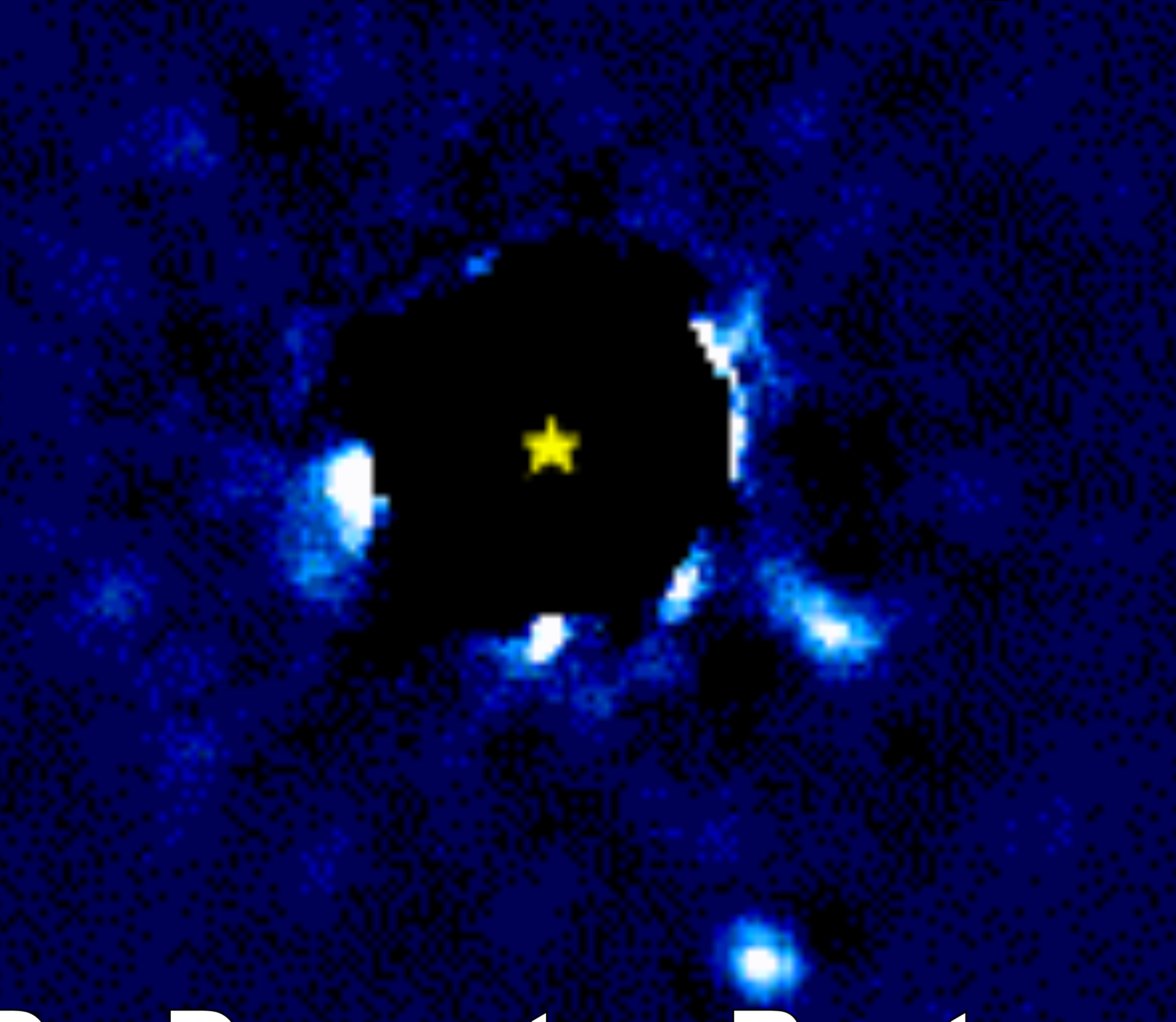
ASTR20A:
Introduction to Astrophysics I



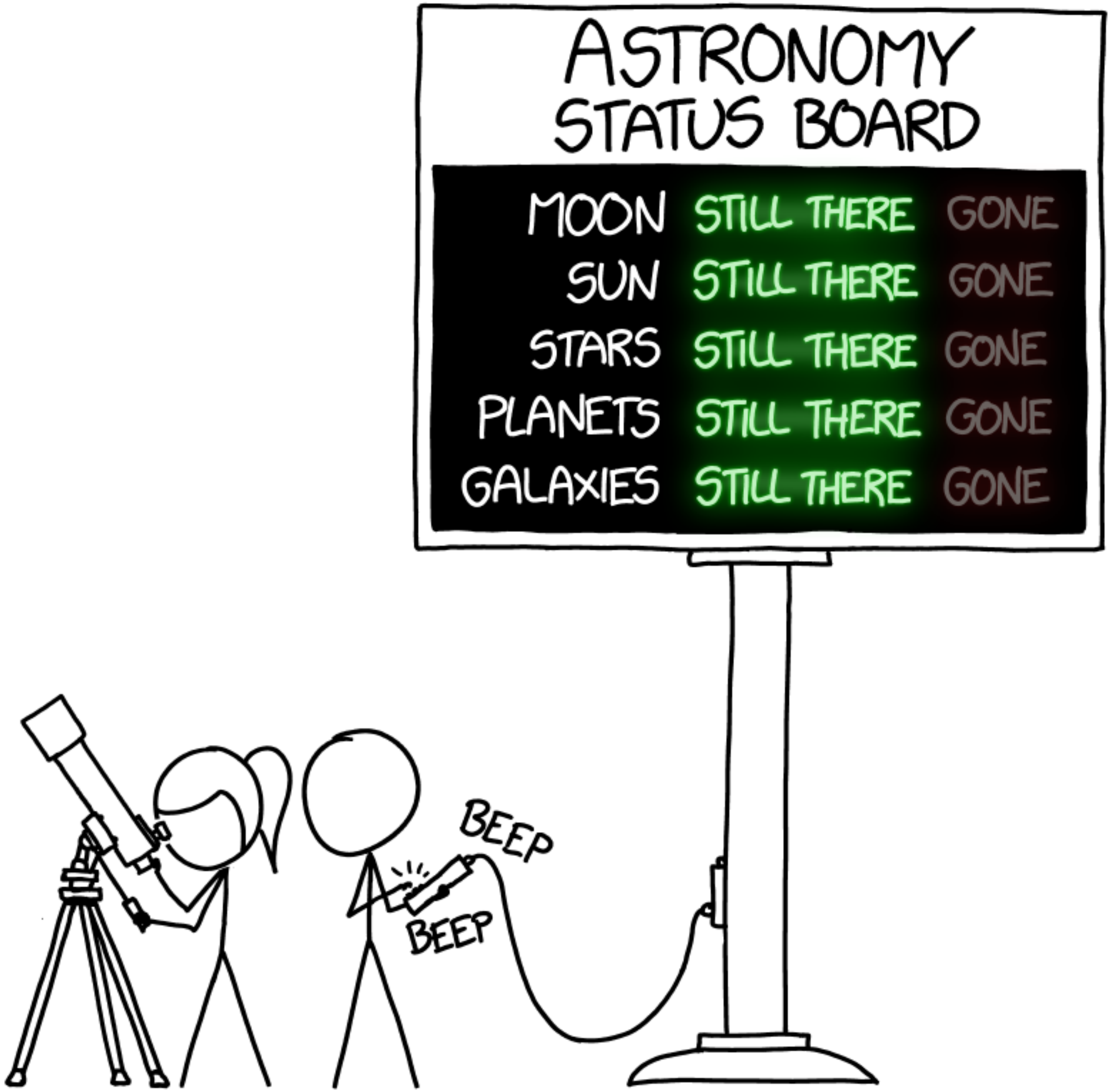
Dr. Devontae Baxter

Lecture 5

Thursday, October 9, 2025

Announcements

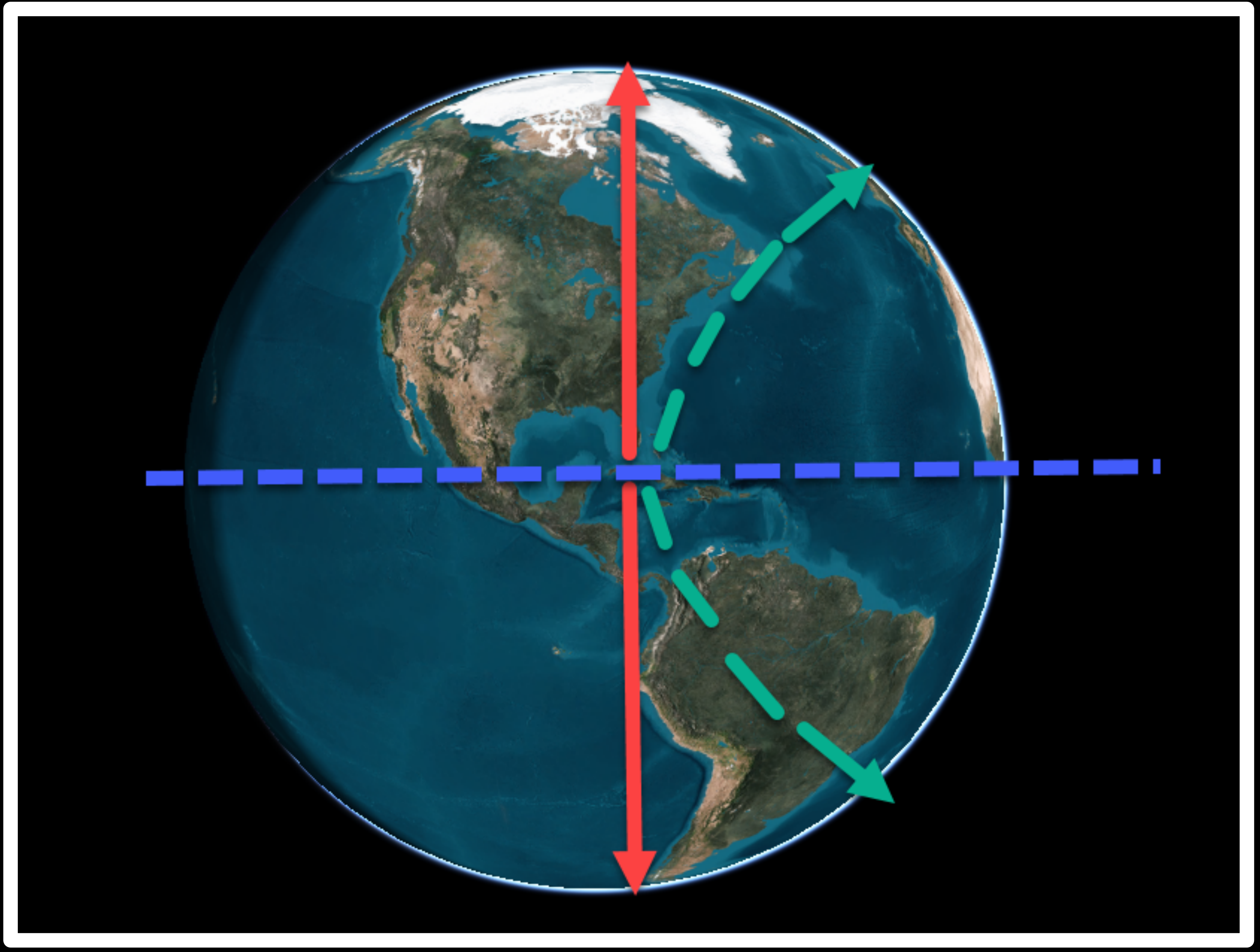
- Homework #2 due **Tuesday, 10/14 by 11:59 pm via Gradescope.** I've included hints to most problems to help you get started.
- Remember that **SERF 383** is reserved for ASTR 20A study session on Mondays from 4-6pm.
- I *highly recommend* that you use this space to work together on the homework.
- Coding exercise #2 due **Sunday, 10/19 by 11:59 pm via Datahub** (this one is a little more involved).



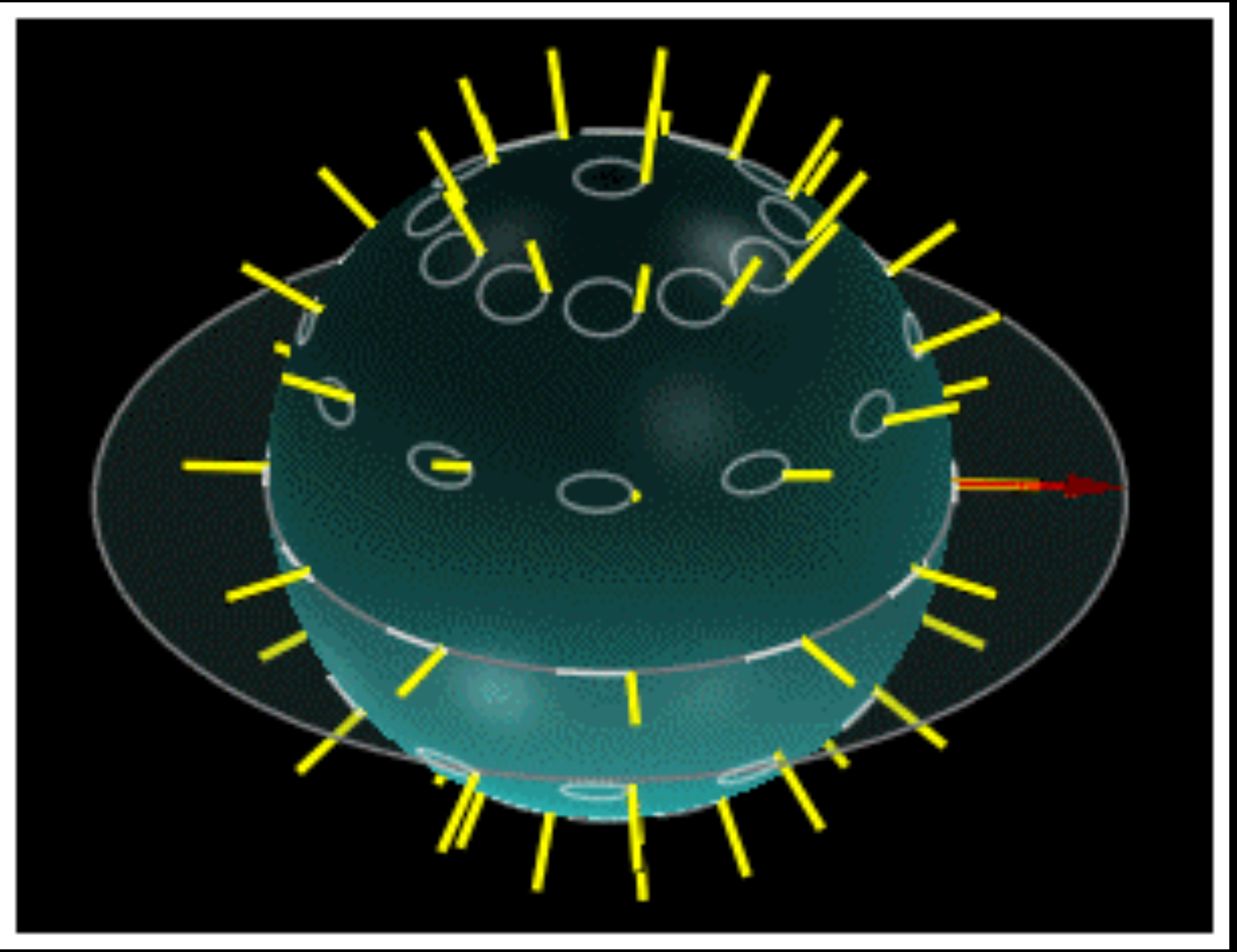
Recap of Lecture 4

Last lecture we demonstrated that the rotation and revolution of the Earth can be explained using the Coriolis Effect and Stellar Aberration, respectively.

