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003

# A MULTIGRID ALGORITHM FOR THE LOWEST-ORDER RAVIART–THOMAS MIXED TRIANGULAR FINITE ELEMENT METHOD*

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**Abstract.** An optimal order multigrid method for the lowest-order Raviart–Thomas mixed triangular finite element is developed. The algorithm and the convergence analysis are based on the equivalence between Raviart–Thomas mixed methods and certain nonconforming methods. Both the Dirichlet and singular Neumann boundary value problems for second-order elliptic equations are discussed.

**Key words.** elliptic boundary value problem, multigrid method, nonconforming finite element, mixed finite element, Raviart–Thomas finite element

**AMS(MOS) subject classifications.** 65N30, 65F10

**Introduction.** Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$, $f \in L^2(\Omega)$, and $\mathbf{A}$ be a sufficiently smooth two-by-two symmetric matrix-valued function on $\overline{\Omega}$ satisfying the following condition. There exists $\alpha > 0$ such that

$$\sum_{i,j} \mathbf{A}(x)\xi \cdot \xi \geq \alpha\, \xi \cdot \xi \quad \forall x \in \overline{\Omega},\ \forall \xi \in \mathbb{R}^2. \tag{0.1}$$

Throughout this paper, we use boldfaced capital letters to denote matrix-valued functions of two independent variables and boldfaced small letters to denote vector-valued functions. The Dirichlet boundary value problem

$$\begin{aligned} -\text{div}\,(\mathbf{A}\,\mathbf{grad}\,u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{0.2}$$

can be discretized in various ways. One of these is the mixed method of Raviart–Thomas (cf. [19]). In this paper we develop a multigrid method for finding the approximate solution of the Raviart–Thomas discretization in the lowest-order case. We will also show how to modify this method in order to handle the singular Neumann boundary value problem.

This problem is significant for two reasons. First, the existing literature on multigrid methods for mixed finite elements (cf. [24], [25]) does not cover the lowest-order Raviart–Thomas triangular finite element. Secondly, the lowest-order Raviart–Thomas finite element is part of the PEERS (cf. [3]) mixed finite element for elasticity problems. The results of this paper should be useful in the development of a multigrid algorithm for the more complicated PEERS element.

It is well known that (0.2) has a unique solution $u$ in $H_0^1(\Omega) \cap H^2(\Omega)$, and we have the following elliptic regularity estimate (cf. [16]).

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There exists $C_{\Omega,\mathbf{A}}$ such that

$$\|u\|_{H^2(\Omega)} \le C_{\Omega,\mathbf{A}} \, \|f\|_{L^2(\Omega)}. \tag{0.3}$$

In this paper we use the following conventions for the Sobolev norms and semi-norms:

$$\|v\|_{H^m(\Omega)} := \Big( \int_\Omega \sum_{|\alpha| \le m} |\partial^\alpha v|^2 dx \Big)^{1/2} \quad \text{and}$$

$$|v|_{H^m(\Omega)} := \Big( \int_\Omega \sum_{|\alpha| = m} |\partial^\alpha v|^2 dx \Big)^{1/2}.$$

A mixed formulation of (0.2) (cf. [19]) is the following.

Find $(\boldsymbol{\sigma}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} \int_\Omega \mathbf{C}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} dx - \int_\Omega u \operatorname{div} \boldsymbol{\tau}\, dx &= 0 \quad \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega), \\ -\int_\Omega v \operatorname{div} \boldsymbol{\sigma} \qquad\qquad &= -\int_\Omega f v\, dx \quad \forall v \in L^2(\Omega), \end{aligned} \tag{0.4}$$

where $\mathbf{C} := \mathbf{A}^{-1}$ is the compliance tensor, and

$$H(\mathrm{div};\Omega) := \{\boldsymbol{\tau} \colon \boldsymbol{\tau} \in (L^2(\Omega))^2, \operatorname{div} \boldsymbol{\tau} \in L^2(\Omega)\}$$

with norm defined by

$$\|\boldsymbol{\tau}\|^2_{H(\mathrm{div};\Omega)} = \sum_{i=1}^{2} \|\tau_i\|^2_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{\tau}\|^2_{L^2(\Omega)}.$$

If $u$ is the unique solution of (0.2), then $(\boldsymbol{\sigma}, u)$ is the unique solution of (0.4), provided we take $\boldsymbol{\sigma}$ to be $-\mathbf{A}\,\mathbf{grad}\, u$.

Let $\{\mathcal{T}_k\}_{k=1}^\infty$ be a family of triangulations of $\Omega$, where $\mathcal{T}_{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}_k$. Define $h_k := \max\{\operatorname{diam} T : T \in \mathcal{T}_k\}$. Following Raviart–Thomas (cf. [19]), we have the following sequence of discretizations of the mixed formulation (0.4). (We use notation similar to that in [2].)

Find $(\boldsymbol{\sigma}_k, u_k) \in RT_0^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{T}_k)$ such that

$$\begin{aligned} \int_\Omega \mathbf{C}\boldsymbol{\sigma}_k \cdot \boldsymbol{\tau}\, dx - \int_\Omega u_k \operatorname{div} \boldsymbol{\tau}\, dx &= 0 \quad \forall \boldsymbol{\tau} \in RT_0^0(\mathcal{T}_k), \\ -\int_\Omega v \operatorname{div} \boldsymbol{\sigma}_k\, dx \qquad\qquad &= -\int_\Omega f v\, dx \quad \forall v \in M_{-1}^0(\mathcal{T}_k). \end{aligned} \tag{0.5}$$

Here

$$\begin{aligned} RT_0^0(\mathcal{T}_k) = \Big\{ \boldsymbol{\tau} \colon \boldsymbol{\tau} \in L^2(\Omega)^2, \boldsymbol{\tau}|_T = \begin{bmatrix} a_T \\ c_T \end{bmatrix} + b_T \begin{bmatrix} x \\ y \end{bmatrix} \ \forall T \in \mathcal{T}_k, \\ \text{where } a_T, b_T, c_T \in \mathbb{R} \text{ and the normal component of } \boldsymbol{\tau} \text{ is} \\ \text{continuous across the interelement boundaries} \Big\}, \end{aligned}$$

and $M_{-1}^0(\mathcal{T}_k) = \{v: v \in L^2(\Omega), v|_T \in \mathcal{P}_0(T) \quad \forall T \in \mathcal{T}_k\}$. Hereafter, we let $\mathcal{P}_n(T)$ denote the space of polynomials on $T$ of degree less than or equal to $n$. Note that vector functions of the form $\left[\begin{smallmatrix} a_T \\ c_T \end{smallmatrix}\right] + b_T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ have the property that their normal components along any straight line are constants.

Equation (0.5) is the lowest-order Raviart–Thomas mixed finite element method. We have the following well-known discretization error estimates (cf. [14], [15], [19]).

There exists a unique solution $(\boldsymbol{\sigma}_k, u_k)$ of (0.5) and a positive constant $C$ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{L^2(\Omega)} + \|u - u_k\|_{L^2(\Omega)} \leq Ch_k\|u\|_{H^2(\Omega)}. \tag{0.6}$$

If $u \in H^3(\Omega)$, then

$$\|P_k^0 u - u_k\|_{L^2(\Omega)} \leq C\, h_k^2\, \|u\|_{H^3(\Omega)}, \tag{0.7}$$

where $P_k^0$ is the $L^2$-orthogonal projection onto $M_{-1}^0(\mathcal{T}_k)$. Throughout this paper, unless otherwise stated, $C$ (with or without subscripts) will denote a generic positive constant independent of the mesh parameter $k$.

We will develop an optimal-order multigrid method for the problem (0.5). Let $n_k = \dim\, (RT_0^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{T}_k))$. Our full multigrid algorithm will yield an approximate solution $(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k)$ to (0.5) in $\mathcal{O}(n_k)$ steps such that

$$\|\boldsymbol{\sigma}_k - \hat{\boldsymbol{\sigma}}_k\|_{L^2(\Omega)} + \|u_k - \hat{u}_k\|_{L^2(\Omega)} \leq C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{0.8}$$

If $f \in H^1(\Omega)$, then the estimate on $\|u_k - \hat{u}_k\|_{L^2(\Omega)}$ can be improved to

$$\|u_k - \hat{u}_k\|_{L^2(\Omega)} \leq C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{0.9}$$

The system (0.5) is symmetric and nonsingular but not positive definite, which is inconvenient for multigrid methods. To overcome this difficulty, we will exploit the fact that (0.5) is equivalent to a positive definite problem related to a nonconforming finite element. We will construct a multigrid algorithm for the nonconforming finite element method. The multigrid solution to (0.5) is then obtained as a by-product. We refer the reader to [18], [22], [24], and [25] for different approaches. In the case of rectangular Raviart–Thomas mixed finite element methods, the reader should also consult [7].

The rest of the paper is organized as follows. We derive the equivalence between the lowest-order Raviart–Thomas mixed method and a nonconforming method in §1. In §2 we analyze this nonconforming method and obtain stronger estimates that imply (0.6) and (0.7). The mesh-dependent norms and intergrid transfer operators are defined in §3, followed by the multigrid algorithm in §4. Section 5 contains the convergence analysis of the multigrid algorithm. The singular Neumann problem is discussed in §6.

**1. Equivalence with a nonconforming method.** The equivalence of the general Raviart–Thomas mixed methods and certain nonconforming methods was discussed in [2]. In order to set the notation, and for the benefit of readers unfamiliar with it, we rederive the equivalence here for the lowest-order case.

Let $\mathcal{E}_k$ denote the set of edges in $\mathcal{T}_k$, $\mathcal{E}_k^\partial = \{e \in \mathcal{E}_k : e \subseteq \partial\Omega\}$, and $\mathcal{E}_k^0 = \mathcal{E}_k \backslash \mathcal{E}_k^\partial$. The space $M_{-1}^0(\mathcal{E}_k^0)$ is the set of all functions on $\cup\mathcal{E}_k$ which restrict to constant functions on each edge $e \in \mathcal{E}_k^0$ and vanish on $\mathcal{E}_k^\partial$.

Let

$$RT^0_{-1}(\mathcal{T}_k) := \Big\{\boldsymbol{\tau}: \boldsymbol{\tau} \in \big(L^2(\Omega)\big)^2, \boldsymbol{\tau}|_T = \begin{bmatrix} a_T \\ c_T \end{bmatrix} + b_T \begin{bmatrix} x \\ y \end{bmatrix} \forall T \in \mathcal{T}_k, \text{ where } a_T, b_T, c_T \in \mathbb{R}\Big\}.$$

Consider the following problem:

Find $(\boldsymbol{\sigma}^*_k, u^*_k, \lambda_k) \in RT^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{E}^0_k)$ such that for all $(\boldsymbol{\tau}, v, \mu) \in RT^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{E}^0_k)$

(1.1)

$$\begin{aligned}
&\text{(i)} \quad \int_\Omega \mathbf{C}\,\boldsymbol{\sigma}^*_k \cdot \boldsymbol{\tau}\,dx - \sum_{T\in\mathcal{T}_k} \int_T u^*_k \operatorname{div} \boldsymbol{\tau}\,dx + \sum_{T\in\mathcal{T}_k} \int_{\partial T} \lambda_k\, \boldsymbol{\tau} \cdot \mathbf{n}_T\,ds = 0 \\
&\text{(ii)} \quad -\sum_{T\in\mathcal{T}_k} \int_T v \operatorname{div} \boldsymbol{\sigma}^*_k\,dx = -\int_\Omega f v\,dx \\
&\text{(iii)} \quad \sum_{T\in\mathcal{T}_k} \int_{\partial T} \mu\, \boldsymbol{\sigma}^*_k \cdot \mathbf{n}_T\,ds = 0.
\end{aligned}$$

Here $\mathbf{n}_T$ denotes the unit outer normal of triangle $T$.

If $(\boldsymbol{\sigma}^*_k, u^*_k, \lambda_k)$ is a solution to (1.1), then (1.1.iii) implies that $\boldsymbol{\sigma}^*_k \in RT^0_0(\mathcal{T}_k)$. By choosing $\boldsymbol{\tau} \in RT^0_0(\mathcal{T}_k)$ in (1.1.i), the term $\sum_{T\in\mathcal{T}_k} \int_{\partial T} \lambda_k\, \boldsymbol{\tau} \cdot \mathbf{n}_T\,ds$ vanishes. Therefore $(\boldsymbol{\sigma}^*_k, u^*_k) \in RT^0_0(\mathcal{T}_k) \times M^0_{-1}(\mathcal{T}_k)$ satisfies (0.5), and hence $\boldsymbol{\sigma}^*_k = \boldsymbol{\sigma}_k$ and $u^*_k = u_k$. Conversely, if we let $u^*_k = u_k$ and $\boldsymbol{\sigma}^*_k = \boldsymbol{\sigma}_k$, then (1.1.ii) and (1.1.iii) are automatically satisfied, and using the first equation in (0.5), $\lambda_k$ is uniquely solvable from (1.1.i). Hence the systems (1.1) and (0.5) are equivalent; therefore, from now on we drop the superscripts in (1.1).

The system (1.1) can be reduced to a positive definite system in the following manner. Let $\boldsymbol{\nabla}_k : M^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{E}^0_k) \longrightarrow RT^0_{-1}(\mathcal{T}_k)$ be defined by

$$\text{(1.2)} \quad \int_\Omega \boldsymbol{\nabla}_k(v,\mu) \cdot \boldsymbol{\tau}\,dx = \sum_{T\in\mathcal{T}_k} \int_{\partial T} \mu\, \boldsymbol{\tau} \cdot \mathbf{n}_T\,ds - \sum_{T\in\mathcal{T}_k} \int_T v \operatorname{div} \boldsymbol{\tau}\,dx \quad \forall\, \boldsymbol{\tau} \in RT^0_{-1}(\mathcal{T}_k).$$

Let $P^0_{RT,\mathbf{C}}$ be the orthogonal projection from $(L^2(\Omega))^2$ onto $RT^0_{-1}(\mathcal{T}_k)$ with respect to the inner product

$$\text{(1.3)} \qquad [\boldsymbol{\sigma}, \boldsymbol{\tau}] := \int_\Omega \mathbf{C}\,\boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,dx.$$

The norm corresponding to $[\cdot,\cdot]$ is defined by

$$\text{(1.4)} \qquad \|\boldsymbol{\tau}\|_{\mathbf{C}} := [\boldsymbol{\tau}, \boldsymbol{\tau}]^{1/2}.$$

Note that $\|\cdot\|_{\mathbf{I}} = \|\cdot\|_{L^2(\Omega)}$ and $P^0_{RT,\mathbf{I}}$ is the $L^2$-projection onto $RT^0_{-1}(\mathcal{T}_k)$. In general, $\|\cdot\|_{\mathbf{C}}$ is equivalent to the $L^2$-norm by the positive-definiteness of $\mathbf{C}$.

