UNIFIED HYBRIDIZATION OF DISCONTINUOUS GALERKIN, MIXED, AND CONTINUOUS GALERKIN METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS[*]

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Abstract. We introduce a unifying framework for hybridization of finite element methods for second order elliptic problems. The methods fitting in the framework are a general class of mixed-dual finite element methods including hybridized mixed, continuous Galerkin, nonconforming, and a new, wide class of hybridizable discontinuous Galerkin methods. The distinctive feature of the methods in this framework is that the only globally coupled degrees of freedom are those of an approximation of the solution defined only on the boundaries of the elements. Since the associated matrix is sparse, symmetric, and positive definite, these methods can be efficiently implemented. Moreover, the framework allows, in a single implementation, the use of different methods in different elements or subdomains of the computational domain, which are then automatically coupled. Finally, the framework brings about a new point of view, thanks to which it is possible to see how to devise novel methods displaying very localized and simple mortaring techniques, as well as methods permitting an even further reduction of the number of globally coupled degrees of freedom.

Key words. discontinuous Galerkin methods, mixed methods, continuous methods, hybrid methods, elliptic problems

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1. Introduction. We introduce a new *unifying framework* for hybridization of finite element methods for second order elliptic problems. This framework is unifying in the sense that it includes as particular cases hybridized versions of mixed methods [4, 11, 26], the continuous Galerkin (CG) method [31], and a new, wide class of hybridizable discontinuous Galerkin (DG) methods. The unifying framework allows us to (i) significantly reduce the number of the globally coupled degrees of freedom of DG methods, (ii) use different methods in different parts of the computational domain and automatically couple them, and (iii) devise novel methods employing new mortaring techniques. We develop the unifying framework on the following model elliptic boundary value problem of second order written in mixed form:

$$\boldsymbol{q} + a\,\mathbf{grad}\,u = 0 \qquad\qquad \text{on } \Omega, \tag{1.1a}$$

$$\operatorname{div}\boldsymbol{q} + d\,u = f \qquad\qquad \text{on } \Omega, \tag{1.1b}$$

$$u = g \qquad\qquad \text{on } \partial\Omega. \tag{1.1c}$$

Here $\Omega \subset \mathbb{R}^n$ is a polyhedral domain ($n \geq 2$), $d(\boldsymbol{x})$ is a scalar nonnegative function, and $a(\boldsymbol{x})$ is a matrix valued function that is symmetric and uniformly positive definite on Ω. In addition, we assume that the function g is the restriction of a smooth scalar function on $\partial\Omega$ and that the functions f, d, and a are smooth on $\overline{\Omega}$. These assumptions can be vastly generalized, but we take them for the sake of a transparent presentation of the design of our unifying framework.

1.1. The structure of the methods of the unifying framework. Let us begin the description of our results by arguing that what makes possible the construction of the unified framework is that all the numerical methods fitting in it are constructed by using a discrete version of a single property of the exact solution of problem (1.1). This property is a characterization of the values of the exact solution u on the interior boundaries of each of the elements K of any triangulation of the domain Ω, $\mathcal{T}_h$. Let us describe it.

If on the border of the element K, ∂K, we set $u = \lambda + g$, where

$$(1.2) \qquad \lambda = \begin{cases} u & \text{on } \partial K \setminus \partial\Omega, \\ 0 & \text{on } \partial K \cap \partial\Omega, \end{cases} \quad \text{and} \quad g = \begin{cases} 0 & \text{on } \partial K \setminus \partial\Omega, \\ g & \text{on } \partial K \cap \partial\Omega, \end{cases}$$

by the *linearity* of the problem, we have that

$$(1.3) \qquad (\boldsymbol{q}, u) = (\mathbf{Q}\lambda + \mathbf{Q}g + \mathbf{Q}f, \mathsf{U}\lambda + \mathsf{U}g + \mathsf{U}f) \qquad \text{in } \Omega,$$

where the so-called *local solvers* $(\mathbf{Q}(\cdot), \mathsf{U}(\cdot))$ are defined on the element $K \in \mathcal{T}_h$ as follows. For any single-valued functions m on $L^2(\partial K)$ and f on $L^2(K)$, the functions $(\mathbf{Q}\mathsf{m}, \mathsf{U}\mathsf{m})$ and $(\mathbf{Q}f, \mathsf{U}f)$ are the solutions of

$$(1.4a) \quad c\,\mathbf{Q}\mathsf{m} + \mathbf{grad}\,\mathsf{U}\mathsf{m} = 0, \quad \text{div}\,\mathbf{Q}\mathsf{m} + d\,\mathsf{U}\mathsf{m} = 0 \quad \text{on } K, \quad \mathsf{U}\mathsf{m} = \mathsf{m} \quad \text{on } \partial K,$$

$$(1.4b) \quad c\,\mathbf{Q}f + \mathbf{grad}\,\mathsf{U}f = 0, \quad \text{div}\,\mathbf{Q}f + d\,\mathsf{U}f = f \quad \text{on } K, \quad \mathsf{U}f = 0 \quad \text{on } \partial K,$$

where $c = a^{-1}$ for each element $K \in \mathcal{T}_h$.

Conversely, the above property holds if and only if (see, for example, [46]) the normal component of $\mathbf{Q}\lambda + \mathbf{Q}g + \mathbf{Q}f$ across interelement boundaries is continuous. We thus see that this *transmission* condition, which we formally express as

$$(1.5) \qquad [\![\mathbf{Q}\lambda + \mathbf{Q}g + \mathbf{Q}f]\!] = 0,$$

completely *characterizes* the function λ. Here $[\![\cdot]\!]$ denotes the jump of the normal component of the a vector accross ∂K.

The finite element methods of the unified framework are those that can be expressed as a discrete version of the above property. In this way, the only globally coupled degrees of freedom *are bound* to be those describing the approximation to λ. Thus, each of those method provides an approximate solution of the form

$$(1.6) \qquad (\boldsymbol{q}_h, u_h) = (\boldsymbol{\mathcal{Q}}\lambda_h + \boldsymbol{\mathcal{Q}}g_h + \boldsymbol{\mathcal{Q}}f, \mathcal{U}\lambda_h + \mathcal{U}g_h + \mathcal{U}f),$$

where λ_h, respectively, g_h, is an approximation in some finite-dimensional space M_h, respectively, M_h, of the values of u on the faces of the elements lying in the interior, respectively, in the border of Ω, and $(\boldsymbol{\mathcal{Q}}\mathsf{m}, \mathcal{U}\mathsf{m})$ and $(\boldsymbol{\mathcal{Q}}f, \mathcal{U}f)$ are discrete versions of the exact local solvers (1.4)—we keep the same notation for the sake of simplicity. Moreover, the methods are such that λ_h *can* be determined by a discrete version of transmission condition (1.5), which we write as follows:

$$(1.7) \qquad a_h(\lambda_h, \mu) = b_h(\mu) \qquad \text{for all } \mu \in M_h.$$

In [26], where the hybridization of mixed methods was considered, the equation determining λ_h was called the *jump condition*. In our setting, it is called the *conservativity condition* to reflect the incorporation into the framework of DG and CG methods.

Note that all the methods in the unified framework provide approximations for $(\boldsymbol{q}, u)$ in the interior of the elements $K \in \mathcal{T}_h$, $(\boldsymbol{q}_h, u_h)$, as well as an approximation of u on the interior border of the elements λ_h; this is why they are called hybrid. This is in agreement with the definition of hybrid methods proposed in [22, p. 421]: "we may define more generally as a *hybrid* method any finite element method based on a formulation where one unknown is a function, or some of its derivatives, on the set Ω, and the other unknown is the trace of some of its derivatives of the same function, or the trace of the function itself, along the boundaries of the set K." Here K denotes a typical element of the triangulation. A long list of hybrid methods can be found in [22, 12, 51].

Of course, not every finite element method displays the above roughly described structure; in particular, it might not even be a hybrid method. However, many such methods can be rewritten as hybrid methods; this process is what can be called the hybridization of a finite element method. We say that we can hybridize a given finite element method if we can find a *hybrid* method (part) of whose solution *coincides* with the solution of the given method. The original finite element method is called *hybridizable*, and the hybrid method is then said to be a *hybridization* of the original method; for short, we call it a hybridized method. Next, we give a brief overview of the hybridization techniques of relevance for our purposes.

1.2. Hybridization of finite element methods. The first hybridization of a finite element method was proposed in 1965 [39] for a numerical method for solving the equations of linear elasticity. Perhaps because it was then intended as an implementation technique, the distinction between hybridization and *static condensation*, a widely known algebraic manipulation for size reduction of *already* assembled matrices, is seldom made in the engineering literature. However, in 1985 [4], hybridization was shown to be more than an implementation trick as it was proven that the new unknown λ_h, also interpreted to be the *Lagrange multiplier* associated with a continuity condition on the approximate flux, contains *extra* information about the exact solution. This was used to enhance the accuracy of the approximation by means of a local postprocessing [4, 11, 35]; see also [10].

After yet another two decades, a new perspective on hybridization emerged [26], and the characterization of the approximate trace λ_h as the solution of weak formulation (1.7) was introduced; this was done in the setting of the hybridization of the Raviart–Thomas (RT) and Brezzi–Douglas–Marini (BDM) mixed methods of arbitrary degree. The special case of the lowest order RT method had been previously considered in [21] within the framework of a study of the equivalence of mixed and nonconforming methods. In [26], it was shown that formulation (1.7) not only simplifies the task of assembling the stiffness matrix for the multiplier but can be used to establish unsuspected links between apparently unrelated mixed methods. It was also shown that it allows the devising and analysis of new, variable degree versions of those methods [27].

This new hybridization approach was later extended to finite element methods for the stationary Stokes equations using spaces of exactly divergence-free velocities; it was intended as an effective technique to bypass the extremely difficult construction of such spaces. It was successfully applied to a DG method [15] and to a mixed method for Stokes flow [28, 29]. For a review of these results, see [30]. Recently [31],

this hybridization approach was applied to the CG method to pave the way for the computation of an $H(\mathrm{div})$-conforming approximation of the flux from the CG solution.

