ASTR20A:
Introduction to Astrophysics I

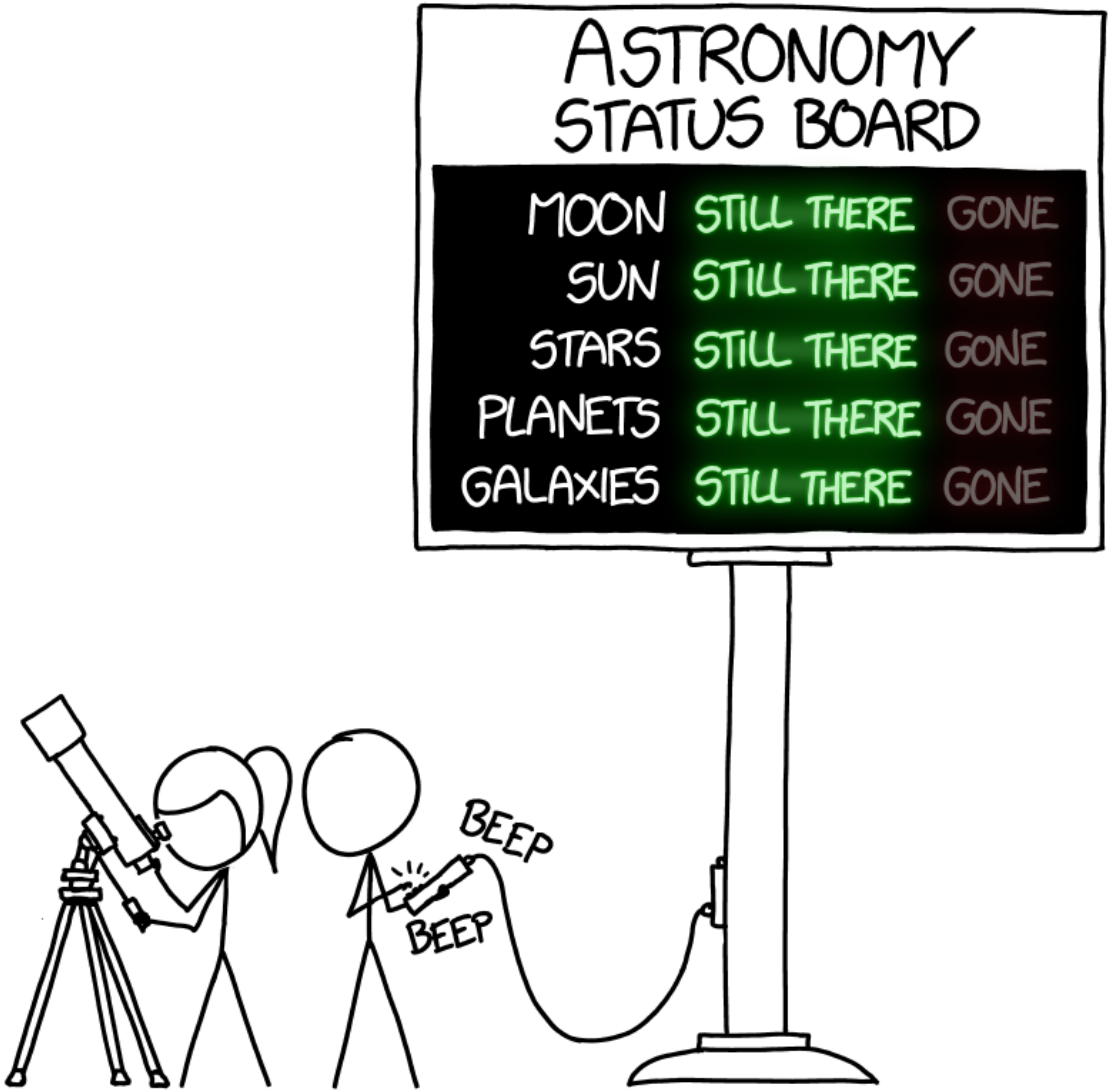
Dr. Devontae Baxter

Lecture 6

Tuesday, October 14, 2025

Announcements

- Homework #2 due **Wednesday, 10/15 by 11:59 pm via Gradescope.**
- Coding exercise #2 due **Sunday, 10/19 by 11:59 pm via Datahub** (this one is a little more involved).
- **Homework #3 due Tuesday, 10/21 by 11:59 pm via Gradescope.**
- Remember that **SERF 329** is reserved for ASTR 20A study session on Mondays from 4-6pm.
    - I *highly recommend* that you use this space to work together on the homework.

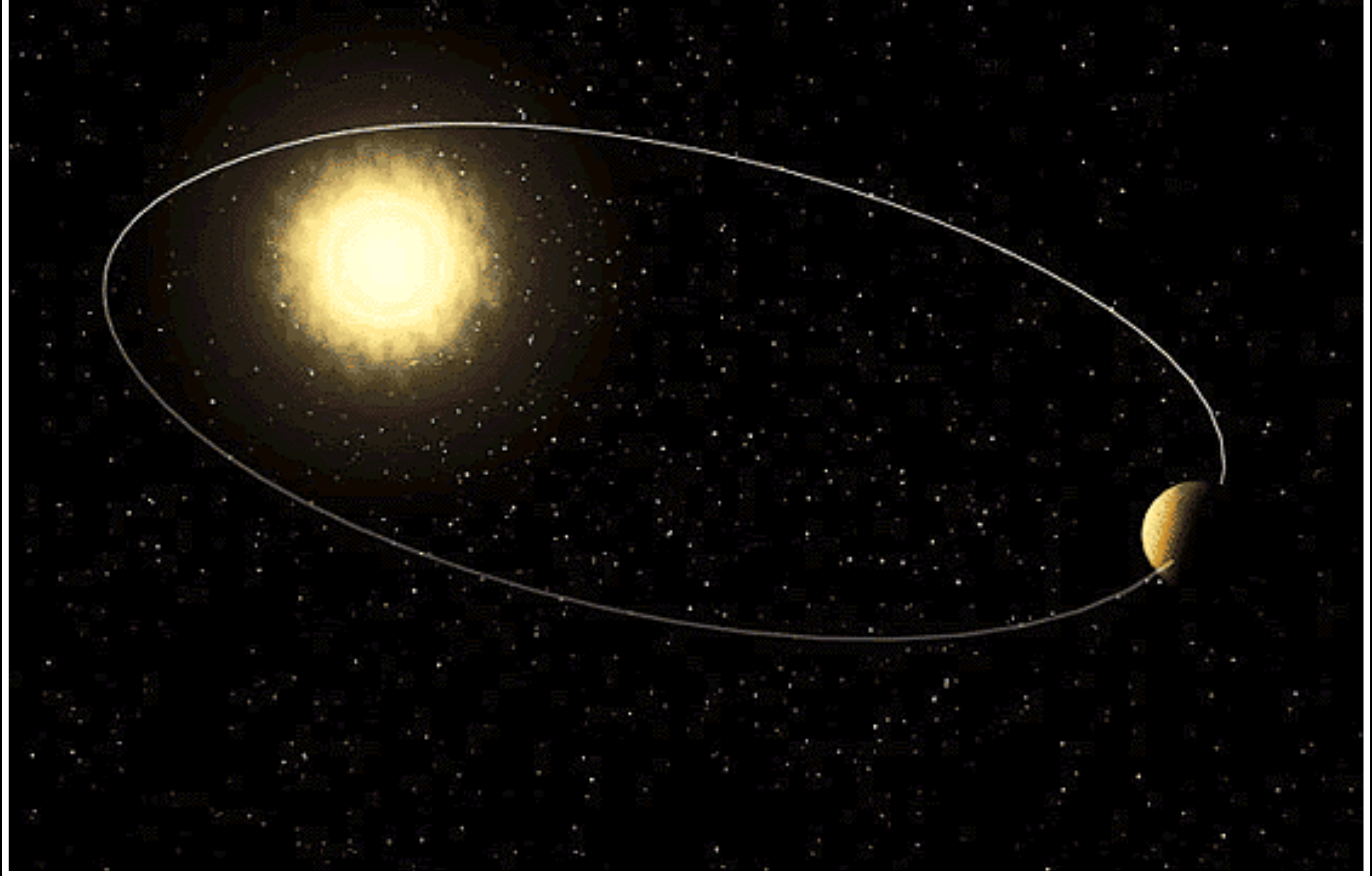



Questions?

