/0.39/0.39/0.9	0.92/0.92/0.92	0.39	0.92	1/0.97	0.4/0.97	0.95	0.95
	p=1000	0.78/0.88/0.92	0.4/0.4/0.4/0.92	0.94/0.94/0.94	0.4	0.94	1/0.96	0.41/0.96	0.95	0.93
	p=1500	0.73/0.81/0.93	0.4/0.4/0.4/0.92	0.95/0.95/0.95	0.4	0.95	1/0.95	0.41/0.95	0.94	0.93
		Sparsistency								
hsnr	p=250	21.3/14.8/16.5	199/199/199/16.3	11.3/3.1/10.1	199	3.1	47.7/74.1	47.2/74.1	32.1	35.5
	p=500	26.8/15.6/14.8	199/199/199/14.5	10.7/3/7.8	199	3	47.2/87.8	80.7/87.8	30.9	34.5
	p=1000	29.9/19.2/13.7	199/199/199/13.6	9.6/3/5.1	199	3	45.9/99	96.1/99	27.7	34
	p=1500	29.3/23.6/13.5	199/199/199/13.3	8.8/3/4.1	198.9	3	45.4/106.3	98/106.3	27.1	36.8
msnr	p=250	12/5.9/7.9	199/199/199/7.6	3.2/1.5/3	199	1.5	32.7/49.7	57.8/49.7	27.7	27.8
	p=500	16.3/6.4/5.7	199/199/199/5.6	3.1/0.8/2.7	199	0.8	30.5/54.7	103.2/54.7	26.2	25.8
	p=1000	18.6/9.8/4.4	199/199/199/4.4	2.9/0.3/2.5	199	0.3	27.7/57.1	106.3/57.1	22.8	20.7
	p=1500	19.6/13.9/3.9	199/199/199/3.9	2.9/0.2/2.3	198.9	0.2	27.2/59	106.4/59	20.7	19.8
lsnr	p=250	1.1/0.3/1.1	199/199/199/1.1	0.2/0/0.1	199	0	11.1/14.7	72.2/14.7	11.3	9.5
	p=500	2.6/0.4/0.8	199/199/199/0.8	0.1/0/0.1	199	0	11.1/14.8	117.1/14.8	12.5	9.1
	p=1000	5/1.5/0.6	199/199/199/0.6	0.1/0/0	199	0	10.6/16.6	114.9/16.6	12.5	10.5
	p=1500	6.5/3.4/0.5	199/199/199/0.5	0.1/0/0	198.9	0	10/17.6	113.5/17.6	12.8	11.8
		Number of extra variables								
hsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
msnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
lsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-

Table S50: The performance of BOSS for high dimensional data, Dense, ρ=0.5, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	9/11/11	62/62/62/13	34/147/59	62	147	44/23	6/23	5	27
	p=500	10/7/13	47/47/47/14	45/127/87	47	127	71/29	9/29	1	45
	p=1000	15/10/24	32/32/32/28	51/104/89	32	104	100/52	11/52	3	90
	p=1500	18/13/35	21/21/21/38	53/88/79	21	88	89/57	8/57	21	79
msnr	p=250	13/14/13	66/66/66/15	19/49/21	66	49	24/17	15/17	10	20
	p=500	8/6/8	52/52/52/8	10/54/13	52	54	21/15	32/15	6	16
	p=1000	10/5/5	46/46/46/5	7/61/11	46	61	22/15	34/15	3	18
	p=1500	12/8/4	44/44/44/4	6/63/13	44	63	24/17	34/17	2	21
lsnr	p=250	10/9/10	156/156/156/11	9/10/9	156	10	4/6	99/6	7	8
	p=500	10/6/7	147/147/147/7	5/5/5	147	5	2/3	135/3	5	5
	p=1000	17/6/6	142/142/142/6	4/3/4	142	3	1/2	135/2	4	5
	p=1500	24/12/5	140/140/140/6	3/2/2	140	2	0/2	134/2	4	4
		Relative efficiency								
hsnr	p=250	0.96/0.95/0.95	0.65/0.65/0.65/0.93	0.78/0.43/0.66	0.65	0.43	0.73/0.86	0.99/0.86	1	0.83
	p=500	0.92/0.94/0.89	0.69/0.69/0.69/0.88	0.7/0.44/0.54	0.69	0.44	0.59/0.78	0.92/0.78	1	0.69
	p=1000	0.9/0.94/0.83	0.78/0.78/0.78/0.81	0.68/0.51/0.55	0.78	0.51	0.52/0.68	0.93/0.68	1	0.54
	p=1500	0.92/0.96/0.8	0.89/0.89/0.89/0.79	0.71/0.58/0.6	0.89	0.58	0.57/0.69	1/0.69	0.89	0.61