Coriolis Effect: Deflection of a projectile that is moving relative to a rotating frame.

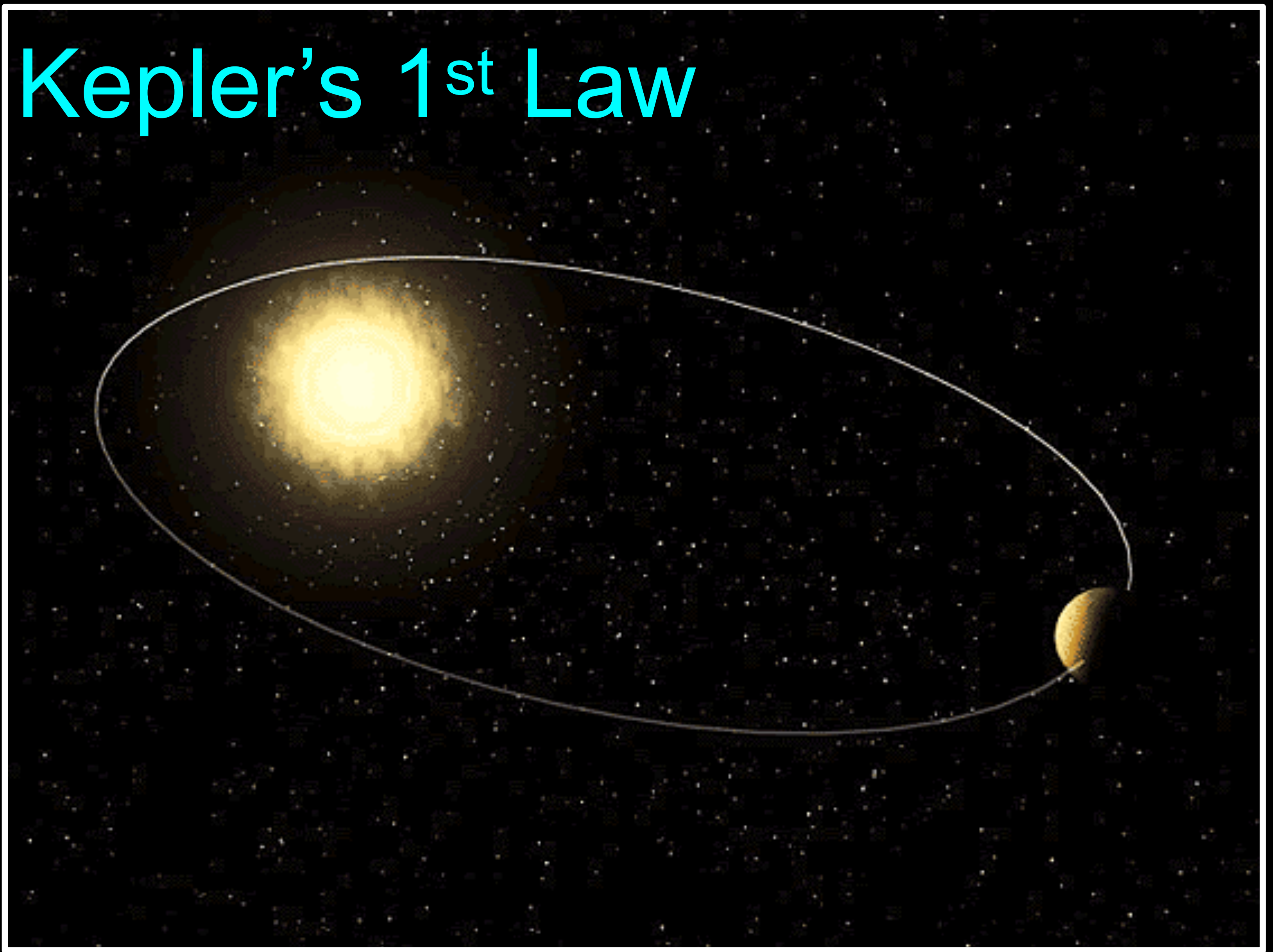


Stellar aberration: Apparent shift in the position of stars due to the Earth's orbital velocity.

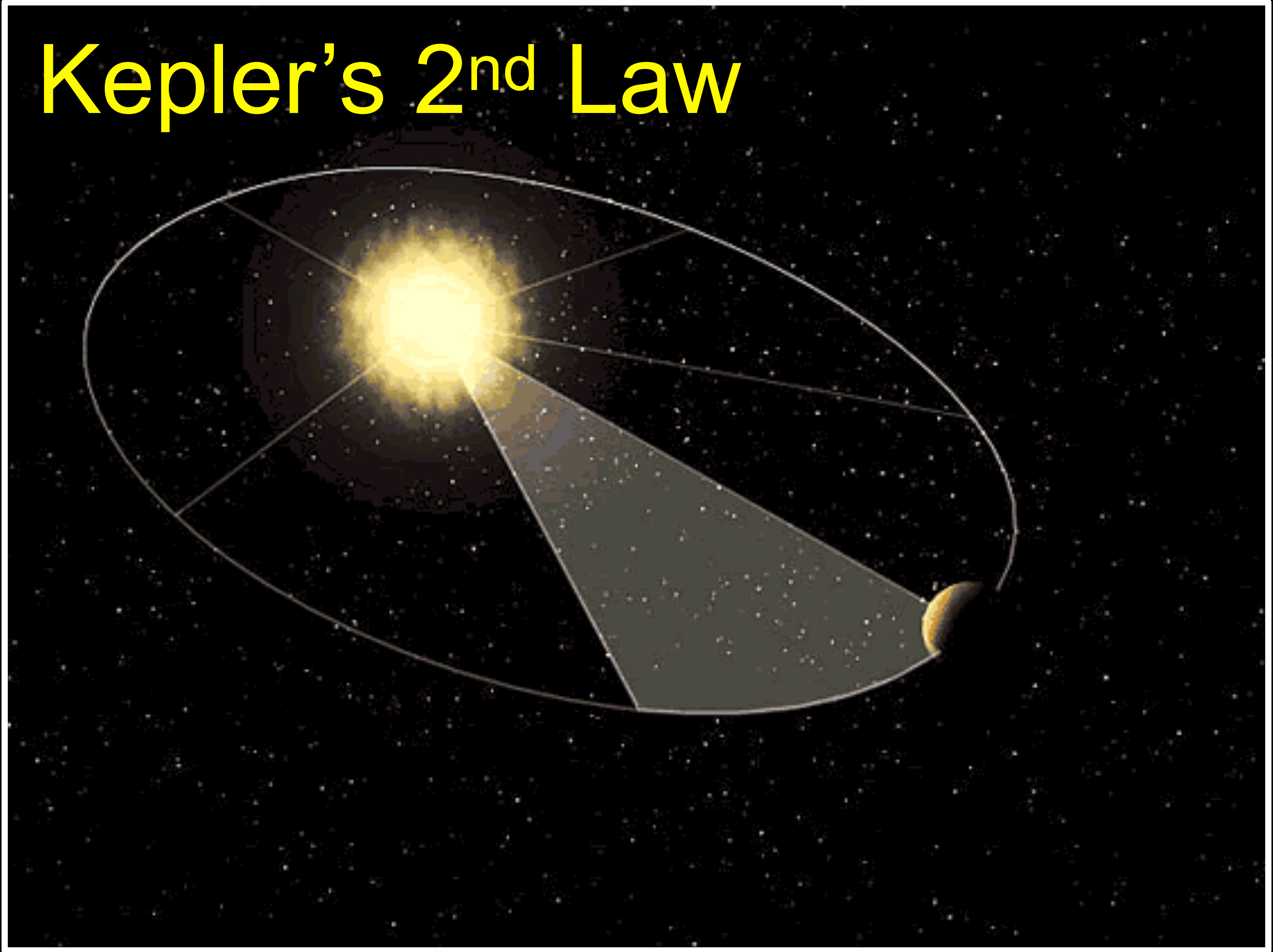


Recap of Lecture 4

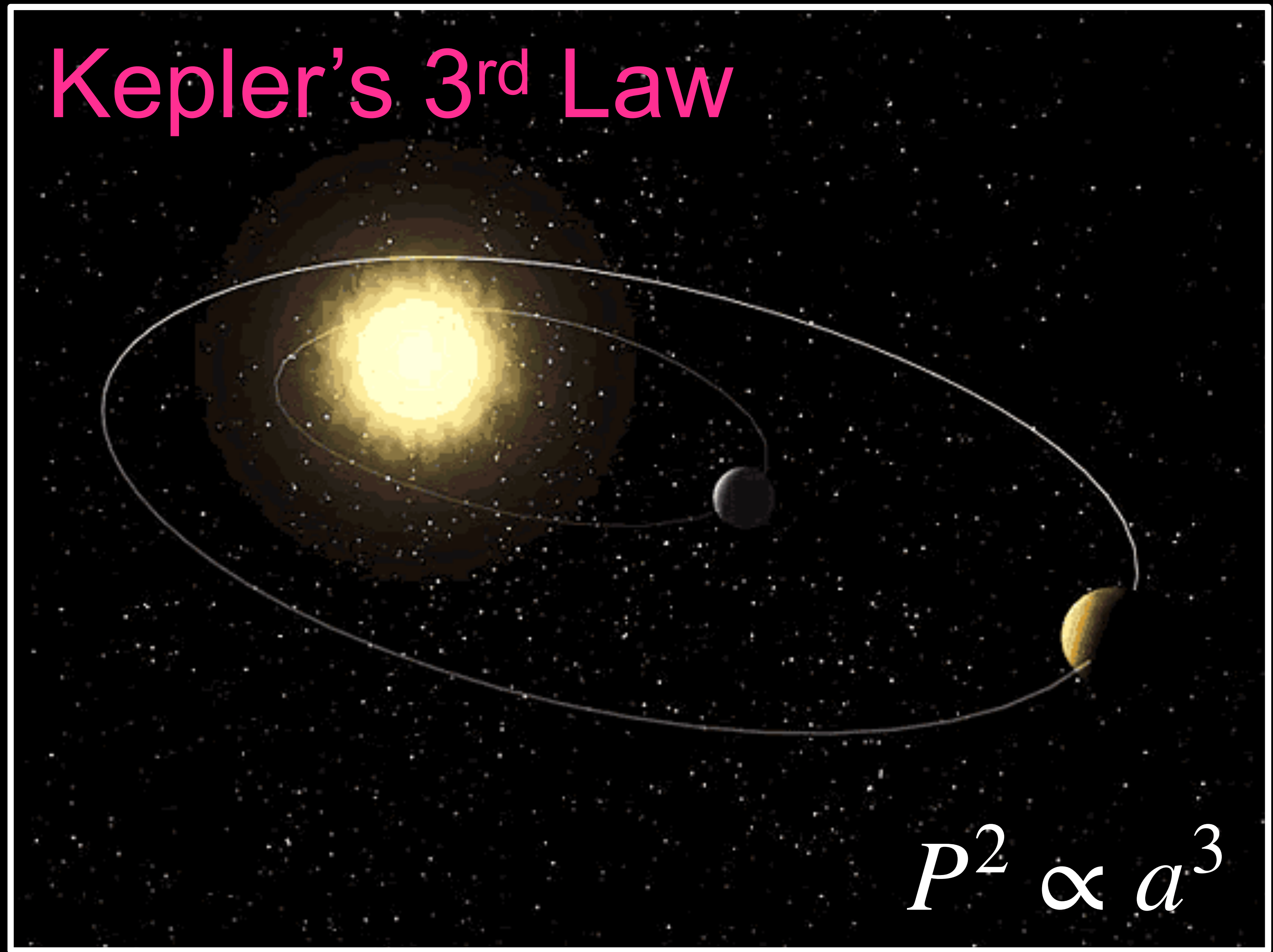
We also reviewed Kepler's Laws of Planetary Motion



Planets orbit in ellipses with the Sun at one focus



Planets sweep out equal areas in equal times.



The square of the orbital period is proportional to the cube of the semi-major axis.

Today, we will use Newton's Laws of Motion to derive Kepler's Laws of Planetary Motion.

Deriving Kepler's Laws

Newton derived the law of universal gravitation **by requiring that the force of gravity result in planetary orbits that obey Kepler's laws of planetary motion.**

In other words, Newton solved the problem in the difficult direction: he deduced the form of the law of gravitation starting from the observations.

We will take the more straightforward route by proving that Kepler's laws *follow as a consequence* of Newton's Laws of Motion.

Even though it may seem numerically incongruous, the derivations will flow more smoothly if **we begin by deriving Kepler's second law, then go on to the first and third laws.**

Deriving Kepler's Second Law

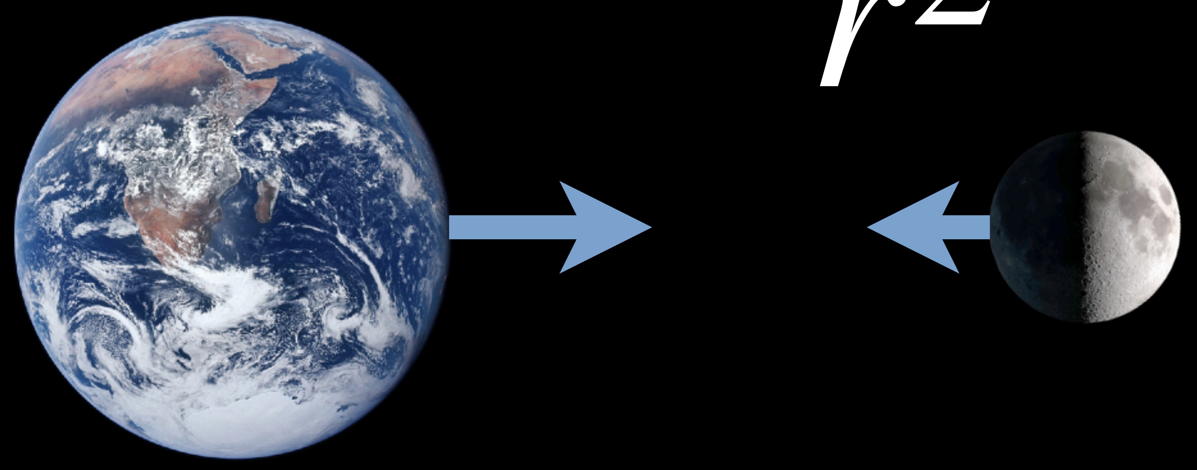
Before we get started, let's review the concept of a **central force.**

By definition, a **central force** is a force directed along the line joining a particle to a fixed point, with a magnitude that depends only on the distance r from that point.

What are some examples of central forces?