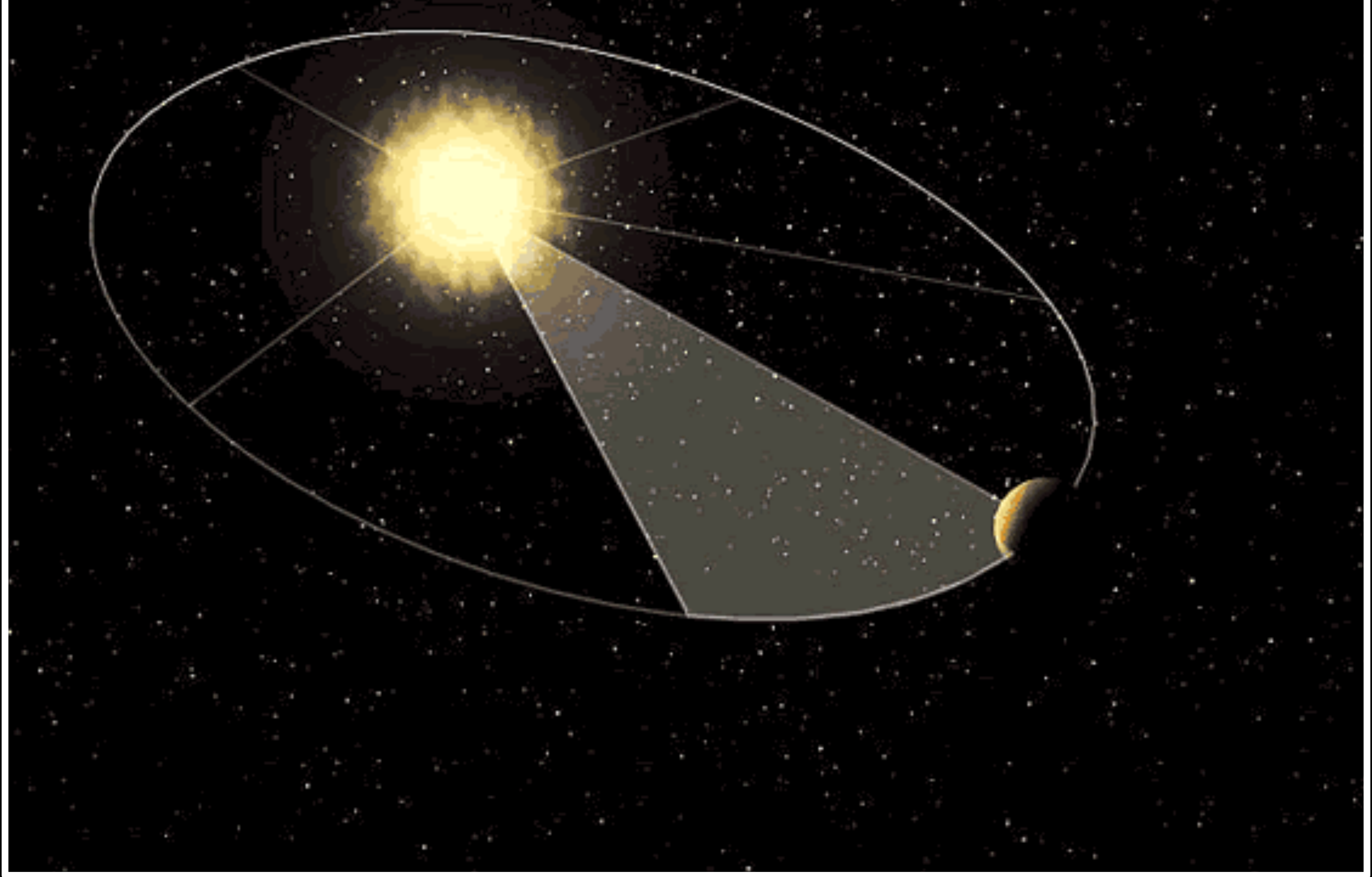
Recap of Lecture 5

In the previous lecture, we applied Newton's laws of motion and universal gravitation to derive Kepler's laws of planetary motion.

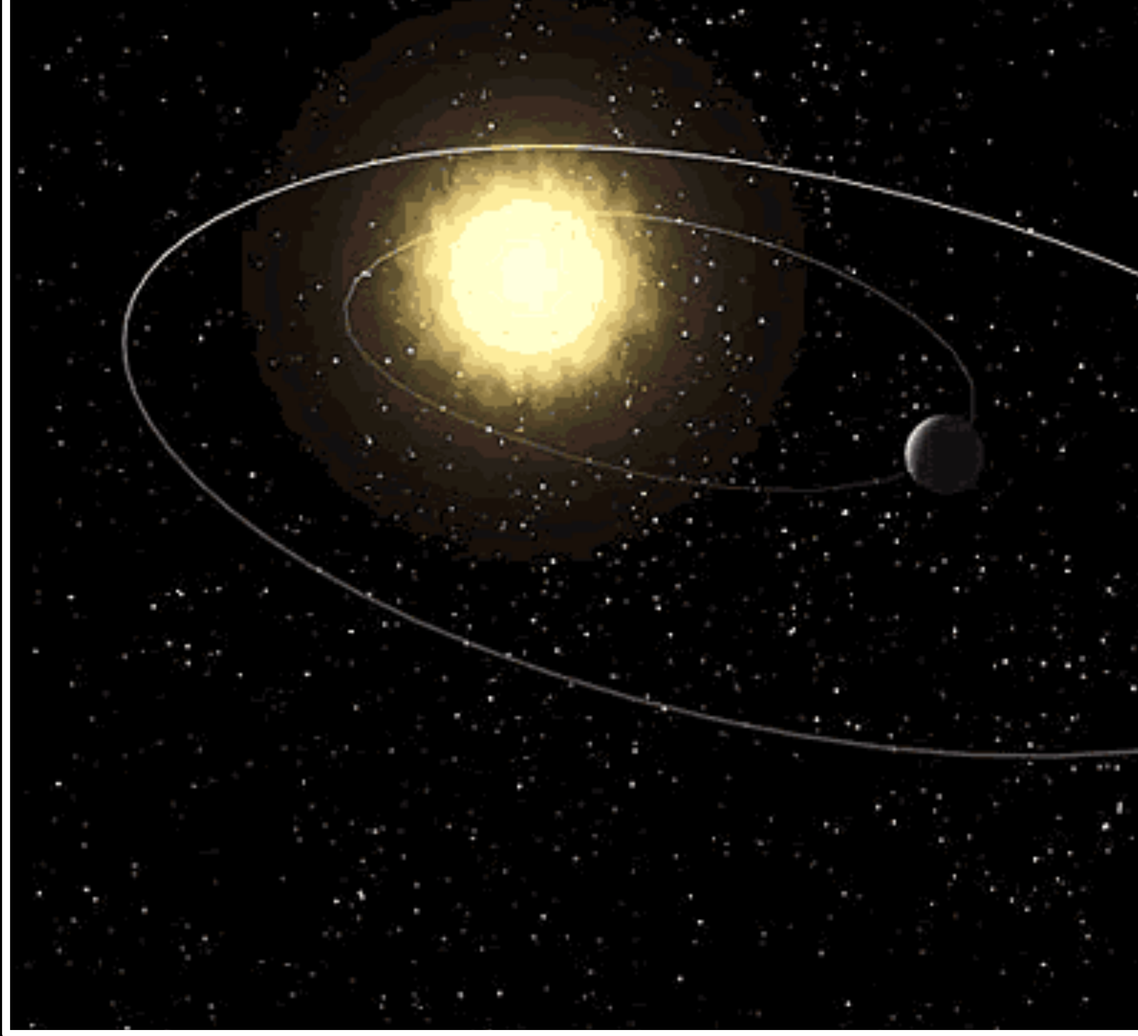
Kepler's 1st Law



Kepler's 2nd Law



Kepler's 3rd Law



$P^2 \propto a^3$

We showed that $r(\theta)$ is equivalent to the equation of an ellipse with the Sun at one focus

We proved that

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{m} = \text{constant}$$

We related $\frac{dA}{dt}$ to $\frac{A_{\text{ellipse}}}{P_{\text{orbit}}}$ to show that $P^2 \propto Ka^3$

Today, we will discuss orbital energetics, orbital speed, and the Earth–Moon system.