1.3. Hybridization of DG methods. In this paper, we continue this effort and show how to hybridize a large class of DG methods. Thus, we show that their approximate solution $(\boldsymbol{q}_h, u_h)$ can be expressed as in (1.6) and that the approximate trace λ_h, which is nothing but the so-called numerical trace $\widehat{u}_h$ on the interelement boundaries (see [5]) satisfies weak formulation (1.7). In other words, we identify a class of DG methods whose globally coupled degrees of freedom are those of the numerical trace $\widehat{u}_h$ *only*; this results in an efficient implementation of these methods, as we argue below. In this way, the main disadvantages of DG methods for elliptic problems compared to other methods, namely, a higher number of globally coupled degrees of freedom for the same mesh and a lower sparsity of the corresponding stiffness matrices, are eliminated to a significant extent.

The simplest examples of such methods are obtained by using a DG method to define the local solvers and by taking what could be called the corresponding natural choice for the space M_h for the approximate trace λ_h. For example, we can use the local discontinuous Galerkin (LDG) method to define the local solvers and construct a hybridizable DG method. Surprisingly, it turns out that the resulting DG method is not an LDG method but one of the DG methods considered in [17]; see Corollary 3.2. A similar result holds for the hybridizable DG methods whose local solvers are the interior penalty (IP) method, that is, the resulting method is not the original IP method but the IP-like method considered in [38]; see Corollary 3.4. This is in sharp contrast with the RT, BDM, and CG methods, each of which can be hybridized by using as local solvers the RT, BDM, and CG methods, respectively.

It is interesting to note that the only known DG methods that turn out to be hybridizable by our technique are the following: a subset of the methods considered in [17], the minimal dissipation DG methods considered in [20], the minimal dissipation LDG method analyzed in [24], and the DG method considered in [38] and then rewritten as an IP method in [37]. With the exception of *some* LDG methods, *none* of the DG methods considered in the unified analysis of DG methods carried out in [5] is a hybridizable DG method. The reason is, roughly speaking, as follows. For all methods considered in [5], the variable $\boldsymbol{q}_h$ is easily eliminated from the equations due to the fact that the numerical trace $\widehat{u}_h$ is *independent* of $\boldsymbol{q}_h$ or $\mathbf{grad}u_h$; a *primal formulation* can then be found solely in terms of u_h. In contrast, in our approach, we eliminate *both* $\boldsymbol{q}_h$ and u_h from the equations and obtain a formulation in terms of $\widehat{u}_h$ only, namely, (1.7). For this, it turns out that we need $\widehat{u}_h$ to be *dependent* on $\boldsymbol{q}_h$ or $\mathbf{grad}u_h$, except for a few special LDG methods.

1.4. Properties of the algebraic system of hybridizable DG methods. As pointed out above, since the degrees of freedom of the functions μ in the finite element space M_h are associated with the borders of the elements only, the stiffness matrix associated with weak formulation (1.7) of the numerical trace $\widehat{u}_h = \lambda_h$ is significantly smaller than the one associated to the original variables $(\boldsymbol{q}_h, u_h)$. Moreover, the actual computation of the approximate solution of DG methods becomes competitive with that of hybridized mixed methods. For example, as we show below, on triangulations made of simplexes, the stiffness matrix associated with weak formulation (1.7) of any hybridizable DG method has the *same* size, block structure, and sparsity as the corresponding hybridized BDM [11] and RT [49] mixed methods; see [26] for details. Even more, it was recently proved (see [25, Property (iii) of Theorem 2.4]) that the stiffness matrices of the hybridized BDM and RT meth-

ods and the so-called *single face* hybridizable DG method are, in fact, *identical* provided $d = 0$.

1.5. New automatic coupling of different methods and mortaring techniques. One of the main features of the unified framework is that it allows for a single implementation of a vast class of finite element methods including DG, mixed, nonconforming, and CG methods and for their automatic coupling. Since it can be done even in the presence of nonmatching meshes, the unified framework provides a novel *coupling and mortaring* technique. This induces a *paradigm shift* in the way we view different finite element methods fitting in the framework, especially when considering adaptive algorithms. Indeed, since all these methods can be implemented within a single framework, the issue is now to investigate which method to use in what part of the domain in order to fully exploit its individual advantages. Let us briefly compare our new mortaring technique with the already established ones. Mortaring techniques (see the pioneering work [9]) were introduced to accommodate methods that can be defined in separate subdomains that could have been independently meshed. This technique introduces an auxiliary space for a Lagrange multiplier associated with a continuity constraint on the approximate solution. The resulting system could be written either as a saddle point problem, symmetric but indefinite [8], or as a nonconforming finite element approximation, which leads to a symmetric positive definite system; see, for example, [9, 42]. This classical mortaring is a powerful technique to achieve flexibility in the meshing and the choice of the finite element approximation. The work in this direction also includes coupling of mixed and CG [53], mixed and mixed finite element methods [2, 45], and DG and mixed methods [40].

However, this mortaring approach is very different from ours, since instead of enforcing the continuity of the approximation to u, we enforce a continuity condition on the approximation to the flux $\boldsymbol{q}$. The way of coupling and mortaring provided by the unified framework represents a simpler alternative to the above-mentioned mortaring techniques, as well as to earlier works on the coupling of CG and DG methods implicitly contained in [5] and explicitly emphasized in [48], as well as to the coupling of DG and mixed methods introduced in [23] and in [50].

1.6. Devising new methods. The unified framework provides a new point of view for constructing new methods. We provide three main examples of such methods. The first one is a family of methods well suited for hp-adaptivity and for dealing with nonmatching meshes. On each element $K \in \mathcal{T}_h$, it uses local solvers obtained from the RT, BDM, LDG, or CG methods by means of a suitable modification of the definition of the numerical trace of the flux of *some faces* of K only. For example, by modifying the numerical trace of the CG-H method on the element faces lying on the nonmatching interface, we allow the method to handle nonmatching grids. This method represents an alternative to the coupling of DG and CG methods proposed in [48].

The second example is a variable-degree RT method that can be used on some classes of nonconforming meshes. The third example is called the embedded DG (EDG) method; it was introduced in the setting of shell problems in [43]. An EDG method is obtained from an already existing hybridizable method by simply modifying the space M_h. This capability can be used as a new mortaring technique for dealing with nonmatching meshes, as we are going to see. Moreover, some EDG methods give rise to a stiffness matrix whose size and sparsity is exactly equal to that of the *statically condensed* stiffness matrix of the CG method, while retaining the stabilization mechanisms typical of DG methods; see [43]. As a consequence, EDG methods can immediately be incorporated into existing commercial codes. Related to EDG meth-

ods are the so-called multiscale DG methods [44, 14], which were introduced with a similar intention but a different approach.

1.7. Possibilities and recent developments. The unified framework could be used to establish a single a priori and a single a posteriori error analysis of all the methods fitting in it. It could be used to compare different methods or to establish new relations between them just as the unsuspected relation between the RT and the BDM methods in [26] was recently uncovered by comparing their hybridized versions. The framework could also be used to further explore the relation between mixed and nonconforming methods like the relation between the RT method of lowest order and a nonconforming method established in [4] and exploited in [47]. This work was later generalized in [1], where links between a variety of mixed and nonconforming methods were established; see also the references therein. Finally, the unifying framework can be used to devise new preconditioners based on, for example, substructuring techniques. However, in this paper, none of the above-mentioned issues will be investigated.

On the other hand, several discoveries induced by the unifying framework have already taken place. In particular, new DG methods which are more accurate and efficient than any other known DG method have been uncovered. Indeed, by exploiting the structure of the unified framework, a new DG method called the *single face, hybridizable* (SFH) DG method was constructed, which lies *in between* the RT and BDM methods; see [25]. It is the first known DG method, using polynomials of degree k for both $\boldsymbol{q}_h$ and u_h, proven to converge with order $k+1$ in *both* variables; all other DG methods converge with order k in the flux only. Moreover, the SFH method shares with the RT and BDM methods their remarkable superconvergence properties; this allow for the element–by–element computation of a new approximation $u_h^\star$ converging with order $k+2$. These results were then extended to other hybridizable DG methods in [33]. Therein, it was shown that in order to achieve the above-mentioned convergence properties, the interelement jumps of both unknowns have to be penalized essentially in the same way. This goes against the established belief that the interelement jumps of u_h need to be strongly penalized, while the interelement jumps of $\boldsymbol{q}_h$ need not be.

Also recently, a study of EDG methods obtained from hybridizable DG methods by forcing the numerical trace to be *continuous* has been carried out in [32]. It was proven that these EDG methods *lose* the above-mentioned convergence properties because the numerical trace $\widehat{\boldsymbol{q}}_h$ is not single valued. Moreover, numerical evidence was provided indicating that this loss of accuracy of the EDG method is not compensated by the computational advantage of having a reduced amount of globally coupled degrees of freedom. Hybridizable DG methods, with properly chosen penalization parameters, are thus more efficient than their EDG counterparts.

1.8. Organization of the paper. The paper is organized as follows. In section 2, we describe the general structure of the hybridized finite element methods and prove that the approximate trace λ_h is characterized as the solution of a weak formulation of the form (1.7); see Theorem 2.1. We then provide sufficient conditions for the existence and uniqueness of the solution λ_h; see Theorem 2.4. Further in this section we give some implementation details and compare the memory requirements of hybridizable methods with those of some classical DG methods. In section 3, we give several examples of hybridizable finite element methods. These include mixed methods using RT and BDM finite element spaces, a large variety of DG, CG, and some nonconforming finite element methods. In section 4, we build on the results of

the previous section and construct the above-mentioned novel hybridizable methods. Finally, in section 5, we conclude the paper with a few extensions and some final remarks.