Using (1.2), (1.1), and (1.3) we have for any $(v, \mu) \in M_{-1}^{0}(\mathcal{T}_k) \times M_{-1}^{0}(\mathcal{E}_k^0)$

$$
\begin{aligned}
&[P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(u_k,\lambda_k), P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(v,\mu)] \\
&\quad = \int_\Omega \nabla_k(u_k,\lambda_k)\cdot P_{RT,\mathbf{C}}^{0}\,\mathbf{A}\,\nabla_k(v,\mu)\,dx \\
&\quad = \sum_{T\in\mathcal{T}_k}\int_{\partial T}\lambda_k\,P_{RT,\mathbf{C}}^{0}\,\mathbf{A}\,\nabla_k(v,\mu)\cdot\mathbf{n}_T\,ds - \sum_{T\in\mathcal{T}_k}\int_T u_k\cdot\operatorname{div}\left(P_{RT,\mathbf{C}}^{0}\,\mathbf{A}\,\nabla_k(v,\mu)\right)dx \\
&\quad = -\int_\Omega \mathbf{C}\,\boldsymbol{\sigma}_k\cdot P_{RT,\mathbf{C}}^{0}\big(\mathbf{A}\,\nabla_k(v,\mu)\big)dx \\
&\quad = -\int_\Omega \boldsymbol{\sigma}_k\cdot\nabla_k(v,\mu)dx \\
&\quad = -\sum_{T\in\mathcal{T}_k}\int_{\partial T}\mu\,\boldsymbol{\sigma}_k\cdot\mathbf{n}_T\,ds + \sum_{T\in\mathcal{T}_k}\int_T v\operatorname{div}\boldsymbol{\sigma}_k\,dx \\
&\quad = \int_\Omega fv\,dx.
\end{aligned}
$$

Conversely, suppose $(u_k, \lambda_k) \in M_{-1}^{0}(\mathcal{T}_k) \times M_{-1}^{0}(\mathcal{E}_k^0)$ satisfies

$$
[P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(u_k,\lambda_k), P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(v,\mu)] = \int_\Omega fv\,dx \tag{1.5}
$$

for all $(v,\mu) \in M_{-1}^{0}(\mathcal{T}_k) \times M_{-1}^{0}(\mathcal{E}_k^0)$. If we define $\boldsymbol{\sigma}_k$ by

$$
\boldsymbol{\sigma}_k := -P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(u_k,\lambda_k), \tag{1.6}
$$

then $(\boldsymbol{\sigma}_k, u_k, \lambda_k)$ will satisfy (1.1).

The system in (1.5) is obviously symmetric and nonnegative. To prove the definiteness, we assume that

$$
[P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(v,\mu), P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(v,\mu)] = 0.
$$

Then

$$
P_{RT,\mathbf{C}}^{0}\mathbf{A}\nabla_k(v,\mu) = 0,
$$

which implies that

$$
\int_\Omega \nabla_k(v,\mu)\cdot\boldsymbol{\tau}dx = 0 \quad \forall\boldsymbol{\tau}\in RT_{-1}^{0}(\mathcal{T}_k).
$$

In other words,

$$
\sum_{T\in\mathcal{T}_k}\int_{\partial T}\mu\boldsymbol{\tau}\cdot\mathbf{n}_T\,ds - \sum_{T\in\mathcal{T}_k}\int_T v\operatorname{div}\boldsymbol{\tau}\,dx = 0 \quad \forall\boldsymbol{\tau}\in RT_{-1}^{0}(\mathcal{T}_k). \tag{1.7}
$$

Let $\boldsymbol{\tau} = \mathbf{0}$, except on triangle $T \in \mathcal{T}_k$. From (1.7) we find that

$$
\int_{\partial T}(\mu - v|_T)\boldsymbol{\tau}\cdot\mathbf{n}_T = 0. \tag{1.8}
$$

Since $\boldsymbol{\tau} \cdot \mathbf{n}_T$ can be assigned an arbitrary constant value on each side of $T$, it follows from (1.8) that

$$\mu = v|_T \quad \text{on } \partial T. \tag{1.9}$$

Since (1.9) holds on every $T \in \mathcal{T}_k$ and since $\mu = 0$ along $\partial\Omega$, we have $\mu \equiv 0 \equiv v$. Therefore, (1.5) is a positive definite system equivalent to the system (1.1).

Finally, we relate (1.5) to a nonconforming finite element method in the following way. We define

$$N^1(\mathcal{T}_k^0) = CR^1(\mathcal{T}_k^0) \oplus B^3(\mathcal{T}_k), \tag{1.10}$$

where

$$\begin{aligned} CR^1(\mathcal{T}_k^0) = \{\psi : \psi \in L^2(\Omega), \psi|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_k,& \\ \psi \text{ is continuous at the midpoints of interelement boundaries}& \\ \text{and } \psi = 0 \text{ at the midpoints of edges along } \partial\Omega\}& \end{aligned} \tag{1.11}$$

(cf. [13]), and

$$\begin{aligned} B^3(\mathcal{T}_k) = \{\psi : \psi|_T \in \mathcal{P}_3(T) \text{ and vanishes on } \partial T& \\ \text{(i.e., a cubic bubble function on } T) \quad \forall T \in \mathcal{T}_k\}.& \end{aligned} \tag{1.12}$$

The spaces $N^1(\mathcal{T}_k^0)$ are nonconforming since $CR^1(\mathcal{T}_k^0) \not\subseteq H^1(\Omega)$, and they are nonnested because $CR^1(\mathcal{T}_{k-1}^0) \not\subseteq CR^1(\mathcal{T}_k^0)$ and $B^3(\mathcal{T}_{k-1}) \not\subseteq B^3(\mathcal{T}_k)$. Observe that $\dim\big(M_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k^0)\big) = \dim N^1(\mathcal{T}_k^0) =$ (the number of internal edges of $\mathcal{T}_k$) + (the number of triangles in $\mathcal{T}_k$). It is easy to check that $\tilde{n}_k = \dim N^1(\mathcal{T}_k^0) \sim C\,4^k \sim n_k = \dim\big(RT_0^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{T}_k)\big)$.

By Green's formula,

$$\sum_{T\in\mathcal{T}_k} \int_T \mathbf{grad}\,\psi \cdot \boldsymbol{\tau}\,dx = \sum_{T\in\mathcal{T}_k} \int_{\partial T} \psi(\boldsymbol{\tau} \cdot \mathbf{n}_T)\,ds - \sum_{T\in\mathcal{T}_k} \int_T \psi \operatorname{div}\boldsymbol{\tau}\,dx. \tag{1.13}$$

Let $\mathcal{S}_k : N^1(\mathcal{T}_k^0) \longrightarrow M_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k^0)$ be defined by

$$\mathcal{S}_k(\psi) = (v, \mu), \tag{1.14}$$

where

$$\begin{aligned} v|_T &= \frac{1}{|T|}\int_T \psi\,dx \quad (\text{i.e., } v = P_k^0\psi) \quad \text{and} \\ \mu|_e &= \psi(m_e). \end{aligned} \tag{1.15}$$

Here $m_e$ is the midpoint of edge $e$.

$\mathcal{S}_k$ is clearly an isomorphism (Ker $\mathcal{S}_k = \{0\}$). By comparing (1.2) and (1.13) we have

$$\int_\Omega \mathbf{grad}_k\psi \cdot \boldsymbol{\tau}\,dx = \int_\Omega \boldsymbol{\nabla}_k\mathcal{S}_k(\psi) \cdot \boldsymbol{\tau}\,dx \quad \forall \boldsymbol{\tau} \in RT_{-1}^0(\mathcal{T}_k), \tag{1.16}$$

where

$$\mathbf{grad}_k \psi|_T = \mathbf{grad}(\psi|_T). \tag{1.17}$$

Therefore, it follows that

$$P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \psi = P^0_{RT,\mathbf{C}}\mathbf{A}\ \boldsymbol{\nabla}_k \mathcal{S}_k(\psi). \tag{1.18}$$

Let $\psi_k \in N^1(\mathcal{T}_k^0)$ be defined by

$$\mathcal{S}_k \psi_k = (u_k, \lambda_k), \tag{1.19}$$

and let $\mathcal{S}_k \chi = (v, \mu)$. Then using (1.18), (1.5), and (1.15), we have

$$\begin{aligned}
&[P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \psi_k, P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \chi] \\
&\quad = [P^0_{RT,\mathbf{C}}\mathbf{A}\boldsymbol{\nabla}_k(u_k, \lambda_k), P^0_{RT,\mathbf{C}}\mathbf{A}\boldsymbol{\nabla}_k(v, \mu)] \\
&\quad = \int_\Omega f v\, dx \\
&\quad = \int_\Omega f(P^0_k \chi)\, dx.
\end{aligned}$$

Therefore, the Raviart–Thomas system (0.5) is equivalent to the following symmetric positive definite problem.

Find $\psi_k \in N^1(\mathcal{T}_k^0)$ such that

$$[P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \psi_k, P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \chi] = \int_\Omega f(P^0_k \chi)\, dx \quad \forall\, \chi \in N^1(\mathcal{T}_k^0). \tag{1.20}$$

From (1.19), (1.18), (1.15), and (1.6), the solutions $(\boldsymbol{\sigma}_k, u_k)$ of (0.5) are related to $\psi_k$ via the following formulas.

$$\boldsymbol{\sigma}_k = -P^0_{RT,\mathbf{C}}\mathbf{A}\ \mathbf{grad}_k \psi_k, \tag{1.21}$$

$$u_k = P^0_k \psi_k. \tag{1.22}$$

Our strategy for the development of a multigrid method for (0.5) is to first construct a multigrid algorithm for (1.20) and then apply the formulas (1.21) and (1.22). For more information on nonconforming multigrid methods, we refer the reader to [5], [8], [9], [10], and [11].

As a final remark of this section, we note that since there is no continuity requirement on $RT^0_{-1}(\mathcal{T}_k)$, the projection $P^0_{RT,\mathbf{C}}$ can be executed by inverting a block diagonal matrix with $3 \times 3$ blocks corresponding to the triangles in $\mathcal{T}_k$. The number of operations involved is $\mathcal{O}(n_k)$, and it can also be done in parallel.

**2. Analysis of the nonconforming method.** In this section we analyze the nonconforming method (1.20) directly, even though some of the discretization error estimates can be deduced from the corresponding estimates for the mixed problem (0.5). There are two reasons for doing this. First of all, this is a good way to acquaint the reader with this nonconforming method. Secondly, as a result, we derive an error estimate for (0.5), (cf. (2.44) below) that is, stronger than the estimate (0.7) obtained by mixed finite element analysis. This stronger estimate will enable us to obtain an

$\mathcal{O}(h_k^2)$ $L^2$ convergence rate for the multigrid method under the weaker assumption that $f \in H^1(\Omega)$ (instead of the assumption that $u \in H^3(\Omega)$). We refer the readers to [13] and [20] for a discussion of nonconforming finite element methods for second-order equations.

The continuous problem (0.2) has the following variational formulation.

Find $u \in H_0^1(\Omega)$ such that

$$[\mathbf{A}\,\mathbf{grad}\,u, \mathbf{A}\,\mathbf{grad}\,v] = F(v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.1}$$

where $F(v) = \int_\Omega fv\,dx$.

The positive semidefinite symmetric bilinear form for $a_k(\cdot,\cdot)$ on $H_0^1(\Omega) + N^1(\mathcal{T}_k^0)$ is defined by

$$a_k(\chi_1, \chi_2) = [P_{RT,\mathbf{C}}^0 \mathbf{A}\,\mathbf{grad}_k \chi_1, P_{RT,\mathbf{C}}^0 \mathbf{A}\,\mathbf{grad}_k \chi_2]. \tag{2.2}$$

The problem (1.20) can be rewritten as the following.

Find $\psi_k \in N^1(\mathcal{T}_k^0)$ such that

$$a_k(\psi_k, \chi) = \big(F \circ P_k^0\big)\chi \quad \forall\, \chi \in N^1(\mathcal{T}_k^0). \tag{2.3}$$

The nonconforming energy *seminorm* $\|\cdot\|_k$ on $H_0^1(\Omega) + N^1(\mathcal{T}_k^0)$ is defined to be

$$\|\psi\|_k := a_k(\psi,\psi)^{1/2}. \tag{2.4}$$

Note that $\|\psi\|_k = \|P_{RT,\mathbf{C}}^0 \mathbf{A}\,\mathbf{grad}_k \psi\|_{\mathbf{C}}$. From §1, we know that $a_k(\cdot,\cdot)$ is positive definite on$N^1(\mathcal{T}_k^0)$. In fact, the following lemma holds.

LEMMA 2.1. *There exists a positive constant $C$ such that*

$$\|\psi\|_k \le C\,\|\mathbf{grad}_k \psi\|_{L^2(\Omega)} \quad \forall\, \psi \in H_0^1(\Omega) + N^1(\mathcal{T}_k^0), \tag{2.5}$$

$$C\,\|\mathbf{grad}_k \psi\|_{L^2(\Omega)} \le \|\psi\|_k \quad \forall\, \psi \in N^1(\mathcal{T}_k^0). \tag{2.6}$$

*Proof.* Using the smoothness of $\mathbf{A}$ and the compactness of $\overline{\Omega}$, the first inequality is obvious.

To prove the second inequality, observe that

$$\begin{aligned}
\|\psi\|_k &= [P_{RT,\mathbf{C}}^0 \mathbf{A}\;\mathbf{grad}_k \psi, P_{RT,\mathbf{C}}^0 \mathbf{A}\;\mathbf{grad}_k \psi]^{1/2} \\
&= \sup_{\substack{\boldsymbol{\tau} \in RT_{-1}^0(\mathcal{T}_k) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\big|[P_{RT,\mathbf{C}}^0 \mathbf{A}\;\mathbf{grad}_k \psi, \boldsymbol{\tau}]\big|}{[\boldsymbol{\tau},\boldsymbol{\tau}]^{1/2}} \\
&= \sup_{\substack{\boldsymbol{\tau} \in RT_{-1}^0(\mathcal{T}_k) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\big|\int_\Omega \mathbf{grad}_k \psi \cdot \boldsymbol{\tau}\,dx\big|}{[\boldsymbol{\tau},\boldsymbol{\tau}]^{1/2}} \\
&\ge C \sup_{\substack{\boldsymbol{\tau} \in RT_{-1}^0(\mathcal{T}_k) \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{\big|\int_\Omega \mathbf{grad}_k \psi \cdot \boldsymbol{\tau}\,dx\big|}{\|\boldsymbol{\tau}\|_{L^2(\Omega)}} \\
&= C\,\|P_{RT,\mathbf{I}}^0\;\mathbf{grad}_k \psi\|_{L^2(\Omega)}.
\end{aligned}$$

Recall from the paragraph after (1.4) that $P^0_{RT,\mathbf{I}}$ represents the $L^2$-orthogonal projection onto $RT^0_{-1}(\mathcal{T}_k)$.