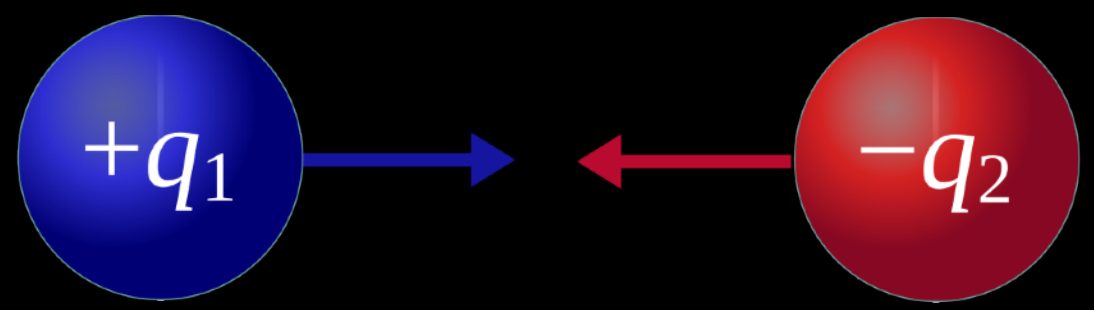
Gravitational Force

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$



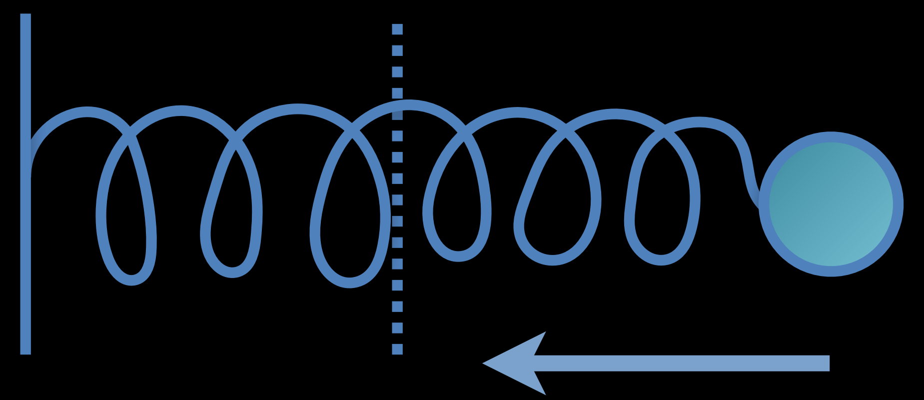
Electrostatic Force

$$\vec{F} = \frac{kq_1q_2}{r^2}\hat{r}$$



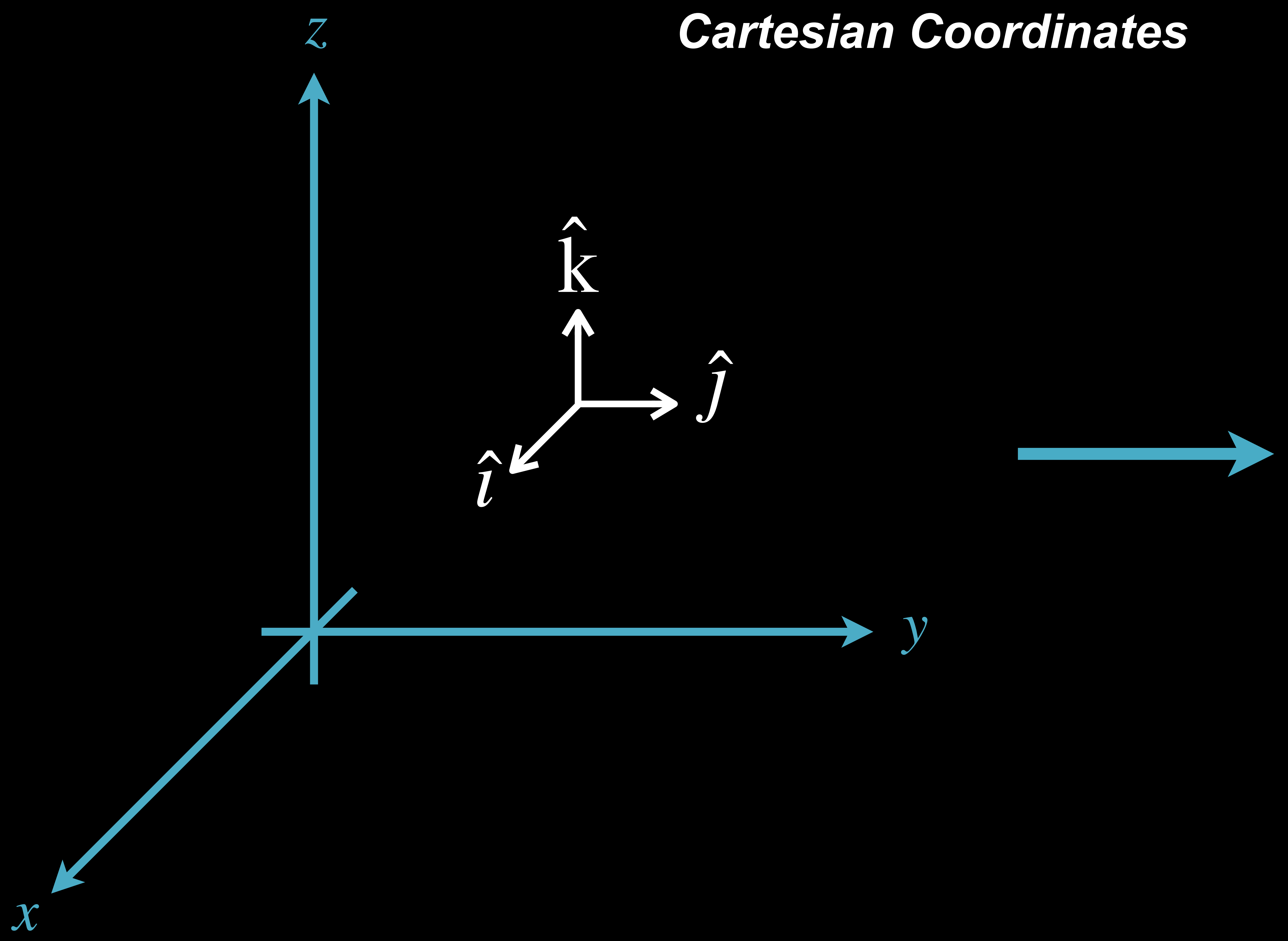
Spring Force

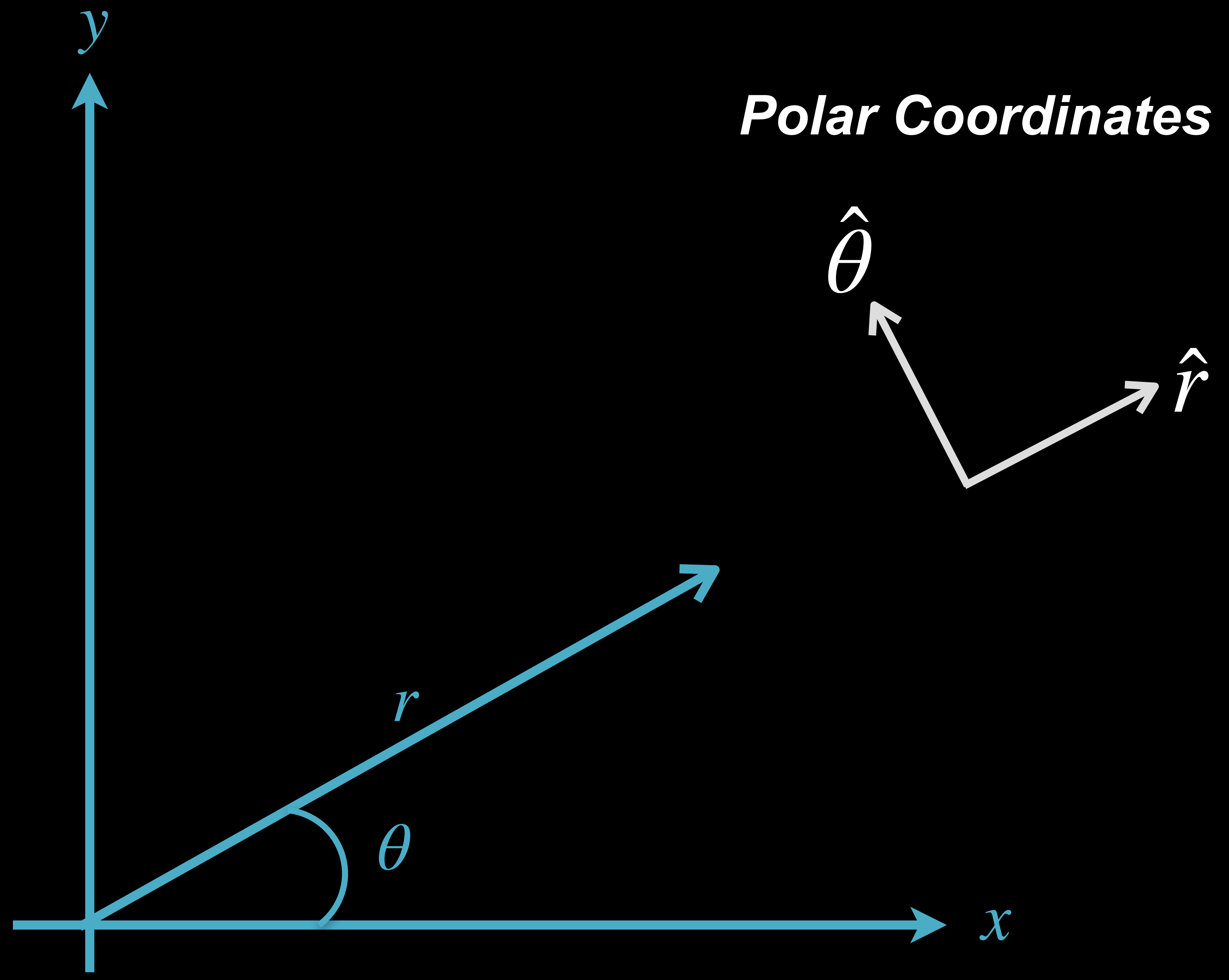
$$\vec{F} = -kr\hat{r}$$



Deriving Kepler's Second Law

While analyzing the motion of a particle responding to a ***central force***, it is *convenient* to be able to switch from *Cartesian Coordinates* to *Polar Coordinates.*





In polar coordinates, the position of an object is described in terms an angle theta (which can increase/decrease with time) and the distance from a fixed point (r).

Note that the carets (^) indicate unit vectors

Deriving Kepler's Second Law

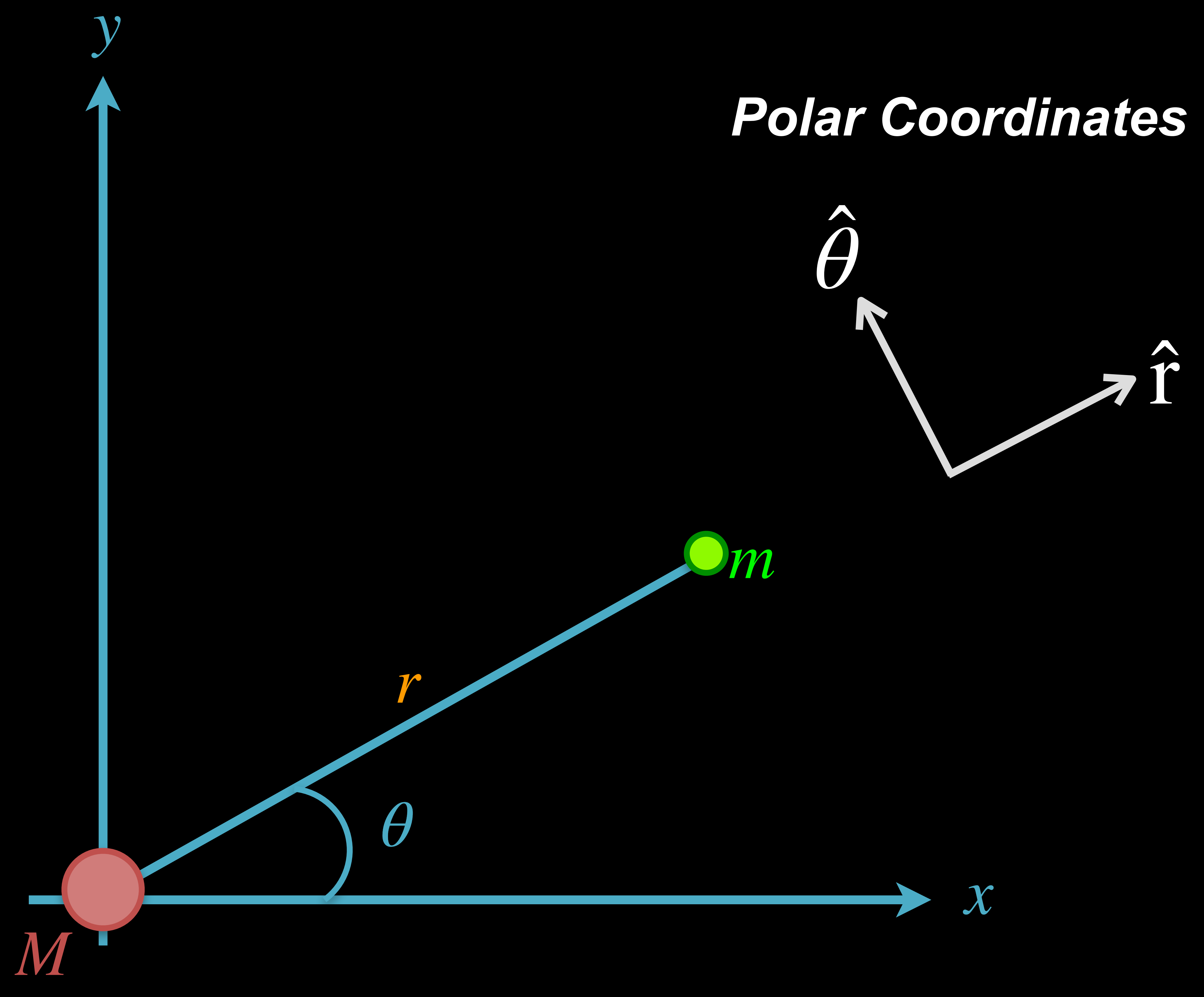
Let's put a mass M at the origin and another mass (m) at a distance r

Each position (x, y) can be expressed using polar coordinates (r, θ):

$$x = r\cos\theta$$
$$y = r\sin\theta$$

There corresponding unit vectors are:

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$
$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$



As the angle θ changes, so do the unit vectors $\hat{r}$ and $\hat{\theta}$.

Note that the carets (^) indicate unit vectors

Deriving Kepler's Second Law

What do we get when we take the **dot product** between these $\hat{r}$ and $\hat{\theta}$?