Orbital Energetics

Suppose a particle of mass m is placed a distance r from a massive object of mass M, and given an initial velocity v. Will its orbit be **open (**circle or ellipse**,** $e < 1$**)** or **closed** (parabola or hyperbola, $e \geq 1$)?

The answer to this question depends on the **orbital energy** of the particle — i.e., the balance between its **kinetic** and **potential** energies:

Eccentricity	Energy	Orbit shape
$e = 0$	$E < 0$	circular
$0 < e < 1$	$E < 0$	elliptical
$e = 1$	$E = 0$	parabolic
$e > 1$	$E > 0$	hyperbolic

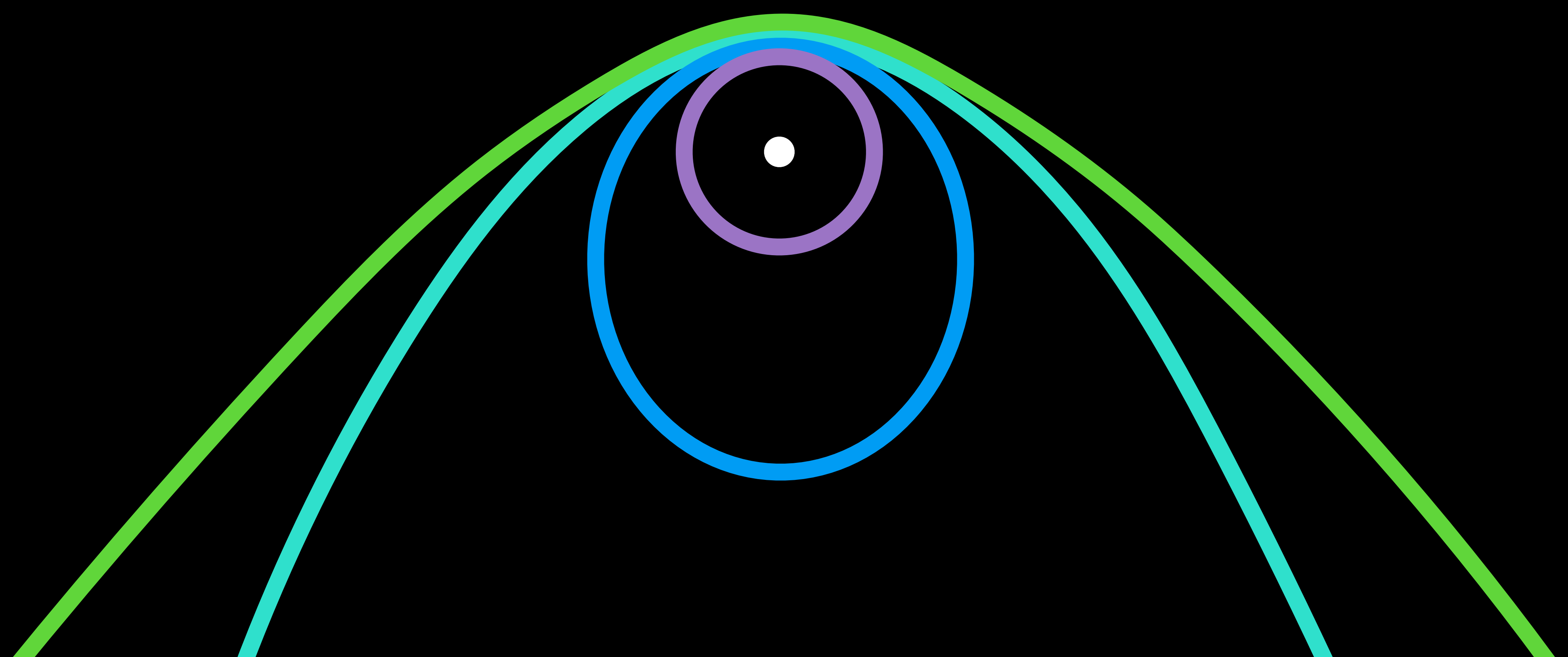
$$E = K + U$$

where,

$$K = \frac{1}{2}m\mathrm{v}^2 \quad \text{and} \quad U = -\frac{GMm}{r}$$

Kinetic Energy

Gravitational Potential Energy



Orbital Energetics

We can relate the kinetic energy to orbital parameters using the following relation we derived in the previous lecture we proved that:

$$\frac{L}{GMm}\vec{\mathrm{v}} = \hat{\theta} + \vec{\mathrm{e}}$$

Squaring both sides, we find

$$(\frac{L}{GMm})^2\vec{\mathrm{v}}\cdot\vec{\mathrm{v}} = \hat{\theta}\cdot\hat{\theta} + 2e\hat{\theta}\cdot\hat{j} + e^2\hat{j}\cdot\hat{j}$$

$$(\frac{L}{GMm})^2\mathrm{v}^2 = 1 + 2e\cos\theta + e^2$$

Recall,

$$\theta\cdot\hat{j} = \cos\theta$$

Orbital Energetics

Solving for the velocity and substituting it into the kinetic energy equation gives:

$$\mathrm{v}^2 = \left(\frac{GMm}{L}\right)^2 (1 + 2e\cos\theta + e^2)$$

$$\boxed{K(\theta) = \frac{1}{2}m\mathrm{v}^2 = \frac{1}{2}m\left(\frac{GMm}{L}\right)^2 (1 + 2e\cos\theta + e^2)}$$

Orbital Energetics

In a similar way, we can use the expression that we derived for $r(\theta)$ to rewrite the gravitational potential energy in terms of the particle's orbital parameters.

$$r(\theta) = \frac{L^2}{GMm^2(1 + e\cos\theta)}$$

$$U(\theta) = -\frac{GMm}{r} = -\frac{GMm(GMm^2(1 + e\cos\theta))}{L^2}$$

$$\boxed{U(\theta) = -\frac{GMm}{r} = -\frac{(GM)^2m^3}{L^2}(1 + e\cos\theta)}$$

Orbital Energetics

Therefore, the total energy of a particle of mass m in orbit around a more massive object of mass M can be expressed in terms of its orbital parameters as:

$$E = K + U = \frac{1}{2}m\mathrm{v}^2 - \frac{GMm}{r}$$

$$E = K + U = \frac{1}{2}m\left(\frac{GMm}{L}\right)^2(1 + 2e\cos\theta + e^2) - \frac{(GM)^2m^3}{L^2}(1 + e\cos\theta)$$

Simplifying and rearranging we find,

$$\boxed{E = K + U = \left(\frac{GMm}{L}\right)^2\frac{m}{2}(e^2 - 1)}$$

Therefore, the total energy of the particle is **constant**!

This implies that the total energy of this isolated two-body system is **conserved**!

Orbital Energetics

We can also rewrite this equation in terms of the orbital eccentricity, expressing e as a function of the **total energy** E and angular momentum L:

$$\boxed{e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}}$$

Orbital Energetics

From this equation, we can identify three distinct limiting cases: $e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$

1. **Hyperbolic orbits ($e > 1$):**
   - The kinetic energy of the particle is **greater than** the gravitational potential energy.
   - This results in an "open orbit" in which the particle is not gravitationally bound to the mass M.
2. **Parabolic orbits ($e = 1$):**
   - The kinetic and gravitational potential energies are **exactly balanced**.
   - The particles moves on parabolic trajectory with its speed equal the escape speed $v_{\text{esc}}(r) = \sqrt{\frac{2GM}{r}}$.
3. **Elliptical orbits ($e < 1$):**
   - The kinetic energy is **less than** the gravitational potential energy.
   - This results in a "closed orbit" in which the particle is gravitationally bound to the mass M.
   - Circular orbits are special cases of closed orbits with $e = 0$.