msnr	p=250	0.98/0.96/0.98	0.66/0.66/0.66/0.96	0.92/0.74/0.91	0.66	0.74	0.89/0.94	0.96/0.94	1	0.92
	p=500	0.98/0.99/0.98	0.7/0.7/0.7/0.98	0.96/0.69/0.94	0.7	0.69	0.88/0.92	0.8/0.92	1	0.91
	p=1000	0.94/0.98/0.98	0.71/0.71/0.71/0.99	0.97/0.64/0.93	0.71	0.64	0.85/0.89	0.77/0.89	1	0.88
	p=1500	0.91/0.94/0.98	0.71/0.71/0.71/0.98	0.96/0.63/0.9	0.71	0.63	0.82/0.87	0.76/0.87	1	0.84
lsnr	p=250	0.95/0.96/0.95	0.41/0.41/0.41/0.94	0.96/0.95/0.95	0.41	0.95	1/0.99	0.52/0.99	0.98	0.97
	p=500	0.92/0.96/0.95	0.41/0.41/0.41/0.95	0.97/0.96/0.96	0.41	0.96	1/0.99	0.43/0.99	0.97	0.97
	p=1000	0.86/0.95/0.95	0.42/0.42/0.42/0.95	0.97/0.97/0.97	0.42	0.97	1/0.98	0.43/0.98	0.97	0.96
	p=1500	0.81/0.89/0.95	0.42/0.42/0.42/0.95	0.97/0.98/0.97	0.42	0.98	1/0.98	0.43/0.98	0.96	0.96
		Sparsistency								
hsnr	p=250	21.5/15.1/17.4	199/199/199/16.5	10.9/3.2/9	199	3.2	58.7/96.1	44.4/96.1	27.1	63
	p=500	22.6/15.5/15.7	199/199/199/15.6	9.1/3.1/5.8	199	3.1	52.3/106.6	63/106.6	30.3	66.7
	p=1000	20.8/15.7/12.5	199/199/199/11.5	7.3/3/4.1	199	3	30.2/92.9	87/92.9	38.3	19.8
	p=1500	16.1/16.1/9.8	199/199/199/8.6	6/3/3.6	198.9	3	29.4/71.9	94.5/71.9	28.8	11
msnr	p=250	9.2/5/6.6	199/199/199/5.2	3.2/1.8/3.1	199	1.8	28.2/49	52.3/49	15.2	25
	p=500	7.7/4.8/4.4	199/199/199/4.1	3.1/1.1/2.9	199	1.1	26.1/43.2	93.3/43.2	10.4	18.5
	p=1000	10.2/5.4/3.8	199/199/199/3.8	3/0.4/2.8	199	0.4	25.9/47	103.5/47	9.2	21.1
	p=1500	11.1/8/3.7	199/199/199/3.7	3/0.3/2.6	198.9	0.3	24.9/48.7	103.2/48.7	10.1	22.5
lsnr	p=250	0.6/0.2/1.1	199/199/199/0.9	0.1/0/0.1	199	0	8.3/9.2	65.2/9.2	7.2	6.2
	p=500	1.4/0.2/0.7	199/199/199/0.6	0.1/0/0	199	0	8.1/9.6	112.3/9.6	7.7	5.8
	p=1000	3.3/0.6/0.5	199/199/199/0.5	0.1/0/0	199	0	8/10.1	112.4/10.1	8.4	7.2
	p=1500	4.2/1.6/0.4	199/199/199/0.4	0/0/0	198.9	0	8.1/10.6	110.5/10.6	9.3	7.2
		Number of extra variables								
hsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
msnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
lsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-

Table S51: The performance of BOSS for high dimensional data, Dense, ρ=0.9, n=200

		BOSS C_p/AICc/CV	FS EBIC/HDBIC/HDHQ/CV	FSstop EBIC/HDBIC/HDHQ	FStrim HDBIC	FSstoptrim HDBIC	lasso AICc/CV	Gamma lasso AICc/CV	SparseNet CV	rlasso CV
		% worse than the best possible BOSS								
hsnr	p=250	10/21/12	79/79/79/12	68/113/90	79	113	75/39	19/39	14	42
	p=500	27/24/18	61/61/61/21	68/95/82	61	95	113/71	32/71	52	80
	p=1000	45/21/26	37/37/37/29	51/67/58	37	67	129/123	30/123	55	125
	p=1500	34/17/25	22/22/22/26	37/51/42	22	51	101/103	17/103	36	106
msnr	p=250	14/13/9	94/94/94/9	17/49/23	94	49	46/43	33/43	15	46
	p=500	10/9/7	85/85/85/7	15/46/22	85	46	49/50	49/50	18	48
	p=1000	8/7/7	79/79/79/7	15/43/25	79	43	45/46	53/46	25	44
	p=1500	7/7/7	76/76/76/7	15/41/26	76	41	41/42	55/42	33	41
