Model Predictive Control

sentry5588, MIT License

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1 Introduction

This note documents the Model Predictive Control (MPC) method for the two wheel balancing robot. Many people has built similar self-balancing robots. Almost all of them uses PID control as the control strategy. For position estimation, some uses Kalman filters, and others uses complementary filters.

The purpose of using MPC in this robot is not trying to invent a new MPC technique, which usually the case for research papers. But rather I intend to 1) practice MPC and 2) test how well MPC behaves compares to other control schemes.

2 Robot Coordinates

As shown in Figure 1a, θ_k and ω_k denotes the angular position at time step k, respectively. Counter-clockwise rotation is positive. Angular acceleration is denoted by $\dot{\omega}_k$. u_k is the horizontal force with positive to the right. Robot specifications can be found in Table 1

3 Problem Formulation

The model is a 1-input-2-output model.

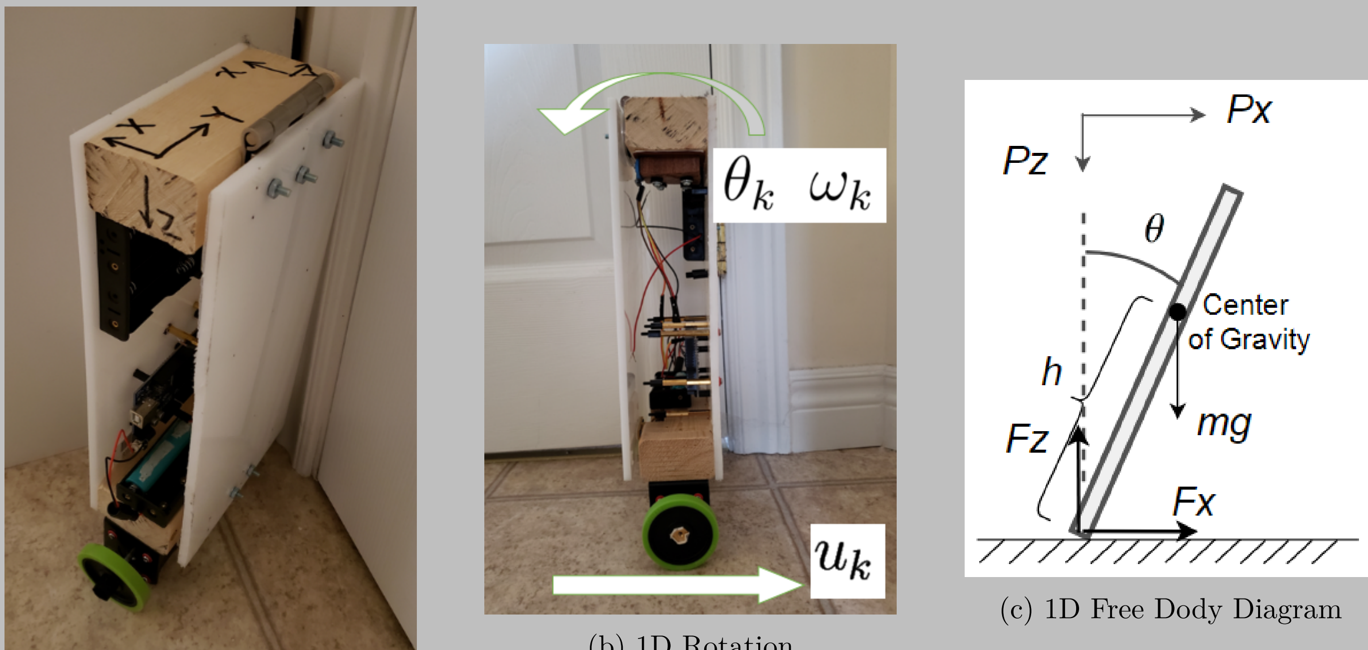
$$x_{k+1} = f(x_k, u_k) \tag{1}$$
$$y_k = C(x_k)x_k \tag{2}$$

$x_k = [\theta_k \;\; \omega_k \;\; \dot{\omega}_k]^T$ is the system state. Linearize Equ 1, we have

$$x_{k+1} = A(x_k)x_k + B(x_k)u_k \tag{3}$$
$$y_k = C(x_k)x_k \tag{4}$$

Specification	Notation	Value
Center of Gravity (CoG)	h	?? 0.2 m
Mass	m	?? 1.1 kg
Moment of Inertia	I	?? 0.8 kg · m^2