It remains to show that there exists a positive constant $C$ such that

$$\|P^0_{RT,\mathbf{I}}\mathbf{grad}_k\psi\|_{L^2(\Omega)} \geq C\,\|\mathbf{grad}_k\psi\|_{L^2(\Omega)}.$$

Let $P^0_{RT,\mathbf{I}}\ \mathbf{grad}_k\psi = 0$ on $T \in \mathcal{T}_k$. Then $\int_T \mathbf{grad}\ \psi \cdot \boldsymbol{\tau}\, dx = 0$ for any $\boldsymbol{\tau} = \left[\begin{smallmatrix} a_T \\ c_T \end{smallmatrix}\right] + b_T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. Applying Green's formula, we have

$$\psi(m_{e_i}) = \frac{1}{|T|}\int_T \psi\, dx, \tag{2.7}$$

where $e_1$, $e_2$, and $e_3$ are the edges of $T$. Recall that $\psi|_T = \mathcal{L}_T + \mathcal{B}_T$ ("linear part + third degree bubble function"). From (2.7) and the quadrature formula for quadratic polynomials on a triangle (cf. [12]) we have

$$\begin{aligned}
\frac{1}{|T|}\int_T \psi\, dx &= \frac{1}{|T|}\left\{\int_T \mathcal{L}_T\, dx + \int_T \mathcal{B}_T\, dx\right\} \\
&= \frac{1}{|T|}\left\{\frac{|T|}{3}\sum_{i=1}^{3}\mathcal{L}_T(m_{e_i}) + \int_T \mathcal{B}_T\, dx\right\} \\
&= \frac{1}{3}\sum_{i=1}^{3}\psi(m_{e_i}) + \frac{1}{|T|}\int_T \mathcal{B}_T\, dx \\
&= \frac{1}{|T|}\int_T \psi\, dx + \frac{1}{|T|}\int_T \mathcal{B}_T\, dx.
\end{aligned}$$

Therefore,

$$\int_T \mathcal{B}_T\, dx = 0,$$

which implies that $\mathcal{B}_T = 0$. Hence, $\psi$ is a constant by (2.7).

By a standard homogeneity argument, there exists a positive constant $C$ such that

$$\int_T \|P^0_{RT,\mathbf{I}}\mathbf{grad}\,\psi\|^2\, dx \geq C\int_T \|\mathbf{grad}_k\psi\|^2\, dx.$$

Summing over all triangles in $\mathcal{T}_k$, we have

$$\|P^0_{RT,\mathbf{I}}\mathbf{grad}_k\psi\|_{L^2(\Omega)} \geq C\|\mathbf{grad}_k\psi\|_{L^2(\Omega)}. \qquad \square$$

Let $\Pi_k : H^1_0(\Omega)\cap H^2(\Omega) \longrightarrow N^1(\mathcal{T}^0_k)$ be defined by the properties that $\Pi_k w \in CR^1(\mathcal{T}^0_k)$ and $\Pi_k w$ agrees with $w$ at the midpoints of $\mathcal{T}_k$. Then a standard interpolation error estimate (cf. [12]) implies that for all $w \in H^1_0(\Omega)\cap H^2(\Omega)$

$$\|w - \Pi_k w\|_{L^2(\Omega)} + h_k\|\mathbf{grad}_k(w - \Pi_k w)\|_{L^2(\Omega)} \leq C\,h_k^2\,|w|_{H^2(\Omega)}. \tag{2.8}$$

From (2.5) we have

$$\|w - \Pi_k w\|_k \leq C\,h_k\,|w|_{H^2(\Omega)}. \tag{2.9}$$

Since the Raviart–Thomas space $RT^0_{-1}(\mathcal{T}_k)$ contains as a subspace piecewise constant vector functions, a standard interpolation error estimate yields

$$\|P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}\,w - \mathbf{A}\,\mathbf{grad}\,w\|_{\mathbf{C}} \le C\,h_k\,\|w\|_{H^2(\Omega)} \quad \forall w \in H^2(\Omega). \tag{2.10}$$

The following lemma is crucial in the error analysis of our nonconforming method. Its proof is based on arguments similar to those used in [13].

LEMMA 2.2. *There exists a positive constant $C$ such that*

$$\left|\sum_{T\in\mathcal{T}_k}\int_{\partial T}(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n}_T)\,\eta|_T\,ds\right| \le C\,h_k\,\|\zeta\|_{H^2(\Omega)}\,\|\mathbf{grad}_k\eta\|_{L^2(\Omega)}, \tag{2.11}$$

*and*

$$\left|\sum_{T\in\mathcal{T}_k}\int_{\partial T}(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n}_T)(\lambda-\Pi_k\lambda|_T)\,ds\right| \le C\,h_k^2\,\|\zeta\|_{H^2(\Omega)}\,|\lambda|_{H^2(\Omega)} \tag{2.12}$$

*for all $\zeta \in H^2(\Omega)$, $\eta \in N^1(\mathcal{T}_k^0)$, and $\lambda \in H^2(\Omega)\cap H^1_0(\Omega)$.*

*Proof.* Let $e \in \mathcal{E}_k^0$ (i.e., $e$ is an internal edge), $T_1$ and $T_2 \in \mathcal{T}_k$ have $e$ as a common boundary, and $\eta_i = \eta|_{T_i}(i = 1, 2)$. Let $m$ be the midpoint of $e$ and $\mathbf{n}$ be the outward unit normal of $T_1$ along $e$. The contribution of edge $e$ to the summation on the left-hand side of (2.11) is given by

$$\int_e(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,[\eta]_e\,ds,$$

where $[\eta]_e = \eta_1|_e - \eta_2|_e$ measures the jump of $\eta$ across $e$. It can be rewritten as

$$\int_e(\mathbf{A}(m)\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,[\eta]_e\,ds + \int_e\Big(\big(\mathbf{A}-\mathbf{A}(m)\big)\,\mathbf{grad}\,\zeta\cdot\mathbf{n}\Big)\,[\eta]_e\,ds. \tag{2.13}$$

Since $[\eta]_e(m) = 0$, the first term in (2.13) can be bounded by

$$C\inf_{p\in\mathcal{P}_1(T_1\cup T_2)}\int_e\|\mathbf{grad}(\zeta-p)\|\,|[\eta]_e|\,ds,$$

which (using the trace theorem (cf. [1]), the Bramble–Hilbert lemma (cf. [6]), and a standard homogeneity argument) can be bounded by

$$C\,h_k\,|\zeta|_{H^2(T_1\cup T_2)}(|\eta|_{H^1(T_1)} + |\eta|_{H^1(T_2)}). \tag{2.14}$$

By the smoothness of $\mathbf{A}$, the second term in (2.13) can be bounded by

$$C\,h_k\int_e\|\mathbf{grad}\,\zeta\|\,|[\eta]_e|\,ds,$$

which (via the trace theorem and a standard homogeneity argument) is bounded by

$$C\,h_k(|\zeta|_{H^1(T_1\cup T_2)} + h_k|\zeta|_{H^2(T_1\cup T_2)})\,\big(|\eta|_{H^1(T_1)} + |\eta|_{H^1(T_2)}\big). \tag{2.15}$$

Combining (2.13) and (2.14), we have

$$\left|\int_e (\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})[\eta]_e\,ds\right| \le C\,h_k\,\|\zeta\|_{H^2(T_1\cup T_2)}\big(|\eta|_{H^1(T_1)}+|\eta|_{H^1(T_2)}\big). \tag{2.16}$$

Let $e\in\mathcal{E}_k^\partial$ (i.e., $e$ is a boundary edge) and $T\in\mathcal{T}_k$ be the triangle containing $e$. Let $\mathbf{n}$ be the outward normal of $T$ along $e$. The contribution of $e$ to the summation on the left-hand side of (2.11) is given by

$$\int_e (\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,\eta|_e\,ds.$$

An argument similar to the one above yields

$$\left|\int_e (\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,\eta|_e\,ds\right| \le C\,h_k\,\|\zeta\|_{H^2(T)}\,|\eta|_{H^1(T)}. \tag{2.17}$$

Summing over all edges in $\mathcal{T}_k$, (2.16), (2.17), and the Cauchy–Schwarz inequality implies (2.11).

We now turn to the proof of (2.12). Let $e\in\mathcal{E}_k^0$. The contribution of $e$ to the left-hand side of (2.12) is given by

$$\int_e \big(\mathbf{A}(m)\,\mathbf{grad}\,\zeta\cdot\mathbf{n}\big)\,[\lambda-\Pi_k\lambda]_e\,ds+\int_e\big((\mathbf{A}-\mathbf{A}(m))\,\mathbf{grad}\,\zeta\cdot\mathbf{n}\big)\,[\lambda-\Pi_k\lambda]_e\,ds. \tag{2.18}$$

Since $[\lambda]_e=0$ and $[\Pi_k\lambda]_e(m)=0$, the first term in (2.18) can be bounded by

$$C\inf_{p\in\mathcal{P}_1(T_1\cup T_2)}\int_e\|\mathbf{grad}(\zeta-p)\|\,\big|[\lambda-\Pi_k\lambda]_e\big|\,ds,$$

which in view of the trace theorem, the Bramble–Hilbert lemma, and a standard homogeneity argument is bounded by

$$C\,h_k^2\,|\zeta|_{H^2(T_1\cup T_2)}|\lambda|_{H^2(T_1\cup T_2)}. \tag{2.19}$$

The second term in (2.18) can be bounded by

$$C\,h_k\int_e\|\mathbf{grad}\zeta\|\,\big|[\lambda-\Pi_k\lambda]_e\big|\,ds,$$

which in turn can be bounded by

$$C\,h_k^2\left(|\zeta|_{H^1(T_1\cup T_2)}+h_k\,|\zeta|_{H^2(T_1\cup T_2)}\right)|\lambda|_{H^2(T_1\cup T_2)}.$$

Hence

$$\left|\int_e(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,[\lambda-\Pi_k\lambda]_e\,ds\right|\le C\,h_k^2\,\|\zeta\|_{H^2(T_1\cup T_2)}\,|\lambda|_{H^2(T_1\cup T_2)}. \tag{2.20}$$

If $e\in\mathcal{E}_k^\partial$, then a similar argument shows that

$$\left|\int_e(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n})\,(\lambda-\Pi_k\lambda)|_e\,ds\right|\le C\,h_k^2\,\|\zeta\|_{H^2(T)}\,|\lambda|_{H^2(T)}. \tag{2.21}$$

Inequality (2.12) is obtained by summing up (2.20) and (2.21) over all the edges. □

To estimate the discretization error, we follow ideas of Scott (cf. [21] and [23]).

PROPOSITION 2.1. *If $u$ is the solution of the continuous problem* (0.2) *and $\psi_k$ is the solution of* (2.3)*, then there exists a positive constant $C$ such that*

$$\|u-\psi_k\|_k \leq C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{2.22}$$

*Proof.* By duality and the Schwarz inequality,

$$\begin{aligned}
\|u-\psi_k\|_k &\leq \|u-\Pi_k u\|_k + \|\Pi_k u-\psi_k\|_k \\
&= \|u-\Pi_k u\|_k + \sup_{\substack{\chi\in N^1(\mathcal{T}_k^0)\\ \chi\neq 0}} \frac{|a_k(\Pi_k u-\psi_k,\chi)|}{\|\chi\|_k} \\
&\leq 2\,\|u-\Pi_k u\|_k + \sup_{\substack{\chi\in N^1(\mathcal{T}_k^0)\\ \chi\neq 0}} \frac{|a_k(u-\psi_k,\chi)|}{\|\chi\|_k}.
\end{aligned}$$

By (2.3), integration by parts, and equation (0.2), we have

$$\begin{aligned}
&a_k(u-\psi_k,\chi) \\
&\quad= [P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}_k u, P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}_k\chi] - \int_\Omega f(P^0_k\chi)\,dx \\
&\quad= \int_\Omega \mathbf{A}\,\mathbf{grad}_k u\cdot \mathbf{grad}_k\chi\,dx - \int_\Omega f(P^0_k\chi)\,dx \\
&\qquad\quad + [P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}_k u - \mathbf{A}\,\mathbf{grad}_k u, \mathbf{A}\,\mathbf{grad}_k\chi] \\
&\quad= \sum_T \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,u\cdot\mathbf{n}_T)\,\chi|_T\,ds + \int_\Omega f(\chi-P^0_k\chi)\,dx \\
&\qquad\quad + [P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}_k u - \mathbf{A}\,\mathbf{grad}_k u, \mathbf{A}\,\mathbf{grad}_k\chi].
\end{aligned} \tag{2.23}$$

From (2.11) and (2.6) we obtain

$$\left|\sum_T \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,u\cdot\mathbf{n}_T)\chi|_T\,ds\right| \leq C\,h_k\,\|u\|_{H^2(\Omega)}\,\|\chi\|_k. \tag{2.24}$$

Standard interpolation error estimates and (2.6) yield

$$\left|\int_\Omega f(\chi-P^0_k\chi)\,dx\right| \leq C\,h_k\,\|f\|_{L^2(\Omega)}\,\|\chi\|_k. \tag{2.25}$$

By the Schwarz inequality, (2.10), and (2.6), we have

$$\left|[P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}_k u - \mathbf{A}\,\mathbf{grad}_k u, \mathbf{A}\,\mathbf{grad}_k\chi]\right| \leq C\,h_k\,\|u\|_{H^2(\Omega)}\,\|\chi\|_k. \tag{2.26}$$

Hence (2.9), (2.23)–(2.26), and the elliptic regularity result (0.3) imply that (2.22) holds. □

The following corollary is a straightforward consequence of Proposition 2.1, interpolation error estimates (2.8), (2.9), Lemma 2.1, and the elliptic regularity estimate (0.3).

COROLLARY. *There exists a positive constant $C$ such that*

$$\|\mathbf{grad}_k(u-\psi_k)\|_{L^2(\Omega)} \leq C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{2.27}$$

PROPOSITION 2.2. *If $u$ is a solution of the continuous problem* (0.2) *and $\psi_k$ is the solution of* (2.3), *then there exists a positive constant $C$ such that*

$$\|u-\psi_k\|_{L^2(\Omega)} \leq C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{2.28}$$

*If $f \in H^1(\Omega)$, then the estimate can be improved to*

$$\|u-\psi_k\|_{L^2(\Omega)} \leq C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{2.29}$$

*Proof.* The proof is based on a duality argument. Let $\zeta \in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of the elliptic boundary value problem

$$\begin{aligned} -\mathrm{div}\,(\mathbf{A}\,\mathbf{grad}\,\zeta) &= u-\psi_k \quad \text{in } \Omega,\\ \zeta &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.30}$$

By elliptic regularity we have

$$\|\zeta\|_{H^2(\Omega)} \leq C\, \|u-\psi_k\|_{L^2(\Omega)}. \tag{2.31}$$

$$\begin{aligned} \|u-\psi_k\|^2_{L^2(\Omega)} &= \int_\Omega -(\mathrm{div}\,\mathbf{A}\,\mathbf{grad}\,\zeta)(u-\psi_k)\,dx\\ &= \int_\Omega \mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{grad}_k(u-\psi_k)\,dx\\ &\qquad + \sum_{T\in\mathcal{T}_k}\int_{\partial T}(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n}_T)\psi_k|_T\,ds. \end{aligned} \tag{2.32}$$

The first term on the right-hand side of (2.32) is estimated by

$$\left|\int_\Omega \mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{grad}_k(u-\psi_k)\,dx\right| \leq C\,\|\zeta\|_{H^1(\Omega)}\,\|\mathbf{grad}_k(u-\psi_k)\|_{L^2(\Omega)}. \tag{2.33}$$

By (2.11), the second term can be estimated by

$$\begin{aligned} &\left|\sum_{T\in\mathcal{T}_k}\int_{\partial T}(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n}_T)\psi_k|_T\,ds\right|\\ &\quad\leq C\,h_k\,\|\zeta\|_{H^2(\Omega)}\,\|\mathbf{grad}_k\psi_k\|_{L^2(\Omega)}\\ &\quad\leq C\,h_k\,\|\zeta\|_{H^2(\Omega)}\,(\|\mathbf{grad}_k(\psi_k-u)\|_{L^2(\Omega)}+\|\mathbf{grad}\,u\|_{L^2(\Omega)}). \end{aligned} \tag{2.34}$$

Therefore, by using (2.27) and (0.3), we see that the right-hand side of equality (2.32) is dominated by $C\,h_k\,\|u-\psi_k\|_{L^2(\Omega)}\,\|f\|_{L^2(\Omega)}$ and hence (2.28) holds.