Example Problem

A small particle of mass (m) is on a *circular* orbit (i.e., $v_0 = \sqrt{GM/R}$) of radius (R) around a much larger mass (M). Suppose that we suddenly *decrease* the speed at which the mass m is moving, by a factor α (i.e., $v_1 = \dfrac{v_0}{\alpha}$, with $\alpha > 1$).

For the new orbit, find expressions for the following quantities in terms of α:

1. semi-major axis (a)

2. semi-minor axis (b)

What quantities are unchanged before & after this interaction?

Example Problem (continued)

Angular momentum *before* braking:

$$L_0 = mv_0R = \sqrt{GMm^2R}$$

Kinetic Energy *before* braking:

$$K_0 = \frac{1}{2}mv_0^2 = \frac{GMm}{2R}$$

Total Energy *before* braking:

$$E_0 = K_0 + U_0 = -\frac{GMm}{R} + \frac{GMm}{2R}$$

$$E_0 = K_0 + U_0 = -\frac{GMm}{2R}$$

Angular momentum *after* braking:

$$L_1 = mv_1R = m\frac{v_0}{\alpha}R = \frac{1}{\alpha}L_0$$

Kinetic Energy *after* braking:

$$K_1 = \frac{1}{2}mv_1^2 = \frac{1}{2}m\frac{v_0^2}{\alpha^2} = \frac{1}{\alpha^2}K_0$$

Total Energy *after* braking:

$$E_1 = K_1 + U_1 = \frac{1}{\alpha^2}K_0 + U_0$$

$$E_1 = K_1 + U_1 = E_0(2 - \frac{1}{\alpha^2})$$

Example Problem (continued)

We know that the total energy (E) is related to the eccentricity (e) by

$$e = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

Since the mass (m) was initially on a circular orbit ($e = 0$), the equation from above becomes:

$$\frac{2E_0L_0^2}{G^2M^2m^3} = -1$$

Example Problem (continued)

After braking, this quantity becomes

$$\frac{2E_1L_1^2}{G^2M^2m^3} = \ldots = \frac{1}{\alpha^2}\left(\frac{1}{\alpha^2} - 2\right)$$

Inserting this into the equation for eccentricity, we find

$$e = \left(1 - \frac{2}{\alpha^2} + \frac{1}{\alpha^4}\right)^{1/2} = \ldots = \left(1 - \frac{1}{\alpha^2}\right)$$

Rearranging, we see that

$$\frac{1}{\alpha^2} = 1 - e$$

Example Problem (continued)

Relating the eccentricity to $r(\theta)$ (see Lecture 5) we find:

$$r(\theta) = \frac{L_1^2}{GMm^2(1 + e\cos\theta)} = \frac{L_0^2}{\alpha^2 GMm^2(1 + e\cos\theta)}$$

By definition, at closest approach this equation becomes:

$$r_{\text{peri}}(\theta = 0) = a(1 - e) = \frac{L_0^2}{\alpha^2 GMm^2(1 + e)} = \frac{R}{\alpha^2(1 + e)}$$

After substituting and simplifying, we find that the semi-major axis of the new orbit is:

$$\boxed{a = \frac{R}{\alpha^2(1 - e^2)} = \ldots = \frac{R\alpha^2}{2\alpha^2 - 1}}$$

Likewise, the semi-minor axis of the new orbit is:

$$\boxed{b = a(1 - e^2)^{1/2} = \ldots = \frac{R}{(2\alpha^2 - 1)^{1/2}}}$$


Questions?

Orbital Speed

In general, there is no simple equation for a planet's distance from the Sun, r or its velocity v as a function of time. Calculating position as a function of time requires solving a very complex transcendental equation.

As such, we cannot easily predict a planet's distance r **at a given time** (e.g., "Where will Mars be in 42 days?") using a *simple formula.*

However, we can express the orbital speed v as a function of distance r using **the "vis viva equation".**

The "*vis viva*" Equation

The vis viva equation:

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

The Latin term *vis viva* translates to "living force," an archaic term for twice the kinetic energy (2K).

relates a spacecraft's **orbital speed** v to is distance r from the central body and semi-major axis a of its orbit.

Note: this equation can be derived by combining the equation for the conic section (see Lecture 5) with the equation for kinetic energy in terms of eccentricity and angular momentum (see Slide 5).

The "*vis viva*" Equation

By combining the vis viva equation with **Kepler's Third Law**, we replace GM with a term involving the orbital period (P) and semi-major axis (a).

$$v(r) = \frac{2\pi a}{P}\left(\frac{2a}{r} - 1\right)^{1/2}$$

This implies that the orbital angular speed $\omega = v/r$ of a planet is

$$\omega(r) = \frac{2\pi}{P}\frac{a}{r}\left(\frac{2a}{r} - 1\right)^{1/2}$$

An Application of *vis viva*: Hohmann Transfer Orbit

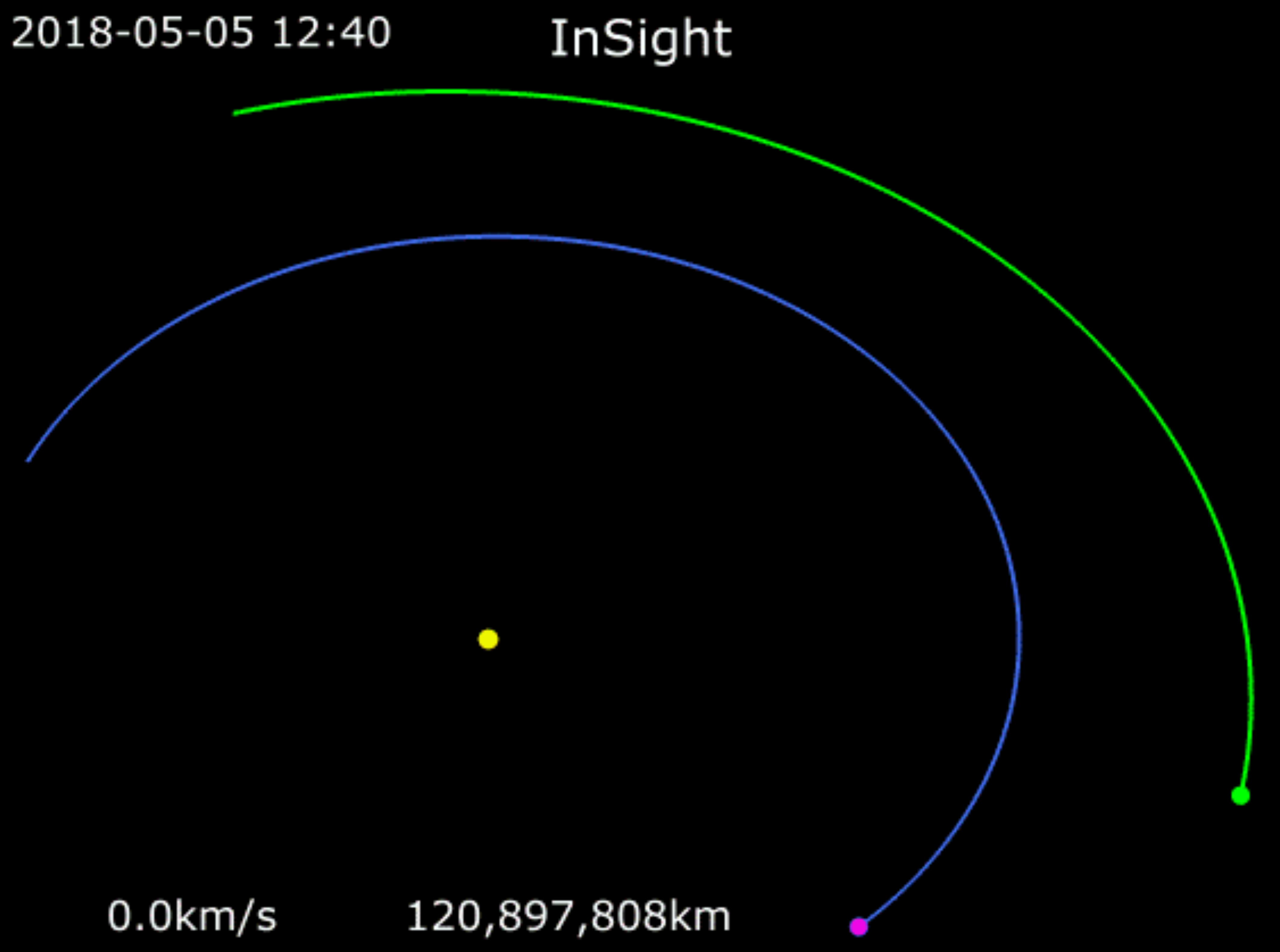
A **Hohmann transfer** is the most fuel-efficient way to move a spacecraft between two circular orbits around the same central body. It uses a “transfer orbit”, which is an ellipse with its perihelion at departure point and its aphelion at the arrival point.