lsnr	p=250	15/16/11	227/227/227/11	15/38/20	227	38	14/14	143/14	10	11
	p=500	15/15/11	220/220/220/11	15/37/22	220	37	13/13	200/13	9	10
	p=1000	19/15/12	213/213/213/12	16/36/24	213	36	12/13	198/13	9	10
	p=1500	24/17/13	209/209/209/13	17/34/24	209	34	12/14	194/14	10	10
		Relative efficiency								
hsnr	p=250	1/0.91/0.99	0.62/0.62/0.62/0.99	0.66/0.52/0.58	0.62	0.52	0.63/0.8	0.92/0.8	0.97	0.78
	p=500	0.93/0.95/1	0.73/0.73/0.73/0.98	0.71/0.61/0.65	0.73	0.61	0.56/0.69	0.9/0.69	0.78	0.66
	p=1000	0.83/1/0.97	0.89/0.89/0.89/0.94	0.8/0.73/0.77	0.89	0.73	0.53/0.54	0.93/0.54	0.78	0.54
	p=1500	0.87/1/0.94	0.96/0.96/0.96/0.93	0.85/0.77/0.82	0.96	0.77	0.58/0.58	1/0.58	0.86	0.57
msnr	p=250	0.96/0.96/1	0.56/0.56/0.56/1	0.93/0.73/0.89	0.56	0.73	0.75/0.76	0.82/0.76	0.95	0.75
	p=500	0.97/0.98/1	0.58/0.58/0.58/1	0.93/0.73/0.87	0.58	0.73	0.71/0.71	0.72/0.71	0.91	0.72
	p=1000	0.99/1/1	0.6/0.6/0.6/1	0.93/0.75/0.85	0.6	0.75	0.74/0.73	0.7/0.73	0.86	0.74
	p=1500	1/1/1	0.61/0.61/0.61/1	0.93/0.76/0.85	0.61	0.76	0.76/0.75	0.69/0.75	0.81	0.76
lsnr	p=250	0.96/0.95/0.99	0.34/0.34/0.34/0.99	0.96/0.8/0.92	0.34	0.8	0.97/0.97	0.45/0.97	1	1
	p=500	0.95/0.95/0.98	0.34/0.34/0.34/0.99	0.95/0.8/0.89	0.34	0.8	0.97/0.96	0.36/0.96	1	0.99
	p=1000	0.92/0.95/0.98	0.35/0.35/0.35/0.98	0.94/0.81/0.89	0.35	0.81	0.97/0.96	0.37/0.96	1	0.99
	p=1500	0.88/0.94/0.97	0.35/0.35/0.35/0.98	0.94/0.82/0.88	0.35	0.82	0.98/0.97	0.37/0.97	1	1
		Sparsistency								
hsnr	p=250	17.2/13.2/19.3	199/199/199/16.3	6.4/3.2/4.7	199	3.2	55.8/93.9	49.7/93.9	42.5	77.7
	p=500	13.1/11.7/16.3	199/199/199/13.6	5/3.1/3.8	199	3.1	38.3/79.6	71.1/79.6	36.1	55.5
	p=1000	6.6/9.7/10.4	199/199/199/8	4.2/3/3.6	199	3	9.8/19.2	62.7/19.2	15.1	9.2
	p=1500	4.9/7.9/6.9	199/199/199/5.8	3.9/2.9/3.5	199	2.9	12.5/12.6	63.7/12.6	20.8	3.2
msnr	p=250	5.1/3.8/6.7	199/199/199/4.6	2.8/1.3/2.5	199	1.3	17/28.6	52.8/28.6	14.9	17.2
	p=500	3.6/3.6/4.6	199/199/199/4	2.7/1.1/2.3	199	1.1	7.4/8.9	82/8.9	16.8	3.5
	p=1000	3.6/3.6/3.8	199/199/199/3.6	2.5/1/1.9	199	1	8.3/8.3	93.8/8.3	17.8	2.3
	p=1500	3.9/3.9/3.6	199/199/199/3.5	2.4/1/1.8	199	1	10.4/10.5	97.2/10.5	11.8	2.6
lsnr	p=250	0.9/0.8/2	199/199/199/1.6	0.7/0.1/0.6	199	0.1	6.6/6.8	64.6/6.8	3.8	3.1
	p=500	1/0.8/1.5	199/199/199/1.3	0.6/0.1/0.4	199	0.1	7.5/7.3	112.2/7.3	3.9	3.2
	p=1000	1.5/0.9/1.2	199/199/199/1.2	0.5/0/0.3	199	0	9.1/9.2	115.1/9.2	4.9	3.9
	p=1500	2.2/1/1.1	199/199/199/1.1	0.5/0/0.3	199	0	9.3/10.3	112.6/10.3	5.6	4.7
		Number of extra variables								
hsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
msnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
lsnr	p=250	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1000	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-
	p=1500	-/-/-	-/-/-/-	-/-/-	-	-	-/-	-/-	-	-

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