If we assume that $f \in H^1(\Omega)$, the estimate of $\|u - \psi_k\|_{L^2(\Omega)}$ can be improved by analyzing the right-hand side of (2.32) more carefuly. It can be rewritten as

(2.35)

$$\begin{aligned}
&\|u - \psi_k\|^2_{L^2(\Omega)} \\
&= \int_\Omega \mathbf{A}\,\mathbf{grad}\,\zeta \cdot \mathbf{grad}_k(u - \psi_k)\,dx \\
&\quad - \sum_{T\in\mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\zeta \cdot \mathbf{n}_T)(u - \Pi_k u|_T + \Pi_k u|_T - \psi_k|_T)\,ds \\
&= [\mathbf{A}\,\mathbf{grad}\,\zeta - P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,\zeta, \mathbf{A}\,\mathbf{grad}_k(u - \psi_k)] \\
&\quad + [P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,\zeta - P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\,\Pi_k\zeta, \mathbf{A}\,\mathbf{grad}_k(u - \psi_k)] \\
&\quad + a_k(u - \psi_k, \Pi_k\zeta) - \sum_{T\in\mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\zeta \cdot \mathbf{n}_T)(u - \Pi_k u|_T + \Pi_k u|_T - \psi_k|_T)\,ds.
\end{aligned}$$

By the Schwarz inequality, (2.8), (2.10), and (2.27) it follows that

$$\begin{aligned}
&|[\mathbf{A}\,\mathbf{grad}\,\zeta - P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,\zeta, \mathbf{A}\,\mathbf{grad}_k(u - \psi_k)] \\
&\quad + [P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\,\zeta - P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,\Pi_k\zeta, \mathbf{A}\,\mathbf{grad}_k(u - \psi_k)]| \\
&\le C\,h_k\,\|\zeta\|_{H^2(\Omega)}\,\|\mathbf{grad}_k(u - \psi_k)\|_{L^2(\Omega)} \\
&\le C\,h_k^2\,\|\zeta\|_{H^2(\Omega)}\,\|f\|_{L^2(\Omega)}.
\end{aligned} \tag{2.36}$$

Furthermore, from (0.2) and (2.3) we have

$$\begin{aligned}
&a_k(u - \psi_k, \Pi_k\zeta) \\
&\quad = \sum_{T\in\mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,u \cdot \mathbf{n}_T)\,(\Pi_k\zeta)|_T\,ds + \int_\Omega f(\Pi_k\zeta - P^0_k\Pi_k\zeta)\,dx \\
&\qquad + [P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}\,u - \mathbf{A}\,\mathbf{grad}\,u, \mathbf{A}\,\mathbf{grad}_k\,\Pi_k\zeta].
\end{aligned} \tag{2.37}$$

By (2.12) and the interpolation error estimate (2.8), it follows that

$$\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,u \cdot \mathbf{n}_T)\,\Pi_k\zeta\,ds\right| \\
&\quad = \left|\sum_{T\in\mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,u \cdot \mathbf{n}_T)\,(\Pi_k\zeta - \zeta)\,ds\right| \\
&\quad \le C\,h_k^2\,\|u\|_{H^2(\Omega)}\,\|\zeta\|_{H^2(\Omega)}.
\end{aligned} \tag{2.38}$$

From standard interpolation error estimates and (2.8) we have

(2.39)

$$\begin{aligned}
&\left|\int_\Omega f(\Pi_k\zeta - P^0_k\Pi_k\zeta)\,ds\right| \\
&= \left|\int_\Omega (f - P^0_k f)(\Pi_k\zeta - P^0_k\Pi_k\zeta)\,ds\right| \\
&\le \|f - P^0_k f\|_{L^2(\Omega)}\Big(\|\Pi_k\zeta - \zeta\|_{L^2(\Omega)} + \|\zeta - P^0_k\zeta\|_{L^2(\Omega)} + \|P^0_k(\zeta - \Pi_k\zeta)\|_{L^2(\Omega)}\Big) \\
&\le C\,h_k^2\,|f|_{H^1(\Omega)}\,\|\zeta\|_{H^2(\Omega)},
\end{aligned}$$

and

(2.40)

$$
\begin{aligned}
&|[P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,u - \mathbf{A}\,\mathbf{grad}\,u, \mathbf{A}\,\mathbf{grad}_k\,\Pi_k\zeta]| \\
&= \big|[P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}u - \mathbf{A}\,\mathbf{grad}u, \mathbf{A}\,\mathbf{grad}_k\Pi_k\zeta - P^0_{RT,\mathbf{C}}\mathbf{A}\,\mathbf{grad}\,\zeta]\big| \\
&\le \|P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,u - \mathbf{A}\ \mathbf{grad}\,u\|_{\mathbf{C}} \\
&\quad \times \Big\{\|\mathbf{A}\,\mathbf{grad}\,\Pi_k\zeta - \mathbf{A}\,\mathbf{grad}\,\zeta\|_{\mathbf{C}} + \|\mathbf{A}\,\mathbf{grad}\,\zeta - P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,\zeta\|_{\mathbf{C}}\Big\} \\
&\le C\,h_k^2\,\|u\|_{H^2(\Omega)}\,\|\zeta\|_{H^2(\Omega)}.
\end{aligned}
$$

From Lemma 2.2, (2.8), (2.27), and the elliptic regularity estimate (0.3) we also have

$$
\begin{aligned}
&\left|\sum_{T\in\mathcal{T}_k}\int_{\partial T}(\mathbf{A}\,\mathbf{grad}\,\zeta\cdot\mathbf{n}_T)\,(u - \Pi_k u|_T + \Pi_k u|_T - \psi_k|_T)\,ds\right| \\
&\le C\,h_k^2\,\|\zeta\|_{H^2(\Omega)}\,|u|_{H^2(\Omega)} + C\,h_k\,\|\zeta\|_{H^2(\Omega)}\,\|\mathbf{grad}_k(\Pi_k u - \psi_k)\|_{L^2(\Omega)} \qquad (2.41)\\
&\le C\,h_k^2\,\|\zeta\|_{H^2(\Omega)}\,\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

Combining (2.35)–(2.41) along with the elliptic regularity estimates (0.3) and (2.31), we obtain (2.29). □

We can deduce the error estimates for the Raviart–Thomas finite element method from the error estimates for the nonconforming finite element method. The estimates (0.6) and (0.7) are immediate consequences of the following proposition. Note that (2.44) below is stronger than (0.7) since, on a convex polygonal domain, $f \in H^1(\Omega)$ does not necessarily imply that $u \in H^3(\Omega)$ (cf. [16]).

PROPOSITION 2.3. *If $u$ is the solution of the elliptic boundary value problem* (0.2), $\boldsymbol{\sigma} = -\mathbf{A}\,\mathbf{grad}\,u$, *and* $(\boldsymbol{\sigma}_k, u_k)$ *solves the discretized system* (0.5), *then there exists a positive constant $C$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{L^2(\Omega)} \le C\,h_k\,\|f\|_{L^2(\Omega)}, \tag{2.42}$$

$$\|u - u_k\|_{L^2(\Omega)} \le C\,h_k\,\|f\|_{L^2(\Omega)}. \tag{2.43}$$

*If $f \in H^1(\Omega)$, then*

$$\|P^0_k u - u_k\|_{L^2(\Omega)} \le C\,h_k^2\,\|f\|_{H^1(\Omega)}. \tag{2.44}$$

*Proof.* To prove (2.42), we use (1.21), (2.10), the equivalence between $\|\cdot\|_{\mathbf{C}}$ and $\|\cdot\|_{L^2}$, (2.27), and the elliptic regularity estimate (0.3) to obtain

$$
\begin{aligned}
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{L^2(\Omega)} &= \|-\mathbf{A}\,\mathbf{grad}\,u + P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\psi_k\|_{L^2(\Omega)} \\
&\le \|-\mathbf{A}\,\mathbf{grad}\,u + P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}\,u\|_{L^2(\Omega)} + C\,\|\mathbf{grad}_k(\psi_k - u)\|_{L^2(\Omega)} \\
&\le C\,h_k\,\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

To prove (2.43), we use (1.22), standard interpolation error estimates, (2.28), and the elliptic regularity estimate (0.3) to get

$$
\begin{aligned}
\|u - u_k\|_{L^2(\Omega)} &\le \|u - P^0_k u\|_{L^2(\Omega)} + \|P^0_k(u - \psi_k)\|_{L^2(\Omega)} \\
&\le C\,h_k\,|u|_{H^1(\Omega)} + C\,h_k\,\|f\|_{L^2(\Omega)} \\
&\le C\,h_k\,\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

Finally, to prove (2.44), we use (1.22) and (2.29) to obtain

$$\begin{aligned}\|P_k^0 u - u_k\|_{L^2(\Omega)} &= \|P_k^0(u-\psi_k)\|_{L^2(\Omega)}\\ &\le C\,h_k^2\,\|f\|_{H^1(\Omega)}.\end{aligned}$$ □

We end this section with a lemma, which indicates that estimates on $N^1(\mathcal{T}_k^0)$ can be done on the component spaces $CR^1(\mathcal{T}_k)$ and $B^3(\mathcal{T}_k)$ separately. It is a simple consequence of a standard homogeneity argument and Lemma 2.1.

LEMMA 2.3. *Let $\psi = \psi_1 + \psi_2$ be an arbitrary element of $N^1(\mathcal{T}_k^0)$, where $\psi_1 \in CR^1(\mathcal{T}_k^0)$ and $\psi_2 \in B^3(\mathcal{T}_k)$. Then*

$$C\Big(\|\psi_1\|_{L^2(\Omega)} + \|\psi_2\|_{L^2(\Omega)}\Big) \le \|\psi\|_{L^2(\Omega)} \le C\Big(\|\psi_1\|_{L^2(\Omega)} + \|\psi_2\|_{L^2(\Omega)}\Big), \tag{2.45}$$

$$\begin{aligned}&C\Big(\|\mathbf{grad}_k\psi_1\|_{L^2(\Omega)} + \|\mathbf{grad}_k\psi_2\|_{L^2(\Omega)}\Big)\\ &\quad\le \|\mathbf{grad}_k\psi\|_{L^2(\Omega)}\\ &\quad\le C\Big(\|\mathbf{grad}_k\psi_1\|_{L^2(\Omega)} + \|\mathbf{grad}_k\psi_2\|_{L^2(\Omega)}\Big),\end{aligned} \tag{2.46}$$

$$C\Big(\|\psi_1\|_k + \|\psi_2\|_k\Big) \le \|\psi\|_k \le C\Big(\|\psi_1\|_k + \|\psi_2\|_k\Big). \tag{2.47}$$

**3. Mesh-dependent norms and intergrid transfer operators.** In this section we introduce mesh-dependent norms, which are essential in the proofs of multigrid convergence. Since natural injection does not work for nonnested finite element spaces, we also need to choose appropriate intergrid transfer operators. We define these operators by averaging and establish some estimates that are important for the convergence analysis.

We define the mesh-dependent inner product $(\cdot,\cdot)_k$ on $N^1(\mathcal{T}_k^0)$ by

$$(\psi,\chi)_k := h_k^2\left(\sum_m \psi(m)\chi(m) + \sum_{T\in\mathcal{T}_k}\left(\frac{1}{|T|}\int_T \psi\,dx\right)\left(\frac{1}{|T|}\int_T \chi\,dx\right)\right), \tag{3.1}$$

where $m$ ranges over the midpoints of internal edges of $\mathcal{T}_k$. A standard homogeneity argument implies that

$$C\,\|\psi\|_{L^2(\Omega)} \le (\psi,\psi)_k^{1/2} \le C\,\|\psi\|_{L^2(\Omega)}. \tag{3.2}$$

Let $A_k : N^1(\mathcal{T}_k^0) \longrightarrow N^1(\mathcal{T}_k^0)$ be defined by

$$a_k(\psi,\chi) = (A_k\psi,\chi)_k \quad \forall\,\psi,\chi \in N^1(\mathcal{T}_k^0). \tag{3.3}$$

The operator $A_k$ is symmetric positive definite with respect to $(\cdot,\cdot)_k$ because $a_k(\cdot,\cdot)$ is a symmetric positive definite bilinear form on $N^1(\mathcal{T}_k^0)$, and standard inverse estimates (cf. [12]) and (3.2) imply that

$$\text{spectral radius of } A_k \le C\,h_k^{-2}. \tag{3.4}$$

The mesh-dependent norm $|||\cdot|||_{s,k}$ is defined by

$$|||\psi|||_{s,k}^2 = (A_k^s \psi, \psi)_k. \tag{3.5}$$

Observe that $|||\psi|||_{0,k} = (\psi,\psi)_k^{1/2}$ is equivalent to $\|\psi\|_{L^2(\Omega)}$, $|||\psi|||_{1,k} = \|\psi\|_k$, and, from (3.4),

$$|||\psi|||_{s,k} \le C\, h_k^{-1}\, |||\psi|||_{s-1,k}. \tag{3.6}$$

We also have a generalized Cauchy–Schwarz inequality

$$|a_k(\psi,\chi)| \le |||\psi|||_{1+t,k}\, |||\chi|||_{1-t,k}. \tag{3.7}$$

The approximation property of the multigrid algorithm is formulated in terms of the mesh-dependent norms, but the proof is based on a duality argument using the elliptic regularity of the continuous problem, which involves norms of Sobolev spaces. A relation between these two types of norms is therefore needed.

