

# Chapter 1 - The Real and Complex Number Systems

**Problem 1.** If  $r$  is rational ( $r \neq 0$ ) and  $x$  is irrational, prove that  $r+x$  and  $rx$  are irrational.

*Proof.* Write  $r = \frac{m}{n}$ , where  $m$  and  $n$  are nonzero integers. Suppose  $r+x$  were rational. Then there exists integers  $p$  and  $q$ , with  $q \neq 0$  such that

$$r+x = \frac{p}{q} \quad (1)$$

Then  $x$  can be expressed as

$$x = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q} \quad (2)$$

contradicting our assumption about  $x$ . So  $r+x$  must be irrational.

Now, suppose  $rx$  were rational. Then there exists integers  $p$  and  $q$ , with  $q \neq 0$  such that

$$x = \frac{p}{q} \cdot \frac{n}{m} \quad (3)$$

since  $m \neq 0$ . Then  $x$  can be written as

$$x = \frac{pn}{qm} \in \mathbb{Q} \quad (4)$$

contradicting our assumption about  $x$ . So  $rx$  must be irrational.  $\square$

**Problem 2.** Prove that there is no rational number whose square is 12.

*Proof.* Suppose there exists a rational number  $r \in \mathbb{Q}$  such that  $r^2 = 12$ . We can write  $r = \frac{m}{n}$  where  $m$  and  $n$  share no common factors. Then

$$\frac{m^2}{n^2} = 3 \cdot 4 \implies m^2 = 3 \cdot 4n^2 \quad (5)$$

Thus  $m^2$  is divisible by 3. This implies  $m$  is divisible by 3 (otherwise  $m^2$  would not be). Hence,  $m^2$  is divisible by 9, and so is the right hand side of (5). This implies that  $4n^2$  is divisible by 3. Since 4 is not divisible by 3, it follows that  $n^2$ , and thus  $n$  is divisible by 3. This contradicts the fact that  $m$  and  $n$  share no common factors. Thus, there can be no rational number that satisfies  $r^2 = 12$ .  $\square$

**Problem 3.** Prove Proposition 1.15: The axioms for multiplication in a field imply the following statements.

- (a) If  $x \neq 0$  and  $xy = xz$  then  $y = z$
- (b) If  $x \neq 0$  and  $xy = x$  then  $y = 1$
- (c) If  $x \neq 0$  and  $xy = 1$  then  $y = 1/x$
- (d) If  $x \neq 0$  then  $1/(1/x) = x$ .

*Proof.* (a)

$$y = (1/x)xy = (1/x)xz = 1z = z \quad (6)$$

(b) Take  $z = 1$  in (a)

(c) Take  $z = 1/x$  in (a)

(d) This follows from (c) if we replace  $x$  with  $1/x$  and  $y$  with  $x$  □

**Problem 4.** Let  $E$  be a nonempty subset of an ordered set; suppose  $\alpha$  is a lower bound of  $E$  and  $\beta$  is an upper bound of  $E$ . Prove that  $\alpha \leq \beta$ .

*Proof.* Suppose  $\alpha > \beta$ . Let  $x \in E$ ; since  $\alpha$  is a lower bound of  $E$ , we must have  $\alpha \leq x$ . Since  $\alpha > \beta$  and  $>$  is transitive, it then follows that  $\beta < x$ . But this contradicts the fact that  $\beta$  is an upper bound of  $E$ . So  $\alpha > \beta$  must be false, i.e.  $\alpha \leq \beta$ . □

**Problem 5.** Let  $A$  be a nonempty set of real numbers which is bounded below. Let  $-A$  be the set of all numbers  $-x$ , where  $x \in A$ . Prove that

$$\inf A = -\sup(-A) \quad (7)$$

*Proof.* Since  $A$  is bounded below,  $\gamma = \inf A$  exists in  $\mathbb{R}$ , and  $\gamma \leq x$  for all  $x \in A$ . This implies  $-\gamma \geq -x$  for all  $x \in A$ , or  $-\gamma \geq y$  for all  $y \in -A$ . So  $-\gamma$  is an upper bound of  $-A$ . Let  $\kappa < -\gamma$ , then  $-\kappa > \gamma$ , so that  $-\kappa$  is not a lower bound of  $A$ . Hence, there exists  $x \in A$  such that  $-\kappa > x$  or  $\kappa < -x \in -A$ . Hence  $\kappa$  is not an upper bound of  $-A$ . Then by definition,  $-\gamma$  is the supremum of  $-A$ , i.e.

