# Model Predictive Control

sentry5588

2020

## 1 Introduction

This note documents the Model Predictive Control (MPC) method for the two wheel balancing robot. Many people has built similar self-balancing robots. Almost all of them uses PID control as the control strategy. For position estimation, some uses Kalman filters, and others uses complementary filters.

The purpose of using MPC in this robot is not trying to invent a new MPC technique, which usually the case for research papers. But rather I intend to 1) practice MPC and 2) test how well MPC behaves compares to other control schemes.

## 2 Robot Coordinates

As shown in Figure 1a, $\theta_k$ and $\omega_k$ denotes the angular position at time step $k$, respectively. Counter-clockwise rotation is positive. Angular acceleration is denoted by $\dot{\omega}_k$. $u_k$ is the horizontal force with positive to the right. Robot specifications can be found in Table 1

## 3 Problem Formulation

The model is a 1-input-2-output model.

$$x_{k+1} = f(x_k, u_k) \tag{1}$$
$$y_k = C(x_k)x_k \tag{2}$$

$x_k = [\theta_k \;\; \omega_k \;\; \dot{\omega}_k]^T$ is the system state. Linearize Equ 1, we have

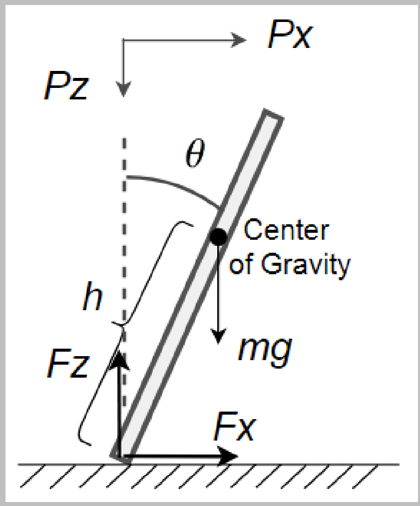
$$x_{k+1} = A(x_k)x_k + B(x_k)u_k \tag{3}$$
$$y_k = C(x_k)x_k \tag{4}$$

| Specification | Notation | Value |
|---|---|---|
| Center of Gravity (CoG) | $h$ | ?? 0.2 m |
| Mass | $m$ | ?? 1.1 kg |
| Moment of Inertia | I | ?? 0.8 kg $\cdot$ m$^2$ |

Table 1: Robot specification

(a) Robot Coordinates


(b) 1D Rotation




(c) 1D Free Dody Diagram

Figure 1: Two Wheel Balancing Robot

where $A(x_k)$ and $B(x_k)$ are Jacobians and given by

$$A(x_k) = \frac{\partial f}{\partial x_k}, B(x_k) = \frac{\partial f}{\partial u_k} \tag{5}$$

Figure 1c is the 1-D free body diagram. In the next subsection, I will develop the nonlinear continuous-time dynamics.

## 3.1 Nonlinear Continuous-Time Model

### 3.1.1 Derived from Newton's 2nd law

Summing the forces in figure 1c in the horizontal direction we have

$$F_x = m\ddot{p}_x \tag{6}$$

where $p_x$ is the horizontal velocity of the CoG. Summing the forces in figure 1c in the vertical direction we have

$$mg - F_z = m\ddot{p}_z \tag{7}$$

where $p_x$ is the horizontal velocity of the CoG. Summing the torques in figure 1c around the center of gravity we have

$$F_x h\cos(\theta) + F_z h\sin(\theta) + mg \cdot 0 = \ddot{\theta} I \tag{8}$$

The contact point between the robot and the ground has 0 vertical velocity. It's vertical velocity is a combination of the rod rotation around CoG, $\dot{\theta} h\sin(\theta)$, and vertical velocity of CoG, $\dot{p}_z$. Therefore

$$-\dot{\theta} h\sin(\theta) + \dot{p}_z = 0 \tag{9}$$

Take time derivative of (9)

$$-\ddot{\theta}h\sin(\theta) - \dot{\theta}^2 h\cos(\theta) + \ddot{p}_z = 0 \tag{10}$$

$$\ddot{p}_z = \ddot{\theta}h\sin(\theta) + \dot{\theta}^2 h\cos(\theta) \tag{11}$$

Substitute $\ddot{p}_z$ in (7) with (11)

$$mg - F_z = m(\ddot{\theta}h\sin(\theta) + \dot{\theta}^2 h\cos(\theta)) \tag{12}$$

$$mg - F_z = m\ddot{\theta}h\sin(\theta) + m\dot{\theta}^2 h\cos(\theta) \tag{13}$$

$$F_z = mg - m\ddot{\theta}h\sin(\theta) - m\dot{\theta}^2 h\cos(\theta) \tag{14}$$

Substitute $F_z$ with above equation in (8)

$$F_x h\cos(\theta) + (mg - m\ddot{\theta}h\sin(\theta) - m\dot{\theta}^2 h\cos(\theta))h\sin(\theta) + mg\cdot 0 = \ddot{\theta}I \tag{15}$$

$$F_x h\cos(\theta) + mgh\sin(\theta) - m\ddot{\theta}h^2\sin^2(\theta) - m\dot{\theta}^2 h^2\cos(\theta)\sin(\theta) = \ddot{\theta}I \tag{16}$$

$$(I + mh^2\sin^2(\theta))\ddot{\theta} + mh^2\cos(\theta)\sin(\theta)\dot{\theta}^2 = F_x h\cos(\theta) + mgh\sin(\theta) \tag{17}$$

So the dynamics can be written as

$$F_x = m\ddot{p}_x \tag{18}$$

$$(I + mh^2\sin^2(\theta))\ddot{\theta} + mh^2\cos(\theta)\sin(\theta)\dot{\theta}^2 = F_x h\cos(\theta) + mgh\sin(\theta) \tag{19}$$

**Check dynamics at special points for (19)**

When $\theta = 0$, i.e. the vertical up position, (19) becomes

$$I\ddot{\theta} = F_x h \tag{20}$$

When $\theta = \pi/2$, i.e. the horizontal position pointing to the left (assume single contact point with the ground) (19) becomes

$$(I + mh^2)\ddot{\theta} = mgh \tag{21}$$

When $\theta = -\pi/2$, i.e. the horizontal position pointing to the right (assume single contact point with the ground) (19) becomes

$$(I + mh^2)\ddot{\theta} = -mgh \tag{22}$$

When $\theta = \pi/4$, i.e. the horizontal position pointing to the left (assume single contact point with the ground) (19) becomes

$$(I + \frac{1}{2}mh^2)\ddot{\theta} + \frac{1}{2}mh^2\dot{\theta}^2 = \frac{\sqrt{2}}{2}F_x h + \frac{\sqrt{2}}{2}mgh \tag{23}$$

### 3.1.2 Derived from Lagrangian mechanics

The derivation follows [1]. The Lagrangian is $L = KE - PE$ where $KE$ and $PE$ are the kinematic energy and potential energy, respectively. The equation of motion can be determined by applying Lagrange mechanics in two generalized coordinate $p_x$ and $\theta$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{p}_x}\right) - \frac{\partial L}{\partial p_x} = \frac{\partial W}{\partial p_x}, \quad \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial W}{\partial \theta} \tag{24}$$

The kinematic energy and potential energy are

$$KE = \frac{1}{2}m\dot{p}_x + \frac{1}{2}I\dot{\theta}, \quad PE = mgh\cos(\theta) \tag{25}$$

# References

[1] Thomas Peacock and Nicolas Hadjiconstantinou. Dynamics and control i. *MIT OpenCourseWare*, 2007.