2. The general framework of hybridization. In this section, we display the structure of hybridized finite element methods for second order elliptic problem (1.1). We begin by presenting the exact definition of the linear forms appearing in the weak formulation of the form (1.7), determining the approximate trace λ_h. We then provide sufficient conditions for the existence and uniqueness of λ_h and show that the assembly of the corresponding matrix equation can be done in a typical finite element fashion. We end by describing the sparsity structure of the stiffness matrix and comparing it with that of the stiffness matrices of the hybridized RT, IP, and LDG methods.

2.1. Notation. We use the notation used in [5]; let us recall it. Let $\mathcal{T}_h$ be a collection of disjoint elements that partition Ω. The shape of the elements is not important in this general framework. Moreover, triangulation $\mathcal{T}_h$ need not be conforming (we say that a triangulation $\mathcal{T}_h$ is conforming if whenever the intersection of the boundaries of any two elements has nonzero $(n-1)$-Lebesgue measure, the intersection is a face of each of the elements). So, $\mathcal{T}_h$ can be a collection of simplices, quadrilaterals, cubes, or a mixture of them which are not required to align across element interfaces. An interior "face" of $\mathcal{T}_h$ is any planar set e of positive $(n-1)$-dimensional measure of the form $e = \partial K^+ \cap \partial K^-$ for some two elements K^+ and K^- of the collection $\mathcal{T}_h$. (We use the word "face" even when $n = 2$.) We say that e is a boundary face if there is an element K of $\mathcal{T}_h$ such that $e = \partial K \cap \partial \Omega$ and the $(n-1)$-Lebesgue measure of e is not zero. Let $\mathcal{E}_h^\circ$ and $\mathcal{E}_h^\partial$ denote the set of interior and boundary faces of $\mathcal{T}_h$, respectively. We denote by $\mathcal{E}_h$ the union of all the faces in $\mathcal{E}_h^\circ$ and $\mathcal{E}_h^\partial$. In all our examples, elements of $\mathcal{E}_h^\circ$ and $\mathcal{E}_h^\partial$ are affine sets, although that is not required for the considerations in this section.

Finite element methods based on the mesh $\mathcal{T}_h$ typically use some finite-dimensional polynomial approximation spaces on each element of $\mathcal{T}_h$. On an element K, we denote by $\boldsymbol{V}(K)$ the polynomial space in which the flux $\boldsymbol{q}$ is approximated and by $W(K)$ the space in which the scalar solution u is approximated. The corresponding global finite element spaces are defined by

$$(2.1) \qquad \boldsymbol{V}_h = \{\boldsymbol{v} \ : \ \boldsymbol{v}|_K \in \boldsymbol{V}(K)\} \quad \text{and} \quad W_h = \{w \ : \ w|_K \in W(K)\}.$$

On an interior face $e = \partial K^+ \cap \partial K^-$, we consider scalar and vector functions that are, in general, double valued. For any discontinuous (scalar or vector) function q in W_h or $\boldsymbol{V}_h$, the trace $q|_e$ is a double-valued function, whose two branches are denoted by $(q|_e)_{K^+}$ and $(q|_e)_{K^-}$. To simplify the notation, we often shorten these to q_{K^+} and q_{K^-}, respectively. These branches are defined by $q_{K^\pm}(\boldsymbol{x}) = \lim_{\epsilon \downarrow 0} q(\boldsymbol{x} - \epsilon \boldsymbol{n}_{K^\pm})$ for all $\boldsymbol{x}$ in e. Here and elsewhere, $\boldsymbol{n}$ denotes the double-valued function of unit normals on $\mathcal{E}_h$, so on any face $e \subseteq \partial K$, $\boldsymbol{n}_K$ denotes the unit outward normal of K. The same notations are used for vector functions. For any double-valued vector function $\boldsymbol{r}$ on an interior face e, we define the *jump* of its normal component across the face e by

$$[\![\boldsymbol{r}]\!]_e := \boldsymbol{r}_{K^+} \cdot \boldsymbol{n}_{K^+} + \boldsymbol{r}_{K^-} \cdot \boldsymbol{n}_{K^-}.$$

On any face e of K lying on the boundary, we set

$$[\![\boldsymbol{r}]\!]_e := \boldsymbol{r}_K \cdot \boldsymbol{n}_K.$$

To simplify the exposition, we use $[\![r]\!]$ to denote the single-valued function on the entire set $\mathcal{E}_h$, which is equal to $[\![r]\!]_e$ on every face $e \in \mathcal{E}_h$. Similarly, for any $e \in \mathcal{E}_h^\circ$, we define

$$\{\!\{\xi\}\!\}_e = \frac{1}{2}(\xi_{K^+} + \xi_{K^-}), \qquad \{\!\{\boldsymbol{q}\}\!\}_e = \frac{1}{2}(\boldsymbol{q}_{K^+} + \boldsymbol{q}_{K^-}), \qquad [\![\xi]\!]_e = \xi_{K^+}\boldsymbol{n}_{K^+} + \xi_{K^-}\boldsymbol{n}_{K^-}.$$

For a boundary face e in $\mathcal{E}_h^\partial$, the operator $\{\!\{\cdot\}\!\}_e$ is also considered to be the identity, so that we can put together local operators $\{\!\{\cdot\}\!\}_e$ to form a global operator $\{\!\{\cdot\}\!\}$ on $\mathcal{E}_h$, just as we did for $[\![\cdot]\!]$.

Our notation for inner products is standard: For functions u and v in $L^2(D)$, we write $(u,v)_D = \int_D uv\, dx$ if D is a domain of $\mathbb{R}^n$ and $\langle u,\, v\rangle_D = \int_D uv\, dx$ if D is a domain of $\mathbb{R}^{n-1}$. To emphasize the mesh-dependent nature of certain integrals, we introduce the notation

$$(v,w)_{\mathcal{T}_h} = \sum_{K\in\mathcal{T}_h} (v,w)_K \qquad \text{and} \qquad \langle\mu,\,\lambda\rangle_{\mathcal{E}} = \sum_{e\in\mathcal{E}}\langle\mu,\,\lambda\rangle_e$$

for functions v, w and μ, λ defined on Ω and $\mathcal{E}_h$, respectively. Here $\mathcal{E}$ is any subset of $\mathcal{E}_h$.

2.2. The general structure of the methods. To describe the structure of the methods fitting in the unified framework, we mimic the characterization of the exact solution given in the Introduction.

Thus, we begin by choosing the space M_h of approximate traces, by taking the approximation to λ, λ_h, in

$$(2.2) \qquad\qquad M_h := \{\mu \in \mathsf{M}_h :\ \mu = 0 \text{ on } \partial\Omega\}$$

and by setting $g_h = I_h g$, where I_h is a suitably defined interpolation operator with image in M_h. Recall that g is the extension by zero of the Dirichlet data on $\partial\Omega$ to $\mathcal{E}_h^\circ$; see (1.2).

Next, we introduce a discrete version of local solvers (1.4a) and (1.4b). The first local solver maps each function m in M_h to the function $(\mathsf{Q}\mathsf{m}, \mathsf{U}\mathsf{m})$ on Ω, whose restriction to any mesh element K is in $\boldsymbol{V}(K) \times W(K)$ and satisfies the following discretization of (1.4a):

$$(2.3a) \qquad (c\,\mathsf{Q}\mathsf{m}, \boldsymbol{v})_K - (\mathsf{U}\mathsf{m}, \operatorname{div}\boldsymbol{v})_K = -\langle\mathsf{m},\, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} \qquad\qquad \text{for all } \boldsymbol{v}\in\boldsymbol{V}(K),$$

$$(2.3b) \qquad -(\mathbf{grad}\,w, \mathsf{Q}\mathsf{m})_K + \langle w,\, \widehat{\mathsf{Q}}\mathsf{m}\cdot\boldsymbol{n}\rangle_{\partial K} + (d\,\mathsf{U}\mathsf{m}, w)_K = 0 \text{ for all } w\in W(K).$$

Here $\widehat{\mathsf{Q}}\mathsf{m}$ represents the numerical trace of the flux, which is, in general, a double-valued function on $\mathcal{E}_h^\circ$. In inner products involving $\widehat{\mathsf{Q}}\mathsf{m}$ over a single simplex boundary ∂K, the integrand is assumed to be branch $(\widehat{\mathsf{Q}}\mathsf{m})_K$ from that simplex. In all examples we consider in this paper, numerical flux $\widehat{\mathsf{Q}}\mathsf{m}$ is either expressed explicitly in terms of $(\mathsf{Q}\mathsf{m}, \mathsf{U}\mathsf{m})$ or is an unknown function. In the examples where the latter case arises, we introduce the space in which the unknown $\widehat{\mathsf{Q}}\mathsf{m}$ lies and add new equations to render the resulting formulation uniquely solvable. At this point, however, the precise definition of $\widehat{\mathsf{Q}}\mathsf{m}$ is not essential, as we are solely interested in displaying the structure of the method for any $\widehat{\mathsf{Q}}\mathsf{m}$. Below, we formally require $\mathsf{m} \mapsto (\mathsf{Q}\mathsf{m}, \widehat{\mathsf{Q}}\mathsf{m}, \mathsf{U}\mathsf{m})$ to be a well-defined linear map; see Assumption 2.1.

The second local solver is a discretization of the second boundary value problem in (1.4b). It associates to any $f \in L^2(\Omega)$ the pair $(\mathsf{Q}f, \mathsf{U}f)$, whose restriction to each

element K is defined as the function in $\boldsymbol{V}(K) \times W(K)$ satisfying

(2.4a)
$$(c\,\boldsymbol{Q}f, \boldsymbol{v})_K - (\mathcal{U}f, \operatorname{div}\boldsymbol{v})_K = 0 \qquad\qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K),$$

(2.4b)
$$-(\mathbf{grad}w, \boldsymbol{Q}f)_K + \left\langle w\,, \widehat{\boldsymbol{Q}}f \cdot \boldsymbol{n} \right\rangle_{\partial K} + (d\,\mathcal{U}f, w)_K = (f, w)_K \quad \text{ for all } w \in W(K).$$

Just as for the first local solver, we leave undefined the numerical trace $\widehat{\boldsymbol{Q}}f$.