$$-\inf A = \sup(-A) \quad (8)$$

which is equivalent to (7) □

**Problem 6.**

**Problem 7.**

**Problem 8.** Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:*  $-1$  is a square.

*Proof.* Let  $>$  be an order on  $\mathbb{C}$ . Assume this turns  $\mathbb{C}$  into an ordered field. Since  $i \neq 0$ , we must either have  $i > 0$  or  $i < 0$ . First, assume  $i > 0$ . Then  $-1 = i^2 > 0$ . Add 1 to both sides to obtain  $0 > 1$ . But  $1 = (-1)(-1) > 0$  since  $-1$  is positive, resulting in a contradiction. Now assume  $i < 0$ . Add  $-i$  to both sides to obtain  $0 < -i$ . Hence  $-1 = (-i)^2 > 0$ . Again, we have  $0 > 1$ , but  $1 = (-1)(-1) > 0$  resulting in another contradiction. So  $\mathbb{C}$  cannot be an ordered field under this order. □

**Problem 9.**

**Problem 10.**

**Problem 11.**

**Problem 12.**

**Problem 13.**

**Problem 14.**

**Problem 15.**

**Problem 16.** (Incomplete) Suppose  $k \geq 3$ ,  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ ,  $|\mathbf{x} - \mathbf{y}| = d > 0$ , and  $r > 0$ . Prove:

(a) If  $2r > d$ , there are infinitely many  $\mathbf{z} \in \mathbb{R}^k$  such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r \quad (9)$$

(b) If  $2r = d$ , there is exactly one such  $\mathbf{z}$ .

(c) If  $2r < d$ , there is no such  $\mathbf{z}$ .

How must these statements be modified if  $k$  is 2 or 1?

*Proof.* (a) Since  $2r > d$  it follows that  $r^2 - (d/2)^2 > 0$ . Then let  $\epsilon = \sqrt{r^2 - (d/2)^2} > 0$ . Suppose there is a vector  $\mathbf{q} \in \mathbb{R}^k$  such that  $|\mathbf{q}| = \epsilon$  and  $\mathbf{q} \cdot (\mathbf{y} - \mathbf{x}) = 0$ . Then  $\mathbf{z} = \mathbf{x} + \frac{1}{2}(\mathbf{y} - \mathbf{x}) + \mathbf{q}$  will meet the desired requirements: expanding out  $|\mathbf{z} - \mathbf{x}|^2 = (\mathbf{z} - \mathbf{x}) \cdot (\mathbf{z} - \mathbf{x})$  and  $|\mathbf{z} - \mathbf{y}|^2 = (\mathbf{z} - \mathbf{y}) \cdot (\mathbf{z} - \mathbf{y})$  and using the fact that  $\mathbf{q} \cdot \mathbf{y} = \mathbf{q} \cdot \mathbf{x}$  we obtain the identical expression

$$\frac{1}{4}(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + \mathbf{q} \cdot \mathbf{q} = \left(\frac{d}{2}\right)^2 + \epsilon^2 = r^2 \quad (10)$$

as desired. We now must prove that there are infinitely many such  $\mathbf{q}$ .

Since  $\mathbf{y} \neq \mathbf{x}$ , there exists an integer  $j$  satisfying  $1 \leq j \leq k$  such that  $x_j - y_j \neq 0$  □

**Problem 17.**

**Problem 18.** If  $k \geq 2$  and  $\mathbf{x} \in \mathbb{R}^k$ , prove that there exists  $\mathbf{y} \in \mathbb{R}^k$  such that  $\mathbf{y} \neq \mathbf{0}$  but  $\mathbf{x} \cdot \mathbf{y} = 0$ . Is this also true if  $k = 1$ ?

*Proof.* If  $\mathbf{x} = \mathbf{0}$ , then take  $\mathbf{y}$  to be any nonzero vector in  $\mathbb{R}^k$ . Otherwise, if  $\mathbf{x} \neq \mathbf{0}$  there is an integer  $\ell$  satisfying  $1 \leq \ell \leq k$  such that  $x_\ell \neq 0$ . Choose an integer  $j \neq \ell$  that satisfies  $1 \leq j \leq k$  (this is possible, since  $k \geq 2$ ). Then let  $\mathbf{y}$  be the vector defined as

$$y_j = x_\ell, \quad y_\ell = -x_j, \quad y_i = 0 \text{ for } i \neq \ell, j \quad (11)$$

$y_j = x_\ell \neq 0$  implies that  $\mathbf{y} \neq \mathbf{0}$  and the inner product  $\mathbf{x} \cdot \mathbf{y}$  satisfies:

$$\sum_{i=1}^k y_i x_i = y_j x_j + y_\ell x_\ell = x_\ell x_j - x_j x_\ell = 0 \quad (12)$$

When  $k = 1$ , the inner product corresponds to standard scalar multiplication. The result fails for  $x = 1$ , because if  $xy = 0$ , we have:

$$0 = xy = 1y = y \tag{13}$$

by the multiplication axioms for a field. Thus the result does not hold in  $\mathbb{R}^1$ .  $\square$

**Problem 19.**

**Problem 20.**