Example based on the InSight Mission to Mars:

1. **Departure (Perihelion):** The **vis-viva equation** is used to calculate the Δv needed to boost the spacecraft above Earth’s orbital speed, placing it onto the transfer orbit toward Mars.

2. **Arrival (Aphelion):** The vis-viva equation is used again at the aphelion of the transfer orbit to determine the Δv required to **match Mars’ orbital speed** and enter orbit.

- This 260-day trip required perfect timing as the spacecraft has to arrive at aphelion just as Mars arrives.
- The limited windows for these trips are called **Launch Windows** and occur only every **2.1 years** for Mars.



The Virial Theorem: Describing Many-Body Systems

- For a system with only **two spherical bodies** (e.g., a star and a planet), there is a **simple analytic solution** for the orbit $r(\theta)$ and orbital energy $E(\theta)$.
- In a system with **more than two bodies**, such as **star clusters or galaxies**, **no simple analytic solutions exist** for the trajectories or energies of *individual* bodies.
- **Despite this complexity**, it is possible to obtain **useful statistical results** describing the system's **average global properties**.
- **The Virial Theorem*** provides such a result:
    - For a stable, bound system, the time-averaged total kinetic energy $\langle K \rangle$ is related to the time-averaged total potential energy $\langle U \rangle$ by:

$$2\langle K \rangle + \langle U \rangle = 0$$

In ASTR 20B you will use this theorem to estimate the total mass of distant galaxies!

*Derivation provided at the end the slide deck


Questions?

Earth-Moon System

So far, we have treated massive bodies as if they were **point masses,** which is good **first-order approximation** for understanding how objects in the Solar System interact.

- However, a more accurate description must account for the fact that **stars, planets, and satellites are *extended bodies*** that are **not perfectly spherical**.
- In addition, we have to consider the effects of other gravitational sources besides the Sun, which can perturb otherwise perfectly elliptical orbits.

Today, we will focus on how these **interactions and perturbations** influence the dynamics of the **Earth–Moon System** in particular.

The Shape of the Earth: An Oblate Spheroid

A **rapidly rotating "plastic" body** will distort from a perfect sphere, becoming an **oblate spheroid** — a shape flattened at the poles and bulging at the equator.

This happens because the **centrifugal force** is greatest at the equator, where rotational speed is highest (see Lecture 4) — think of how a spinning lump of pizza dough flattens into a disk as it rotates.

Due to its angular velocity, the Earth experiences a centrifugal force that causes it to become an **oblate spheroid**, with a slight **bulge at the equator**.

$$R_{\text{equator}} \sim 6378 \text{ km}$$

$$R_{\text{pole}} \sim 6357 \text{ km}$$

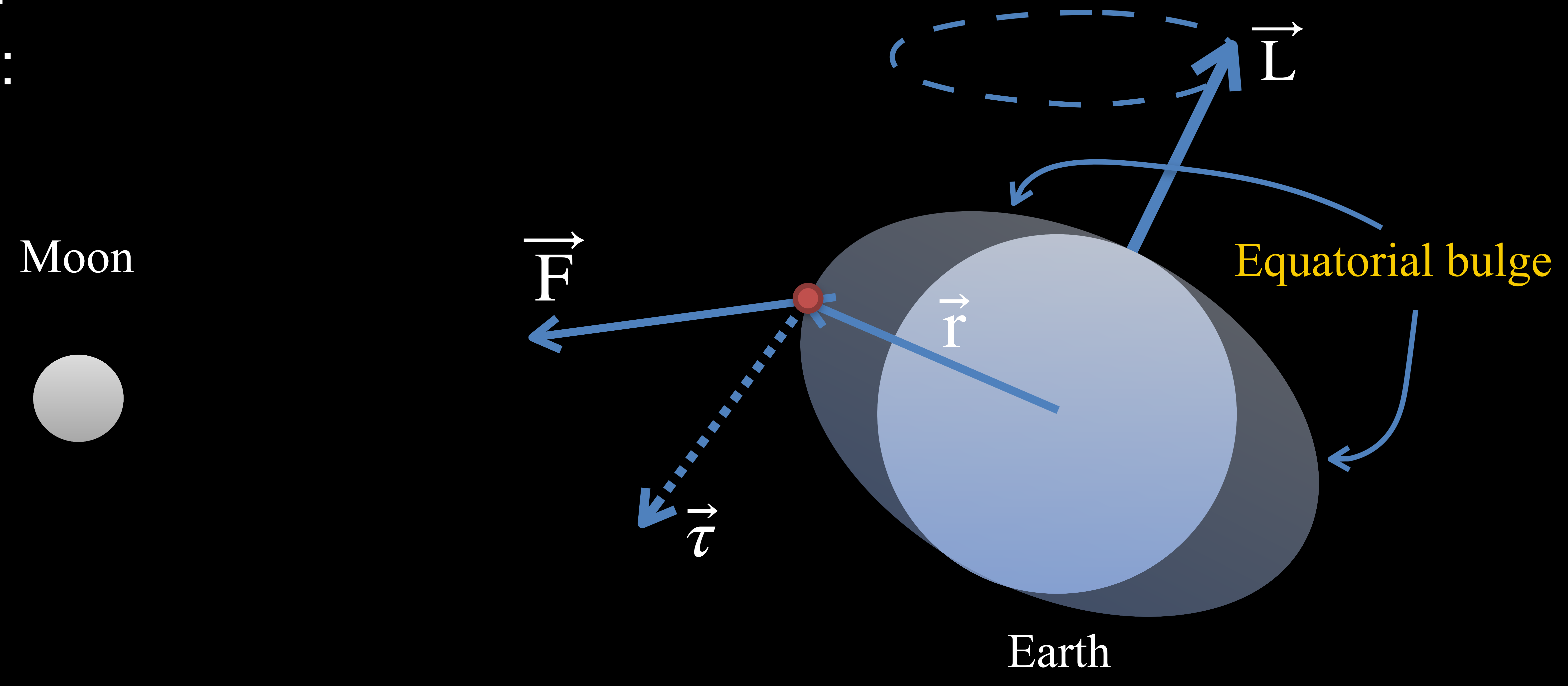
The difference is about **0.3%.**

Earth-Moon System: Precession

Let's imagine that we place a small mass m within the Earth's equatorial bulge.

The gravitational force exerted on the mass m by the moon is:

