演習問題 No. 6 の解答

■$\boxed{1}$ (1) $\displaystyle\int_1^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx = \Big[\sin^{-1}\frac{x}{2}\Big]_1^{\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\frac{1}{2} = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$

(2) 部分積分により

$$\begin{aligned}\int_0^1 x^3 e^{-x^2}\,dx &= \int_0^1 x^2(-\frac{1}{2}e^{-x^2})'\,dx = \Big[x^2(-\frac{1}{2}e^{-x^2})\Big]_0^1 + \int_0^1 xe^{-x^2}\,dx \\ &= -\frac{1}{2}e^{-1} + \Big[-\frac{1}{2}e^{-x^2}\Big]_0^1 = \frac{1}{2} - e^{-1}\end{aligned}$$

(3) $\tan^{-1} x = t$ とおけば，$\dfrac{dt}{dx} = \dfrac{1}{1+x^2}$ より，

$$\int_0^{\sqrt{3}} \frac{\tan^{-1} x}{x^2+1}\,dx = \int_0^{\pi/3} t\,dt = \Big[\frac{t^2}{2}\Big]_0^{\pi/3} = \frac{\pi^2}{18}$$

■$\boxed{2}$ (1) $\displaystyle\sum_{k=1}^{n} \frac{1}{n+3k} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{1+3(k/n)}$ より，関数 $\dfrac{1}{1+3x}$ を考えると，

$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{n+3k} = \int_0^1 \frac{1}{1+3x}\,dx = \Big[\frac{1}{3}\log(1+3x)\Big]_0^1 = \frac{2}{3}\log 2$$

(2) $\displaystyle\sum_{k=1}^{n} \frac{1}{\sqrt{3n^2+nk}} = \sum_{k=1}^{n} \frac{1}{n}\frac{1}{\sqrt{3+(k/n)}}$ より，関数 $\dfrac{1}{\sqrt{3+x}}$ を考えると，

$$\lim_{n\to\infty}\sum_{k=1}^{n} \frac{1}{\sqrt{n^2+nk}} = \int_0^1 \frac{1}{\sqrt{3+x}}\,dx = \Big[2\sqrt{3+x}\Big]_0^1 = 2(2-\sqrt{3})$$

■$\boxed{3}$ (1) $x^2 = t$ とおけば，$\dfrac{dt}{dx} = 2x$ より，

$$\int \frac{x}{x^4+1}\,dx = \int \frac{1}{t^2+1}\cdot\frac{1}{2}\,dt = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1} x^2$$

(2) $\sqrt{x} = t$ とおけば，$x = t^2$，$\dfrac{dx}{dt} = 2t$ より，

$$\int e^{\sqrt{x}}\,dx = 2\int te^t\,dt = 2(t-1)e^t = 2(\sqrt{x}-1)e^{\sqrt{x}}$$

(3) $\displaystyle\int \frac{1}{\sqrt{6x-x^2}}\,dx = \int \frac{1}{\sqrt{9-(x-3)^2}}\,dx = \sin^{-1}\Big(\frac{x-3}{3}\Big)$

(4) 部分積分により，

$$\begin{aligned}\int(\sin^{-1} x)^2\,dx &= x(\sin^{-1} x)^2 - \int x2\sin^{-1} x\frac{1}{\sqrt{1-x^2}}\,dx \\ &= x(\sin^{-1} x)^2 - 2\Big((-\sqrt{1-x^2})\sin^{-1} x - \int(-\sqrt{1-x^2})\frac{1}{\sqrt{1-x^2}}\,dx\Big) \\ &= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x\end{aligned}$$

(5) $\displaystyle\int \frac{1}{x^7+x}\,dx = \int \frac{1}{(1+x^{-6})x^7}\,dx$ より，$1+x^{-6} = t$ とおけば，$\dfrac{dt}{dx} = -\dfrac{6}{x^7}$ となるから，

$$\int \frac{1}{x^7+x}\,dx = -\frac{1}{6}\int\frac{1}{t}\,dt = -\frac{1}{6}\log|t| = -\frac{1}{6}\log\Big(1+\frac{1}{x^6}\Big)$$

■$\boxed{4}$ (1) $\displaystyle\int_0^{\pi} = \int_0^{\pi/2} + \int_{\pi/2}^{\pi}$ と分割し，最後の積分に変数変換 $x = \pi - t$ を用いて，

$$\begin{aligned}\int_0^{\pi} xf(\sin x)\,dx &= \int_0^{\pi/2} xf(\sin x)\,dx + \int_{\pi/2}^{0} (\pi - t)f(\sin(\pi - t))(-1)\,dt \\ &= \int_0^{\pi/2} xf(\sin x)\,dx + \pi\int_0^{\pi/2} f(\sin t)\,dt - \int_0^{\pi/2} tf(\sin t)\,dt \\ &= \pi\int_0^{\pi/2} f(\sin x)\,dx\end{aligned}$$

(2) $\cos^2 x = 1 - \sin^2 x$ より，(1) の結果を用いて，

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \pi\Big[-\tan^{-1}(\cos x)\Big]_0^{\pi/2} = \pi\tan^{-1}1 = \frac{\pi^2}{4}$$

■$\boxed{5}$ 変数変換 $9 - x = t$ を用いて，

$$I = \int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx = \int_9^0 \frac{\sqrt{9-t}}{\sqrt{9-t}+\sqrt{t}}\,(-1)\,dt = \int_0^9 \frac{\sqrt{9-t}}{\sqrt{t}+\sqrt{9-t}}\,dt$$

となるから，

$$\begin{aligned}2I &= \int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx + \int_0^9 \frac{\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\,dx = \int_0^9 dx = 9 \\ I &= \int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx = \frac{9}{2}\end{aligned}$$