Chapter 1 - The Real and Complex Number Systems

Problem 1. If r is rational ($r \neq 0$) and x is irrational, prove that $r+x$ and rx are irrational.

Proof. Write $r = \frac{m}{n}$, where m and n are nonzero integers. Suppose $r+x$ were rational. Then there exists integers p and q, with $q \neq 0$ such that

$$r + x = \frac{p}{q} \tag{1}$$

Then x can be expressed as

$$x = \frac{p}{q} - \frac{m}{n} = \frac{pn - mq}{qn} \in \mathbb{Q} \tag{2}$$

contradicting our assumption about x. So $r + x$ must be irrational.

Now, suppose rx were rational. Then there exists integers p and q, with $q \neq 0$ such that

$$x = \frac{p}{q} \cdot \frac{n}{m} \tag{3}$$

since $m \neq 0$. Then x can be written as

$$x = \frac{pn}{qm} \in \mathbb{Q} \tag{4}$$

contradicting our assumption about x. So rx must be irrational. $\square$

Problem 2. Prove that there is no rational number whose square is 12.

Proof. Suppose there exists a rational number $r \in \mathbb{Q}$ such that $r^2 = 12$. We can write $r = \frac{m}{n}$ where m and n share no common factors. Then

$$\frac{m^2}{n^2} = 3 \cdot 4 \implies m^2 = 3 \cdot 4n^2 \tag{5}$$

Thus m^2 is divisible by 3. This implies m is divisible by 3 (otherwise m^2 would not be). Hence, m^2 is divisible by 9, and so is the right hand side of (5). This implies that $4n^2$ is divisible by 3. Since 4 is not divisible by 3, it follows that n^2, and thus n is divisible by 3. This contradicts the fact that m and n share no common factors. Thus, there can be no rational number that satisfies $r^2 = 12$. $\square$

Problem 3. Prove Proposition 1.15: The axioms for multiplication in a field imply the following statements.

(a) If $x \neq 0$ and $xy = xz$ then $y = z$
(b) If $x \neq 0$ and $xy = x$ then $y = 1$
(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$
(d) If $x \neq 0$ then $1/(1/x) = x$.

Proof. (a)

$$y = (1/x)xy = (1/x)xz = 1z = z \tag{6}$$

(b) Take $z = 1$ in (a)
(c) Take $z = 1/x$ in (a)
(d) This follows from (c) if we replace x with $1/x$ and y with x □

Problem 4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.

Proof. Suppose $\alpha > \beta$. Let $x \in E$; since α is a lower bound of E, we must have $\alpha \leq x$. Since $\alpha > \beta$ and $>$ is transitive, it then follows that $\beta < x$. But this contradicts the fact that β is an upper bound of E. So $\alpha > \beta$ must be false, i.e. $\alpha \leq \beta$. □

Problem 5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A) \tag{7}$$

Proof. Since A is bounded below, $\gamma = \inf A$ exists in $\mathbb{R}$, and $\gamma \leq x$ for all $x \in A$. This implies $-\gamma \geq -x$ for all $x \in A$, or $-\gamma \geq y$ for all $y \in -A$. So $-\gamma$ is an upper bound of $-A$. Let $\kappa < -\gamma$, then $-\kappa > \gamma$, so that $-\kappa$ is not a lower bound of A. Hence, there exists $x \in A$ such that $-\kappa > x$ or $\kappa < -x \in -A$. Hence κ is not an upper bound of $-A$. Then by definition, $-\gamma$ is the supremum of $-A$, i.e

$$-\inf A = \sup(-A) \tag{8}$$

which is equivalent to (7) □

Problem 6.

Problem 7.

Problem 8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint*: -1 is a square.

Proof. Let $>$ be an order on $\mathbb{C}$. Assume this turns $\mathbb{C}$ into an ordered field. Since $i \neq 0$, we must either have $i > 0$ or $i < 0$. First, assume $i > 0$. Then $-1 = i^2 > 0$. Add 1 to both sides to obtain $0 > 1$. But $1 = (-1)(-1) > 0$ since -1 is positive, resulting in a contradiction. Now assume $i < 0$. Add $-i$ to both sides to obtain $0 < -i$. Hence $-1 = (-i)^2 > 0$. Again, we have $0 > 1$, but $1 = (-1)(-1) > 0$ resulting in another contradiction. So $\mathbb{C}$ cannot be an ordered field under this order. □

Problem 9.

Problem 10.

Problem 11.

Problem 12.

Problem 13.

Problem 14.

Problem 15.

Problem 16.

Problem 17.

Problem 18.

Problem 19.

Problem 20.