Obviously, while the functions $(\boldsymbol{Q}f, \mathcal{U}f)|_K$ and $(\boldsymbol{Q}\mathsf{m}, \mathcal{U}\mathsf{m})|_K$ are in $\boldsymbol{V}(K) \times W(K)$, the space in which $\widehat{\boldsymbol{Q}}f$ and $\widehat{\boldsymbol{Q}}\mathsf{m}$ lie will vary from example to example. Now we make our assumption about the local solvers.

Assumption 2.1 (existence and uniqueness of the local solvers). For every m in M_h, there is a unique set of functions of m, $(\boldsymbol{Q}\mathsf{m}, \widehat{\boldsymbol{Q}}\mathsf{m}, \mathcal{U}\mathsf{m})$ depending linearly on m and satisfying (2.3). Furthermore, for every f in $L^2(\Omega)$, there is a unique set of functions $(\boldsymbol{Q}f, \widehat{\boldsymbol{Q}}f, \mathcal{U}f)$ depending linearly on f and satisfying (2.4).

Each of the methods under consideration define an approximation to $(\boldsymbol{q}, u)$,

$$(2.5) \qquad (\boldsymbol{q}_h, u_h) = (\boldsymbol{Q}\lambda_h + \boldsymbol{Q}g_h + \boldsymbol{Q}f, \mathcal{U}\lambda_h + \mathcal{U}g_h + \mathcal{U}f) \in (\boldsymbol{V}_h \times W_h),$$

where λ_h is *assumed* to be *determined* by the following discrete version of transmission condition (1.5):

$$(2.6) \qquad \left\langle \mu\,, \left[\!\left[\widehat{\boldsymbol{Q}}\lambda_h + \widehat{\boldsymbol{Q}}g_h + \widehat{\boldsymbol{Q}}f\right]\!\right] \right\rangle_{\mathcal{E}_h} = 0 \qquad \text{for all } \mu \in M_h.$$

If we define the numerical flux by

$$(2.7) \qquad\qquad\qquad \widehat{\boldsymbol{q}}_h := \widehat{\boldsymbol{Q}}\lambda_h + \widehat{\boldsymbol{Q}}g_h + \widehat{\boldsymbol{Q}}f,$$

and if the (extension by zero to $\mathcal{E}_h$ of the) function $[\![\widehat{\boldsymbol{q}}_h]\!]|_{\mathcal{E}_h^\circ}$ belongs to the space M_h, then condition (2.6) is simply stating that $[\![\widehat{\boldsymbol{q}}_h]\!]|_{\mathcal{E}_h^\circ} = 0$ pointwise, that is, the normal component of the numerical trace $\widehat{\boldsymbol{q}}_h$ is single valued, or, adopting the terminology of [5], the function $\widehat{\boldsymbol{q}}_h$ is a *conservative* numerical flux. It is for this reason we call (2.6) the conservativity condition. If the function $[\![\widehat{\boldsymbol{q}}_h]\!]|_{\mathcal{E}_h^\circ}$ does not belong to the space M_h, the conservativity condition imposes only the weak continuity of the normal component of the numerical trace $\widehat{\boldsymbol{q}}_h$, which, as a consequence, is not single valued.

It is worth noting that the method just described can be viewed as seeking the approximation $(\boldsymbol{q}_h, u_h, \lambda_h)$ in $\boldsymbol{V}_h \times W_h \times M_h$ satisfying

(2.8a)
$$(c\,\boldsymbol{q}_h, \boldsymbol{r})_{\mathcal{T}_h} - (u_h, \operatorname{div}\boldsymbol{r})_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} \langle \lambda_h\,, \boldsymbol{r} \cdot \boldsymbol{n}\rangle_{\partial K \backslash \partial\Omega} = -\langle g_h\,, \boldsymbol{r} \cdot \boldsymbol{n}\rangle_{\partial\Omega} \text{ for all } \boldsymbol{r} \in \boldsymbol{V}_h,$$

(2.8b)
$$-(\boldsymbol{q}_h, \mathbf{grad}w)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\,, w\rangle_{\partial K} + (d\,u_h, w)_{\mathcal{T}_h} = (f, w)_{\mathcal{T}_h} \qquad \text{for all } w \in W_h,$$

$$(2.8c) \qquad\qquad \sum_{K \in \mathcal{T}_h} \langle \mu, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\rangle_{\partial K} = 0 \qquad\qquad \text{for all } \mu \in M_h.$$

Note that the first two equations are used to define local solvers (2.3) and (2.4), while the last is nothing but conservativity condition (2.6). This type of method is sometimes called a hybrid dual-mixed method. As pointed out in the Introduction, it is

called mixed because we seek approximations for the flux $\boldsymbol{q}_h$, as well as the potential u_h, on Ω. It is called hybrid dual because the approximate trace λ_h associated to the conservativity condition is an approximation for the trace of the potential u on the boundaries of the elements.

Many hybridized finite element methods admit this structure. For example, some classic hybridized mixed methods [4, 26] are obtained by an appropriate choice of the local spaces and by choosing $\widehat{\mathbf{Q}}(\cdot)$ in such a way that we have $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h$. Many DG methods also fall into this form—although not all of them are hybridizable. Indeed, the schemes considered in the unified analysis of DG methods in [5] can be written in our notation as

$$(c\,\boldsymbol{q}_h, \boldsymbol{v})_{\mathcal{T}_h} - \sum_{K \in \mathcal{T}_h} (u_h, \operatorname{div} \boldsymbol{v})_K + \sum_{K \in \mathcal{T}_h} \langle \widehat{u}_h\,,\, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K \backslash \partial \Omega} = -\langle g_h\,,\, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial \Omega},$$

$$-(\mathbf{grad}\, w, \boldsymbol{q}_h)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} \langle w\,,\, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} \rangle_{\partial K} + (d\, u_h, w)_{\mathcal{T}_h} = (f, w)_{\mathcal{T}_h},$$

where $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$ are the so-called *numerical traces* of the DG method. Comparing these equations with (2.8) of our general framework, we immediately realize that $\widehat{u}_h = \lambda_h$ on $\mathcal{E}_h^\circ$. We thus see that, for a finite element method to be hybridizable, its numerical trace $\widehat{u}_h$ *must* be single valued. This implies, in particular, that the DG methods in [5] that are not *adjoint consistent* cannot be hybridized by using our technique. In contrast, the (normal component of the) numerical trace $\widehat{\boldsymbol{q}}_h$ is *not* required to be single valued, since conservativity condition (2.6) does not always ensure a single-valued numerical trace. Thanks to this flexibility, the CG method and the EDG methods turn out to be hybridizable.

This concludes the description of the general structure of the methods. Methods with this structure include a wide class of DG and hybridized mixed and CG methods, as we show in sections 3, 4, and 5.

2.3. The characterization of the variable λ_h. As we see next, the relevance of the methods fitting the previously described general structure resides in the fact that the λ_h can be characterized in terms of a simple weak formulation in which none of the other variables appear.

THEOREM 2.1. *Suppose Assumption 2.1 on the existence and uniqueness of the local solvers holds. Then $\lambda_h \in M_h$ satisfies conservativity condition (2.6) if and only if it satisfies*

$$(2.9) \qquad a_h(\lambda_h, \mu) = b_h(\mu) \quad \text{for all } \mu \in M_h,$$

where

$$a_h(\eta, \mu) = (c\,\mathbf{Q}\eta, \mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\, \mathcal{U}\mu)_{\mathcal{T}_h} + \left\langle 1\,,\, \left[\!\left[(\mathcal{U}\mu - \mu) \left(\widehat{\mathbf{Q}}\eta - \mathbf{Q}\eta \right) \right]\!\right] \right\rangle_{\mathcal{E}_h},$$

$$b_h(\mu) = \left\langle g_h\,,\, \left[\!\left[\widehat{\mathbf{Q}}\mu \right]\!\right] \right\rangle_{\mathcal{E}_h} + (f, \mathcal{U}\mu)_{\mathcal{T}_h} - \left\langle 1\,,\, \left[\!\left[(\mathcal{U}\mu - \mu) \left(\widehat{\mathbf{Q}}f - \mathbf{Q}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h}$$

$$+ \left\langle 1\,,\, \left[\!\left[\mathcal{U}f \left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu \right) \right]\!\right] \right\rangle_{\mathcal{E}_h}$$

$$- \left\langle 1\,,\, \left[\!\left[(\mathcal{U}\mu - \mu) \left(\widehat{\mathbf{Q}}g_h - \mathbf{Q}g_h \right) \right]\!\right] \right\rangle_{\mathcal{E}_h}$$

$$+ \left\langle 1\,,\, \left[\!\left[(\mathcal{U}g_h - g) \left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu \right) \right]\!\right] \right\rangle_{\mathcal{E}_h}$$

for all η and $\mu \in M_h$.