Table 1: Robot specification



(a) Robot Coordinates

(b) 1D Rotation

(c) 1D Free Dody Diagram

Figure 1: Two Wheel Balancing Robot

where $A(x_k)$ and $B(x_k)$ are Jacobians and given by

$$A(x_k) = \frac{\partial f}{\partial x_k}, B(x_k) = \frac{\partial f}{\partial u_k} \tag{5}$$

Figure 1c is the 1-D free body diagram. In the next subsection, I will develop the nonlinear continuous-time dynamics.

3.1 Nonlinear Continuous-Time Model

3.1.1 Derived from Newton's 2nd law

Summing the forces in figure 1c in the horizontal direction we have

$$F_x = m\ddot{p}_x \tag{6}$$

where p_x is the horizontal velocity of the CoG. Summing the forces in figure 1c in the vertical direction we have

$$mg - F_z = m\ddot{p}_z \tag{7}$$

where p_x is the horizontal velocity of the CoG. Summing the torques in figure 1c around the center of gravity we have

$$F_x h \cos(\theta) + F_z h \sin(\theta) + mg \cdot 0 = \ddot{\theta} I \tag{8}$$

The contact point between the robot and the ground has 0 vertical velocity. It's vertical velocity is a combination of the rod rotation around CoG, $\dot{\theta} h \sin(\theta)$, and vertical velocity of CoG, $\dot{p}_z$. Therefore

$$-\dot{\theta} h \sin(\theta) + \dot{p}_z = 0 \tag{9}$$

Take time derivative of (9)

$$-\ddot{\theta}h\sin(\theta) - \dot{\theta}^2 h\cos(\theta) + \ddot{p}_z = 0 \tag{10}$$

$$\ddot{p}_z = \ddot{\theta}h\sin(\theta) + \dot{\theta}^2 h\cos(\theta) \tag{11}$$

Substitute $\ddot{p}_z$ in (7) with (11)

$$mg - F_z = m(\ddot{\theta}h\sin(\theta) + \dot{\theta}^2 h\cos(\theta)) \tag{12}$$

$$mg - F_z = m\ddot{\theta}h\sin(\theta) + m\dot{\theta}^2 h\cos(\theta) \tag{13}$$

$$F_z = mg - m\ddot{\theta}h\sin(\theta) - m\dot{\theta}^2 h\cos(\theta) \tag{14}$$

Substitute F_z with above equation in (8)

$$F_x h\cos(\theta) + (mg - m\ddot{\theta}h\sin(\theta) - m\dot{\theta}^2 h\cos(\theta))h\sin(\theta) + mg \cdot 0 = \ddot{\theta}I \tag{15}$$

$$F_x h\cos(\theta) + mgh\sin(\theta) - m\ddot{\theta}h^2\sin^2(\theta) - m\dot{\theta}^2 h^2\cos(\theta)\sin(\theta) = \ddot{\theta}I \tag{16}$$

$$(I + mh^2\sin^2(\theta))\ddot{\theta} + mh^2\cos(\theta)\sin(\theta)\dot{\theta}^2 = F_x h\cos(\theta) + mgh\sin(\theta) \tag{17}$$

So the dynamics can be written as

$$F_x = m\ddot{p}_x \tag{18}$$

$$(I + mh^2\sin^2(\theta))\ddot{\theta} + mh^2\cos(\theta)\sin(\theta)\dot{\theta}^2 = F_x h\cos(\theta) + mgh\sin(\theta) \tag{19}$$

Check dynamics at special points for (19)
When $\theta = 0$, i.e. the vertical up position, (19) becomes

$$I\ddot{\theta} = F_x h \tag{20}$$

When $\theta = \pi/2$, i.e. the horizontal position pointing to the left (assume single contact point with the ground) (19) becomes

$$(I + mh^2)\ddot{\theta} = mgh \tag{21}$$

When $\theta = -\pi/2$, i.e. the horizontal position pointing to the right (assume single contact point with the ground) (19) becomes

$$(I + mh^2)\ddot{\theta} = -mgh \tag{22}$$

When $\theta = \pi/4$, i.e. the horizontal position pointing to the left (assume single contact point with the ground) (19) becomes

$$(I + \frac{1}{2}mh^2)\ddot{\theta} + \frac{1}{2}mh^2\dot{\theta}^2 = \frac{\sqrt{2}}{2}F_x h + \frac{\sqrt{2}}{2}mgh \tag{23}$$

