Chapter 1 - The Real and Complex Number Systems

Problem 1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. Write $r = \frac{m}{n}$, where m and n are nonzero integers. Suppose $r+x$ were rational. Then there exists integers p and q, with $q \neq 0$ such that

$$r + x = \frac{p}{q} \tag{1}$$

Then x can be expressed as

$$x = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q} \tag{2}$$

contradicting our assumption about x. So $r + x$ must be irrational.

Now, suppose rx were rational. Then there exists integers p and q, with $q \neq 0$ such that

$$x = \frac{p}{q} \cdot \frac{n}{m} \tag{3}$$

since $m \neq 0$. Then x can be written as

$$x = \frac{pn}{qm} \in \mathbb{Q} \tag{4}$$

contradicting our assumption about x. So rx must be irrational. □

Problem 2. Prove that there is no rational number whose square is 12.

Proof. Suppose there exists a rational number $r \in \mathbb{Q}$ such that $r^2 = 12$. We can write $r = \frac{m}{n}$ where m and n share no common factors. Then

$$\frac{m^2}{n^2} = 3 \cdot 4 \implies m^2 = 3 \cdot 4n^2 \tag{5}$$

Thus m^2 is divisible by 3. This implies m is divisible by 3 (otherwise m^2 would not be). Hence, m^2 is divisible by 9, and so is the right hand side of (5). This implies that $4n^2$ is divisible by 3. Since 4 is not divisible by 3, it follows that n^2, and thus n is divisible by 3. This contradicts the fact that m and n share no common factors. Thus, there can be no rational number that satisfies $r^2 = 12$. □

Problem 3. Prove Proposition 1.15: The axioms for multiplication in a field imply the following statements.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$

(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof. (a)

$$y = (1/x)xy = (1/x)xz = 1z = z \tag{6}$$

(b) Take $z = 1$ in (a)

(c) Take $z = 1/x$ in (a)

(d) This follows from (c) if we replace x with $1/x$ and y with x □

Problem 4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Suppose $\alpha > \beta$. Let $x \in E$; since α is a lower bound of E, we must have $\alpha \leq x$. Since $\alpha > \beta$ and $>$ is transitive, it then follows that $\beta < x$. But this contradicts the fact that β is an upper bound of E. So $\alpha > \beta$ must be false, i.e. $\alpha \leq \beta$. □

Problem 5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A) \tag{7}$$

Proof. Since A is bounded below, $\gamma = \inf A$ exists in $\mathbb{R}$, and $\gamma \leq x$ for all $x \in A$. This implies $-\gamma \geq -x$ for all $x \in A$, or $-\gamma \geq y$ for all $y \in -A$. So $-\gamma$ is an upper bound of $-A$. Let $\kappa < -\gamma$, then $-\kappa > \gamma$, so that $-\kappa$ is not a lower bound of A. Hence, there exists $x \in A$ such that $-\kappa > x$ or $\kappa < -x \in -A$. Hence κ is not an upper bound of $-A$. Then by definition, $-\gamma$ is the supremum of $-A$, i.e

$$-\inf A = \sup(-A) \tag{8}$$

which is equivalent to (7) □

Problem 6.

Problem 7.

Problem 8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint*: -1 is a square.

Proof. Let $>$ be an order on $\mathbb{C}$. Assume this turns $\mathbb{C}$ into an ordered field. Since $i \neq 0$, we must either have $i > 0$ or $i < 0$. First, assume $i > 0$. Then $-1 = i^2 > 0$. Add 1 to both sides to obtain $0 > 1$. But $1 = (-1)(-1) > 0$ since -1 is positive, resulting in a contradiction. Now assume $i < 0$. Add $-i$ to both sides to obtain $0 < -i$. Hence $-1 = (-i)^2 > 0$. Again, we have $0 > 1$, but $1 = (-1)(-1) > 0$ resulting in another contradiction. So $\mathbb{C}$ cannot be an ordered field under this order. □

Problem 9.

Problem 10.

Problem 11.

Problem 12.

Problem 13.

Problem 14.

Problem 15.

Problem 16. (Incomplete) Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r \tag{9}$$

(b) If $2r = d$, there is exactly one such $\mathbf{z}$.

(c) If $2r < d$, there is no such $\mathbf{z}$.

How must these statements be modified if k is 2 or 1?