Note that, since λ_h is an approximation of the function u on $\mathcal{E}_h^\circ$, it is natural to expect bilinear form $a_h(\cdot, \cdot)$ to be symmetric. This motivates the following observation. Bilinear form $a_h(\cdot, \cdot)$ is symmetric if and only if numerical trace $\widehat{\mathbf{Q}}\cdot$ is such that

$$(2.10\text{a}) \qquad \left\langle 1, \; \left[\!\left[(\mathcal{U}\mu - \mu)\left(\widehat{\mathbf{Q}}\eta - \mathbf{Q}\eta\right)\right]\!\right] \right\rangle_{\mathcal{E}_h} = \left\langle 1, \; \left[\!\left[(\mathcal{U}\eta - \eta)\left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu\right)\right]\!\right] \right\rangle_{\mathcal{E}_h}$$

for all $\eta, \mu \in M_h$. If we also have

$$(2.10\text{b}) \qquad \left\langle 1, \; \left[\!\left[(\mathcal{U}\mu - \mu)\left(\widehat{\mathbf{Q}}f - \mathbf{Q}f\right)\right]\!\right] \right\rangle_{\mathcal{E}_h} = \left\langle 1, \; \left[\!\left[\mathcal{U}f\left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu\right)\right]\!\right] \right\rangle_{\mathcal{E}_h},$$

then

$$b_h(\mu) = \left\langle g_h, \; \left[\!\left[\widehat{\mathbf{Q}}\mu\right]\!\right] \right\rangle_{\mathcal{E}_h} + (f, \mathcal{U}\mu)_\Omega.$$

All the examples in this paper satisfy the above symmetry conditions.

Now we prove Theorem 2.1. Set

$$(2.11\text{a}) \qquad\qquad a_h(\lambda_h, \mu) = -\left\langle \mu, \; \left[\!\left[\widehat{\mathbf{Q}}\lambda_h\right]\!\right] \right\rangle_{\mathcal{E}_h},$$

$$(2.11\text{b}) \qquad\qquad b_h(\mu) = \left\langle \mu, \; \left[\!\left[\widehat{\mathbf{Q}}g_h + \widehat{\mathbf{Q}}f\right]\!\right] \right\rangle_{\mathcal{E}_h}$$

so that conservativity condition (2.6) takes the form (2.9). Theorem 2.1 then follows from the following result.

LEMMA 2.2 (elementary identities). *We have, for any* $\mathsf{m}, \mu \in \mathsf{M}_h$ *and* $f \in L^2(\Omega)$,

$$(\text{i}) \qquad -\left\langle \mu, \; \left[\!\left[\widehat{\mathbf{Q}}\mathsf{m}\right]\!\right] \right\rangle_{\mathcal{E}_h} = (c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mu)_\Omega + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mu)_\Omega$$
$$+ \left\langle 1, \; \left[\!\left[(\mathcal{U}\mu - \mu)\left(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m}\right)\right]\!\right] \right\rangle_{\mathcal{E}_h},$$

$$(\text{ii}) \qquad -\left\langle \mu, \; \left[\!\left[\widehat{\mathbf{Q}}g_h\right]\!\right] \right\rangle_{\mathcal{E}_h} = -\left\langle g_h, \; \left[\!\left[\widehat{\mathbf{Q}}\mu\right]\!\right] \right\rangle_{\mathcal{E}_h}$$
$$+ \left\langle 1, \; \left[\!\left[(\mathcal{U}\mu - \mu)\left(\widehat{\mathbf{Q}}g_h - \mathbf{Q}g_h\right)\right]\!\right] \right\rangle_{\mathcal{E}_h}$$
$$- \left\langle 1, \; \left[\!\left[(\mathcal{U}g_h - g_h)\left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu\right)\right]\!\right] \right\rangle_{\mathcal{E}_h},$$

$$(\text{iii}) \qquad -\left\langle \mu, \; \left[\!\left[\widehat{\mathbf{Q}}f\right]\!\right] \right\rangle_{\mathcal{E}_h} = -(f, \mathcal{U}\mu)_{\mathcal{T}_h}$$
$$+ \left\langle 1, \; \left[\!\left[(\mathcal{U}\mu - \mu)\left(\widehat{\mathbf{Q}}f - \mathbf{Q}f\right)\right]\!\right] \right\rangle_{\mathcal{E}_h}$$
$$- \left\langle 1, \; \left[\!\left[\mathcal{U}f\left(\widehat{\mathbf{Q}}\mu - \mathbf{Q}\mu\right)\right]\!\right] \right\rangle_{\mathcal{E}_h}.$$

To prove Lemma 2.2, we need some identities which follow from the equations defining the local solvers by integration by parts.

LEMMA 2.3 (relation between jumps and local residuals). *For any* $\mathsf{m}, \mu \in \mathsf{M}_h$, $f \in L^2(\Omega)$, $\boldsymbol{v} \in \boldsymbol{V}_h$, *and* $w \in W_h$, *the following identities hold:*

$$(2.12\text{a}) \qquad (c\,\mathbf{Q}\mathsf{m} + \mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_{\mathcal{T}_h} = + \left\langle 1, \; [\![(\mathcal{U}\mathsf{m} - \mathsf{m})\,\boldsymbol{v}]\!] \right\rangle_{\mathcal{E}_h},$$

$$(2.12\text{b}) \qquad (\operatorname{div}\mathbf{Q}\mathsf{m} + d\,\mathcal{U}\mathsf{m}, w)_{\mathcal{T}_h} = - \left\langle 1, \; \left[\!\left[w\left(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m}\right)\right]\!\right] \right\rangle_{\mathcal{E}_h},$$

$$(2.12\text{c}) \qquad (c\,\mathbf{Q}f + \mathbf{grad}\,\mathcal{U}f, \boldsymbol{v})_{\mathcal{T}_h} = + \left\langle 1, \; [\![\mathcal{U}f\,\boldsymbol{v}]\!] \right\rangle_{\mathcal{E}_h},$$

$$(2.12\text{d}) \qquad (\operatorname{div}\mathbf{Q}f + d\,\mathcal{U}f - f, w)_{\mathcal{T}_h} = - \left\langle 1, \; \left[\!\left[w\left(\widehat{\mathbf{Q}}f - \mathbf{Q}f\right)\right]\!\right] \right\rangle_{\mathcal{E}_h}.$$

Using these identities, we now prove Lemma 2.2.

Proof. Let us prove identity (i) of Lemma 2.2. We have

$$
\begin{aligned}
-\left\langle \mu,\, \left[\!\left[\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} \right]\!\right] \right\rangle_{\mathcal{E}_h} &= -\left\langle \mu,\, \left[\!\left[\mathbf{\mathcal{Q}}\mathsf{m} \right]\!\right] \right\rangle_{\mathcal{E}_h} - \left\langle \mu,\, \left[\!\left[\left(\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} - \mathbf{\mathcal{Q}}\mathsf{m} \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \\
&= (c\,\mathbf{\mathcal{Q}}\mu, \mathbf{\mathcal{Q}}\mathsf{m})_{\mathcal{T}_h} - (\mathcal{U}\mu, \operatorname{div}\mathbf{\mathcal{Q}}\mathsf{m})_{\mathcal{T}_h} \\
&\quad - \left\langle \mu,\, \left[\!\left[\left(\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} - \mathbf{\mathcal{Q}}\mathsf{m} \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.3a),} \\
&= (c\,\mathbf{\mathcal{Q}}\mu, \mathbf{\mathcal{Q}}\mathsf{m})_{\mathcal{T}_h} + (d\,\mathcal{U}\mathsf{m},\, \mathcal{U}\mu)_{\mathcal{T}_h} \\
&\quad + \left\langle 1,\, \left[\!\left[\mathcal{U}\mu \left(\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} - \mathbf{\mathcal{Q}}\mathsf{m} \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \\
&\quad - \left\langle \mu,\, \left[\!\left[\left(\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} - \mathbf{\mathcal{Q}}\mathsf{m} \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.12b).}
\end{aligned}
$$

This proves identity (i) of Lemma 2.2.

Now we prove identity (ii) of Lemma 2.2. To do that, note that, by identity (i) of Lemma 2.2, the bilinear form

$$
B(\mathsf{m}, \mu) = \left\langle \mu,\, \left[\!\left[\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} \right]\!\right] \right\rangle_{\mathcal{E}_h} + \left\langle 1,\, \left[\!\left[\left(\mathcal{U}\mu - \mu \right) \left(\widehat{\mathbf{\mathcal{Q}}}\mathsf{m} - \mathbf{\mathcal{Q}}\mathsf{m} \right) \right]\!\right] \right\rangle_{\mathcal{E}_h}
$$

is symmetric. As a consequence, identity (ii) of Lemma 2.2 follows from equality $B(\mu, g_h) = B(g_h, \mu)$.

Finally, we prove identity (iii) of Lemma 2.2. We have

$$
\begin{aligned}
-\left\langle \mu,\, \left[\!\left[\widehat{\mathbf{\mathcal{Q}}}f \right]\!\right] \right\rangle_{\mathcal{E}_h} &= -\left\langle \mu,\, \left[\!\left[\mathbf{\mathcal{Q}}f \right]\!\right] \right\rangle_{\mathcal{E}_h} - \left\langle \mu,\, \left[\!\left[\left(\widehat{\mathbf{\mathcal{Q}}}f - \mathbf{\mathcal{Q}}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \\
&= (c\,\mathbf{\mathcal{Q}}\mu, \mathbf{\mathcal{Q}}f)_{\mathcal{T}_h} - (\mathcal{U}\mu, \operatorname{div}\mathbf{\mathcal{Q}}f)_{\mathcal{T}_h} \\
&\quad - \left\langle \mu,\, \left[\!\left[\left(\widehat{\mathbf{\mathcal{Q}}}f - \mathbf{\mathcal{Q}}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.3a),} \\
&= -(f, \mathcal{U}\mu)_{\mathcal{T}_h} + (c\,\mathbf{\mathcal{Q}}\mu, \mathbf{\mathcal{Q}}f)_{\mathcal{T}_h} + (d\,\mathcal{U}\mu,\, \mathcal{U}f)_{\mathcal{T}_h} \\
&\quad + \left\langle 1,\, \left[\!\left[\left(\mathcal{U}\mu - \mu \right) \left(\widehat{\mathbf{\mathcal{Q}}}f - \mathbf{\mathcal{Q}}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.12d),} \\
&= -(f, \mathcal{U}\mu)_{\mathcal{T}_h} + (\operatorname{div}\mathbf{\mathcal{Q}}\mu, \mathcal{U}f)_{\mathcal{T}_h} + (d\,\mathcal{U}\mu,\, \mathcal{U}f)_{\mathcal{T}_h} \\
&\quad + \left\langle 1,\, \left[\!\left[\left(\mathcal{U}\mu - \mu \right) \left(\widehat{\mathbf{\mathcal{Q}}}f - \mathbf{\mathcal{Q}}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.4a),} \\
&= -(f, \mathcal{U}\mu)_{\mathcal{T}_h} - \left\langle 1,\, \left[\!\left[\mathcal{U}f \left(\widehat{\mathbf{\mathcal{Q}}}\mu - \mathbf{\mathcal{Q}}\mu \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \\
&\quad + \left\langle 1,\, \left[\!\left[\left(\mathcal{U}\mu - \mu \right) \left(\widehat{\mathbf{\mathcal{Q}}}f - \mathbf{\mathcal{Q}}f \right) \right]\!\right] \right\rangle_{\mathcal{E}_h} \quad \text{by (2.12b).}
\end{aligned}
$$