Recall,

$$[\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1]$$

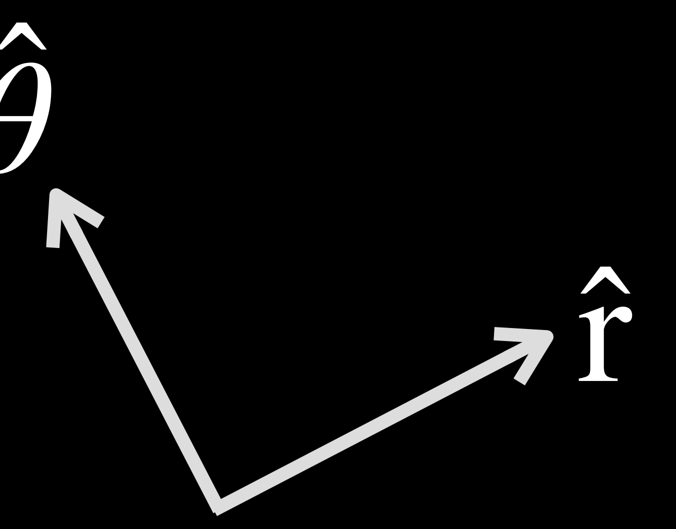
$$\hat{r} \cdot \hat{\theta} = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$$

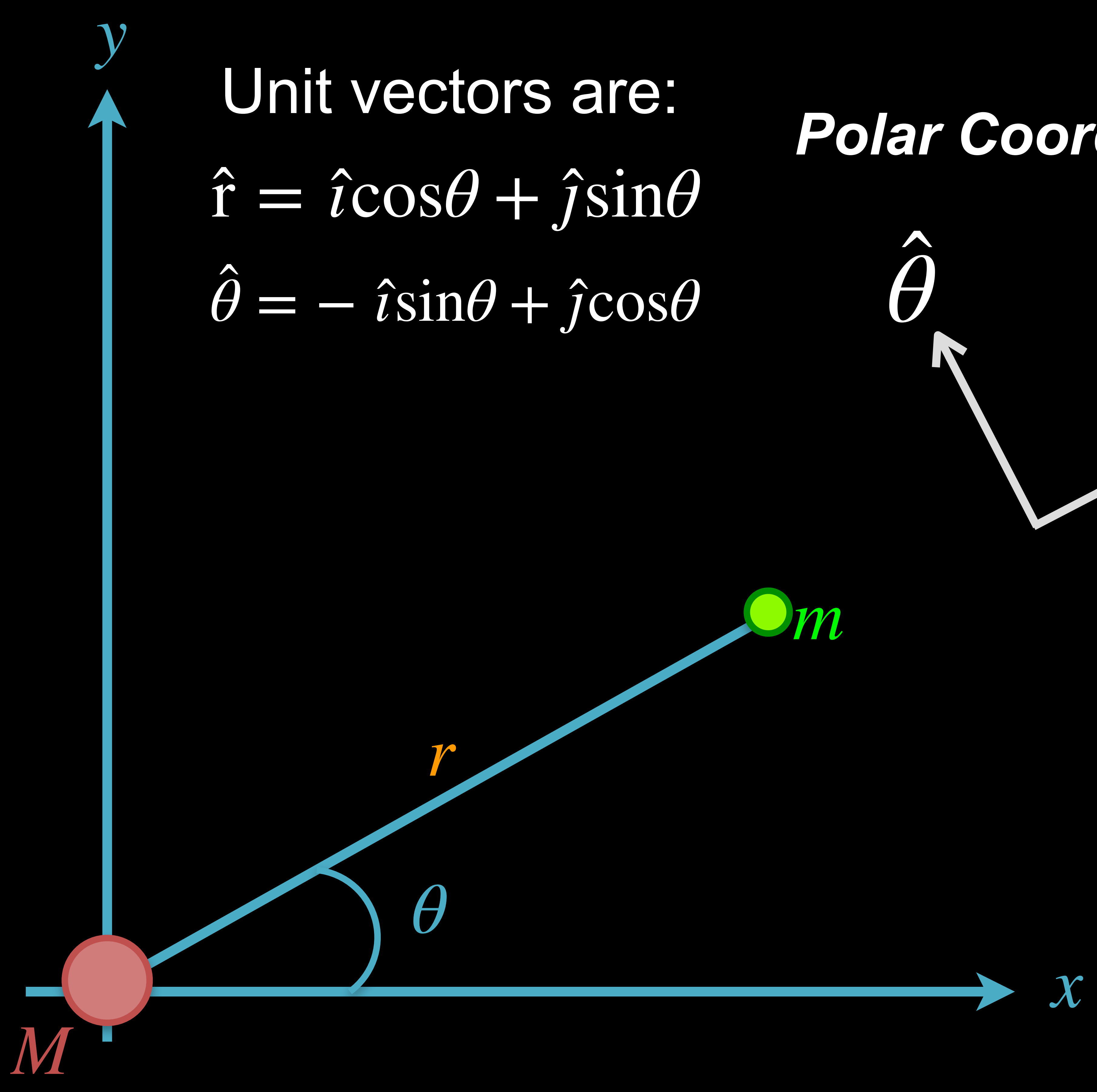
Unit vectors are:

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Polar Coordinates





Note that the carets (^) indicate unit vectors

Deriving Kepler's Second Law

What do we get when we take the **cross product** between these $\hat{r}$ and $\hat{\theta}$?

Recall,

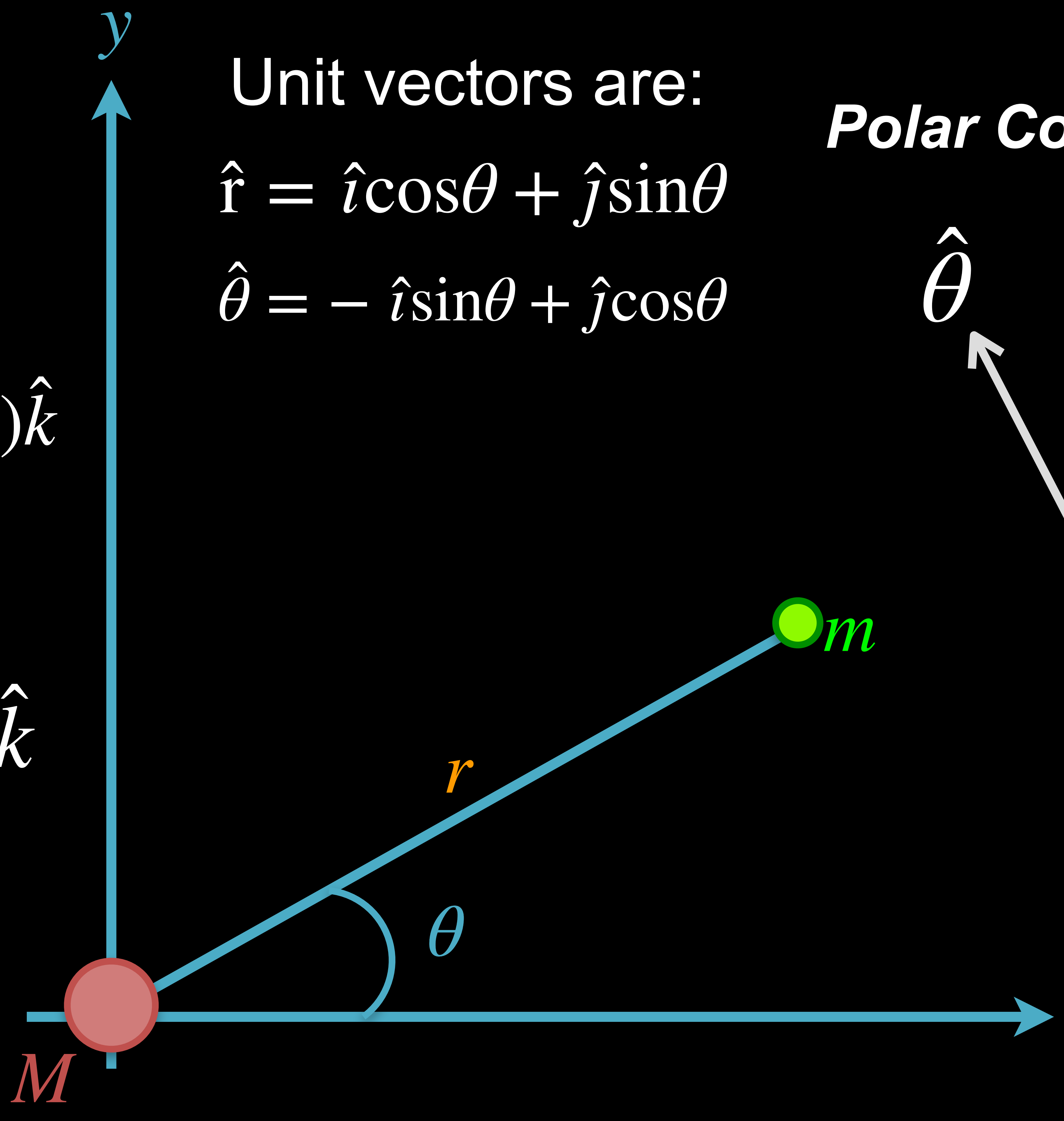
$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

$$\hat{r} \times \hat{\theta} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = (\cos^2 + \sin^2)\hat{k} = \hat{k}$$

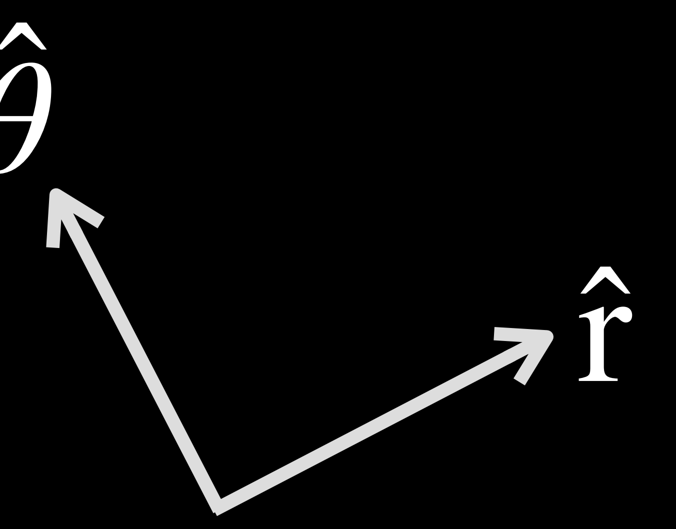
Unit vectors are:

$$\hat{r} = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{\theta} = -\hat{i}\sin\theta + \hat{j}\cos\theta$$



Polar Coordinates



This proves that $\hat{r}$ and $\hat{\theta}$ are *mutually orthogonal*, and *orthogonal* to $\hat{k}$.

Note that the carets (^) indicate unit vectors

Deriving Kepler's Second Law

Taking derivatives of $\hat{\mathrm{r}}$ and $\hat{\theta}$ with respect to θ,

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} = -\hat{\mathrm{r}}$$

$$\hat{\mathrm{r}} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Deriving Kepler's Second Law

We can then apply the chain rule to find the rate of change of the unit vectors.

$$\frac{d\hat{\mathrm{r}}}{dt} = \frac{d\hat{\mathrm{r}}}{d\theta}\frac{d\theta}{dt} = \hat{\theta}\,\frac{d\theta}{dt}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\,\hat{\mathrm{r}}\frac{d\theta}{dt}$$

We just found,

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} = \hat{\theta}$$

$$\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} = -\,\hat{\mathrm{r}}$$

$$\hat{\mathrm{r}} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\,\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Deriving Kepler's Second Law

Note that since these are *unit vectors*, **they change only in direction**, *not in magnitude*.
The velocity of the planet can therefore be expressed in polar coordinates as,

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \underbrace{\frac{d}{dt}\left(r\,\hat{\mathrm{r}}\right)}_{\text{Apply product rule}} = \frac{dr}{dt}\hat{\mathrm{r}} + r\underbrace{\frac{d\hat{\mathrm{r}}}{d\theta}\frac{d\theta}{dt}}_{\text{Chain rule}} \qquad = v_r\hat{\mathrm{r}} + v_t\hat{\theta}$$

Where,

$$v_r = \frac{dr}{dt}$$

Radial velocity

$$v_t = r\frac{d\theta}{dt}$$

Tangential velocity

$$\hat{\mathrm{r}} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\,\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Deriving Kepler's Second Law

The angular momentum $(\overrightarrow{\mathrm{L}})$ of the planet is defined as:

$$\overrightarrow{\mathrm{L}} \equiv \vec{\mathrm{r}} \times \vec{\mathrm{p}}$$

$\vec{\mathrm{p}} = m\vec{\mathrm{v}}$ (linear momentum)

The rate of change of $\overrightarrow{\mathrm{L}}$ is then,

$$\frac{d\overrightarrow{\mathrm{L}}}{dt} = \frac{d}{dt}\left(\vec{\mathrm{r}} \times \vec{\mathrm{p}}\right) = \frac{d\vec{\mathrm{r}}}{dt} \times \vec{\mathrm{p}} + \vec{\mathrm{r}} \times \frac{d\vec{\mathrm{p}}}{dt} = \underbrace{\vec{\mathrm{v}} \times m\vec{\mathrm{v}}}_{\vec{\mathrm{v}} \times \vec{\mathrm{v}} = 0\text{, since the vectors are parallel}} + \vec{\mathrm{r}} \times m\frac{d\vec{\mathrm{v}}}{dt}$$

Recall the vector product rule ,

$$\frac{d}{dt}\left(\mathbf{A} \times \mathbf{B}\right) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}$$

$$\hat{\mathrm{r}} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

Deriving Kepler's Second Law

Using Newton's 2nd law,

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$

The rate of change in angular momentum can be expressed as

$$\frac{d\vec{L}}{dt} = \vec{r} \times m\frac{d\vec{v}}{dt} = \vec{r} \times \vec{F}$$

The cross product in this equation tells us that only the component of the force perpendicular to $\vec{r}$ changes the angular momentum

Deriving Kepler's Second Law

However, for a central force (e.g., gravity) $\overrightarrow{\mathrm{F}}$ is always parallel to $\vec{\mathrm{r}}$!

Therefore,

$$\frac{d\overrightarrow{\mathrm{L}}}{dt} = \vec{\mathrm{r}} \times \overrightarrow{\mathrm{F}} = 0$$

What does this imply about angular momentum for central forces?

Answer: it stays constant in both magnitude and direction for motion under a central force.

Deriving Kepler's Second Law

The **conservation of angular momentum** is directly equivalent to **Kepler's Second Law** — the law of equal areas in equal times.

That is, an increase (decrease) in the orbital radius (r) must be accompanied by a decrease (increase) in the orbital speed (v_t), so that the product (rv_t) — and therefore the **angular momentum** — remains constant.

This ensures that the **area swept out per unit time** stays the same, as stated by **Kepler's Second Law**.

To prove this, let's write the angular momentum in terms of the velocity of the planet.

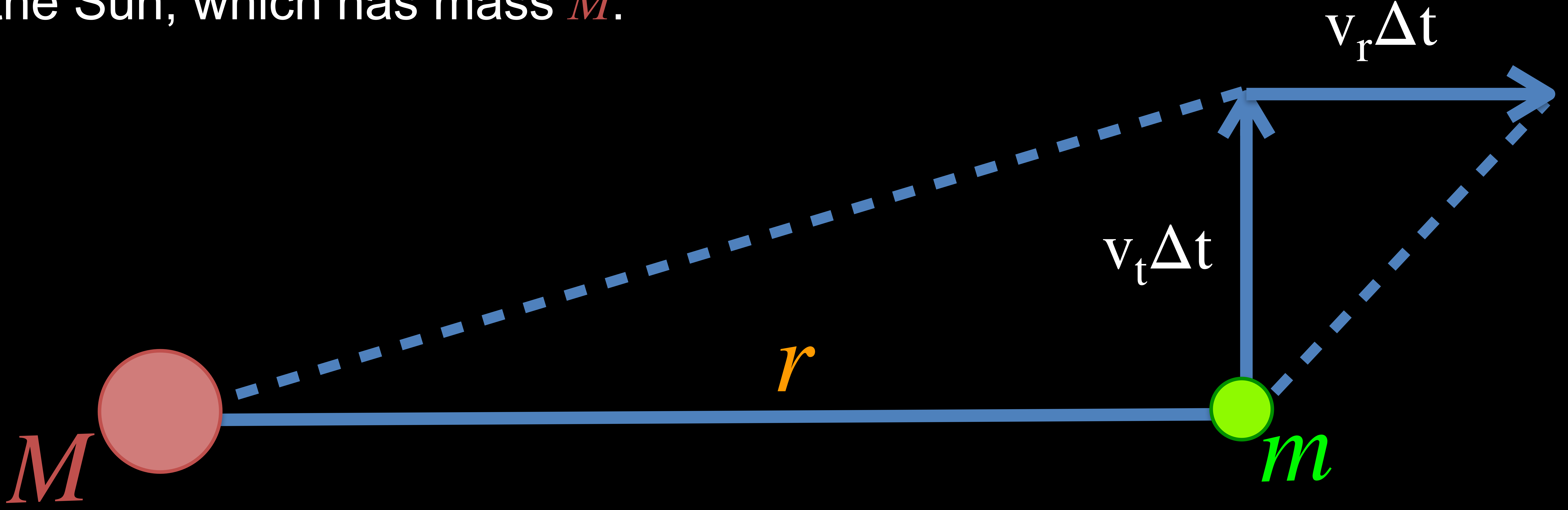
$$\overrightarrow{\mathrm{L}} = \vec{\mathrm{r}} \times m\vec{\mathrm{v}} = mv_t r(\hat{\mathrm{r}} \times \hat{\theta}) = mv_t r\hat{\mathrm{k}} = L\hat{\mathrm{k}}$$

Recall,

$\vec{\mathrm{v}} = v_r\hat{\mathrm{r}} + v_t\hat{\theta}$

Deriving Kepler's Second Law

Now, consider a planet of mass (m); at a time t , it is at a distance r from the Sun, which has mass M.

