演習問題 No. 7 の解答

■$\boxed{1}$ (1) 通分して, 両辺の分子を比較して

$$
\begin{aligned}
1 &= (Ax+B)(x^2+2)(x^2+3)+(Cx+D)(x^2+1)(x^2+3)+(Ex+F)(x^2+1)(x^2+2) \\
&= (A+C+E)x^5+(B+D+F)x^4+(5A+4C+3E)x^3 \\
&\quad +(5B+4D+3F)x^2+(6A+3C+2E)x+(6B+3D+2F)
\end{aligned}
$$

この式がすべての x に対して成り立たなければならないから,

$$
\begin{gathered}
A+C+E=0, \quad B+D+F=0, \quad 5A+4C+3E=0 \\
5B+4D+3F=0, \quad 6A+3C+2E=0, \quad 6B+3D+2F=1
\end{gathered}
$$

が成り立つ．この連立方程式を解く．第 1,3 式より, $A=E, C=-2E$. これを第 5 式に代入して, $6E-6E+2E=0$. よって $A=C=E=0$. 同様に, 第 2,4 式より $B=F, D=-2F$. 第 6 式に代入して $6F-6F+2F=1$. よって $F=\frac{1}{2}$. 従って $B=\frac{1}{2}, D=-1$ となる.

注意: この場合は, 実は次の変形の方が簡単である.

$$
\frac{1}{(x^2+1)(x^2+2)}=\frac{1}{x^2+1}-\frac{1}{x^2+2}
$$

なので

$$
\begin{aligned}
\frac{1}{(x^2+1)(x^2+2)(x^2+3)} &= \frac{1}{(x^2+1)(x^2+3)}-\frac{1}{(x^2+2)(x^2+3)} = \frac{1}{2}\left(\frac{1}{x^2+1}-\frac{1}{x^2+2}\right)-\left(\frac{1}{x^2+2}-\frac{1}{x^2+3}\right) \\
&= \frac{1}{2}\frac{1}{x^2+1}-\frac{1}{x^2+2}+\frac{1}{2}\frac{1}{x^2+3}
\end{aligned}
$$

(2) (1) より,

$$
\begin{aligned}
\int \frac{1}{(x^2+1)(x^2+2)(x^2+3)}\,dx &= \frac{1}{2}\int\frac{1}{x^2+1}\,dx-\int\frac{1}{x^2+2}\,dx+\frac{1}{2}\int\frac{1}{x^2+3}\,dx \\
&= \frac{1}{2}\tan^{-1}x-\frac{1}{\sqrt{2}}\tan^{-1}(\frac{x}{\sqrt{2}})+\frac{1}{2}\times\frac{1}{\sqrt{3}}\tan^{-1}(\frac{x}{\sqrt{3}})
\end{aligned}
$$

■$\boxed{2}$ (1) $\dfrac{2}{(x-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$ として，A, B, C を求めると，$A=1, B=C=-1$ となるから,

$$
\int\frac{2}{(x-1)(x^2+1)}=\int\frac{1}{x-1}\,dx-\int\frac{x+1}{x^2+1}\,dx=\log|x-1|-\frac{1}{2}\log(x^2+1)-\tan^{-1}x
$$

(2) $\dfrac{3x}{x^3+1}=\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2-x+1}$ として，A, B, C を求めると，$A=-1, B=C=1$ となるから,

$$
\begin{aligned}
\int\frac{3x}{x^3+1}\,dx &= -\int\frac{1}{x+1}\,dx+\int\frac{x+1}{x^2-x+1}\,dx=-\log|x+1|+\int\frac{(x-1/2)+3/2}{(x-1/2)^2+3/4}\,dx \\
&= -\log|x+1|+\int\frac{(x-1/2)}{(x-1/2)^2+3/4}\,dx+\frac{3}{2}\int\frac{1}{(x-1/2)^2+3/4}\,dx \\
&= -\log|x+1|+\frac{1}{2}\log|(x-1/2)^2+3/4|+\frac{3}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Big(\frac{x-1/2}{\sqrt{3}/2}\Big) \\
&= -\log|x+1|+\frac{1}{2}\log(x^2-x+1)+\sqrt{3}\tan^{-1}\Big(\frac{2x-1}{\sqrt{3}}\Big)
\end{aligned}
$$

(3) $x^4+x^2+1=(x^2+1)^2-x^2=(x^2-x+1)(x^2+x+1)$ であるから,

$$
\frac{2}{x^4+x^2+1}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+x+1}
$$