LEMMA 3.1. *There exists a positive constant $C$ such that*

$$|||\psi|||_{0,k} \le C\left\{\|P_k^0\psi\|_{L^2(\Omega)} + h_k\,\|\psi\|_k\right\} \quad \forall\,\psi\in N^1(\mathcal{T}_k^0). \tag{3.8}$$

*Proof.* From (3.2), Lemma 2.1, and the fact that our triangulation is quasi uniform, it suffices to show that for $T\in\mathcal{T}_k$,

$$\|\psi\|_{L^2(T)} \le C\left\{h_k^{-1}\left|\int_T \psi\,dx\right| + h_k\,|\psi|_{H^1(T)}\right\}$$

holds for any $\psi$, which is the sum of a linear function and a cubic bubble function on $T$. This is the consequence of a standard homogeneity argument. □

We now turn to the intergrid transfer operators. The coarse-to-fine intergrid transfer operator $I_{k-1}^k : N^1(\mathcal{T}_{k-1}^0) \longrightarrow N^1(\mathcal{T}_k^0)$ is defined such that $I_{k-1}^k : CR^1(\mathcal{T}_{k-1}^0) \longrightarrow CR^1(\mathcal{T}_k^0)$ and $I_{k-1}^k : B^3(\mathcal{T}_{k-1}) \longrightarrow B^3(\mathcal{T}_k)$.

If $\psi\in CR^1(\mathcal{T}_{k-1}^0)$, $I_{k-1}^k\psi \in CR^1(\mathcal{T}_k^0)$ is defined as in [8] and [11] by averaging: Let $m$ be a midpoint of an edge of a triangle in $\mathcal{T}_k$. If $m\in\partial\Omega$, then $\big(I_{k-1}^k\psi\big)(m) = 0$. If $m$ lies in the interior of a triangle in $\mathcal{T}_{k-1}$, then $\big(I_{k-1}^k\psi\big)(m) := \psi(m)$. Otherwise, if $m$ lies on the common edge of two adjacent triangles $T_1$ and $T_2$ in $\mathcal{T}_{k-1}$, then $\big(I_{k-1}^k\psi\big)(m) := \frac{1}{2}\left[\psi|_{T_1}(m) + \psi|_{T_2}(m)\right]$.

Let $\psi\in B^3(\mathcal{T}_{k-1})$. Then $I_{k-1}^k\psi\in B^3(\mathcal{T}_k)$ is determined by

$$\frac{1}{|T|}\int_T I_{k-1}^k\psi\,dx = \frac{1}{|\tilde T|}\int_{\tilde T}\psi\,dx, \tag{3.9}$$

where $T\in\mathcal{T}_k$ is one of the four triangles obtained from subdividing $\tilde T\in\mathcal{T}_{k-1}$.

The fine-to-coarse intergrid transfer operator $I_k^{k-1} : N^1(\mathcal{T}_k^0) \longrightarrow N^1(\mathcal{T}_{k-1}^0)$ is then defined by

$$(I_k^{k-1}\psi,\chi)_{k-1} = (\psi, I_{k-1}^k\chi)_k \quad \forall\,\psi\in N^1(\mathcal{T}_k^0), \chi\in N^1(\mathcal{T}_{k-1}^0). \tag{3.10}$$

The following estimates are essential in the convergence analysis of nonconforming multigrid methods (cf. [10]).

PROPOSITION 3.1. *There exists a positive constant $C$ such that*

$$\|I_{k-1}^k \psi\|_k \le C\,\|\psi\|_{k-1} \quad \forall\,\psi \in N^1(\mathcal{T}_{k-1}^0), \tag{3.11}$$

$$\|P_k^0(I_{k-1}^k\psi) - P_{k-1}^0\psi\|_{L^2(\Omega)} \le C\,h_k\,\|\psi\|_{k-1} \quad \forall\,\psi \in N^1(\mathcal{T}_{k-1}^0), \tag{3.12}$$

$$\begin{aligned}&\|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta\|_{L^2(\Omega)} + h_k\,\|I_{k-1}^k\Pi_{k-1}\zeta - \Pi_k\zeta\|_k \\ &\quad \le C\,h_k^2\,\|\zeta\|_{H^2(\Omega)} \quad \forall\,\zeta \in H^2(\Omega)\cap H_0^1(\Omega).\end{aligned} \tag{3.13}$$

*Proof.* Note that since $\Pi_{k-1}\zeta \in CR^1(\mathcal{T}_{k-1}^0)$, $I_{k-1}^k\Pi_{k-1}\psi$, and $\Pi_k\zeta \in CR^1(\mathcal{T}_k^0)$, inequality (3.13) does not involve the bubble functions. From Lemma 2.3, in order to prove (3.11) and (3.12), it suffices to prove them separately for $\psi \in CR^1(\mathcal{T}_k^0)$ and $\psi \in B^3(\mathcal{T}_k)$.

The proof of (3.11) and (3.12) in the case that $\psi \in CR^1(\mathcal{T}_k^0)$ and the proof of (3.13) can be found in [11]. In fact, there a stronger estimate was obtained:

$$\|I_{k-1}^k\psi - \psi\|_{L^2(\Omega)} \le C\,h_k\,\|\psi\|_{k-1} \quad \forall\,\psi \in N^1(\mathcal{T}_k^0). \tag{3.14}$$

It remains to prove (3.11) and (3.12) in the case that $\psi \in B^3(\mathcal{T}_{k-1})$.

By the equivalence of norms on finite-dimensional vector spaces, there exists a positive constant $C$ such that

$$C\,|\psi|_{H^1(T)} \le \frac{1}{|T|}\left|\int_T \psi\,dx\right| \le C\,|\psi|_{H^1(T)}, \tag{3.15}$$

where $\psi$ is any cubic bubble function on $T \in \mathcal{T}_k$, $k = 1, 2, \cdots$.

Let $T$ be a triangle in $\mathcal{T}_k$, which is obtained by subdividing a triangle $\tilde{T} \in \mathcal{T}_{k-1}$. Using (3.15) and (3.9) we have, for $\psi \in B^3(\mathcal{T}_{k-1})$,

$$\begin{aligned}|I_{k-1}^k\psi|_{H^1(T)} &\le \frac{C}{|T|}\left|\int_T I_{k-1}^k\psi\,dx\right| \\ &= \frac{C}{|\tilde{T}|}\left|\int_{\tilde{T}} \psi\,dx\right| \\ &\le C\,|\psi|_{H^1(\tilde{T})}.\end{aligned}$$

Summing over all triangles, $T \in \mathcal{T}_k$ yields

$$\|\mathbf{grad}_k(I_{k-1}^k\psi)\|_{L^2(\Omega)} \le C\,\|\mathbf{grad}_k\psi\|_{L^2(\Omega)},$$

which implies (3.11) by Lemma 2.1.

The proof of (3.12) for $\psi \in B^3(\mathcal{T}_k)$ is trivial because (3.9) is equivalent to

$$P_k^0 I_{k-1}^k\psi = P_{k-1}^0\psi \quad \forall\,\psi \in B^3(\mathcal{T}_{k-1}). \tag{3.16}$$ □

COROLLARY. *There exists a positive constant $C$ such that*

$$|\!|\!|I_{k-1}^k\psi|\!|\!|_{0,k} \le C\,|\!|\!|\psi|\!|\!|_{0,k-1} \quad \forall\,\psi \in N^1(\mathcal{T}_{k-1}^0). \tag{3.17}$$

*Proof.* If $\psi \in CR^1(\mathcal{T}_{k-1}^0)$, (3.2), (3.6), and (3.14) imply

$$\begin{aligned}|\!|\!|I_{k-1}^k\psi|\!|\!|_{0,k} &\le C\left\{\|I_{k-1}^k\psi - \psi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}\right\} \\ &\le C\left\{h_k|\!|\!|\psi|\!|\!|_{1,k-1} + |\!|\!|\psi|\!|\!|_{0,k-1}\right\} \\ &\le C\,|\!|\!|\psi|\!|\!|_{0,k-1}.\end{aligned}$$

Observe that by a standard homogeneity argument we have

$$C\,\|P_k^0\chi\|_{L^2(\Omega)} \le \|\chi\|_{L^2(\Omega)} \le C\,\|P_k^0\chi\|_{L^2(\Omega)} \quad \forall\,\chi \in B^3(\mathcal{T}_k). \tag{3.18}$$

Therefore, for $\psi \in B^3(\mathcal{T}_{k-1})$, (3.2), (3.16), and (3.18) imply that

$$\begin{aligned}|\!|\!| I_{k-1}^k\psi |\!|\!|_{0,k} &\le C\,\|P_k^0 I_{k-1}^k\psi\|_{L^2(\Omega)} \\ &= C\,\|P_{k-1}^0\psi\|_{L^2(\Omega)} \\ &\le C\,|\!|\!|\psi|\!|\!|_{0,k-1}.\end{aligned}$$ □

As a final remark in this section, we note that there are other choices for the intergrid transfer operators $I_{k-1}^k$ (cf. [5] and [25]). We choose averaging for $\psi \in CR^1(\mathcal{T}_{k-1}^0)$ because it is simple and the corresponding estimates have already been established. In the case of bubble functions, we use (3.9) because it is simple, is equivalent to (3.16), and also satisfies

$$\int_\Omega I_{k-1}^k\psi\,dx = \int_\Omega \psi\,dx \quad \forall\,\psi \in B^3(\mathcal{T}_{k-1}),$$

which is important for the multigrid algorithm of the singular Neumann boundary value problem.

**4. The multigrid algorithm.** We first describe a $W$-cycle multigrid algorithm for the nonconforming method (2.3).

**4.1. The $k$th level iteration.** The $k$th level iteration with initial guess $z_0$ yields $MG(k, z_0, g)$ as an approximate solution to the equation

$$A_k z = g.$$

For $k = 1$, $MG(1, z_0, g)$ is the solution obtained from a direct method. In other words,

$$MG(1, z_0, g) = A_1^{-1}g.$$

For $k > 1$, there are two steps.

*Smoothing Step.* Let $z_l \in N^1(\mathcal{T}_k^0)$ $(1 \le l \le m)$ be defined recursively by the equations

$$z_l = z_{l-1} + \frac{1}{\Lambda_k}(g - A_k z_{l-1}), \qquad 1 \le l \le m, \tag{4.1}$$

where $m$ is a positive integer independent of $k$ and $\Lambda_k := C\,h_k^{-2}$ dominates the spectral radius of $A_k$.

*Correction Step.* Let $\overline{g} := I_k^{k-1}(g - A_k z_m)$. Let $q_i \in N^1(\mathcal{T}_{k-1})$ $(0 \le i \le p,\ p = 2$ or $3)$ be defined recursively by

$$\begin{aligned} q_0 &= 0 \quad \text{and} \\ q_i &= MG(k-1, q_{i-1}, \overline{g}), \qquad 1 \le i \le p. \end{aligned} \tag{4.2}$$

Then $MG(k, z_0, g)$ is defined to be $z_m + I_{k-1}^k q_p$.

**4.2. The full multigrid algorithm.** In the case $k = 1$, the approximate solution $\hat{\psi}_1$ of (2.3) is obtained by a direct method. The approximate solutions $\hat{\psi}_k$ $(k \geq 2)$ of (2.3) are obtained recursively from

$$\psi_0^k = I_{k-1}^k \hat{\psi}_{k-1},$$
$$\psi_l^k = MG(k, \psi_{l-1}^k, f_k), \qquad 1 \leq l \leq r,$$

where

$$f_k \in N^1(\mathcal{T}_k^0) \text{ and } (f_k, \chi)_k = \int_\Omega f(P_k^0 \chi)\, dx \quad \forall \chi \in N^1(\mathcal{T}_k^0),$$
$$\hat{\psi}_k = \psi_r^k,$$

where $r$ is a positive integer independent of $k$.

Motivated by formulas (1.21) and (1.22), the approximate solution $(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k)$ of the Raviart–Thomas method (0.5) is defined by

$$(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k) = (-\mathcal{A}\, P_{RT,\mathbf{C}}^0\, \mathbf{A}\, \mathbf{grad}_k \hat{\psi}_k, P_k^0 \hat{\psi}_k), \tag{4.3}$$

where the average operator $\mathcal{A} : RT_{-1}^0(\mathcal{T}_k) \longrightarrow RT_0^0(\mathcal{T}_k)$ is defined in the following way.

Let $e$ be an edge in $\mathcal{T}_k$ and $\mathbf{n}_e$ be a unit normal of $e$. If $e \in \mathcal{E}_k^\partial$ is an edge of $T \in \mathcal{T}_k$, then $(\mathcal{A}\boldsymbol{\tau} \cdot \mathbf{n}_e)\big|_e = [\boldsymbol{\tau}|_T \cdot \mathbf{n}_e]\big|_e$. Otherwise if $e \in \mathcal{E}_k^0$ is the common edge of $T_1$, $T_2 \in \mathcal{T}_k$, then

$$[\mathcal{A}\boldsymbol{\tau} \cdot \mathbf{n}_e]\big|_e = \tfrac{1}{2}\big[(\boldsymbol{\tau}|_{T_1} \cdot \mathbf{n}_e)\big|_e + (\boldsymbol{\tau}|_{T_2} \cdot \mathbf{n}_e)\big|_e\big]. \tag{4.4}$$

The following lemma is straightforward.

LEMMA 4.1. *There exists a positive constant $C$ such that*

$$\|\mathcal{A}\boldsymbol{\tau}\|_{L^2(\Omega)} \leq C\, \|\boldsymbol{\tau}\|_{L^2(\Omega)} \quad \forall \boldsymbol{\tau} \in RT_{-1}^0(\mathcal{T}_k). \tag{4.5}$$

The standard basis of $N^1(\mathcal{T}_k^0)$ consists of two types of functions. The first type belongs to $CR^1(\mathcal{T}_k^0)$: its value equals one at the midpoint of an internal edge and equals zero at all of the other midpoints of $\mathcal{T}_k$. The second type belongs to $B^3(\mathcal{T}_k)$: its average on one triangle $T \in \mathcal{T}_k$ equals one and it vanishes identically on all of the other triangles in $\mathcal{T}_k$. With respect to this standard basis, the operators $I_{k-1}^k$, $I_k^{k-1}$, and $A_k$ are represented by matrices with $\mathcal{O}(\tilde{n}_k)$ nonzero entries. Since the number of correction steps $p$ is less than four and $\tilde{n}_k \sim C\,4^k$, the total work in the full multigrid algorithm of the nonconforming method is therefore $\mathcal{O}(\tilde{n}_k)$. The proof is the same as the one in [4]. Since the execution of (4.3) is completely local (cf. the remark at the end of §1) and, therefore, $\mathcal{O}(\tilde{n}_k)$, the final work count of computing $(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k)$ is also $\mathcal{O}(\tilde{n}_k) = \mathcal{O}(n_k)$.

**5. Convergence analysis.** The convergence analysis of the nonconforming method is based on the analysis of the two-grid algorithm and a perturbation argument. We refer the reader to [10] for the convergence analysis in a more general setting.

**5.1. Convergence of the two-grid algorithm.** We assume that the residual equation is solved exactly on the coarser grid in the two-grid algorithm. The final output of the $k$th level iteration is, therefore, $z_m + I_{k-1}^k q$, where

$$q = A_{k-1}^{-1}\bar{g} = A_{k-1}^{-1}\big(I_k^{k-1}(g - A_k z_m)\big) = A_{k-1}^{-1}\big(I_k^{k-1} A_k (z - z_m)\big). \tag{5.1}$$

We denote the final error $z - (z_m + I_{k-1}^k q)$ of the two-grid algorithm by $e$ and the intermediate errors $z - z_i$ by $e_i$, for $i = 0, 1, \cdots, m$.