Proof. (a) Since $2r > d$ it follows that $r^2 - (d/2)^2 > 0$. Then let $\epsilon = \sqrt{r^2 - (d/2)^2} > 0$. Suppose there is a vector $\mathbf{q} \in \mathbb{R}^k$ such that $|\mathbf{q}| = \epsilon$ and $\mathbf{q} \cdot (\mathbf{y} - \mathbf{x}) = 0$. Then $\mathbf{z} = \mathbf{x} + \frac{1}{2}(\mathbf{y} - \mathbf{x}) + \mathbf{q}$ will meet the desired requirements: expanding out $|\mathbf{z} - \mathbf{x}|^2 = (\mathbf{z} - \mathbf{x}) \cdot (\mathbf{z} - \mathbf{x})$ and $|\mathbf{z} - \mathbf{y}|^2 = (\mathbf{z} - \mathbf{y}) \cdot (\mathbf{z} - \mathbf{y})$ and using the fact that $\mathbf{q} \cdot \mathbf{y} = \mathbf{q} \cdot \mathbf{x}$ we obtain the identical expression

$$\frac{1}{4}(\mathbf{y} - \mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) + \mathbf{q} \cdot \mathbf{q} = \left(\frac{d}{2}\right)^2 + \epsilon^2 = r^2 \tag{10}$$

as desired. We now must prove that there are infinitely many such $\mathbf{q}$.

Since $|\mathbf{x} - \mathbf{y}| > 0$, we have $\mathbf{x} \neq \mathbf{y}$ and so there exists an integer j satisfying $1 \leq j \leq k$ such that $y_j - x_j \neq 0$. Choose two other distinct integers i, p that satisfy $1 \leq i, p \leq k$, and $i, p \neq j$ (this is possible since $k \geq 3$).

Let $\alpha = (y_j - x_j) \neq 0$, $\beta = (y_i - x_i)$ and $\gamma = (y_p - x_p)$. Let $q_p \in \mathbb{R}$ satisfying:

$$|q_p| \leq \epsilon \left(\frac{\beta^2 + \alpha^2}{\gamma^2 + \beta^2 + \alpha^2} \right)^{1/2} \tag{11}$$

The number on the right hand side of equation (11) is well-defined and positive, hence, there are infinitely many such q_p.

For some such q_p, consider the quadratic equation in the variable q_i:

$$q_i^2 \left(\frac{\beta^2 + \alpha^2}{\alpha^2} \right) + q_i \frac{2 q_p \beta \gamma}{\alpha^2} + q_p^2 \frac{\gamma^2 + \alpha^2}{\alpha^2} - \epsilon^2 = 0 \tag{12}$$

There is at least one solution to this equation if the discriminant is non-negative. After expanding out the discriminant and some algebra, it can be shown that the discriminant is non-negative if and only if (11) holds. Thus at least one number q_i satisfies this equation; let q_i take this value.

Finally, define q_j as:

$$q_j = \frac{-1}{\alpha}(\beta q_i + \gamma q_p) \tag{13}$$

Let $\mathbf{q} \in \mathbb{R}^k$ be a vector satisfying:

$$\mathbf{q}_i = q_i \tag{14}$$

$$\mathbf{q}_p = q_p \tag{15}$$

$$\mathbf{q}_j = q_j \tag{16}$$

$$\mathbf{q}_n = 0 \quad \text{if } n \neq i, p, j \tag{17}$$

Then $\mathbf{q} \cdot (\mathbf{y} - \mathbf{x}) = 0$ by appealing to equation (13) and the definition of γ, α, and β. Furthermore, the equation

$$|\mathbf{q}|^2 = q_i^2 + q_j^2 + q_p^2 = \epsilon^2 \tag{18}$$

is shown to be equivalent to equation (12) after plugging in expression (13) for q_j and some algebra. By construction, this equation holds and so we have $|\mathbf{q}| = \epsilon$. Thus $\mathbf{q}$ satisfies the properties given at the start of the proof. Since there are infinitely many q_p that satisfy equation (11) there are infinitely many such $\mathbf{q}$ and the result follows.

(b) (Solve problem 13 first)

□

Problem 17.

Problem 18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Proof. If $\mathbf{x} = \mathbf{0}$, then take $\mathbf{y}$ to be any nonzero vector in $\mathbb{R}^k$. Otherwise, if $\mathbf{x} \neq \mathbf{0}$ there is an integer ℓ satisfying $1 \leq \ell \leq k$ such that $x_\ell \neq 0$. Choose an integer $j \neq \ell$ that satisfies $1 \leq j \leq k$ (this is possible, since $k \geq 2$). Then let $\mathbf{y}$ be the vector defined as

$$y_j = x_\ell, \quad y_\ell = -x_j, \quad y_i = 0 \text{ for } i \neq \ell, j \tag{19}$$

$y_j = x_\ell \neq 0$ implies that $\mathbf{y} \neq \mathbf{0}$ and the inner product $\mathbf{x} \cdot \mathbf{y}$ satisfies:

$$\sum_{i=1}^{k} y_i x_i = y_j x_j + y_\ell x_\ell = x_\ell x_j - x_j x_\ell = 0 \tag{20}$$

When $k = 1$, the inner product corresponds to standard scalar multiplication. The result fails for $x = 1$, because if $xy = 0$, we have:

$$0 = xy = 1y = y \tag{21}$$

by the multiplication axioms for a field. Thus the result does not hold in $\mathbb{R}^1$. □

Problem 19.

Problem 20.