This completes the proof of Lemma 2.2. □

2.4. Sufficient conditions for the existence and uniqueness of $\boldsymbol{\lambda_h}$. Next, we provide two conditions which are sufficient for the existence and uniqueness of λ_h. The first is a condition on the local solvers, and the second is a condition on the relation between the local solvers, on each element K of triangulation $\mathcal{T}_h$ and the global space M_h of approximate traces. It is worth emphasizing that, by guaranteeing the existence and uniqueness of λ_h, these simple conditions ensure the *automatic* coupling of the different local solvers even across nonmatching meshes. Note that no explicit conditions on triangulation $\mathcal{T}_h$ are involved in these conditions.

Assumption 2.2 (on the positive semidefiniteness of the local solvers). The local solvers and the numerical flux traces in (2.3) and (2.4) are such that, for every $K \in \mathcal{T}_h$, the following holds:

$$(2.13a) \qquad -\left\langle \mu, \widehat{\mathbf{Q}}\mu \cdot \boldsymbol{n} \right\rangle_{\partial K} \geq 0 \qquad \text{for all } \mu \in \mathsf{M}_h.$$

Moreover, there exits a space $M(\partial K)$ containing the set $\{\nu : \nu|_e \in \mathcal{P}_0(e)$ on each face $e \in \mathcal{E}_h^\circ$ lying on $\partial K\}$ such that

$$(2.13b) \qquad \text{if } \left\langle \mu, \widehat{\mathbf{Q}}\mu \cdot \boldsymbol{n} \right\rangle_{\partial K} = 0 \text{ for some } \mu \in \mathsf{M}_h, \text{ then } P_{\partial K}\mu = C_K$$

for some constant C_K, where $P_{\partial K}$ is the $L^2(\partial K)$-orthogonal projection onto $M(\partial K)$.

Note that auxiliary space $M(\partial K)$ is not necessarily finite-dimensional. Its use is only theoretical; it is not used in practice in any way.

Let us argue that (2.13) is a reasonable condition on the positive semidefiniteness of the bilinear forms corresponding to the local solvers. Indeed, taking $\boldsymbol{v} := \mathbf{Q}\mu$ in (2.3a), $\mathsf{m} := \mu$ and $w := \mathcal{U}\mathsf{m}$ in (2.3b), and adding the equations, we get

$$(2.14)$$
$$-\left\langle \mathsf{m}, \widehat{\mathbf{Q}}\mu \cdot \boldsymbol{n} \right\rangle_{\partial K} = (c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mu)_K + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mu)_K + \left\langle \left(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m}\right) \cdot \boldsymbol{n}, \mathcal{U}\mu - \mu \right\rangle_{\partial K}$$
$$=: a_{h,K}(\mathsf{m}, \mu).$$

Thus, (2.13a) ensures that bilinear form $a_{h,K}(\cdot, \cdot)$, which coincides with form $a_h(\cdot, \cdot)$ when Ω is single element K, is positive semidefinite. Further, condition (2.13b) states that those functions $\mathsf{m} \in \mathsf{M}_h$ for which $a_{h,K}(\mathsf{m}, \mathsf{m}) = 0$ yield constants under an appropriate projection. This is a reasonable assumption, since it is a discrete version of a similar property of the exact solution. Indeed, for the exact solution, such a condition readily implies that $\mathbf{Q}\mathsf{m} = 0$ and, by (1.4a), that $\mathsf{m} = \mathcal{U}\mathsf{m} = constant$ on ∂K.

This argument suggests that it is reasonable to expect projection $P_{\partial K}$ to be strongly related to the *identity*, at least in parts of ∂K. The following assumption captures this property. It will allow us to establish a link between the different local solvers and, in so doing, to ensure the uniqueness of the solution of (1.7).

Assumption 2.3 (the "gluing condition"). *If $\mu \in \mathsf{M}_h$, then on every interior face $e = \partial K^+ \cap \partial K^-$, either $\mu = P_{\partial K^+}\mu$ or $\mu = P_{\partial K^-}\mu$.*

We are now ready to state our result.

THEOREM 2.4 (existence and uniqueness of λ_h). *If Assumption* 2.1 *on the existence and the uniqueness of the local solvers, Assumption* 2.2 *on the positive semidefiniteness of the local solvers, and Assumption* 2.3, *the gluing condition, hold, then there is a unique solution λ_h of weak formulation* (2.9).

Proof. By Theorem 2.1, Assumption 2.1 guarantees the existence and the uniqueness of $\widehat{\mathbf{Q}}\lambda_h$. Therefore, system (2.9) is well defined. Since it is a square system, to prove the existence and the uniqueness of its solution, it is enough to show that if $a_h(\mu, \mu) = 0$ for some $\mu \in M_h$, we have that $\mu = 0$.

By Lemma 2.2,

$$a_h(\mu, \mu) = -\left\langle \mu, \left[\!\left[\widehat{\mathbf{Q}}\mu\right]\!\right] \right\rangle_{\mathcal{E}_h} = -\sum_{K \in \mathcal{T}_h} \left\langle \mu, \widehat{\mathbf{Q}}\mu \cdot \boldsymbol{n} \right\rangle_{\partial K}.$$

Now, since $a_h(\mu, \mu) = 0$, by (2.13a) of Assumption 2.2 on on the positive semidefinite-ness of the local solvers, each of the summands on the right-hand side must vanish. Thus,

$$\left\langle \mu \,,\, \widehat{\mathbf{Q}}\mu \cdot \boldsymbol{n} \right\rangle_{\partial K} = 0 \quad \text{for all } K \in \mathcal{T}_h.$$

By condition (2.13b), on any interior face $e = \partial K^+ \cap \partial K^-$, this implies

$$C_{K^+} = P_{\partial K^+}\mu = \frac{1}{|e|}\langle \mu \,,\, 1 \rangle_e = P_{\partial K^-}\mu = C_{K^-},$$

and by Assumption 2.3 (the gluing condition), we conclude that $C_{K^+} = \mu = C_{K^-}$ on the face e. This means that μ is a constant on $\mathcal{E}_h$. Since $\mu = 0$ on $\partial\Omega$, we see that μ is identically equal to zero on $\mathcal{E}_h$. This completes the proof. $\quad\square$

2.5. The sparsity structure of the stiffness matrix for $\boldsymbol{\lambda_h}$. Next, we comment on the sparsity structure of the stiffness matrix associated with weak formulation (1.7). For any given basis of the space of approximate traces M_h, we denote by $[\mu]$ the corresponding vector of coefficients of the representation of μ in a given basis of M_h. Then, weak formulation (2.9)

$$A\,[\lambda_h] = b,$$

where

$$[\mu]^t\, A\,[\lambda_h] = a_h(\lambda_h, \mu) \quad \text{and} \quad [\mu]^t\, b = b_h(\mu).$$

Now, by (2.11),

$$a_h(\eta, \mu) = -\sum_{K \in \mathcal{T}_h} \left\langle \mu \,,\, \widehat{\mathbf{Q}}\eta \cdot \boldsymbol{n} \right\rangle_{\partial K} \quad \text{and} \quad b_h(\mu) = \sum_{K \in \mathcal{T}_h} \left\langle \mu \,,\, \left(\widehat{\mathbf{Q}}f + \widehat{\mathbf{Q}}g_h\right) \cdot \boldsymbol{n} \right\rangle_{\partial K},$$

we have that

$$A = \sum_{K \in \mathcal{T}_h} A_K \quad \text{and} \quad b = \sum_{K \in \mathcal{T}_h} b_K,$$

where A_K and b_K are defined by

$$[\mu]^t\, A_K\,[\eta] = -\left\langle \mu \,,\, \widehat{\mathbf{Q}}\eta \cdot \boldsymbol{n} \right\rangle_{\partial K} \quad \text{and} \quad [\mu]^t\, b_K = \left\langle \mu \,,\, \left(\widehat{\mathbf{Q}}f + \widehat{\mathbf{Q}}g_h\right) \cdot \boldsymbol{n} \right\rangle_{\partial K}.$$

Thus, the matrix equations for the multiplier can be obtained in a typical finite element manner. Moreover, the sparsity of the matrices A_K and b_K can be deduced from the following result.