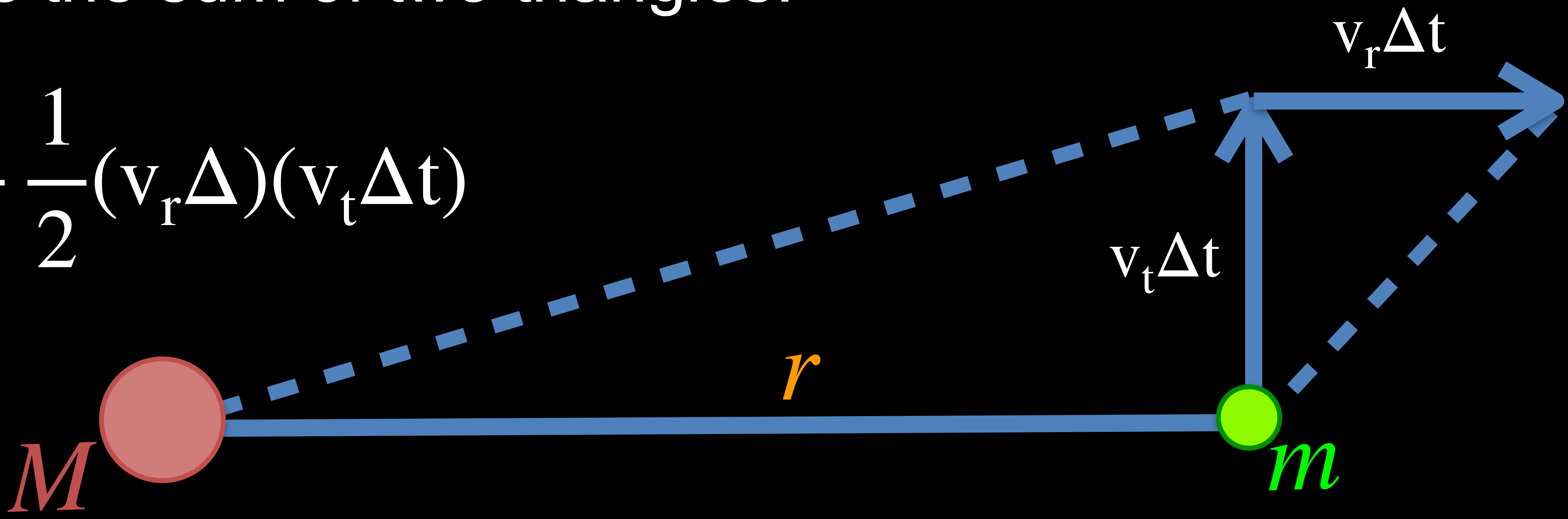


During a brief time interval Δt, a planet moves a distance $v_t\Delta t$ in the tangential direction and $v_r\Delta t$ in the radial direction

Deriving Kepler's Second Law

The area ΔA swept out by the planet–Sun line during this brief interval can be approximated as the sum of two triangles:

$$\Delta A \approx \frac{1}{2}(r)(v_t \Delta t) + \frac{1}{2}(v_r \Delta)(v_t \Delta t)$$



In the limit where $v_t \Delta t \ll r$, the right-hand triangle is vanishingly small compared to the left-hand triangle, and the area swept out becomes:

$$\Delta A \approx \frac{1}{2}(r)(v_t \Delta t)$$

Deriving Kepler's Second Law

Taking the following limit,

$$\lim_{t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r \mathrm{v}_{\mathrm{t}}$$

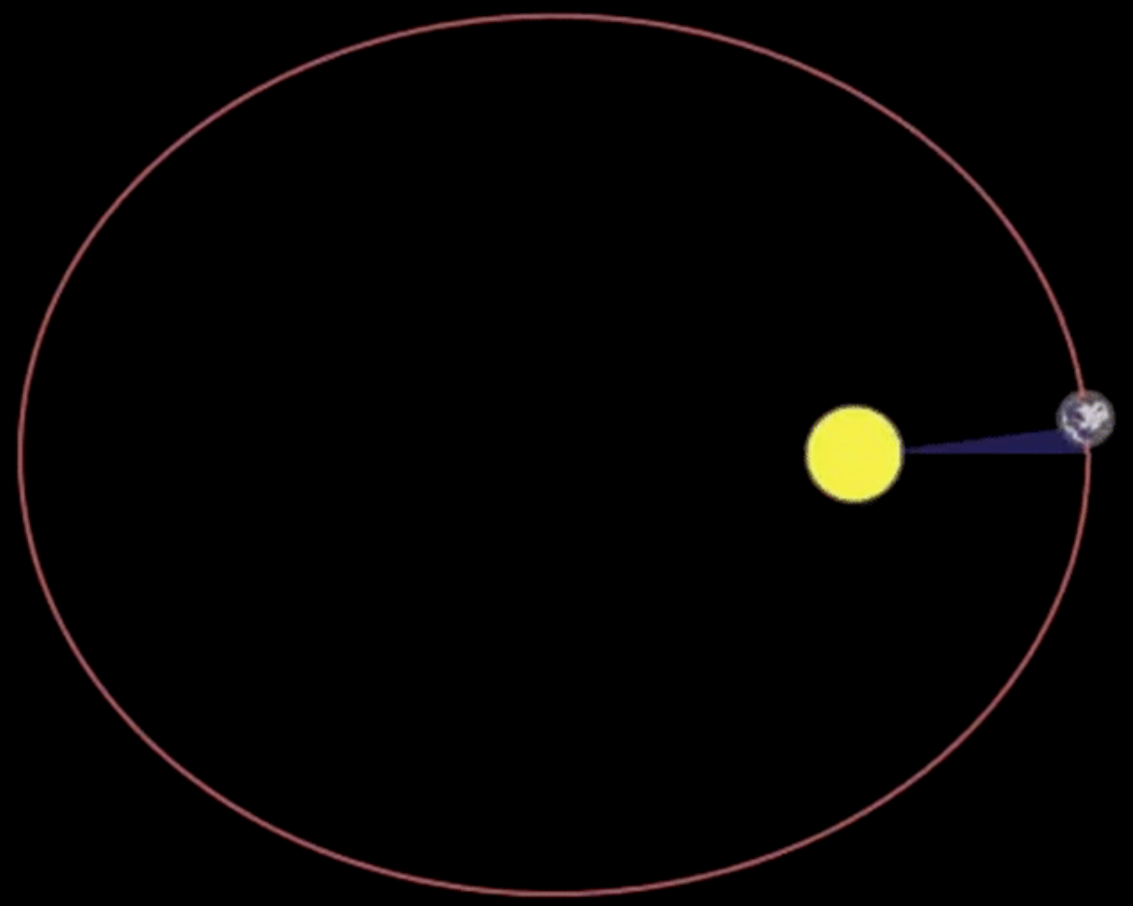
A few slides ago, we showed that $L = m\mathrm{v}_{\mathrm{t}}\mathrm{r}$. This implies the rate at which the planet-sun line sweeps out area can be written as,

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{L}{m}$$

Deriving Kepler's Second Law

Since L and m are constant, so is the rate $\dfrac{dA}{dt}$ at which the planet-Sun line sweeps out area.

$$\boxed{\frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{L}{m}}$$

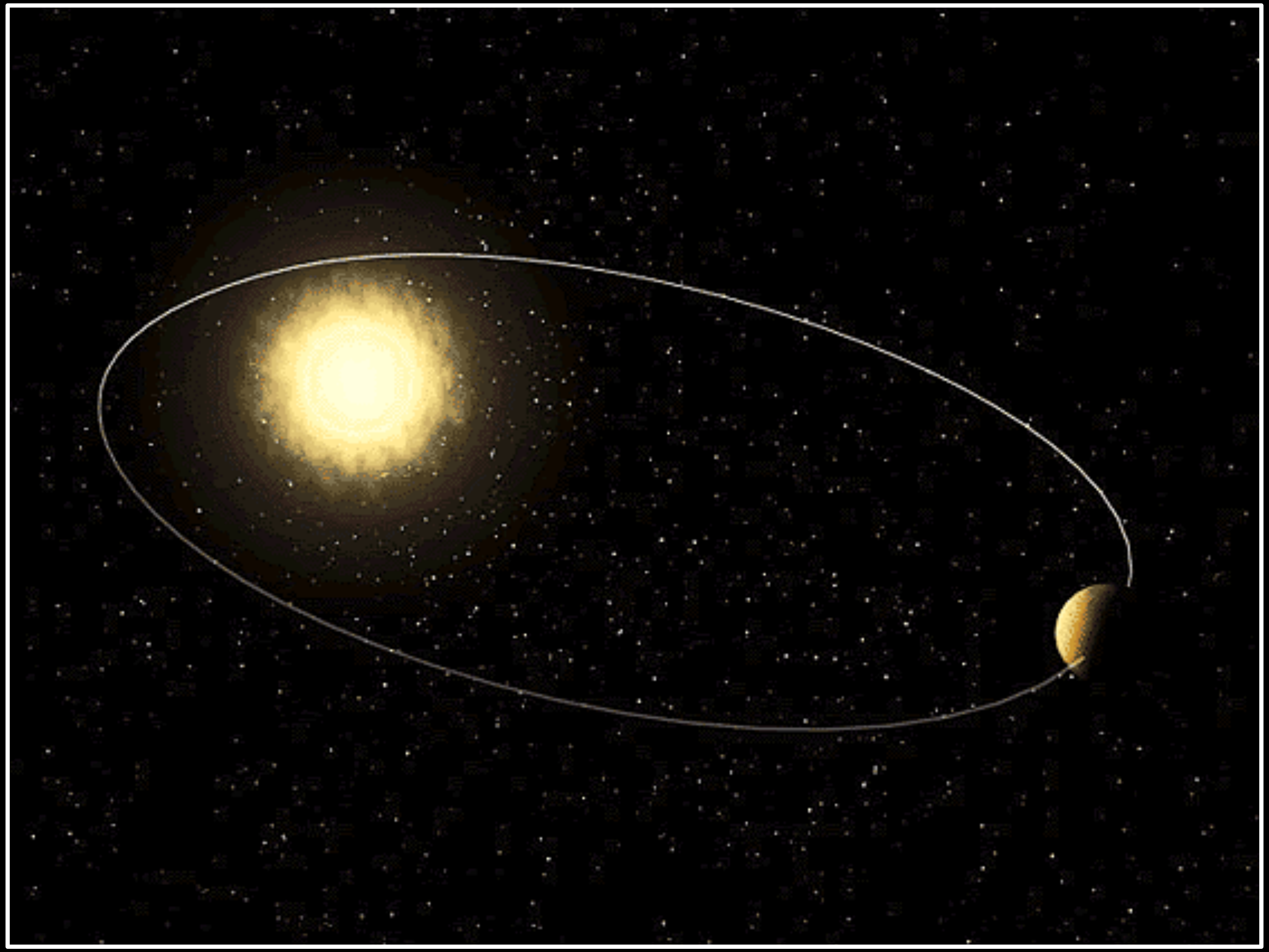


In other words, we have proven that Kepler's second law will be true for a body acting under *any central force*, not just gravity!

Deriving Kepler’s First Law

To prove that Kepler’s 1st law follows from Newton’s law of universal gravitation, will we have to show:

The trajectory $r(\theta)$ of an object with mass m constitutes an ellipse with the larger object of mass M at one focus.



Deriving Kepler's First Law

The angular momentum (L) per unit mass (m) of an orbiting body is:

$$\frac{L}{m} = r\mathrm{v}_{\mathrm{t}} = r^2\frac{d\theta}{dt}$$

(which is *constant*)

If the force acting on the mass m is gravitational, then from Newton's law of universal gravitation and second law of motion

$$\vec{\mathrm{F}} = -\frac{GMm}{r^2}\hat{\mathrm{r}} = m\frac{d\vec{\mathrm{v}}}{dt}$$

Recall,

$$v_t = r\frac{d\theta}{dt} \quad v_r = \frac{dr}{dt}$$

Deriving Kepler's First Law

The orbital acceleration under gravity is therefore,

$$\frac{d\vec{\mathrm{v}}}{dt} = -\frac{GM}{r^2}\hat{\mathrm{r}}$$

But, we've already shown that,

$$\frac{d\hat{\theta}}{dt} = -\hat{\mathrm{r}}\frac{d\theta}{dt} \xrightarrow[\textit{Rearranging terms}]{} \hat{\mathrm{r}} = -\left(\frac{d\theta}{dt}\right)^{-1}\left(\frac{d\hat{\theta}}{dt}\right)$$

Deriving Kepler's First Law

Therefore, we can write the acceleration terms as,

$$\frac{d\vec{\mathrm{v}}}{dt} = -\frac{GM}{r^2}\hat{\mathrm{r}} = \frac{GM}{r^2}\left(\frac{\mathrm{d}\theta}{dt}\right)^{-1}\left(\frac{\mathrm{d}\hat{\theta}}{dt}\right)$$

Using $\frac{L}{m} = r\mathrm{v}_{\mathrm{t}} = r^2\frac{d\theta}{dt}$, we can rewrite this equation in terms of angular momentum,

$$\frac{L}{GMm}\frac{d\vec{\mathrm{v}}}{dt} = \frac{d\hat{\theta}}{dt}$$

Deriving Kepler's First Law

Now, we can integrate this equation with respect to time!

$$\frac{L}{GMm}\frac{d\vec{\mathrm{v}}}{dt} = \frac{d\hat{\theta}}{dt}$$

$$\frac{L}{GMm}\int \frac{d\vec{\mathrm{v}}}{dt}dt = \int \frac{d\hat{\theta}}{dt}dt$$

Constant of integration

$$\frac{L}{GMm}\vec{\mathrm{v}} = \hat{\theta} + \boxed{\vec{\mathrm{e}}}$$

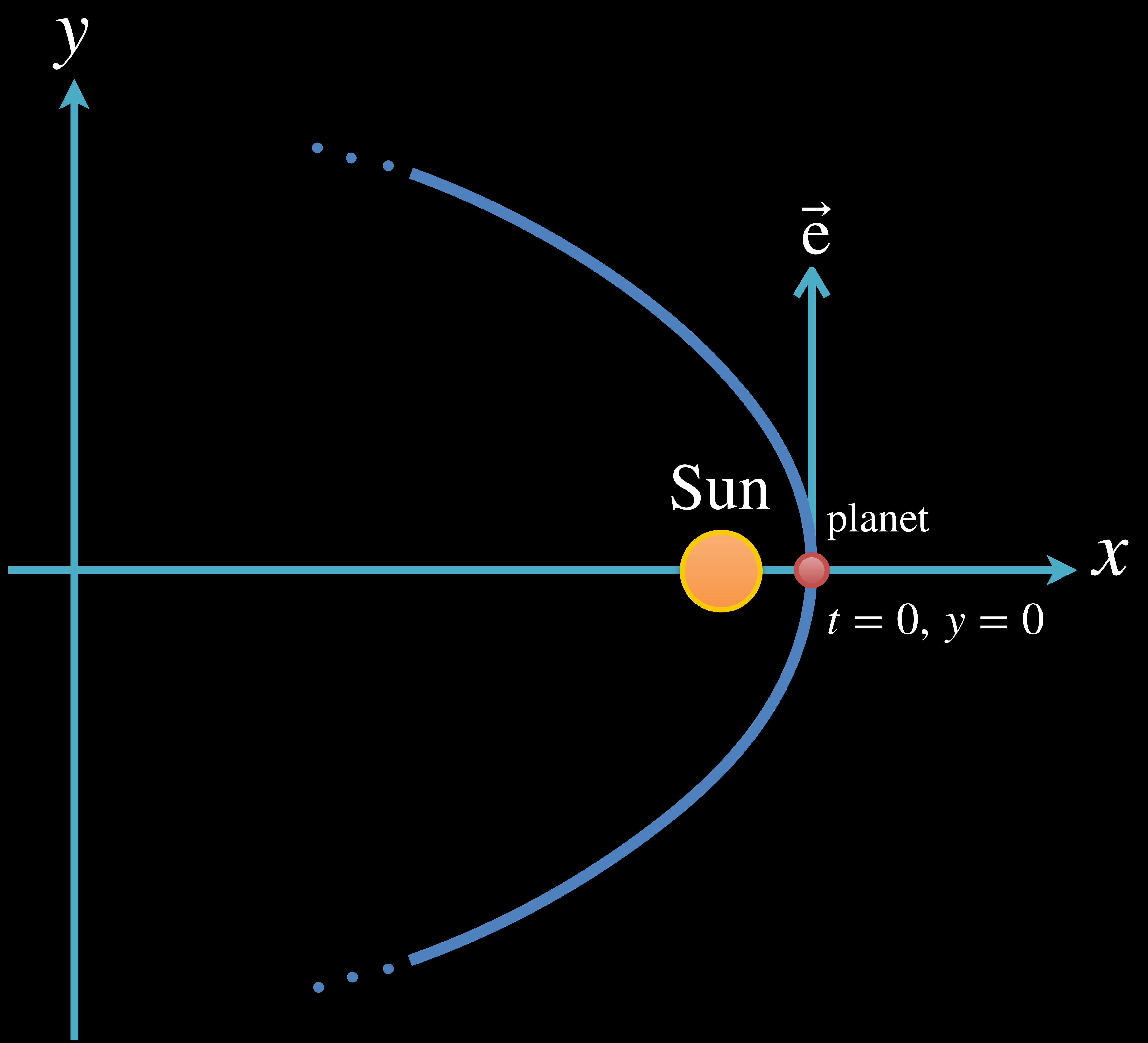
Deriving Kepler's First Law

Let's solve for the constant of integration ($\vec{e}$) by choosing initial conditions that make our lives easier.