$$\vec{F} = -\frac{GM_{\text{moon}}m}{r_{\text{moon}}^2}$$



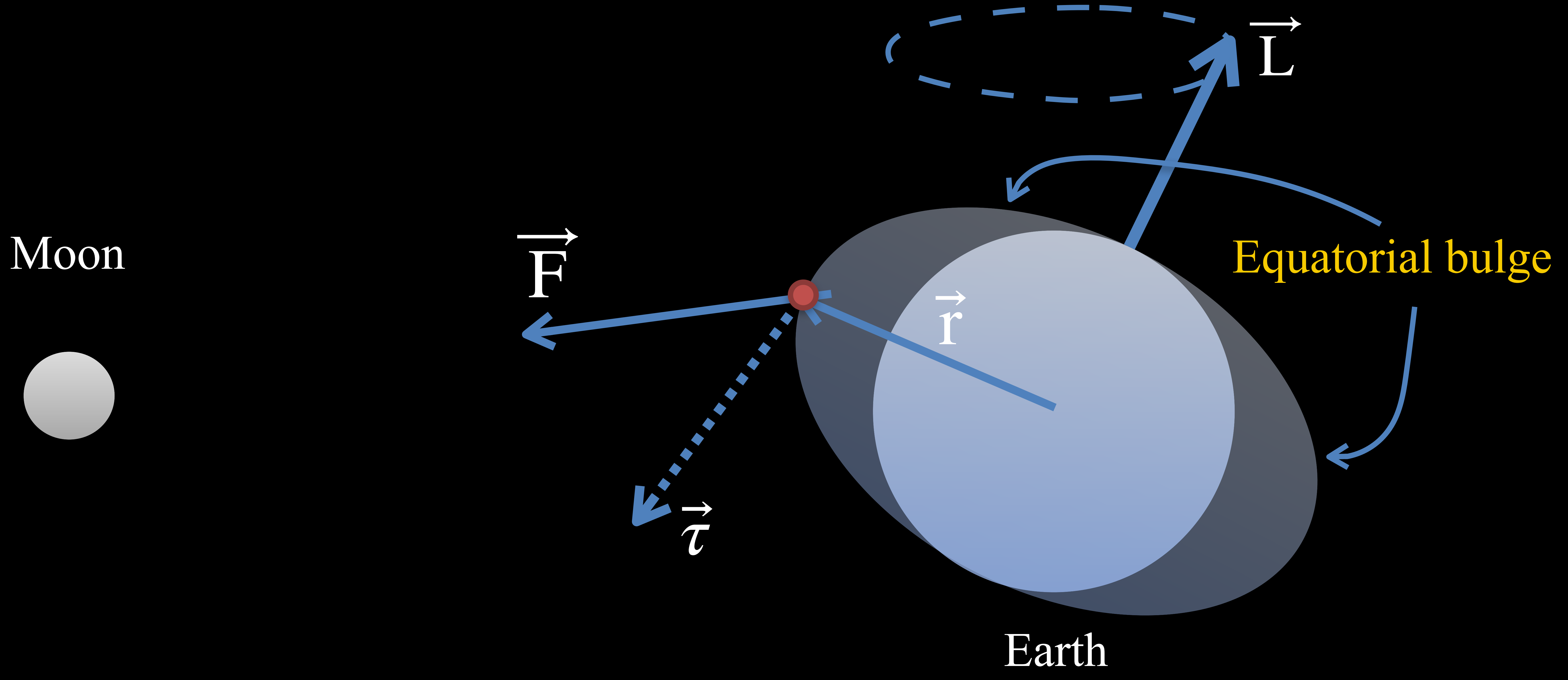
r_{moon} is distance from the small mass to the center of the Moon.

Earth-Moon System: Precession

Due to the **tilt of Earth's axis** with respect to the **Moon's orbital plane**, the gravitational force acting on the mass is **not perfectly aligned** with the position vector $\vec{r}$.

As a result, the mass m experiences a torque!

$$\vec{\tau} = \vec{r} \times \vec{F} \neq 0$$

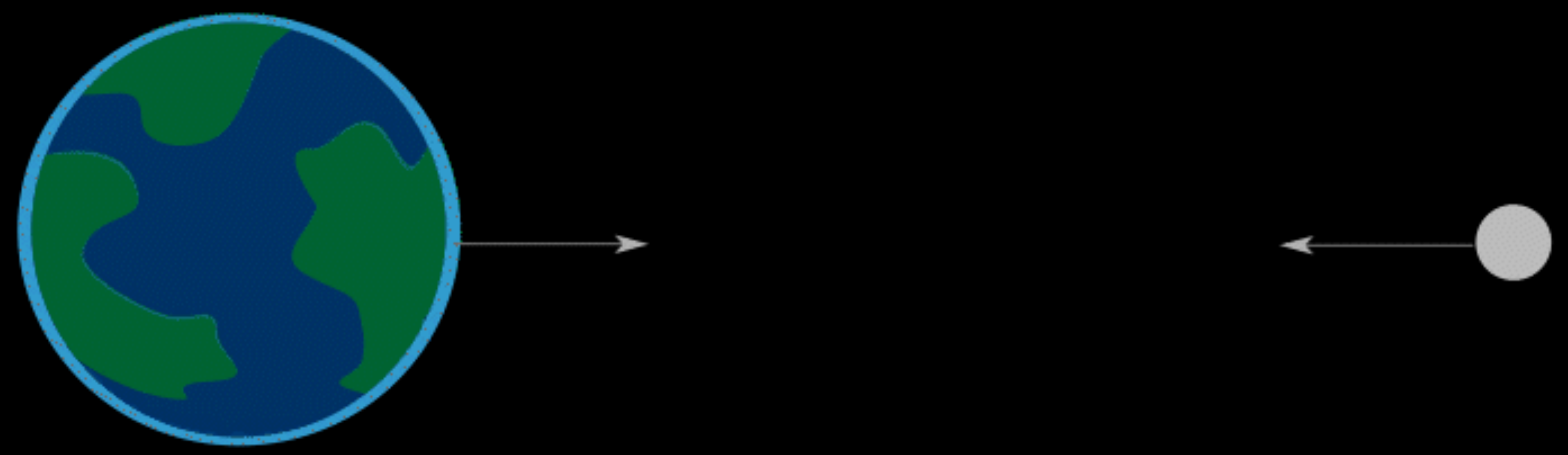


It is the **combined torque from the Moon and the Sun** acting on Earth's **equatorial bulge** that produces the observed precession period of approximately 25,800 years!

Earth-Moon System: Tides

The strength of gravity is the same at all points on the surface of an **isolated spherical body**. However, if another massive object (like a nearby Moon) comes close, its gravity creates an **uneven pull** across the body's surface. These distortions are called **tides**.

Because Earth has a **large, nearby Moon**, its shape is **stretched by the tidal effect of the Moon.**



On Earth, we observe the manifestation of the tides as the daily rise and fall of water levels along the seashore.

Bay of Fundy: Minas Basin Tides Timelapse

Earth-Moon System: Tides

To compute the differential tidal force, let's first consider the gravitational force exerted on a small piece of matter on Earth (e.g., a pebble of drop of ocean water) of mass m:

$$F_{\mathrm{Moon}}(r) = -\frac{GM_{\mathrm{moon}}m}{r_{\mathrm{moon}}^2}$$

Since the distance to the Moon is much large than the size of Earth itself ($r_{\mathrm{moon}} \sim 60R_{\oplus}$), the variation in the Moon's gravitational force across Earth is small.

Therefore, we can use a Taylor series expansion around the Earth's center to estimate the small differences in the Moon's gravitational force on the near and far sides of the Earth.

$$F_{\mathrm{Moon}}(r) \approx F_{\mathrm{Moon}}(r_0) + (r - r_0)\frac{dF_{\mathrm{Moon}}}{dr}\bigg|_{r=r_0}$$

Where,

$$\frac{dF_{\mathrm{Moon}}}{dr} = \frac{2GM_{\mathrm{moon}}m}{r_{\mathrm{moon}}^3}$$

r_{moon} is distance from the small mass to the center of the Moon.