Let $J_k^{k-1} : N^1(\mathcal{T}_k^0) \longrightarrow N^1(\mathcal{T}_{k-1}^0)$ be defined by

$$a_{k-1}(J_k^{k-1}\psi, \chi) = a_k(\psi, I_{k-1}^k \chi) \quad \forall \psi \in N^1(\mathcal{T}_k^0), \quad \chi \in N^1(\mathcal{T}_{k-1}^0). \tag{5.2}$$

As a consequence of (3.11) we have

$$\|J_k^{k-1}\psi\|_{k-1} \le C\,\|\psi\|_k \quad \forall \psi \in N^1(\mathcal{T}_k^0). \tag{5.3}$$

It is immediate from the definition of $q$, $I_k^{k-1}$, and $J_k^{k-1}$ that

$$q = J_k^{k-1} e_m. \tag{5.4}$$

Let the relaxation operator $R_k$ be defined by

$$R_k := I - \frac{1}{\Lambda_k} A_k. \tag{5.5}$$

From the smoothing step (4.1), we obtain

$$e_l = R_k e_{l-1}, \qquad l = 1, 2, \cdots, m. \tag{5.6}$$

Since $\Lambda_k$ dominates the spectral radius of $A_k$, it is obvious that

$$|\!|\!| R_k \chi |\!|\!|_{s,k} \le |\!|\!| \chi |\!|\!|_{s,k} \quad \forall \chi \in N^1(\mathcal{T}_k^0). \tag{5.7}$$

Equations (5.4) and (5.6) imply that

$$\begin{aligned} e &= e_m - I_{k-1}^k q \\ &= e_m - I_{k-1}^k P_k^{k-1} e_m \\ &= (I - I_{k-1}^k J_k^{k-1}) e_m \\ &= (I - I_{k-1}^k J_k^{k-1}) R_k^m e_0. \end{aligned} \tag{5.8}$$

The two-grid convergence analysis will be complete once we have estimated $I - I_{k-1}^k J_k^{k-1}$ (the approximation property) and $R_k^m$ (the smoothing property).

LEMMA 5.1 (Smoothing property). *There exists a positive constant $C$ such that*

$$|\!|\!| R_k^m \chi |\!|\!|_{\beta,k} \le C\, h_k^{-1}\, m^{-1/2}\, |\!|\!| \chi |\!|\!|_{\beta-1,k}, \tag{5.9}$$

*for any $\beta \in \mathbb{R}$.*

*Proof.* By the spectral theorem, there exist eigenvalues $0 < \lambda_1, \lambda_2, \cdots, \lambda_{\tilde{n}_k}$ and corresponding eigenvectors $\phi_1, \phi_2, \cdots, \phi_{\tilde{n}_k} \in N^1(\mathcal{T}_k^0)$ such that

$$A_k \phi_i = \lambda_i \phi_i \tag{5.10}$$

and

$$(\phi_i, \phi_j)_k = \delta_{ij}. \tag{5.11}$$

We can write $\chi = \sum_{i=1}^{\tilde{n}_k} \alpha_i \phi_i$. Then from (3.5), (5.5), (5.10), and (5.11) we have

$$\begin{aligned}
|||R_k^m \chi|||_{\beta,k}^2 &= \Big|\Big|\Big| \left( I - \frac{1}{\Lambda_k} A_k \right)^m \chi \Big|\Big|\Big|_{\beta,k}^2 \\
&= \left( A_k^\beta \left( I - \frac{1}{\Lambda_k} A_k \right)^m \chi, \left( I - \frac{1}{\Lambda_k} A_k \right)^m \chi \right)_k \\
&= \sum_{i=1}^{\tilde{n}_k} \lambda_i^\beta \left( 1 - \frac{\lambda_i}{\Lambda_k} \right)^{2m} \alpha_i^2 \\
&= \Lambda_k \sum_{i=1}^{\tilde{n}_k} \frac{\lambda_i}{\Lambda_k} \left( 1 - \frac{\lambda_i}{\Lambda_k} \right)^{2m} \lambda_i^{\beta-1} \alpha_i^2 \\
&\leq \Lambda_k \sup_{0 \leq x \leq 1} [x(1-x)^{2m}] \sum_{i=1}^{\tilde{n}_k} \lambda_i^{\beta-1} \alpha_i^2 \\
&\leq C h_k^{-2} m^{-1} |||\chi|||_{\beta-1,k}^2.
\end{aligned}$$

□

LEMMA 5.2 (Approximation property). *There exists $C > 0$ such that*

$$|||(I - I_{k-1}^k J_k^{k-1})\chi|||_{0,k} \leq C h_k |||\chi|||_{1,k}, \tag{5.12}$$

$$|||(I - I_{k-1}^k J_k^{k-1})\chi|||_{1,k} \leq C h_k |||\chi|||_{2,k} \tag{5.13}$$

*for all $\chi \in N^1(\mathcal{T}_k^0)$.*

*Proof.* First note that (5.13) is a consequence of (5.12) by duality.

Given $\chi \in N^1(\mathcal{T}_k^0)$, let $\hat{\chi} = (I - I_{k-1}^k J_k^{k-1})\chi$. It follows from (3.11) and (5.3) that

$$\|\hat{\chi}\|_k \leq C \|\chi\|_k. \tag{5.14}$$

Lemma 3.1 and (5.14) imply that

$$\begin{aligned}
|||\hat{\chi}|||_{0,k} &\leq C\big\{\|P_k^0 \hat{\chi}\|_{L^2(\Omega)} + h_k \|\hat{\chi}\|_k\big\} \\
&\leq C\big\{\|P_k^0 \hat{\chi}\|_{L^2(\Omega)} + h_k \|\chi\|_k\big\}.
\end{aligned}$$

To establish (5.12), it therefore suffices to prove

$$\|P_k^0 \hat{\chi}\|_{L^2(\Omega)} \leq C h_k \|\chi\|_k, \tag{5.15}$$

which is established by a duality argument.

There exists a unique $\zeta \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$-\mathrm{div}\,(\mathbf{A}\,\mathbf{grad}\,\zeta) = P_k^0 \hat{\chi} \quad \text{in } \Omega$$

and

$$\zeta = 0 \quad \text{on } \partial\Omega.$$

From the elliptic regularity estimate (0.3) we have

$$\|\zeta\|_{H^2(\Omega)} \leq C \|P_k^0 \hat{\chi}\|_{L^2(\Omega)}. \tag{5.16}$$

Let $\zeta_k \in N^1(\mathcal{T}_k^0)$ and $\zeta_{k-1} \in N^1(\mathcal{T}_{k-1}^0)$ satisfy the discretized equations

$$a_k(\zeta_k, \psi) = \int_\Omega (P_k^0 \hat{\chi})(P_k^0 \psi)\, dx \quad \forall\, \psi \in N^1(\mathcal{T}_k^0), \tag{5.17}$$

$$a_{k-1}(\zeta_{k-1}, \psi) = \int_\Omega (P_k^0 \hat{\chi})(P_{k-1}^0 \psi)\, dx \quad \forall\, \psi \in N^1(\mathcal{T}_{k-1}^0). \tag{5.18}$$

From the interpolation error estimate (2.9), the discretization error estimate (2.22), and the elliptic regularity estimate (5.16) we have

$$\max\{\|\zeta - \Pi_k \zeta\|_k, \|\zeta - \Pi_{k-1}\zeta\|_{k-1}, \|\zeta - \zeta_k\|_k, \|\zeta - \zeta_{k-1}\|_{k-1}\} \le C\, h_k\, \|P_k^0 \hat{\chi}\|_{L^2(\Omega)}.$$

Hence,

$$\max\{\|\zeta_k - \Pi_k \zeta\|_k, \|\zeta_{k-1} - \Pi_{k-1}\zeta\|_{k-1}\} \le C\, h_k\, \|P_k^0 \hat{\chi}\|_{L^2(\Omega)}. \tag{5.19}$$

We are now ready to prove (5.15). From (5.2), (5.17)–(5.19), and (3.11)–(3.13) we find

$$\begin{aligned}
&\|P_k^0 \hat{\chi}\|_{L^2(\Omega)}^2 \\
&= \int_\Omega (P_k^0 \hat{\chi})(P_k^0 \hat{\chi})\, dx \\
&= a_k(\zeta_k, \hat{\chi}) \\
&= a_k\big(\zeta_k, (I - I_{k-1}^k J_k^{k-1})\chi\big) \\
&= a_k(\zeta_k, \chi) - a_k(\zeta_k, I_{k-1}^k J_k^{k-1}\chi) \\
&= a_k(\zeta_k - \Pi_k \zeta, \chi) + a_k(\Pi_k \zeta - I_{k-1}^k \Pi_{k-1}\zeta, \chi) \\
&\qquad + a_k(I_{k-1}^k \Pi_{k-1}\zeta, \chi) - \int_\Omega (P_k^0 \hat{\chi}) P_k^0 (I_{k-1}^k J_k^{k-1}\chi)\, dx \\
&= a_k(\zeta_k - \Pi_k \zeta, \chi) + a_k(\Pi_k \zeta - I_{k-1}^k \Pi_{k-1}\zeta, \chi) + a_{k-1}(\Pi_{k-1}\zeta, J_k^{k-1}\chi) \\
&\qquad - \int_\Omega (P_k^0 \hat{\chi}) P_{k-1}^0 (J_k^{k-1}\chi)\, dx \\
&\qquad - \int_\Omega (P_k^0 \hat{\chi}) \big(P_k^0 I_{k-1}^k (J_k^{k-1}\chi) - P_{k-1}^0 (J_k^{k-1}\chi)\big)\, dx \\
&= a_k(\zeta_k - \Pi_k \zeta, \chi) + a_k(\Pi_k \zeta - I_{k-1}^k \Pi_{k-1}\zeta, \chi) + a_{k-1}(\Pi_{k-1}\zeta - \zeta_{k-1}, J_k^{k-1}\chi) \\
&\qquad - \int_\Omega (P_k^0 \hat{\chi}) \big(P_k^0 I_{k-1}^k (J_k^{k-1}\chi) - P_{k-1}^0 (J_k^{k-1}\chi)\big)\, dx \\
&\le C\, h_k\, \|P_k^0 \hat{\chi}\|_{L^2(\Omega)}\, \|\chi\|_k.
\end{aligned}$$

Hence, (5.15) holds. □

COROLLARY. *There exists a positive constant $C$ such that*

$$\|J_k^{k-1}\psi\|_{L^2(\Omega)} \le C \|\psi\|_{L^2(\Omega)} \quad \forall\, \psi \in N^1(\mathcal{T}_k^0). \tag{5.20}$$

*Proof.* From (3.2), (3.6), (3.8), (3.12), (5.3), and (5.12) we have

$$\begin{aligned}
&\|J_k^{k-1}\psi\|_{L^2(\Omega)} \\
&\quad \le C\big\{\|P_{k-1}^0 J_k^{k-1}\psi\|_{L^2(\Omega)} + h_{k-1}\, \|J_k^{k-1}\psi\|_{k-1}\big\} \\
&\quad \le C\big\{\|P_{k-1}^0 J_k^{k-1}\psi - P_k^0 I_{k-1}^k J_k^{k-1}\psi\|_{L^2(\Omega)} + \|P_k^0 I_{k-1}^k J_k^{k-1}\psi - P_k^0 \psi\|_{L^2(\Omega)} \\
&\qquad\qquad + \|P_k^0 \psi\|_{L^2(\Omega)} + h_{k-1}\|J_k^{k-1}\psi\|_{k-1}\big\} \\
&\quad \le C\big\{h_k \|J_k^{k-1}\psi\|_{k-1} + h_k\|\psi\|_k + \|\psi\|_{L^2(\Omega)} + h_{k-1}\|J_k^{k-1}\psi\|_{k-1}\big\} \\
&\quad \le C\{h_k\|\psi\|_k + \|\psi\|_{L^2(\Omega)}\} \\
&\quad \le C\, \|\psi\|_{L^2(\Omega)}.
\end{aligned}$$

□

THEOREM 5.1 (Convergence of the two-grid algorithm). *There exists a positive constant $C$ such that*

$$\|e\|_k \leq C\, m^{-1/2}\, \|e_0\|_k \tag{5.21}$$

*and*

$$\|e\|_{L^2(\Omega)} \leq C\, m^{-1/2}\, \|e_0\|_{L^2(\Omega)}. \tag{5.22}$$

*Hence, if $m$ is chosen large enough, then the two-grid algorithm converges.*

*Proof.* Using the approximation property (5.13) and the smoothing property (5.9) we have from (5.8) that

$$\begin{aligned} \|e\|_k &= \|(I - I_{k-1}^k J_k^{k-1}) R_k^m e_0\|_k \\ &\leq C\, h_k\, |\!|\!| R_k^m e_0 |\!|\!|_{2,k} \\ &\leq C\, m^{-1/2}\, \|e_0\|_k. \end{aligned}$$

Similarly, using the approximation property (5.12), we have

$$\begin{aligned} \|e\|_{L^2(\Omega)} &= \|(I - I_{k-1}^k J_k^{k-1}) R_k^m e_0\|_{L^2(\Omega)} \\ &\leq C\, |\!|\!|(I - I_{k-1}^k J_k^{k-1}) R_k^m e_0 |\!|\!|_{0,k} \\ &\leq C\, h_k\, |\!|\!| R_k^m e_0 |\!|\!|_{1,k} \\ &\leq C\, m^{-1/2}\, \|e_0\|_{L^2(\Omega)}. \end{aligned}$$

□

THEOREM 5.2 (Convergence of the $k$th level iteration). *There exists a positive constant $C$ such that when the $k$th level iteration is applied to $A_k z = g$, we have*

$$\|z - MG(k, z_0, g)\|_k \leq C\, m^{-1/2}\, \|z - z_0\|_k \quad \textit{and} \tag{5.23}$$

$$\|z - MG(k, z_0, g)\|_{L^2(\Omega)} \leq C\, m^{-1/2}\, \|z - z_0\|_{L^2(\Omega)}, \tag{5.24}$$

*provided that $m$ is large enough. Therefore the $k$th level iteration converges with contraction number independent of the grid level if $m$ is chosen large enough.*