Specfically, let's choose the time $t = 0$ to occur at perihelion passage (closest approach) of the planet *and* orient the axes so that perihelion passage occurs on the positive x-axis.

In this case, both $\hat{\theta}$ and $\vec{v}$ point in the y-direction at $t = 0$; therefore, we can write $\vec{e} = e\hat{j}$.

$$\frac{L}{GMm}\vec{v} = \hat{\theta} + e\hat{j}$$



Deriving Kepler's First Law

Now, we can take the dot product of this equation and the unit vector $\hat{\theta}$:

$$\frac{L}{GMm}\vec{\mathrm{v}} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\theta} + e\hat{\jmath} \cdot \hat{\theta}$$

Working through the dot products we find,

1. $\vec{\mathrm{v}} \cdot \hat{\theta} = [v_r\hat{\mathrm{r}} + v_t\hat{\theta}] \cdot \hat{\theta} = v_t = \dfrac{L}{mr}$
2. $\vec{\theta} \cdot \hat{\theta} = 1$
3. $\hat{\jmath} \cdot \hat{\theta} = -\hat{\jmath} \cdot \hat{\imath}\sin\theta + \hat{\jmath} \cdot \hat{\jmath}\cos\theta = \cos\theta$

Recall,

$$\hat{\mathrm{r}} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta$$

$$\hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta$$

$$\vec{v} = v_r\hat{\mathrm{r}} + v_t\hat{\theta}$$

Therefore, the equation becomes,

$$\frac{L^2}{GMm^2r} = 1 + e\cos\theta$$

Solving for r →

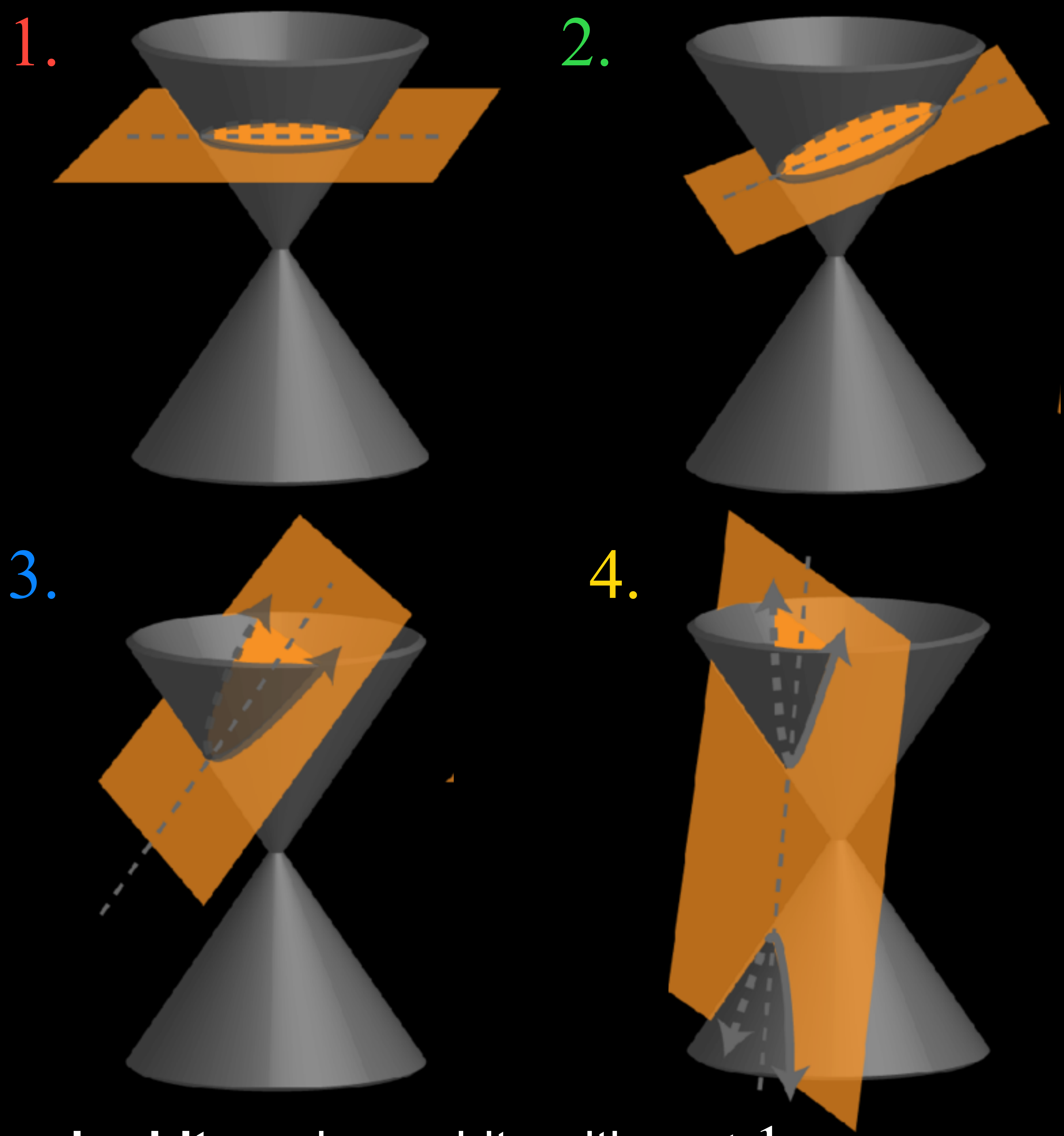
$$\boxed{r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}}$$

Deriving Kepler's First Law

This is the equation of a **conic section**:

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}$$

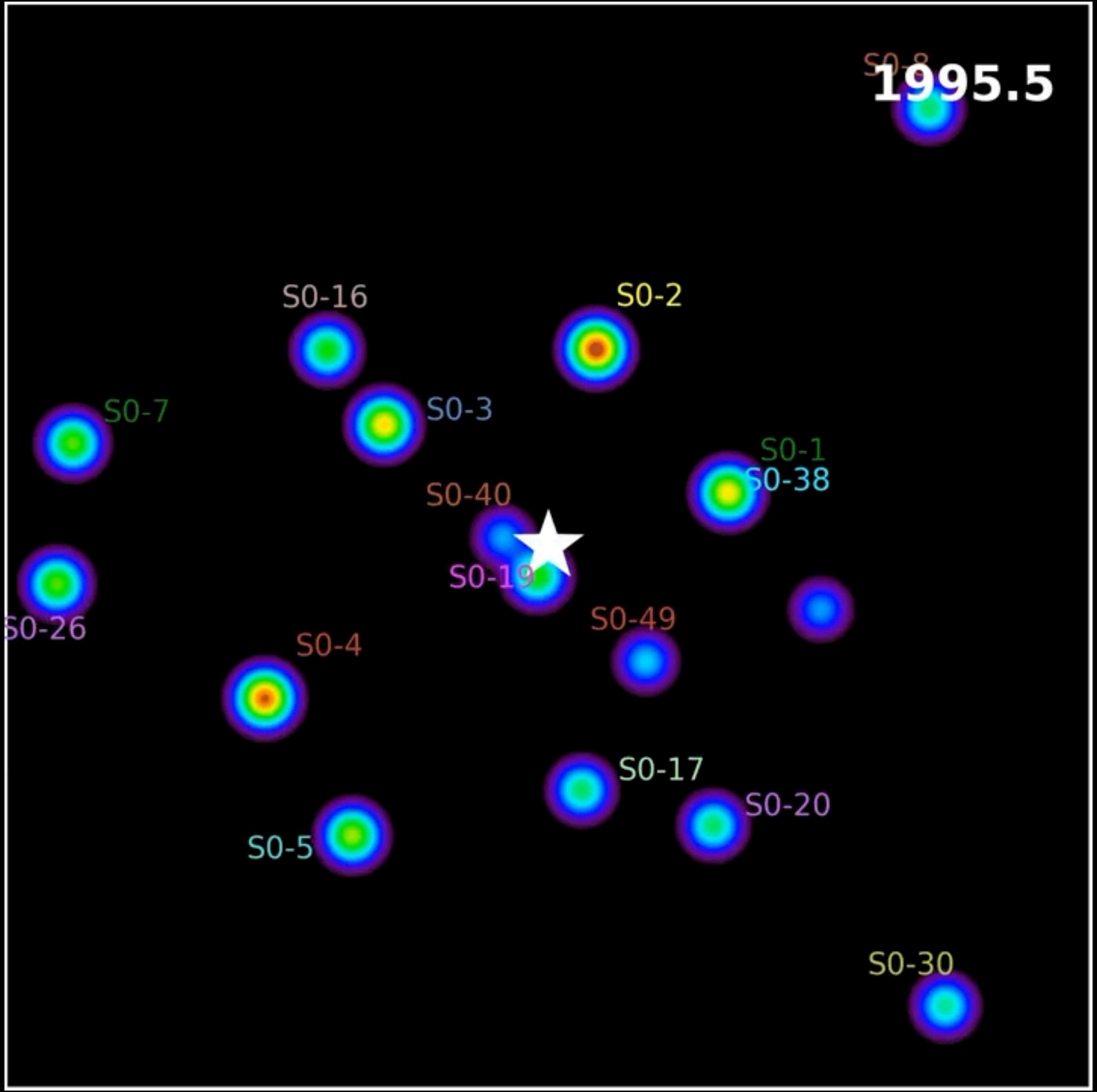
1. Slicing a plane perpendicular to the cone's axis creates a conic section that is a **circle** (i.e, **special case with $e = 0$**).
2. Tilting the slicing plane from the perpendicular by an angle *less than the half-opening angle* of the cone, then the conic section is an **ellipse** ($0 < e < 1$).
3. Tilting the slicing plane from the perpendicular by an angle *exactly equal to the half-opening angle* of the cone the conic section resulting is a **parabola** ($e = 1$).
4. Titling the slicing plane by a larger angle, the conic section that results is a **hyperbola** ($e > 1$).



Kepler's 1st law is a *special case* that only deals with **closed orbits** — i.e., orbits with $e < 1$

Brain Break – Think-pair-share

1. What do you think we're observing in this video?
2. What laws govern the motions of these objects?
3. What is the object in the center of the video?

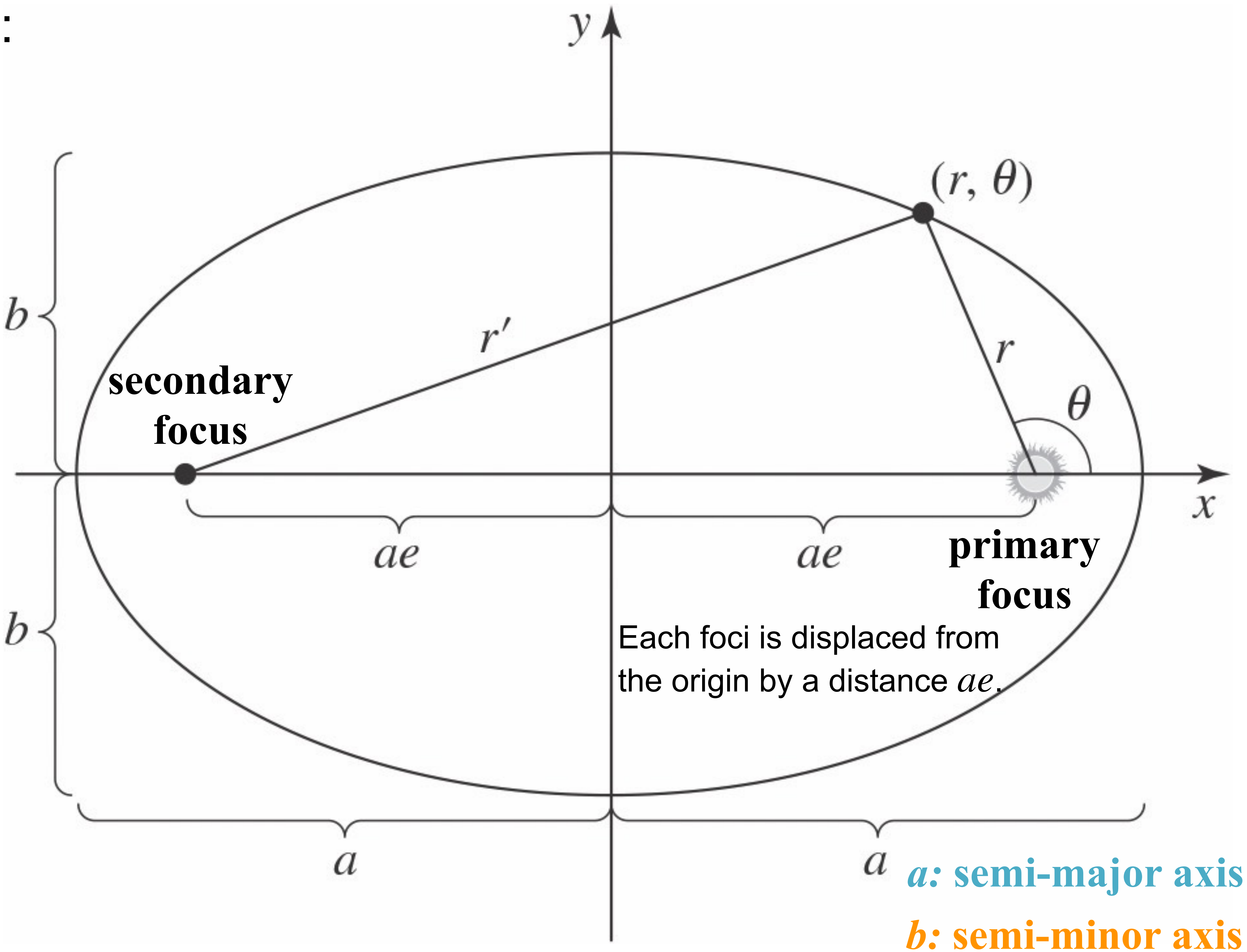


Elliptical Orbits

Let's support the assertion that the constant e represents the eccentricity of an ellipse!