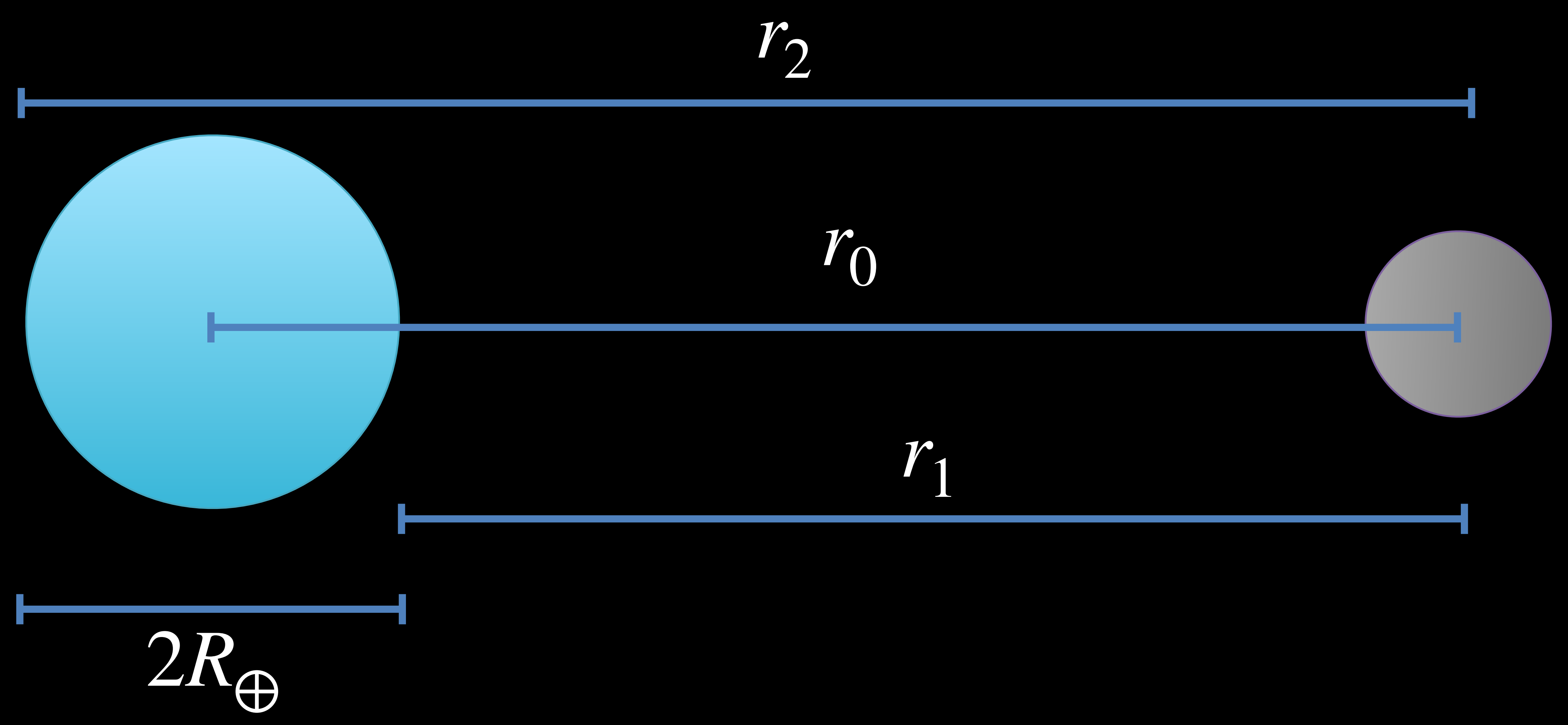
r_0 is the average distance between the center of the Earth and the center of the Moon — about 384,000 km.

Earth-Moon System: Tides

The differential gravitational force near the Earth's center is:

$$\Delta F_{\mathrm{Moon}}(r) = F_{\mathrm{Moon}}(r) - F_{\mathrm{Moon}}(r_0) = (r - r_0)\frac{2GM_{\mathrm{Moon}}m}{r_0^3}$$

To calculate the differential force, we can evaluate forces on the near (r_1) and far (r_2) sides of Earth.



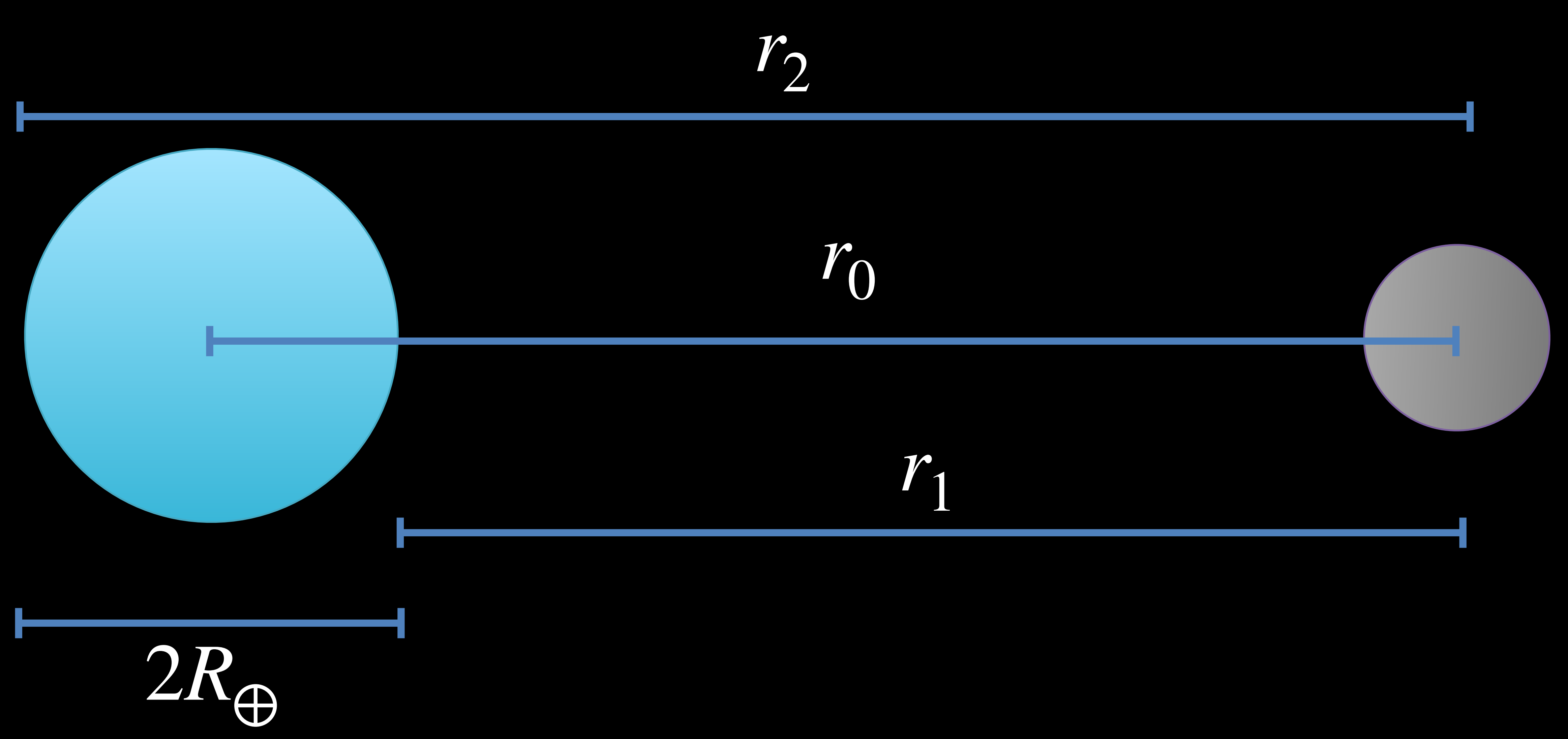
Earth-Moon System: Tides

Near side: $r_1 = r_0 - R_\oplus$

$$\Delta F_{\text{Moon}}(r_1) = -\frac{2GM_{\text{Moon}} m R_\oplus}{r_0^3}$$

Far side: $r_2 = r_0 + R_\oplus$

$$\Delta F_{\text{Moon}}(r_2) = \frac{2GM_{\text{Moon}} m R_\oplus}{r_0^3}$$



They are equal in magnitude, but opposite signs!

What does that imply?