PROPOSITION 2.1. *Suppose Assumption* 2.1 *on the existence and the uniqueness of the local solvers holds. Then*
   (i) *if the support of $\mu \in M_h$ does not intersect ∂K, we have that $[\mu]^t\, b_K = 0$;*
   (ii) *if the support of $\mu \in M_h$ or the support of $\eta \in M_h$ does not intersect ∂K, we have that $[\mu]^t\, A_K\,[\eta] = 0$.*
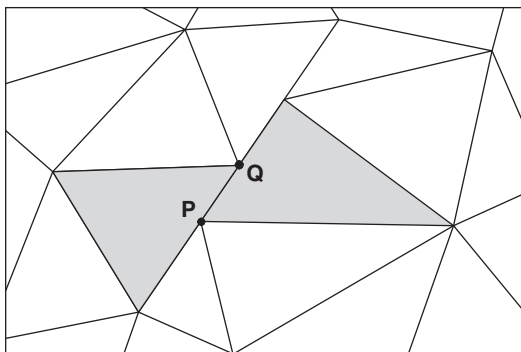
Fig. 2.1. *Interior edge* $e = \mathbf{PQ}$ *and the support of local solver* $(\mathbf{Q}\mathsf{m}, \mathbf{U}\mathsf{m})$ *for any* m *supported on* e. *Numerical trace* $(\widehat{\mathbf{Q}}\mathsf{m})_K$ *is generally nontrivial on the boundary of the two shadowed triangles* K, *but it vanishes on the boundary of other triangles.*

Proof. That $[\mu]^t b_K = 0$ and $[\mu]^t A_K [\eta] = 0$ if the support of μ does not intersect ∂K follows immediately from the definition of b_K and A_K. Let us show that $[\mu]^t A_K [\eta] = 0$ if the support of η does not intersect ∂K. Since we are assuming that the local solvers are well defined, if the support of η does not intersect ∂K, we have, by Assumption 2.1, that $(\widehat{\mathbf{Q}}\eta)_K = 0$ on ∂K, and the result follows. This completes the proof. $\square$

We emphasize that this result, illustrated in Figure 2.1, is possible due to the fact that numerical trace $\widehat{\mathbf{Q}}\cdot$ is *double valued* on all interior faces $e \in \mathcal{E}_h^\circ$. Indeed, take η as in the above proof and further assume that its support intersects $\partial K'$, where the intersection of ∂K and $\partial K'$ is a face e in $\mathcal{E}_h^\circ$. Then $(\widehat{\mathbf{Q}}\eta)_{K'}$ can be nontrivial on e, in general. However, this does not contradict the fact that $(\widehat{\mathbf{Q}}\eta)_K = 0$ on e because the function $\widehat{\mathbf{Q}}\eta$ is double valued on e.

In the remainder of this subsection, we compare the number of globally coupled degrees of freedom and the number of nonzero entries of the stiffness matrix, restricting our attention to the case of a conforming triangulation $\mathcal{T}_h$ (no hanging nodes). First, consider the case in which $\mathsf{M}_h := \mathcal{M}_{h,k}^c$, where

$$\mathcal{M}_{h,k}^c := \{\mu \in \mathcal{C}(\mathcal{E}_h) : \ \mu|_e \in \mathcal{P}_k(e) \text{ for all faces } e \in \mathcal{E}_h\}.$$

Here, $\mathcal{C}(\mathcal{E}_h)$ denotes the space of continuous functions on $\mathcal{E}_h$ and $\mathcal{P}_k(D)$ the set of polynomials of degree at most k on a domain D. Then the sparsity structure of the matrix A is exactly that of the statically condensed stiffness matrix of a CG method using approximations whose restriction to each simplex K is in $\mathcal{P}_k(K)$.

If, instead, we take $\mathsf{M}_h := \mathcal{M}_{h,k}$, where

$$(2.15) \qquad \mathcal{M}_{h,k} = \left\{\mu \in L^2(\mathcal{E}_h) : \ \mu|_e \in \mathcal{P}_k(e) \text{ for all faces } e \in \mathcal{E}_h^\circ\right\},$$

then by choosing basis functions whose support is always contained in a single face, we obtain matrix A, which has a block structure with square blocks of order equal to the dimension of $\mathcal{P}_k(e)$. The number of block rows and block columns is equal to the number of interior faces of triangulation $N_{\text{i.f.}}$, and, on each block row, there are at most $(2n+1)$ blocks that are not equal to zero. In other words, the size and sparsity structure of matrix A is precisely that of the stiffness matrix for the hybridized RT method using M_h as space of approximate traces; see [26]. This means that the order

TABLE 2.1
Comparison between hybridizable DG methods and two typical DG methods on simplicial meshes.

n	k	$R_{\text{d.o.f.}}$	R_{sparsity}		n	k	$R_{\text{d.o.f.}}$	R_{sparsity}	
			IP	LDG				IP	LDG
2	1	**1.00**	**1.20**	**3.00**	3	1	0.67	0.63	**2.16**
	2	**1.33**	**2.13**	**5.33**		2	0.83	0.99	**3.37**
	3	**1.67**	**3.33**	**8.33**		3	**1.00**	**1.42**	**4.86**
	4	**2.00**	**4.80**	**12.00**		4	**1.17**	**1.94**	**6.61**

of matrix A, which is equal to the number of degrees of freedom of λ_h, is given by

$$N_{\text{d.o.f.}} = N_{\text{i.f.}} \dim \mathcal{P}_k(e)$$

and that the number of possibly nonvanishing entries of A is bounded by

$$N_{\text{sparsity}} = N_{\text{i.f.}} (2n+1) (\dim \mathcal{P}_k(e))^2.$$

Let us now compare the size and sparsity structure of this stiffness matrix with those of the IP and the (Schur-complement matrix of the) LDG methods that use polynomials of degree k. The number of globally coupled degrees of freedom for both methods is

$$N_{\text{d.o.f.}}^{\text{IP}} = N_{\text{d.o.f.}}^{\text{LDG}} = N_s \dim \mathcal{P}_k(K),$$

where N_s denotes the number of simplexes of the triangulation. Moreover, the stiffness matrices in question have a block structure with square blocks of order equal to the dimension of $\mathcal{P}_k(K)$. On each block-row, the number of blocks that are not equal to zero are at most $(n+2)$ for the IP method and $((n+1)^2+1)$ for the LDG method; recall that, for the LDG method, the degrees of freedom of the neighbors of the neighbors are also involved. This means that the number of nonzero entries of the corresponding stiffness matrices are (bounded by)

$$N_{\text{sparsity}}^{\text{IP}} = N_s (n+2) (\dim \mathcal{P}_k(K))^2, \quad N_{\text{sparsity}}^{\text{LDG}} = N_s ((n+1)^2 + 1) (\dim \mathcal{P}_k(K))^2.$$

To compare with the hybridized methods, we consider the ratio of the number of globally coupled degrees of freedom $R_{\text{d.o.f.}} := N_{\text{d.o.f.}}^{\text{DG}}/N_{\text{d.o.f.}}$ and the ratio of the number of entries different from zero $R_{\text{sparsity}}^{\text{IP}} := N_{\text{sparsity}}^{\text{IP}}/N_{\text{sparsity}}$ and $R_{\text{sparsity}}^{\text{LDG}} := N_{\text{sparsity}}^{\text{LDG}}/N_{\text{sparsity}}$. Since $N_s/N_{\text{i.f.}} \approx 2/(n+1)$ (up to a lower order term related to the faces on the boundary), then

$$R_{\text{sparsity}}^{\text{IP}} = \frac{2(n+2)}{(n+1)(2n+1)} \left(\frac{k}{n}+1\right)^2, \quad R_{\text{sparsity}}^{\text{LDG}} = \frac{2((n+1)^2+1)}{(n+1)(2n+1)} \left(\frac{k}{n}+1\right)^2.$$

In Table 2.1, we see that in two- or three-space dimensions, the hybridizable methods always have less degrees of freedom and have a stiffness matrix that is sparser than the corresponding LDG methods. The same is valid for the IP method in two-space dimensions and in three-space dimensions for $k \geq 3$. In three-space dimensions, the IP method with $k = 1$ is more advantageous than the corresponding hybridizable DG method; for $k = 2$, its advantages are, however, marginal.

It is interesting to extend the comparison with the IP method for which static condensation of the interior degrees of freedom has been carried out; of course, this

TABLE 2.2
Comparison between hybridizable and the statically-condensed IP methods on simplicial meshes.

n	k	$R_{\text{d.o.f.}}$	R_{sparsity}	n	k	$R_{\text{d.o.f.}}$	R_{sparsity}
2	3	**1.50**	**2.70**	3	4	**1.13**	**1.86**
	4	**1.60**	**3.07**		5	**1.23**	**2.19**
	5	**1.67**	**3.33**		6	**1.32**	**2.49**

can be done only if $k \geq n+1$. In this case, the number of globally coupled degrees of freedom is

$$N_{\text{d.o.f.}}^{\text{sc}-\text{IP}} = N_s \left(\dim \mathcal{P}_k(K) - \dim \mathcal{P}_{k-n-1}(K)\right).$$

The stiffness matrix in question has again a block structure with square blocks of order equal to $(\dim \mathcal{P}_k(K) - \dim \mathcal{P}_{k-n-1}(K))$. On each block-row, the number of blocks that are not equal to zero are $n + 2$. Indeed, it can be shown that the interior degrees of freedom on a given simplex can be expressed in terms of the condensed degrees of freedom of the simplex and those of its neighbors, and that the condensed degrees of freedom can be expressed in terms of the interior degrees of freedom of the simplex *and* those of its neighbors. We then have

$$N_{\text{sparsity}}^{\text{sc}-\text{IP}} = N_s \left(n + 2\right)\left(\dim \mathcal{P}_k(K) - \dim \mathcal{P}_{k-n-1}(K)\right)^2.$$

This implies that the corresponding ratios are

$$R_{\text{d.o.f.}}^{\text{sc}-\text{IP}} = \frac{2}{(n+1)} \left(\frac{k}{n} + 1\right) \left(1 - \Pi_{j=1}^{n} \frac{k-j}{k+j}\right),$$

and

$$R_{\text{sparsity}}^{\text{sc}-\text{IP}} = \frac{2\left(n+2\right)}{(n+1)\left(2n+1\right)} \left(\frac{k}{n} + 1\right)^2 \left(1 - \Pi_{j=1}^{n} \frac{k-j}{k+j}\right)^2.$$

We show some results in Table 2.2. We see that the hybridized methods produce smaller and more sparse matrices than the statically-condensed IP method.