First, let's review the properties of an ellipse:

1. **Foci (plural of focus):** Two fixed points inside the ellipse.
2. **Major axis ($2a$):** The longest diameter of the ellipse, passing through both foci.
3. **Minor axis ($2b$):** The shortest diameter, perpendicular to the major axis at the center.
4. **Center:** The midpoint of the major and minor axes.
5. **Eccentricity (e):** A measure of how "stretched" the ellipse is, defined as $e = \dfrac{c}{a}$, where c is the distance from the center to a focus.



a: semi-major axis

b: semi-minor axis

Elliptical Orbits

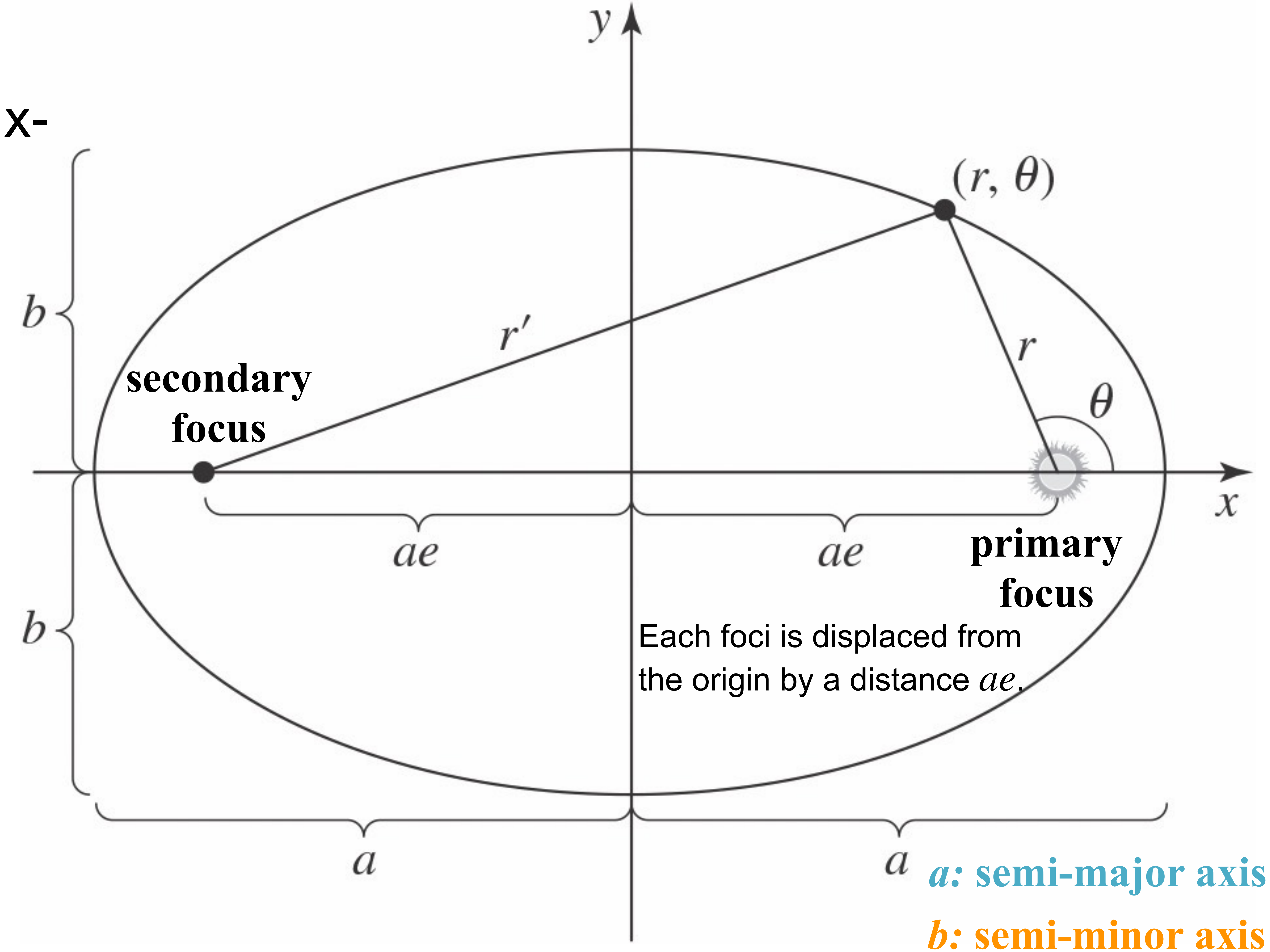
A basic property of an ellipse is that the distance $r + r'$ is constant.

Take two points on the ellipse along the x-axis:

- r at $(x = a, y = 0)$
- r' at $(x = -a, y = 0)$

Therefore, **the sum of the distances to the foci** is constant:

$$r + r' = 2a$$



a: semi-major axis

b: semi-minor axis

Elliptical Orbits

The **perihelion** and **aphelion** distances follow directly from the constant sum of distances to the foci of the ellipse.

At **perihelion** (closest separation) the distance from the Sun, let's call it "q" is:

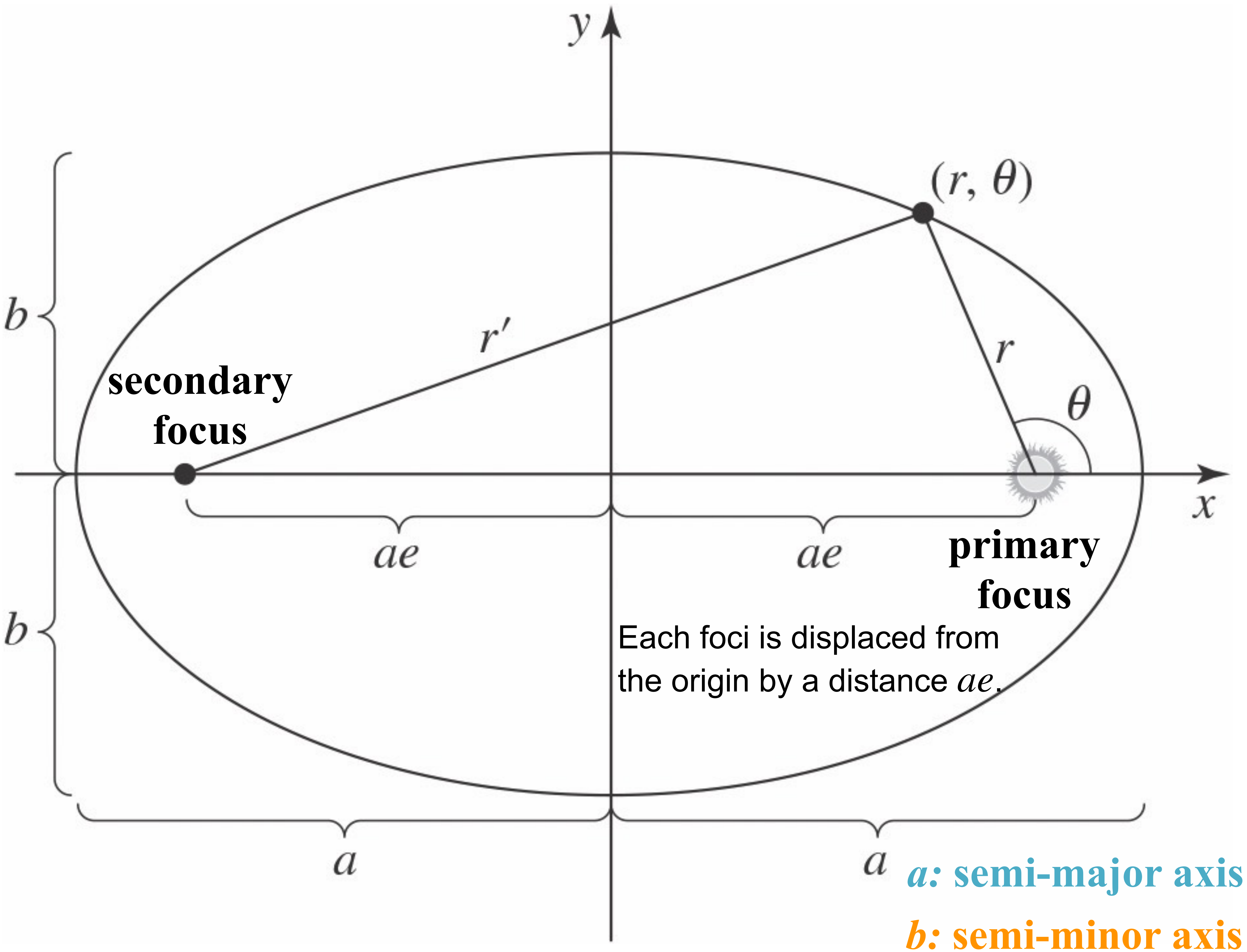
$$q = a - ae = a(1 - e)$$

At **aphelion** (farthest separation) the distance from the Sun, let's call it "Q" is:

$$Q = a + ae = a(1 + e)$$

Such that,

$$Q + q = (a - ae) + (a + ae) = 2a$$

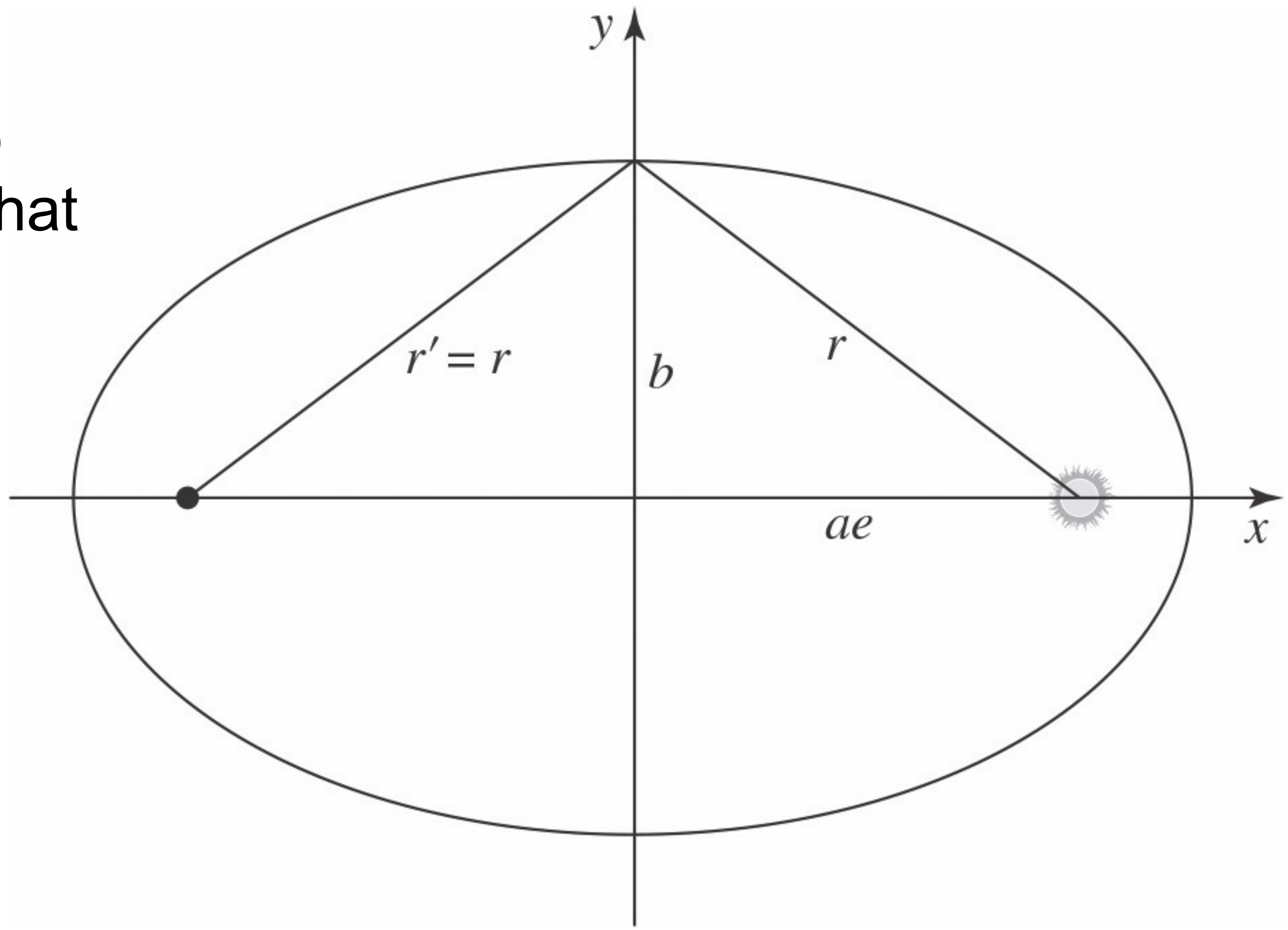


***a*: semi-major axis**

***b*: semi-minor axis**

Elliptical Orbits

Now consider the point of the ellipse that lies on the positive y-axis, where $r = r' = a$.



From the Pythagorean theorem, as applied to the right triangle drawn in the figure, we find that

$$r^2 = b^2 + (ae)^2$$

However, since $r = a$

$$a^2 = b^2 + (ae)^2 \longrightarrow b^2 = a^2(1 + e^2)$$

This allows us to define the eccentricity as,

$$\boxed{e = \left(1 - \frac{b^2}{a^2}\right)^{1/2}}$$

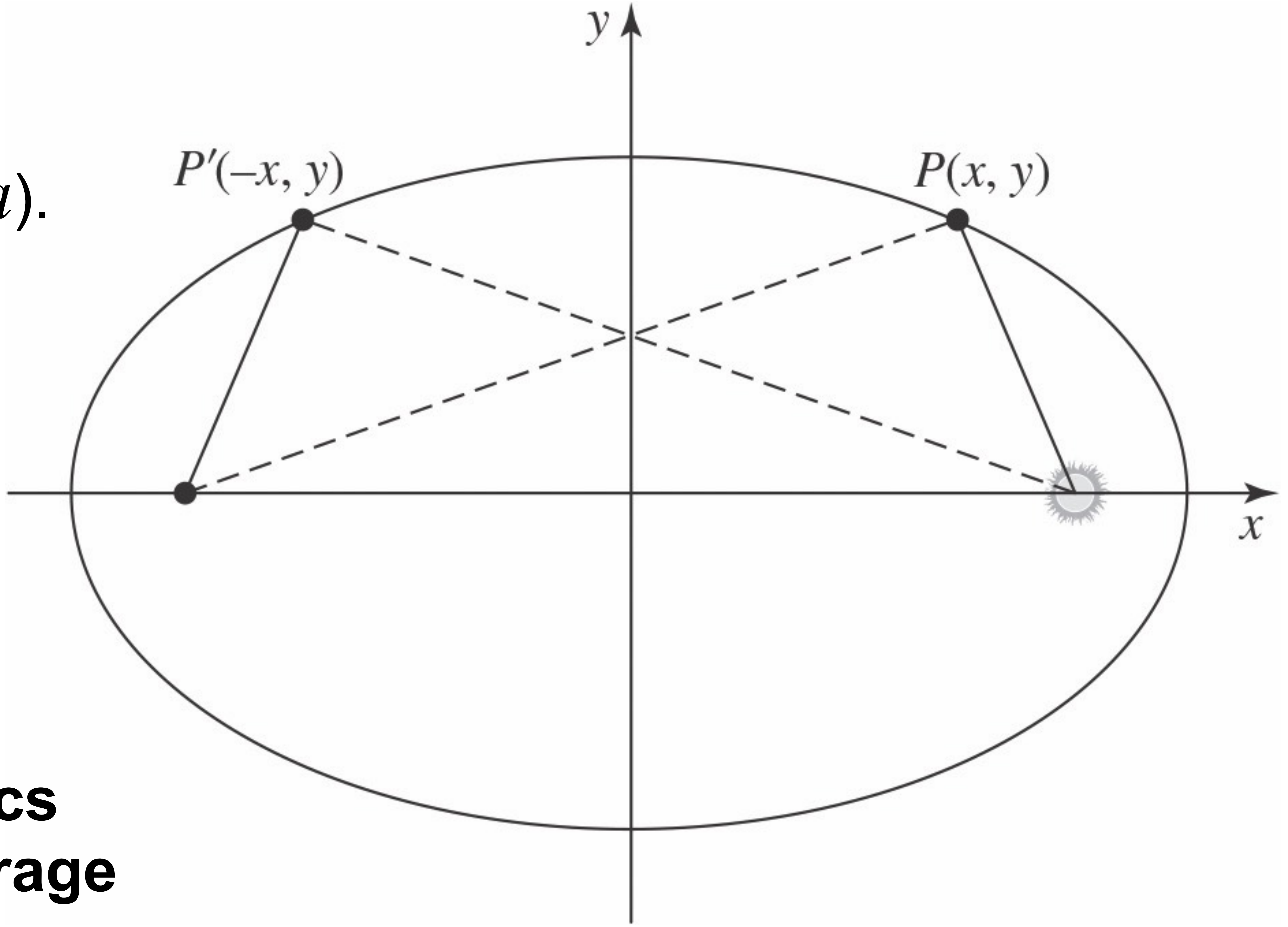
Elliptical Orbits

It can also be shown that the average distance of all points on the ellipse from either focus is equal to the semi-major axis (a).