Earth-Moon System: Tides

The differential gravitational force associated with the Sun can similarly be written as:

$$\Delta F_\odot = \frac{2GM_\odot m R_\oplus}{a_\oplus^3}$$

The ratio of the differential forces due to the Sun and the Moon is:

$$\frac{\Delta F_\odot}{\Delta F_{\text{Moon}}} = \frac{M_\odot}{M_{\text{Moon}}}\left(\frac{r_0}{a_\oplus}\right)^3 = \frac{2.0 \times 10^{30}\ \text{kg}}{7.4 \times 10^{22}\ \text{kg}}\left(\frac{3.8 \times 10^5\ \text{km}}{1.5 \times 10^8\ \text{km}}\right)^3 \approx 0.44$$

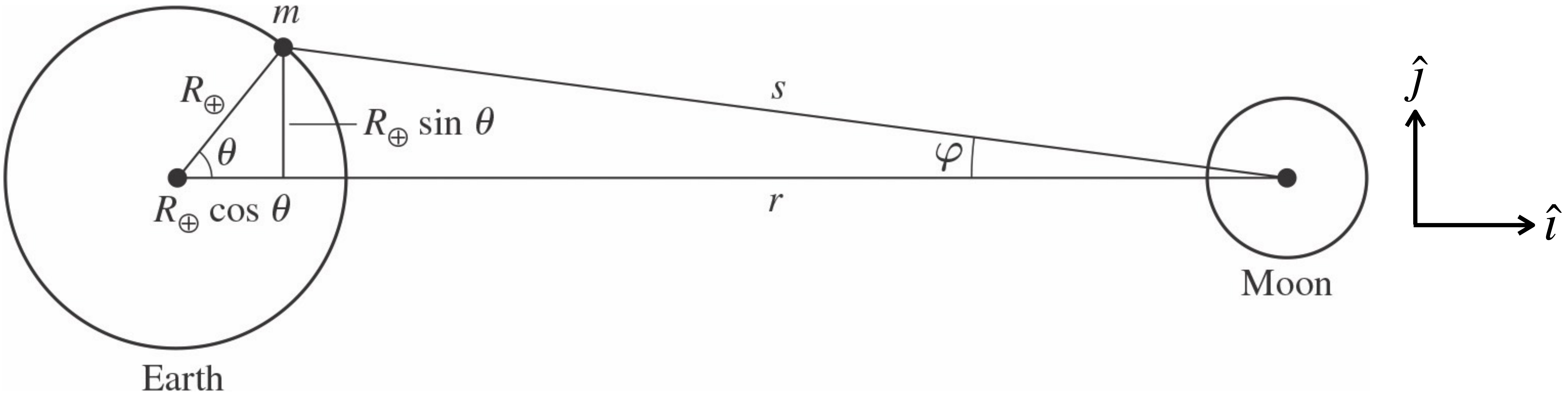
$a_\oplus$ is average Earth-Sun distance.

$R_\oplus$ is radius of the Earth.

Tidal Forces Across Earth's Surface

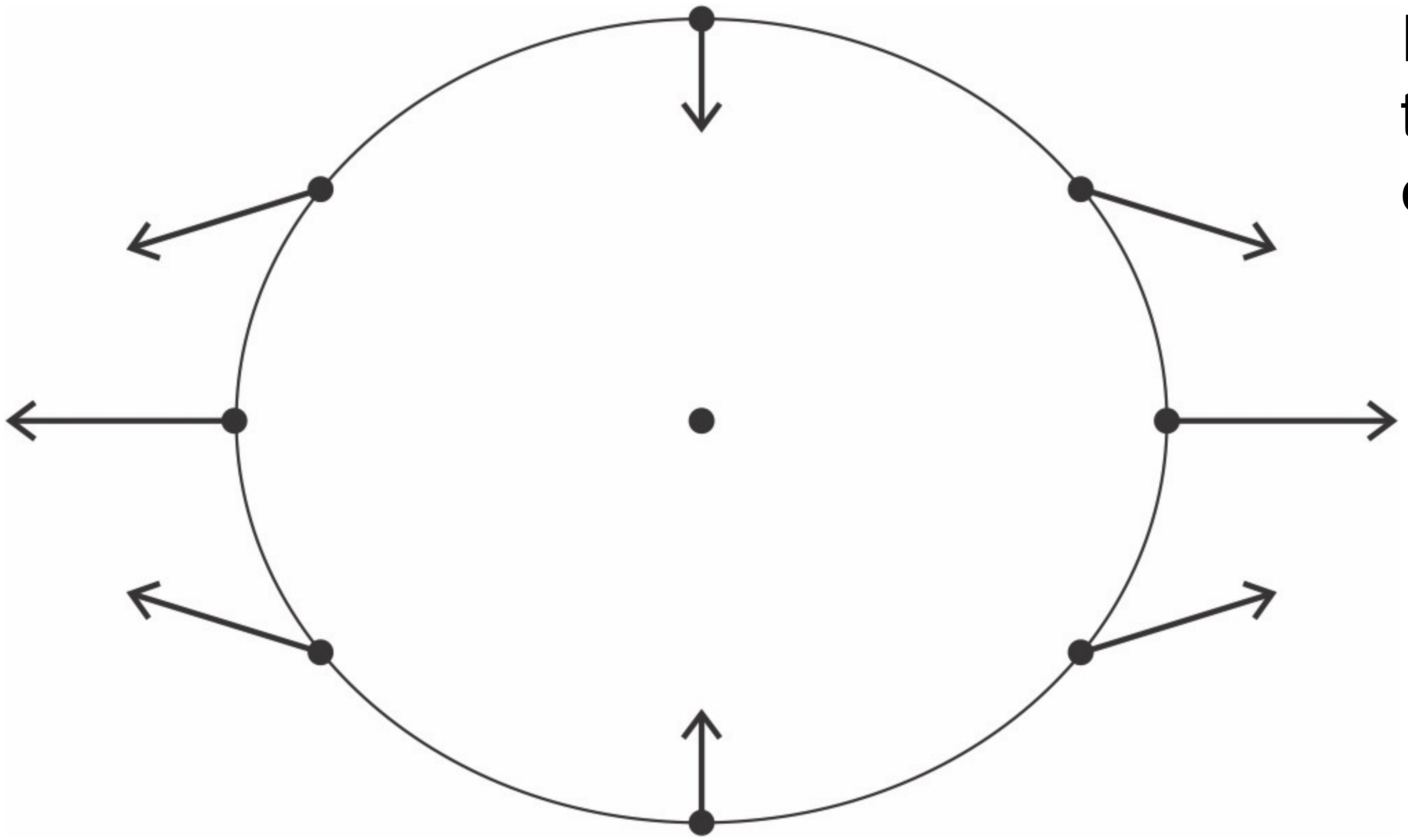
The expression for the tidal force on the Earth's surface caused by the Moon's gravity is:

$$\Delta \vec{F} = \frac{2GM_{\mathrm{Moon}} m R_{\oplus}}{r^3}(2\cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath})$$



Tidal Forces Across Earth’s Surface

The differential force vectors at several locations on the Earth’s surface are shown below:



In addition to stretching the Earth along the x-axis, the differential forces also **compress it along the y-axis.**


Questions?


Brain Break – Think-pair-share
The Moon is receding from the Earth at a rate of 4 centimeters per year.
Assuming this recession rate remains constant, and given that the average separation between Earth and Mars is approximately 225 million kilometers, how long would it take for the Moon to reach the distance of Mars?
What do you think will change in our daily lives without the Moon?

Reminders

- Homework #2 due tomorrow **Wednesday, 10/15 by 11:59 pm via Gradescope.** I've included hints to most problems to help you get started.
- Remember that **SERF 329** is reserved for ASTR 20A study session on Mondays from 4-6pm.
- Coding exercise #2 due **Sunday, 10/19 by 11:59 pm via Datahub** (this one is a little more involved).
- **Homework #3 due Tuesday, 10/21 by 11:59 pm via Gradescope.**
- Log into canvas and submit your answer to the discussion question <u>by the end of the day</u> to receive participation credit.

Virial Theorem Derivation

We define a scalar quantity A (often called the **virial function** or the moment of inertia function) for a system of N particles:

$$A(t) = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} \cdot \vec{r}_i$$

We take the time derivative of A:

Sum of potential energies

Sum of kinetic energies

$$\frac{dA}{dt} = \sum_{i=1}^{N} \left(m_i \frac{d^2\vec{r}_i}{dt^2} \cdot \vec{r}_i + m_i \frac{d\vec{r}_i}{dt} \cdot \frac{d\vec{r}_i}{dt} \right) = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i + \sum_{i=1}^{N} m_i v_i^2$$

The two components in the sum are:

- First term: the net force force (Newton's Second Law,)
- Second term: is twice the total kinetic energy of the system.

Virial Theorem Derivation

Taking the time average:

$$\langle \frac{dA}{dt} \rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} \frac{dA}{dt} dt = \lim_{\tau\to\infty} \frac{A(\tau) - A(0)}{\tau} = 0$$

$$\langle \frac{dA}{dt} \rangle = 2\langle K \rangle + \sum_{i=1}^{N} \langle \overrightarrow{F}_i \cdot \vec{r}_i \rangle = 2\langle K \rangle + \langle U \rangle = 0$$

$$2\langle K \rangle + \langle U \rangle = 0$$