*Proof.* Let $C^* \geq 1$ be a positive constant that dominates the constants in (5.21), (3.11), and (5.3). Let $\gamma = 2\,(C^*)^2\, m^{-1/2}$, and choose $m$ large enough (since $p \geq 2$) that $\gamma^p \leq m^{-1/2}$. Note that, by our choice,

$$\gamma^p \leq \frac{\gamma}{2\,(C^*)^2}. \tag{5.25}$$

We claim that

$$\|z - MG(n, z_0, g)\|_n \leq \gamma\, \|z - z_0\|_n. \tag{5.26}$$

For $n = 1$, this is obviously true since $z = MG(1, z_0, g)$. Assume that (5.26) is true for $n = k - 1$. Then

$$\begin{aligned} z - MG(k, z_0, g) &= z - (z_m + I_{k-1}^k q_p) \\ &= z - (z_m + I_{k-1}^k q) + I_{k-1}^k (q - q_p), \end{aligned} \tag{5.27}$$

where $q = J_k^{k-1} e_m$ (cf. (5.4)) satisfies $A_{k-1} q = \bar{g}$ $\big(= I_k^{k-1}(g - A_k z_m)\big)$ and $q_p$ is the approximation of $q$ obtained by applying the $(k-1)$ level iteration $p$ times. From (3.11), the induction hypothesis, (5.4), (5.3), (5.6), (5.7), and (5.25) it follows that

$$\begin{aligned} \|I_{k-1}^k (q - q_p)\|_k &\le C_* \, \gamma^p \, \|q\|_{k-1} \\ &= C_* \, \gamma^p \, \|J_k^{k-1} e_m\|_{k-1} \\ &\le (C_*)^2 \, \gamma^p \, \|R_k^m e_0\|_k \\ &\le \frac{\gamma}{2} \, \|e_0\|_k. \end{aligned} \tag{5.28}$$

Since $z - (z_m + I_{k-1}^k q)$ is the final error of the two-grid algorithm, it follows from (5.21) and the choice of $\gamma$ that

$$\begin{aligned} \|z - (z_m + I_{k-1}^k q)\|_k &\le C_* \, m^{-1/2} \, \|e_0\|_k \\ &\le \frac{\gamma}{2} \, \|e_0\|_k. \end{aligned} \tag{5.29}$$

Therefore, by combining (5.27)–(5.29), inequality (5.26) holds for $n = k$. Hence (5.23) holds for $n = 1, 2, \cdots$ by induction. The proof of (5.24) is similar. □

THEOREM 5.3 (Full multigrid convergence for the nonconforming method). *If $m$ is chosen so that the $k$th level iteration is a contraction for $k = 1, 2, \cdots$ with contraction number independent of $k$, and the parameter $r$ in the full multigrid algorithm is chosen large enough, then*

$$\|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} + \|\psi_k - \hat{\psi}_k\|_k \le C \, h_k \, \|f\|_{L^2(\Omega)}. \tag{5.30}$$

*If $f \in H^1(\Omega)$, then*

$$\|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} \le C \, h_k^2 \, \|f\|_{H^1(\Omega)}. \tag{5.31}$$

*Proof.* It suffices to show that

$$\|\psi_k - I_{k-1}^k \psi_{k-1}\|_{L^2(\Omega)} + \|\psi_k - I_{k-1}^k \psi_{k-1}\|_k \le C \, h_k \, \|f\|_{L^2(\Omega)}, \tag{5.32}$$

and in the case that $f \in H^1(\Omega)$,

$$\|\psi_k - I_{k-1}^k \psi_{k-1}\|_{L^2(\Omega)} \le C \, h_k^2 \, \|f\|_{H^1(\Omega)}. \tag{5.33}$$

A standard argument (cf. [17, Thm. 7.1, p. 162]) will then prove (5.30) and (5.31).

From the estimates (2.8), (2.9), (2.22), (2.28), (3.11), and (3.13) we have

$$\begin{aligned} &\|\psi_k - I_{k-1}^k \psi_{k-1}\|_{L^2(\Omega)} + \|\psi_k - I_{k-1}^k \psi_{k-1}\|_k \\ &\quad \le \big(\|\psi_k - \Pi_k u\|_{L^2(\Omega)} + \|\psi_k - \Pi_k u\|_k\big) \\ &\qquad + \big(\|\Pi_k u - I_{k-1}^k \Pi_{k-1} u\|_{L^2(\Omega)} + \|\Pi_k u - I_{k-1}^k \Pi_{k-1} u\|_k\big) \\ &\qquad + \big(\|I_{k-1}^k \Pi_{k-1} u - I_{k-1}^k \psi_{k-1}\|_{L^2(\Omega)} + \|I_{k-1}^k \Pi_{k-1} u - I_{k-1}^k \psi_{k-1}\|_k\big) \\ &\quad \le C \, h_k \, \|f\|_{L^2(\Omega)}. \end{aligned}$$

The proof of (5.33) is similar (with (2.28) replaced by (2.29)). □

THEOREM 5.4 (Full multigrid convergence for the Raviart–Thomas mixed method). *Under the same conditions on $m$ and $r$ as in Theorem* 5.3, *we have*

$$\|\boldsymbol{\sigma}_k - \hat{\boldsymbol{\sigma}}_k\|_{L^2(\Omega)} + \|u_k - \hat{u}_k\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)}, \tag{5.34}$$

*and in the case that $f \in H^1(\Omega)$,*

$$\|u_k - \hat{u}_k\|_{L^2(\Omega)} \le C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{5.35}$$

*Proof.* Using (1.21), (1.22), (4.3), Lemmas 4.1 and 2.1, and Theorem 5.3 we have

$$\begin{aligned}
&\|\boldsymbol{\sigma}_k - \hat{\boldsymbol{\sigma}}_k\|_{L^2(\Omega)} + \|u_k - \hat{u}_k\|_{L^2(\Omega)} \\
&\le \| - P^0_{RT,\mathbf{C}} \mathbf{A}\, \mathbf{grad}_k \psi_k + \mathcal{A}\, P^0_{RT,\mathbf{C}} \mathbf{A}\, \mathbf{grad}_k \hat{\psi}_k\|_{L^2(\Omega)} + \|P^0_k \psi_k - P^0_k \hat{\psi}_k\|_{L^2(\Omega)} \\
&= \|\mathcal{A}\, P^0_{RT,\mathbf{C}} \mathbf{A}\, \mathbf{grad}_k (\hat{\psi}_k - \psi_k)\|_{L^2(\Omega)} + \|P^0_k (\psi_k - \hat{\psi}_k)\|_{L^2(\Omega)} \\
&\le C\, \|\mathbf{grad}_k (\hat{\psi}_k - \psi_k)\|_{L^2(\Omega)} + \|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} \\
&\le C\, h_k\, \|f\|_{L^2(\Omega)}.
\end{aligned}$$

To prove (5.35), we use (1.22), (4.3), and (5.31) to obtain

$$\begin{aligned}
\|u_k - \hat{u}_k\|_{L^2(\Omega)} &= \|P^0_k (\psi_k - \hat{\psi}_k)\|_{L^2(\Omega)} \\
&\le \|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} \\
&\le C\, h_k^2\, \|f\|_{H^1(\Omega)}.
\end{aligned}$$

□

**6. The singular Neumann problem.** We shall denote $\{v : v \in H^k(\Omega)$ and $\int_\Omega v\, dx = 0\}$ by $\tilde{H}^k(\Omega)$. Let $f \in \tilde{L}^2(\Omega) (= \tilde{H}^0(\Omega))$, and let $\mathbf{n}$ be the outward unit normal on $\partial\Omega$. The elliptic boundary value problem

$$\begin{aligned}
-\mathrm{div}\, (\mathbf{A}\, \mathbf{grad}\, u) &= f \quad \text{in } \Omega, \\
(\mathbf{A}\, \mathbf{grad}\, u) \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega
\end{aligned} \tag{6.1}$$

has a unique solution $u \in \tilde{H}^2(\Omega)$. Furthermore, there exists a positive constant $C_{\Omega,\mathbf{A}}$ such that

$$\|u\|_{H^2(\Omega)} \le C_{\Omega,\mathbf{A}}\, \|f\|_{L^2(\Omega)} \tag{6.2}$$

(cf. [16]).

A mixed formulation of (6.1) is the following.

Find $(\boldsymbol{\sigma}, u) \in \tilde{H}(\mathrm{div};\Omega) \times \tilde{L}^2(\Omega)$ such that

$$\begin{aligned}
\int_\Omega \mathbf{C}\, \boldsymbol{\sigma} \cdot \boldsymbol{\tau} dx - \int_\Omega u\, \mathrm{div}\, \boldsymbol{\tau}\, dx &= 0 \quad \forall\, \boldsymbol{\tau} \in \tilde{H}(\mathrm{div};\Omega), \\
- \int_\Omega v\, \mathrm{div}\, \boldsymbol{\sigma}\, dx &= - \int_\Omega f v\, dx \quad \forall\, v \in \tilde{L}^2(\Omega),
\end{aligned} \tag{6.3}$$

where $\tilde{H}(\mathrm{div};\Omega) = \{\boldsymbol{\tau} : \boldsymbol{\tau} \in H(\mathrm{div};\Omega), \boldsymbol{\tau} \cdot \mathbf{n} = 0 \text{ along } \partial\Omega\}$.

If $u$ is the unique solution of (6.1) in $\tilde{H}^2(\Omega)$, then $(\boldsymbol{\sigma}, u)$ is the unique solution of (6.3), provided that $\boldsymbol{\sigma}$ is taken to be $-\mathbf{A}\,\mathbf{grad}\,u$.

Let $\widetilde{RT}_0^0(\mathcal{T}_k) = \{\boldsymbol{\tau} : \boldsymbol{\tau} \in RT_0^0(\mathcal{T}_k) \text{ and } \boldsymbol{\tau}\cdot\mathbf{n} = 0 \text{ along } \partial\Omega\}$ and $\tilde{M}_{-1}^0(\mathcal{T}_k) = \{v : v \in M_{-1}^0(\mathcal{T}_k) \text{ and } \int_\Omega v\,dx = 0\}$. The discretized problem for the mixed method is the following.

Find $(\boldsymbol{\sigma}_k, u_k) \in \widetilde{RT}_0^0(\mathcal{T}_k) \times \tilde{M}_{-1}^0(\mathcal{T}_k)$ such that

$$
(6.4)\qquad
\begin{aligned}
&\int_\Omega \mathbf{C}\boldsymbol{\sigma}_k\cdot\boldsymbol{\tau}\,dx - \int_\Omega u_k \operatorname{div}\boldsymbol{\tau}\,dx = 0 \quad \forall \boldsymbol{\tau} \in \tilde{RT}_0^0(\mathcal{T}_k),\\
&-\int_\Omega v \operatorname{div}\boldsymbol{\sigma}_k\,dx = -\int_\Omega fv\,dx \quad \forall v \in \tilde{M}_{-1}^0(\mathcal{T}_k).
\end{aligned}
$$

As in §1, we can transform the mixed method (6.4) into an equivalent nonconforming method. Since the arguments are the same, here we describe the steps involved without proof.

By introducing Lagrange multipliers, the system (6.4) is equivalent to the following problem. Find $(\boldsymbol{\sigma}_k, u_k, \lambda_k) \in RT_{-1}^0(\mathcal{T}_k) \times \tilde{M}_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k)$ such that for all $(\boldsymbol{\tau}, v, \mu) \in RT_{-1}^0(\mathcal{T}_k) \times \tilde{M}_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k)$

$$
(6.5)\qquad
\begin{aligned}
&\text{(i)} && \int_\Omega \mathbf{C}\boldsymbol{\sigma}_k\cdot\boldsymbol{\tau}\,dx - \sum_{T\in\mathcal{T}_k}\int_T u_k \operatorname{div}\boldsymbol{\tau}\,dx + \sum_{T\in\mathcal{T}_k}\int_{\partial T}\lambda_k\boldsymbol{\tau}\cdot\mathbf{n}_T\,ds = 0,\\
&\text{(ii)} && -\sum_{T\in\mathcal{T}_k}\int_T v \operatorname{div}\boldsymbol{\sigma}_k\,dx = -\int_\Omega fv\,dx,\\
&\text{(iii)} && \sum_{T\in\mathcal{T}_k}\int_{\partial T}\mu\boldsymbol{\sigma}_k\cdot\mathbf{n}_T\,ds = 0,
\end{aligned}
$$

where $M_{-1}^0(\mathcal{E}_k)$ is the set of all functions on $\cup\mathcal{E}_k$ which restrict to constant functions on each edge $e \in \mathcal{E}_k$.

Let an artificial gradient operator $\tilde{\boldsymbol{\nabla}}_k : \tilde{M}_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k) \longrightarrow RT_{-1}^0(\mathcal{T}_k)$ be defined by

$$
(6.6)\qquad \int_\Omega \tilde{\boldsymbol{\nabla}}_k(v,\mu)\cdot\boldsymbol{\tau}\,dx = \sum_{T\in\mathcal{T}_k}\int_{\partial T}\mu\boldsymbol{\tau}\cdot\mathbf{n}_T\,ds - \sum_{T\in\mathcal{T}_k}\int_T v\cdot\operatorname{div}\boldsymbol{\tau}\,dx \quad \forall\boldsymbol{\tau}\in RT_{-1}^0(\mathcal{T}_k).
$$

Problem (6.5) is then equivalent to the following positive definite problem.

Find $(u_k, \lambda_k) \in \tilde{M}_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k)$ such that

$$
(6.7)\qquad [P_{RT,\mathbf{C}}^0\mathbf{A}\tilde{\boldsymbol{\nabla}}_k(u_k,\lambda_k), P_{RT,\mathbf{C}}^0\mathbf{A}\,\tilde{\boldsymbol{\nabla}}_k(v,\mu)] = \int_\Omega fv\,dx
$$

for all $(v, \mu) \in \tilde{M}_{-1}^0(\mathcal{T}_k) \times M_{-1}^0(\mathcal{E}_k)$.

The problem (6.7) can be identified with a nonconforming method. The nonconforming spaces are defined by

$$
(6.8)\qquad \tilde{N}^1(\mathcal{T}_k) = \left\{\psi : \psi \in CR^1(\mathcal{T}_k) \oplus B^3(\mathcal{T}_k) \text{ and } \int_\Omega \psi\,dx = 0\right\},
$$

where $CR^1(\mathcal{T}_k) = \{\psi : \psi \in L^2(\Omega),\ \psi|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_k, \psi$ is continuous at the midpoints of interelement boundaries$\}$ and $B^3(\mathcal{T}_k)$ is defined as in (1.12).

Let $\tilde{S}_k : \tilde{N}^1(\mathcal{T}_k) \longrightarrow \tilde{M}^0_{-1}(\mathcal{T}_k) \times M^0_{-1}(\mathcal{E}_k)$ be defined by

$$\tilde{S}_k(\psi) = (v, \mu), \tag{6.9}$$

where $v$ and $\mu$ are defined as in (1.15). Note that $\int_\Omega v\,dx = \int_\Omega \psi\,dx = 0$.

Using the isomorphism $\tilde{S}_k$, (6.7) can be transformed into the following nonconforming method.

Find $\psi_k \in \tilde{N}^1(\mathcal{T}_k)$ such that

$$[P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\psi_k, P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\chi] = \int_\Omega f(P^0_k\chi)\,dx \quad \forall \chi \in \tilde{N}^1(\mathcal{T}_k). \tag{6.10}$$

The solution $\psi_k$ of (6.10) is related to the solution $(\boldsymbol{\sigma}_k, u_k)$ of (6.4) by the formulas

$$\boldsymbol{\sigma}_k = -P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\psi_k, \tag{6.11}$$

$$u_k = P^0_k\,\psi_k. \tag{6.12}$$

Let the positive semidefinite bilinear form for $\tilde{a}_k(\cdot,\cdot)$ on $\tilde{H}^1(\Omega)+\tilde{N}^1(\mathcal{T}_k)$ be defined by

$$\tilde{a}_k(\chi_1,\chi_2) = [P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\chi_1, P^0_{RT,\mathbf{C}}\,\mathbf{A}\,\mathbf{grad}_k\chi_2]. \tag{6.13}$$

The nonconforming problem (6.9) can be rewritten as the following.