This follows from the ellipse definition: the sum of distances to the foci is constant ($r + r' = 2a$).

$$\langle r \rangle = \frac{r + r'}{2}\bigg|_{r=r'=a} = \frac{2a}{2} = a$$

This property is often used in **orbital mechanics** to connect the geometry of the orbit to the **average orbital distance** of a planet or satellite



All points symmetric across x or y axis.

Elliptical Orbits

Deriving Orbital Speeds at Perihelion and Aphelion

Using the Law of cosines we can write,

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

So we have,

$$(r')^2 = (2ae)^2 + r^2 - 2(ae)r\cos(\pi - \theta)$$

Where we've previously shown that ,

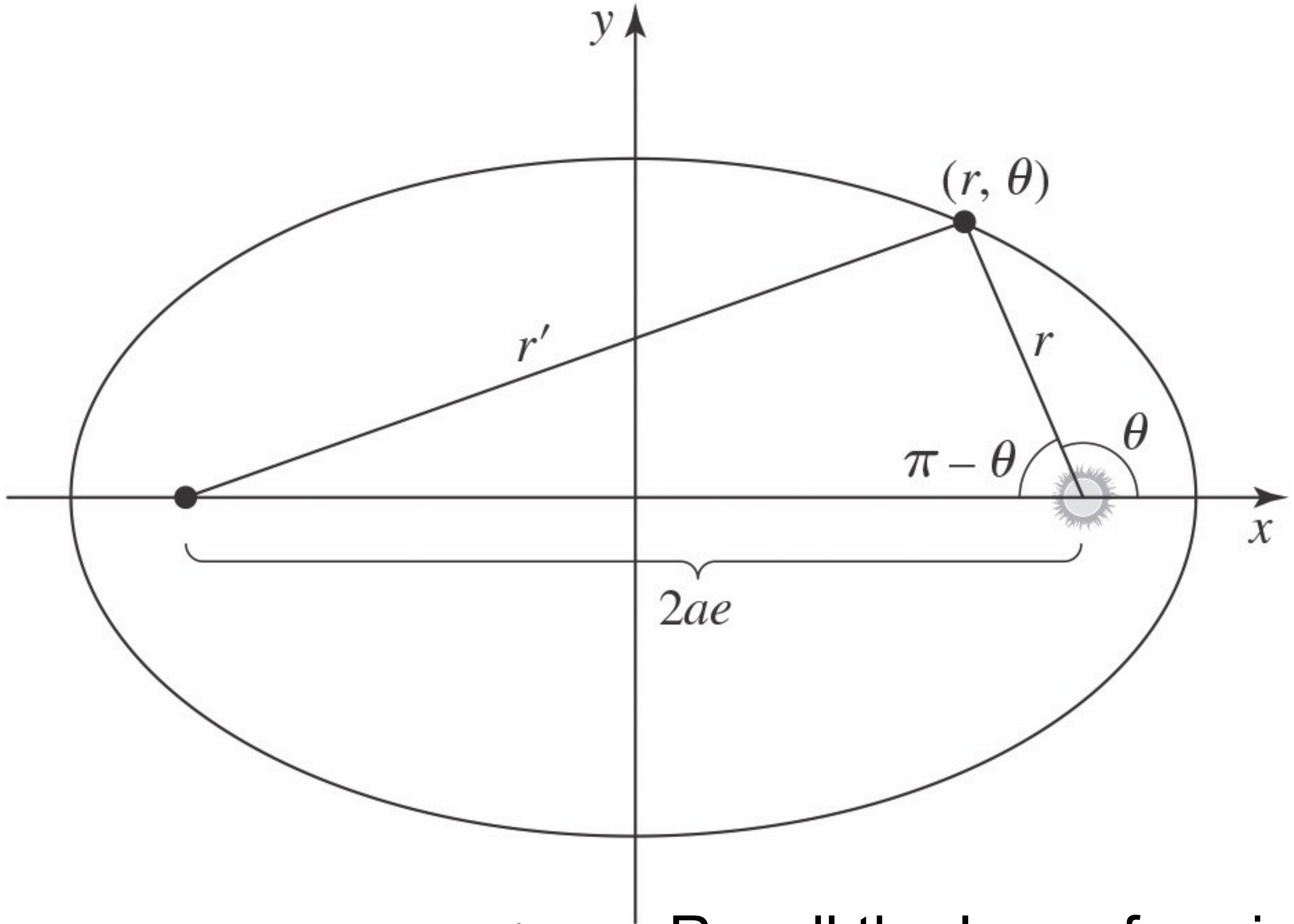
$$r' = 2a - r \longrightarrow (r')^2 = 4a^2 - 4ar + r^2$$

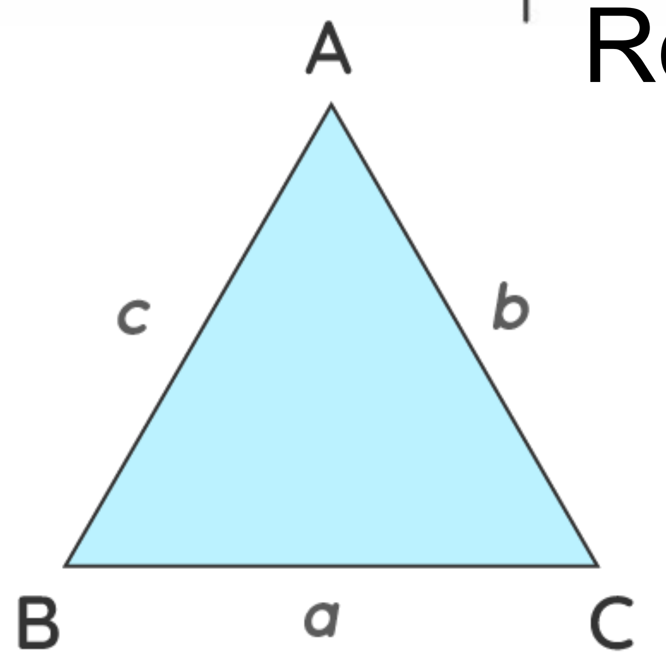
Since cosine is an even function (trig), we can write

$$\cos(\pi - \theta) = -\cos(\theta)$$

Therefore,

$$4a^2e^2 + 4aer\cos\theta = 4a^2 - 4ar$$





Recall the Law of cosines,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = c^2 + a^2 - 2ca \cos B$

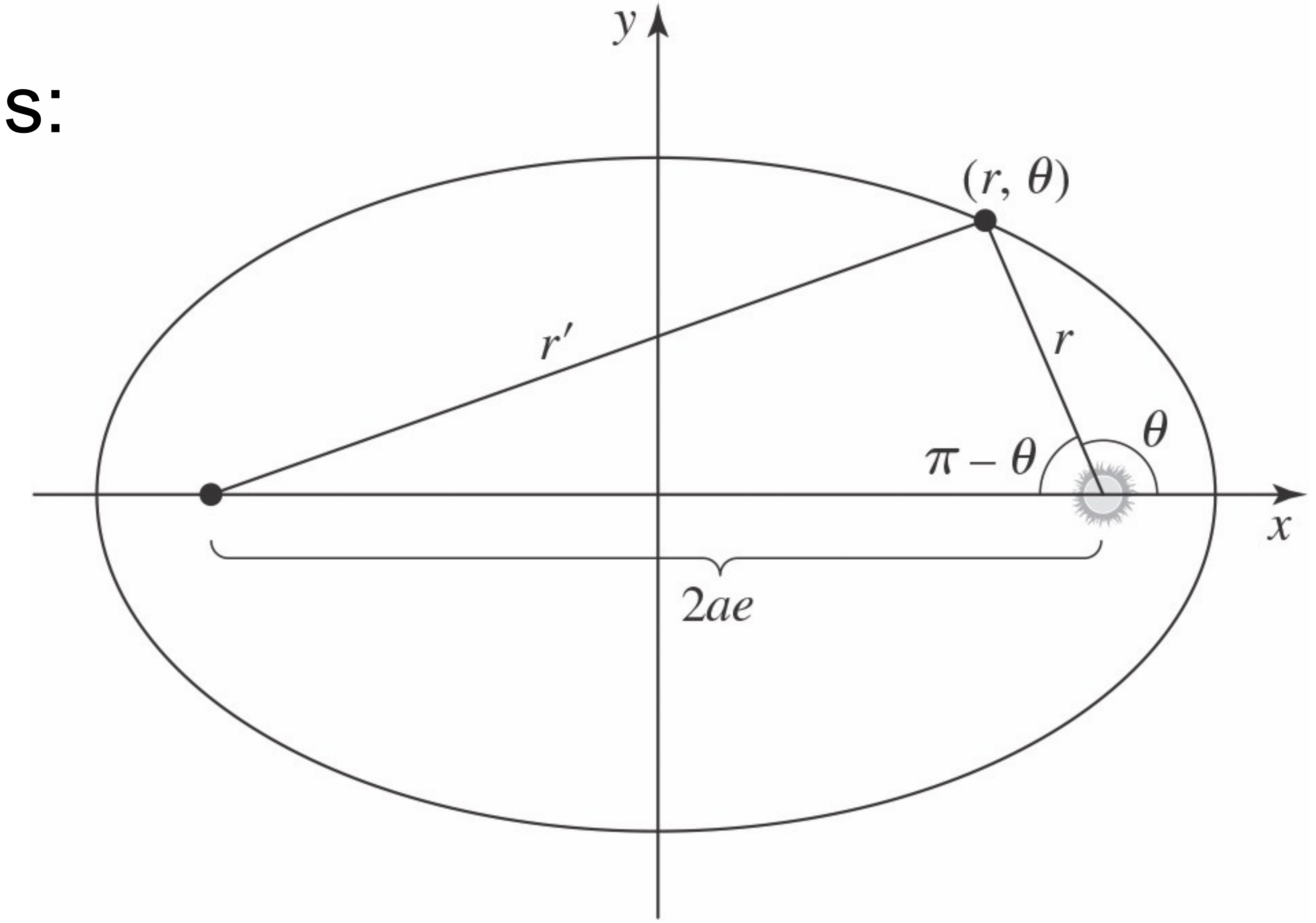
$c^2 = a^2 + b^2 - 2ab \cos C$

Elliptical Orbits

Deriving Orbital Speeds at Perihelion and Aphelion

After dividing by $4a$ and rearranging terms:

$$4a^2e^2 + 4aer\cos\theta = 4a^2 - 4ar$$
$$4aer\cos\theta + 4ar = 4a^2 - 4a^2e^2$$
$$4ar(1 + e\cos\theta) = 4a^2(1 - e^2)$$
$$r(1 + e\cos\theta) = a(1 - e^2)$$



Therefore,

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

This is **the equation for an in polar coordinates, with the origin at one focus.**

Elliptical Orbits

Deriving Orbital Speeds at Perihelion and Aphelion

This equation is equivalent in form to the equation for the shape of an orbit if Newton's law of universal gravitation holds true!

$$r(\theta) = \frac{a(1 - e^2)}{1 + e\cos\theta} \qquad r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}$$

This tells us that a planet's angular momentum is related to the size and shape of its orbit!

$$\frac{L^2}{m^2} = GMa(1 - e^2)$$

Elliptical Orbits

Deriving Orbital Speeds at Perihelion and Aphelion

Since $L = mrv_t$, this relation can also be written in the form:

$$r^2 v_t^2 = \frac{L^2}{m^2} = GMa(1 - e^2)$$

At perihelion, a its velocity is entirely tangential ($v_{\mathrm{peri}} = v_t$) and its distance from the Sun is $q = a(1 - e)$. This implies that for a planet at perihelion,

$$v_{\mathrm{peri}}^2 a^2 (1 - e)^2 = GMa(1 - e^2)$$

Solving for v_{peri}

$$v_{\mathrm{peri}} = \sqrt{\frac{GM}{a}\frac{1 + e}{1 - e}}$$

Elliptical Orbits

Deriving Orbital Speeds at Perihelion and Aphelion

Since $L = mrv_t$, this relation can also be written in the form:

$$r^2 v_t^2 = \frac{L^2}{m^2} = GMa(1 - e^2)$$

At aphelion, a its velocity is entirely tangential ($v_{\text{ap}} = v_t$) and its distance from the Sun is $q = a(1 + e)$. This implies that for a planet at perihelion,

$$v_{\text{ap}}^2 a^2 (1 + e)^2 = GMa(1 - e^2)$$

Solving for v_{ap}

$$v_{\text{ap}} = \sqrt{\frac{GM}{a}\frac{1 - e}{1 + e}}$$

Deriving Kepler's Third Law

Area for an ellipse: $A = \pi ab$

We know that the area swept out is constant.

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m} = \frac{\pi ab}{P}$$

Deriving Kepler's Third Law

Using $b^2 = a^2(1 - e^2)$ and squaring both sides, we find

$$\frac{\pi^2 a^4 (1 - e^2)}{P^2} = \frac{L^2}{4m^2}$$

But we've already shown that,

$$\frac{L^2}{m^2} = GMa(1 - e^2)$$

Deriving Kepler's Third Law

Substituting, we find that

$$\frac{\pi^2 a^4(1-e^2)}{P^2} = \frac{GMa(1-e^2)}{4}$$

Rearranging, we define Kepler's third law!

$$P^2 = \frac{4\pi^2}{GM}a^3 \propto Ka^3$$

Deriving Kepler's Third Law

Including the Sun's acceleration, the general form of Kepler's third law:

$$P^2 = \frac{4\pi^2}{G(M+m)} a^3$$

In the Solar System, $m \ll M$ (e.g., Jupiter $m \approx 1/1000 M_\odot$), so we can use the simplified form:

$$P^2 \approx \frac{4\pi^2}{GM} a^3$$


Questions?

Reminders

- Homework #2 due **Tuesday, 10/14 by 11:59 pm via Gradescope.** I've included hints to most problems to help you get started.

- Remember that **SERF 383** is reserved for ASTR 20A study session on Mondays from 4-6pm.

- I *highly recommend* that you use this space to work together on the homework.

- Coding exercise #2 due **Sunday, 10/19 by 11:59 pm via Datahub** (this one is a little more involved).

- Log into canvas and submit your answer to the discussion question by the end of the day to receive participation credit.