として，A, B, C を求めると，$A=-1$, $B=C=D=1$ となるから，

$$\begin{aligned}
\int \frac{1}{x^4+x^2+1}\,dx &= \int \frac{-x+1}{x^2-x+1}\,dx + \int \frac{x+1}{x^2+x+1}\,dx \\
&= \int \frac{-(x-1/2)+1/2}{(x-1/2)^2+3/4}\,dx + \int \frac{(x+1/2)+1/2}{(x+1/2)^2+3/4}\,dx \\
&= -\frac{1}{2}\log(x^2-x+1) + \frac{1}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Big(\frac{x-1/2}{\sqrt{3}/2}\Big) \\
&\quad + \frac{1}{2}\log(x^2+x+1) + \frac{1}{2}\frac{1}{\sqrt{3}/2}\tan^{-1}\Big(\frac{x+1/2}{\sqrt{3}/2}\Big) \\
&= \frac{1}{2}\log\Big(\frac{x^2+x+1}{x^2-x+1}\Big) + \frac{1}{\sqrt{3}}\tan^{-1}\Big(\frac{2x-1}{\sqrt{3}}\Big) + \frac{1}{\sqrt{3}}\tan^{-1}\Big(\frac{2x+1}{\sqrt{3}}\Big)
\end{aligned}$$

■ $\boxed{3}$ (1) $t=\tan\dfrac{x}{2}$ とおくと，

$$\sin x = \frac{2t}{1+t^2}\,, \qquad \cos x = \frac{1-t^2}{1+t^2}\,, \qquad \frac{dx}{dt} = \frac{2}{1+t^2}$$

となるから，

$$\begin{aligned}
\int \frac{1}{1+\sin x+\cos x}\,dx &= \int \frac{1}{1+\dfrac{2t}{1+t^2}+\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int \frac{1}{t+1}\,dt \\
&= \log(t+1) = \log\Big(\tan\frac{x}{2}+1\Big)
\end{aligned}$$

(2) $t=\tan x$ とおくと，

$$\sin x = \frac{t^2}{1+t^2}\,, \qquad \cos x = \frac{1}{1+t^2}\,, \qquad \frac{dx}{dt} = \frac{1}{1+t^2}$$

となるから，

$$\int \frac{\cos^2 x}{\sin^4 x}\,dx = \int \frac{\dfrac{1}{1+t^2}}{\Big(\dfrac{t^2}{1+t^2}\Big)^2}\frac{1}{1+t^2}\,dt = \int \frac{1}{t^4}\,dt = -\frac{1}{3}\,\frac{1}{t^3} = -\frac{1}{3}\,\frac{1}{\tan^3 x}$$

(3) $t=\sqrt{x+1}$ とおくと，$x=t^2-1$ より，$\dfrac{dx}{dt}=2t$ となるから，

$$\begin{aligned}
\int \frac{\sqrt{x+1}}{x}\,dx &= \int \frac{t}{t^2-1}\,2t\,dt = \int \frac{2t^2}{t^2-1}\,dt = \int \Big(2+\frac{1}{t-1}-\frac{1}{t+1}\Big)\,dt \\
&= 2t+\log\Big|\frac{t-1}{t+1}\Big| = 2\sqrt{x+1} + \frac{|\sqrt{x+1}-1|}{\sqrt{x+1}+1}
\end{aligned}$$

(4) $t=\sqrt{e^x-1}$ とおくと，$e^x=t^2+1$ より，

$$e^x\frac{dx}{dt} = 2t\,, \qquad \frac{dx}{dt} = \frac{2t}{t^2+1}$$

となるから，

$$\int \frac{1}{\sqrt{e^x-1}}\,dx = \int \frac{1}{t}\,\frac{2t}{t^2+1}\,dt = 2\int \frac{1}{t^2+1}\,dt = 2\tan^{-1}t = 2\tan^{-1}(\sqrt{e^x-1})$$

■ $\boxed{4}$ (1) $t=\sqrt[3]{x+1}$ とおくと，$x=t^3-1$ より，$\dfrac{dx}{dt}=3t^2$ となるから，

$$\begin{aligned}
\int_{-1}^{7} \frac{1}{1+\sqrt[3]{x+1}}\,dx &= \int_0^2 \frac{3t^2}{1+t}\,dt \\
&= 3\int_0^2 \Big(t-1+\frac{1}{1+t}\Big)\,dt \\
&= 3\Big[\frac{t^2}{2}-t+\log|1+t|\Big]_0^2 = 3\log 3
\end{aligned}$$

(2) $t=\tan\dfrac{x}{2}$ とおくと，

$$\sin x = \frac{2t}{1+t^2}\,, \qquad \cos x = \frac{1-t^2}{1+t^2}\,, \qquad \frac{dx}{dt} = \frac{2}{1+t^2}$$

となるから，

$$\int_0^{\pi/2} \frac{1}{2+\cos x}\,dx = \int_0^1 \frac{1}{2+\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt = \int_0^1 \frac{2}{3+t^2}\,dt$$

$$= \left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1 = \frac{\pi}{3\sqrt{3}}$$