Find $\psi_k \in \tilde{N}^1(\mathcal{T}_k)$ such that

$$\tilde{a}_k(\psi_k,\chi) = \big(F \circ P^0_k\big)\chi \quad \forall \chi \in \tilde{N}^1(\mathcal{T}_k). \tag{6.14}$$

The nonconforming energy seminorm on $\tilde{H}^1(\Omega) + \tilde{N}^1(\mathcal{T}_k)$ is defined to be

$$\|\psi\|\!\lceil_k := \tilde{a}_k(\psi,\psi)^{1/2}. \tag{6.15}$$

The following analog of Lemma 2.1 holds. There exists a positive constant $C$ such that

$$\|\psi\|\!\lceil_k \le C\,\|\mathbf{grad}_k\psi\|_{L^2(\Omega)} \quad \forall \psi \in \tilde{H}^1(\Omega) + \tilde{N}^1(\mathcal{T}_k), \tag{6.16}$$

$$C\,\|\mathbf{grad}_k\psi\|_{L^2(\Omega)} \le \|\psi\|\!\lceil_k \quad \forall \psi \in \tilde{N}^1(\mathcal{T}_k). \tag{6.17}$$

Let $w_I$ be the nonconforming linear interpolant of $w$ in $\tilde{H}^2(\Omega)$. In other words, $w_I \in CR^1(\mathcal{T}_k)$ and $w_I$ agrees with $w$ at the midpoints of $\mathcal{T}_k$. Then the interpolation operator $\tilde{\Pi}_k : \tilde{H}^2(\Omega) \longrightarrow \tilde{N}^1(\mathcal{T}_k)$ is defined by

$$\tilde{\Pi}_k w = w_I - \frac{1}{|\Omega|}\int_\Omega w_I\,dx. \tag{6.18}$$

From standard interpolation error estimates, we have (cf. [12])

$$\|w - \tilde{\Pi}_k w\|_{L^2(\Omega)} + h_k\|\mathbf{grad}_k(w - \tilde{\Pi}_k w)\|_{L^2(\Omega)} \le C\,h_k^2\,|w|_{H^2(\Omega)} \tag{6.19}$$

and

$$\|w - \tilde{\Pi}_k w\|_k \le C\, h_k\, |w|_{H^2(\Omega)} \tag{6.20}$$

for all $w \in \tilde{H}^2(\Omega)$.

The key to the error estimates for the nonconforming method is the following analog of Lemma 2.2.

LEMMA 6.1. *There exists a positive constant $C$ such that*

$$\left| \sum_{T \in \mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,\zeta \cdot \mathbf{n}_T)\, \eta|_T\, ds \right| \le C\, h_k\, \|\zeta\|_{H^2(\Omega)}\, \|\mathbf{grad}_k \eta\|_{L^2(\Omega)} \tag{6.21}$$

*and*

$$\left| \sum_{T \in \mathcal{T}_k} \int_{\partial T} (\mathbf{A}\,\mathbf{grad}\,\zeta \cdot \mathbf{n}_T)(\lambda - \tilde{\Pi}_k \lambda|_T)\, ds \right| \le C\, h_k^2\, \|\zeta\|_{H^2(\Omega)}\, |\lambda|_{H^2(\Omega)} \tag{6.22}$$

*for all $\zeta \in H^2(\Omega)$ satisfying $(\mathbf{A}\,\mathbf{grad}\,\zeta) \cdot \mathbf{n}_T = 0$ on $\partial\Omega$, $\eta \in \tilde{N}^1(\mathcal{T}_k)$, and $\lambda \in \tilde{H}^2(\Omega)$.*

The proofs of Lemma 6.1 and the following discretization error estimates are simple modifications of their analogs in §2.

PROPOSITION 6.1. *If $u \in \tilde{H}^2(\Omega)$ is the solution of the continuous problem* (6.1) *and $\psi_k$ is the solution of* (6.14), *then there exists a positive constant $C$ such that*

$$\|u - \psi_k\|\tilde{}_k \le C\, h_k\, \|f\|_{L^2(\Omega)}, \tag{6.23}$$

$$\|\mathbf{grad}_k(u - \psi_k)\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)}, \tag{6.24}$$

$$\|u - \psi_k\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{6.25}$$

*If $f \in \tilde{H}^1(\Omega)$, then the last estimate can be improved to*

$$\|u - \psi_k\|_{L^2(\Omega)} \le C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{6.26}$$

Using the relations (6.11) and (6.12), we obtain the discretization error estimates for the mixed method (6.4).

PROPOSITION 6.2. *If $u \in \tilde{H}^2(\Omega)$ is the solution of the elliptic boundary value problem* (6.1), *$\boldsymbol{\sigma} = -\mathbf{A}\,\mathbf{grad}\, u$ and $(\boldsymbol{\sigma}_k, u_k)$ solve the discretized system* (6.4), *then there exists a positive constant $C$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_k\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)}, \tag{6.27}$$

$$\|u - u_k\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{6.28}$$

*If $f \in \tilde{H}^1(\Omega)$, then*

$$\|P_k^0 u - u_k\|_{L^2(\Omega)} \le C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{6.29}$$

The mesh-dependent inner product $(\cdot,\cdot)\tilde{}_k$ on $\tilde{N}^1(\mathcal{T}_k)$ is defined by

$$(\psi, \chi)\tilde{}_k := h_k^2 \left( \sum_m \psi(m)\, \chi(m) + \sum_{T \in \mathcal{T}_k} \left( \frac{1}{|T|} \int_T \psi\, dx \right) \left( \frac{1}{|T|} \int_T \chi\, dx \right) \right), \tag{6.30}$$

where $m$ ranges over the midpoints of the edges of $\mathcal{T}_k$. Let $\tilde{A}_k : \tilde{N}^1(\mathcal{T}_k) \longrightarrow \tilde{N}^1(\mathcal{T}_k)$ be defined by

$$\tilde{a}_k(\psi, \chi) = (\tilde{A}_k\psi, \chi)\tilde{}_k \quad \forall\, \psi, \chi \in \tilde{N}^1(\mathcal{T}_k). \tag{6.31}$$

The estimate (3.2) holds in the present context; therefore, by a standard inverse estimate we have

$$\text{spectral radius of } \tilde{A}_k \le C\, h_k^{-2}. \tag{6.32}$$

The mesh-dependent norm $|\!|\!| \cdot |\!|\!|\tilde{}_{s,k}$ is defined by

$$|\!|\!|\psi|\!|\!|\tilde{}_{s,k} = \sqrt{(\tilde{A}_k^s\psi, \psi)\tilde{}_k}. \tag{6.33}$$

Inequalities (3.6) and (3.7) also hold for these norms. We have the following analog of (3.8). There exists a positive constant $C$ such that

$$|\!|\!|\psi|\!|\!|\tilde{}_{0,k} \le C\left\{\|P_k^0\psi\|_{L^2(\Omega)} + h_k\|\psi\|\tilde{}_k\right\} \quad \forall\, \psi \in \tilde{N}^1(\mathcal{T}_k). \tag{6.34}$$

The intergrid transfer operator $\tilde{I}_{k-1}^k : CR^1(\mathcal{T}_{k-1}) \oplus B^3(\mathcal{T}_{k-1}) \longrightarrow CR^1(\mathcal{T}_k) \oplus B^3(\mathcal{T}_k)$ is defined as follows. If $\psi \in B^3(\mathcal{T}_{k-1})$, then $\tilde{I}_{k-1}^k\psi = I_{k-1}^k\psi$(cf. (3.9)). If $\psi \in CR^1(\mathcal{T}_{k-1})$, then there are two cases. If $m$ lies on the common edge of two adjacent triangles $T_1$ and $T_2$ in $\mathcal{T}_{k-1}$, then $\big(\tilde{I}_{k-1}^k v\big)(m) := \frac{1}{2}\big[v|_{T_1}(m) + v|_{T_2}(m)\big]$. If $m$ lies in the interior of a triangle in $\mathcal{T}_{k-1}$ or on $\partial\Omega$, then $\big(\tilde{I}_{k-1}^k v\big)(m) := v(m)$. Observe that by this choice of intergrid transfer operator, we have

$$\int_\Omega \tilde{I}_{k-1}^k\psi\, dx = \int_\Omega \psi\, dx \quad \forall\, \psi \in CR^1(\mathcal{T}_{k-1}) \oplus B^3(\mathcal{T}_{k-1}). \tag{6.35}$$

Therefore, $\tilde{I}_{k-1}^k$ maps $\tilde{N}^1(\mathcal{T}_{k-1})$ into $\tilde{N}^1(\mathcal{T}_k)$.

The fine-to-coarse intergrid transfer operator $\tilde{I}_k^{k-1} : \tilde{N}^1(\mathcal{T}_k) \longrightarrow \tilde{N}^1(\mathcal{T}_{k-1})$ is then defined by

$$(\tilde{I}_k^{k-1}\psi, \chi)\tilde{}_{k-1} = (\psi, \tilde{I}_{k-1}^k\chi)\tilde{}_k \quad \forall\, \psi \in \tilde{N}^1(\mathcal{T}_k),\ \chi \in \tilde{N}^1(\mathcal{T}_{k-1}). \tag{6.36}$$

A simple modification of the proof of Proposition 3.1 and its corollary yield the following proposition.

PROPOSITION 6.3. *There exists a positive constant $C$ such that*

$$\|\tilde{I}_{k-1}^k\psi\|\tilde{}_k \le C\, \|\psi\|\tilde{}_{k-1} \quad \forall\, \psi \in \tilde{N}^1(\mathcal{T}_{k-1}), \tag{6.37}$$

$$|\!|\!|\tilde{I}_{k-1}^k\psi|\!|\!|\tilde{}_{0,k} \le C\, |\!|\!|\psi|\!|\!|\tilde{}_{0,k-1} \quad \forall\, \psi \in \tilde{N}^1(\mathcal{T}_{k-1}), \tag{6.38}$$

$$\|P_k^0(\tilde{I}_{k-1}^k\psi) - P_{k-1}^0\psi\|_{L^2(\Omega)} \le C\, h_k\, \|\psi\|\tilde{}_{k-1} \quad \forall\, \psi \in \tilde{N}^1(\mathcal{T}_{k-1}), \tag{6.39}$$

$$\begin{aligned} &\|\tilde{I}_{k-1}^k\tilde{\Pi}_{k-1}\zeta - \tilde{\Pi}_k\zeta\|_{L^2(\Omega)} + h_k\, \|\tilde{I}_{k-1}^k\tilde{\Pi}_{k-1}\zeta - \tilde{\Pi}_k\zeta\|\tilde{}_k \\ &\quad \le C\, h_k^2\, \|\zeta\|_{H^2(\Omega)} \quad \forall\, \zeta \in \tilde{H}^2(\Omega). \end{aligned} \tag{6.40}$$

The $W$-cycle multigrid algorithm ($k$th level iteration and full multigrid algorithm) for the nonconforming method (6.14) is as described in §4, with $A_k$, $I_{k-1}^k$, $I_k^{k-1}$, $N^1(\mathcal{T}_k^0)$, and $(\cdot,\cdot)_k$ replaced by $\tilde{A}_k$, $\tilde{I}_{k-1}^k$, $\tilde{I}_k^{k-1}$, $\tilde{N}^1(\mathcal{T}_k)$, and $(\cdot,\cdot)\tilde{}_k$, respectively.

The approximate solution $(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k)$ of the Raviart–Thomas method (6.4) can then be obtained from the approximate solution $\hat{\psi}_k$ of (6.14) via

$$(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k) = (-\tilde{A} P^0_{RT,\mathbf{C}}\, \mathbf{A}\, \mathbf{grad}_k \hat{\psi}_k, P^0_k \hat{\psi}_k), \tag{6.41}$$

where $\tilde{A} : RT^0_{-1}(\mathcal{T}_k) \longrightarrow \widetilde{RT}^0_0(\mathcal{T}_k)$ is defined as in (4.4) for $e \in \mathcal{E}^0_k$. We have

$$\|\tilde{A}\boldsymbol{\tau}\|_{L^2(\Omega)} \le C\, \|\boldsymbol{\tau}\|_{L^2(\Omega)} \quad \forall \boldsymbol{\tau} \in RT^0_{-1}(\mathcal{T}_k) \tag{6.42}$$

for some positive constant $C$.

The cost of computing $\hat{\psi}_k$ (respectively, $(\hat{\boldsymbol{\sigma}}_k, \hat{u}_k)$) is bounded by a constant times the dimension of $\tilde{N}^1(\mathcal{T}_k)$ (respectively, $\widetilde{RT}^0_0(\mathcal{T}_k) \times \tilde{M}^0_{-1}(\mathcal{T}_k)$).

The key ingredients in the convergence analysis are the smoothing and approximation properties. The proof of the smoothing property is identical with that of Lemma 5.1. The proof of the approximation property relies on elliptic regularity, interpolation, and discretization error estimates, along with the fundamental estimates on the intergrid transfer operator, which we have been established in (6.2), (6.19)–(6.20), (6.23)–(6.26), and (6.37)–(6.40). Hence the approximation property holds in the present context.

The convergence of the two-grid algorithm, the $k$th level iteration, and the full multigrid algorithm now follow in similar fashion. Here we only state the full multigrid results.

THEOREM 6.1 (Full multigrid convergence for the nonconforming method). *If $m$ is chosen so that the $k$th level iteration is a contraction for $k = 1, 2, \cdots$ with contraction number independent of $k$, and the parameter $r$ in the full multigrid algorithm is chosen large enough, then*

$$\|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} + \|\psi_k - \hat{\psi}_k\|_{\Gamma_k} \le C\, h_k\, \|f\|_{L^2(\Omega)}. \tag{6.43}$$

*If $f \in \tilde{H}^1(\Omega)$, then*

$$\|\psi_k - \hat{\psi}_k\|_{L^2(\Omega)} \le C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{6.44}$$

THEOREM 6.2 (Full multigrid convergence for the Raviart–Thomas mixed method). *Under the same conditions on $m$ and $r$ as in Theorem 6.1, we have*

$$\|\boldsymbol{\sigma}_k - \hat{\boldsymbol{\sigma}}_k\|_{L^2(\Omega)} + \|u_k - \hat{u}_k\|_{L^2(\Omega)} \le C\, h_k\, \|f\|_{L^2(\Omega)} \tag{6.45}$$

*and in the case that $f \in \tilde{H}^1(\Omega)$,*

$$\|u_k - \hat{u}_k\|_{L^2(\Omega)} \le C\, h_k^2\, \|f\|_{H^1(\Omega)}. \tag{6.46}$$

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