3.1.2 Derived from Lagrangian mechanics

The derivation follows [Peacock and Hadjiconstantinou, 2007]. The Lagrangian is $L = KE - PE$ where KE and PE are the kinematic energy and potential energy, respectively. W is the virtual work. The equation of motion can be determined by applying Lagrange mechanics in two generalized coordinate p_x and θ

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{p}_x}\right) - \frac{\partial L}{\partial p_x} = \frac{\partial W}{\partial p_x}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial W}{\partial \theta} \tag{24}$$

The contact point displacement relative to the center of gravity is $h\theta\cos(\theta)$. Therefore the absolute velocity of the contact point is $p_x + h\theta\cos(\theta)$. The kinematic energy, potential energy and virtual work are

$$KE = \frac{1}{2}m\dot{p}_x^2 + \frac{1}{2}I\dot{\theta}^2, \quad PE = mgh\cos(\theta), \quad W = F_x(p_x + h\theta\cos(\theta)) + F_x h\theta\cos(\theta) = F_x p_x + 2F_x h\theta\cos(\theta) \tag{25}$$

Substitute (24) with (25), in p_x direction we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{p}_x}\left(\frac{1}{2}m\dot{p}_x^2 + \frac{1}{2}I\dot{\theta}^2\right)\right) \tag{26}$$

$$-\frac{\partial}{\partial p_x}\left(\frac{1}{2}m\dot{p}_x^2 + \frac{1}{2}I\dot{\theta}^2 - mgh\cos(\theta)\right) = \frac{\partial}{\partial p_x}F_x p_x + \frac{\partial}{\partial p_x}2F_x h\theta\cos(\theta) \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{p}_x + 0) - (0 + 0 - 0) = F_x + 0 \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}m\dot{p}_x = F_x \tag{29}$$

$$m\ddot{p}_x = F_x \tag{30}$$

Substitute (24) with (25), in θ direction we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\dot{p}_x^2 + \frac{1}{2}I\dot{\theta}^2 - mgh\cos(\theta)\right)\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\dot{p}_x^2 + \frac{1}{2}I\dot{\theta}^2 - mgh\cos(\theta)\right) \tag{31}$$

$$= \frac{\partial}{\partial \theta}F_x p_x + \frac{\partial}{\partial \theta}2F_x h\theta\cos(\theta) \tag{32}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(0 + I\dot{\theta} - 0) - (0 + 0 + mgh\sin(\theta)) = -2F_x h\theta\sin(\theta) + 2F_x h\cos(\theta) \tag{33}$$

$$(0 + I\ddot{\theta} - 0) - (0 + 0 + mgh\sin(\theta)) = -2F_x h\theta\sin(\theta) + 2F_x h\cos(\theta) \tag{34}$$

$$I\ddot{\theta} - mgh\sin(\theta) = -2F_x h\theta\sin(\theta) + 2F_x h\cos(\theta) \tag{35}$$

$$I\ddot{\theta} = 2F_x h\cos(\theta) + mgh\sin(\theta) - 2F_x h\theta\sin(\theta) \tag{36}$$

3.2 Model Linearization and discretization

I follow [Zhakatayev et al., 2017] to linearize and discretize the nonlinear continuous-time dynamics in [**?**]

References

[Peacock and Hadjiconstantinou, 2007] Peacock, T. and Hadjiconstantinou, N. (2007). Dynamics and control i. *MIT OpenCourseWare.*

[Zhakatayev et al., 2017] Zhakatayev, A., Rakhim, B., Adiyatov, O., Baimyshev, A., and Varol, H. A. (2017). Successive linearization based model predictive control of variable stiffness actuated robots. In *2017 IEEE International Conference on Advanced Intelligent Mechatronics (AIM)*, pages 1774–1779. IEEE.