The same argument could be made for DG methods on n-dimensional rectangular finite elements. In this case, the DG approximations could be based on polynomials of degree k (instead of polynomials of degree k in each variable in the case of continuous elements). Then the ratio between the degrees of freedom (and the sparsity) will be lower, since instead of the factor $N_s/N_{\text{i.f.}} \approx 2/(n+1)$, we have the factor $N_r/N_{\text{i.f.}} \approx 2/2^n$.

A complete comparison of methods would require factoring in the costs of solving the algebraic problem. While greater sparsity or lesser number of degrees of freedom often yields faster solution methods, definitive conclusions can be made only after numerical experiments with specific direct or iterative methods; see [16] for such studies on older methods.

3. Examples of hybridizable methods. In this section, we give several examples of methods fitting the general structure described in the previous section. We restrict ourselves to methods that use the same local solver in all the elements K of triangulation $\mathcal{T}_h$. Throughout this section, we *assume* that $\mathcal{T}_h$ is a *conforming simplicial triangulation.*

To define each of the methods, we have only to specify (1) the numerical trace of the flux $\widehat{\mathbf{Q}}\cdot$, (2) the local spaces $\boldsymbol{V}(K)$, $W(K)$, and (3) the space of approximate traces M_h. We then verify that the local solvers are well posed and discuss the conservativity condition by using Theorem 2.1. We use Theorem 2.4 to verify the existence and the uniqueness of the approximate trace λ_h and end by relating these results to relevant, earlier material.

Our examples are summarized Tables 3.1 and 3.2; some of them are schematicaly related in Figure 3.1. The first column of the tables consists of method names. We adopt the following convention: Suppose that we define the local solver on each element by using a numerical method previously known as the "N" method. Then we call the resulting hybridized formulation an "N-hybridizable method" or, in short, an "N-H" method. For example, if we use the well-known IP method to define the local solvers, then any hybridized formulation with such local solvers is denoted as IP-H. We also say that a finite element method is an N-H method if there is a hybridization of the method that is an N-H method.

In columns 2–4 of Table 3.1, we give the spaces of the local solvers and the approximate trace. In the fifth column, we indicate whether the method gives a single-valued flux trace $\widehat{\boldsymbol{q}}_h$ so the conservativity condition is satisfied in a strong form or $\widehat{\boldsymbol{q}}_h$ is double-valued so the methods leads to a weak conservativity condition. In the last two columns of Table 3.1, we define the numerical traces of the fluxes $\widehat{\mathbf{Q}}\mathsf{m}$ and $\widehat{\mathbf{Q}}f$. The weak formulations for the approximate traces obtained via Theorem 2.1 for each type of method are listed in Table 3.2.

3.1. The RT-H method. This method is obtained by using the RT method to define the local solvers. The three ingredients of the RT-H method are as follows:

1. For each $K \in \mathcal{T}_h$, we take

$$\widehat{\mathbf{Q}}\mathsf{m} = \mathbf{Q}\mathsf{m}, \qquad \widehat{\mathbf{Q}}f = \mathbf{Q}f \qquad \text{on } \partial K;$$

2. The finite element space $\boldsymbol{V}(K) \times W(K)$ is defined as Raviart–Thomas space of degree k:

$$\boldsymbol{V}(K) = \mathcal{P}_k(K)^n + \boldsymbol{x}\,\mathcal{P}_k(K), \quad W(K) = \mathcal{P}_k(K), \quad k \geq 0,$$

where $\mathcal{P}_k(K)^n$ denotes the set of vector functions whose components are in $\mathcal{P}_k(K)$;

3. We define the space of approximate traces as

$$M_h = \mathcal{M}_{h,k}.$$

The fact that the local solvers are well defined can be established by realizing that they are defined by using exactly the RT mixed finite element method. Indeed, if we insert the expression of numerical traces $\widehat{\mathbf{Q}}\mathsf{m}$ and $\widehat{\mathbf{Q}}f$ into the equations defining the local solvers, we see that they are nothing but the RT discretizations of exact local problems (1.4), as claimed. Since the RT method is well defined (see [49, 12]) local solvers $(\mathbf{Q}\mathsf{m}, \mathcal{U}\mathsf{m})$ and $(\mathbf{Q}f, \mathcal{U}f)$ are also well defined.

Note that conservativity condition (2.6) forces numerical trace $\widehat{\boldsymbol{q}}_h$ to be single valued. Indeed, because (extension by zero from $\mathcal{E}_h^\circ$ to $\mathcal{E}_h$ of) $[\![\widehat{\mathbf{Q}}\lambda_h + \widehat{\mathbf{Q}}g_h + \widehat{\mathbf{Q}}f]\!]$ and test functions μ belong to the same space, conservativity condition (2.6) forces equality

$$[\![\widehat{\boldsymbol{q}}_h]\!] = [\![\boldsymbol{q}_h]\!] = \left[\!\!\left[\widehat{\mathbf{Q}}\lambda_h + \widehat{\mathbf{Q}}g_h + \widehat{\mathbf{Q}}f\right]\!\!\right] = 0 \quad \text{on} \quad \mathcal{E}_h^\circ,$$

TABLE 3.1
Summary of the examples.

Method	$V(K)$	$W(K)$	M_h	Conservativity	$\hat{Q}m$	$\hat{Q}f$
RT-H	$\mathcal{P}_k(K)^n + x\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{M}_{h,k}$	strong	Qm	Qf
BDM-H	$\mathcal{P}_k(K)^n$	$\mathcal{P}_{k-1}(K)$	$\mathcal{M}_{h,k}$	strong	Qm	Qf
LDG-H	$\mathcal{P}_k(K)^n$	$\mathcal{P}_{k-1}(K)$	$\mathcal{M}_{h,k}$	strong	$Qm + \tau(Um - m)\,n$	$Qf + \tau(Uf)\,n$
LDG-H	$\mathcal{P}_k(K)^n$	$\mathcal{P}_k(K)$	$\mathcal{M}_{h,k}$	strong	$Qm + \tau(Um - m)\,n$	$Qf + \tau(Uf)\,n$
LDG-H	$\mathcal{P}_{k-1}(K)^n$	$\mathcal{P}_k(K)$	$\mathcal{M}_{h,k}$	strong	$Qm + \tau(Um - m)\,n$	$Qf + \tau(Uf)\,n$
IP-H	$\mathcal{P}_k(K)^n$	$\mathcal{P}_k(K)$	$\mathcal{M}_{h,k}$	strong	$-a\mathbf{grad}\,Um + \tau(Um - m)\,n$	$-a\mathbf{grad}\,Uf + \tau(Uf)\,n$
NC-H	$\mathcal{P}_{k-1}(K)^2$, k odd	$\mathcal{P}_k(K)$	$\mathcal{M}_{h,k-1}$	strong	a new unknown variable	a new unknown variable
CG-H	$\mathcal{P}_{k-1}(K)^n$	$\mathcal{P}_k(K)$	$\mathcal{M}^c_{h,k}$	weak	a new unknown variable	a new unknown variable

TABLE 3.2
Weak formulations for the approximate trace.

Method	$a_h(\eta, \mu)$	$b_h(\mu)$
RT-H	$(c\,Q\eta, Q\mu)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, Q\mu\cdot n\rangle_{\partial\Omega}$
BDM-H	$(c\,Q\eta, Q\mu)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, Q\mu\cdot n\rangle_{\partial\Omega}$
LDG-H	$(c\,Q\eta, Q\mu)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h} +$ $\langle 1, [\![(U\mu - \mu)(\tau(U\eta - \eta)n]\!]\rangle_{\varepsilon_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, Q\mu\cdot n + \tau\,U\mu\rangle_{\partial\Omega}$
IP-H[†]	$(a\mathbf{grad}\,U\mu, \mathbf{grad}\,U\eta)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h} +$ $\langle 1, [\![((\eta - U\eta)a\mathbf{grad}\,U\mu + (\mu - U\mu)a\mathbf{grad}\,U\eta]\!]\rangle_{\varepsilon_h} +$ $\langle 1, [\![(U\mu - \mu)(\tau(U\eta - \eta)n]\!]\rangle_{\varepsilon_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, -a\mathbf{grad}\,U\mu\cdot n + \tau\,U\mu\rangle_{\partial\Omega}$
NC-H[†]	$(a\mathbf{grad}\,U\eta, \mathbf{grad}\,U\mu)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, \hat{Q}\mu\cdot n\rangle_{\partial\Omega}$
CG-H[†]	$(a\mathbf{grad}\,U\eta, \mathbf{grad}\,U\mu)_{\mathcal{T}_h} + (d\,U\eta, U\mu)_{\mathcal{T}_h}$	$(f, U\mu)_{\mathcal{T}_h} + \langle g_h, [\![\hat{Q}\mu]\!]\rangle_{\varepsilon_h}$

[†]We assume that $a(x)$ is a constant on each element.
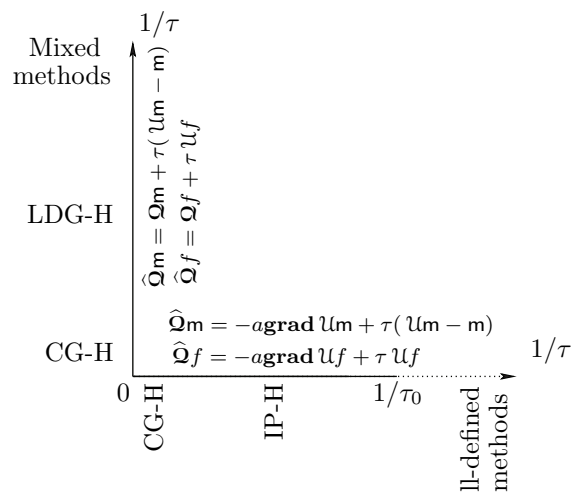
FIG. 3.1. *Relations between some hybridizable methods in terms of stabilization parameter τ.*

so the normal component of numerical trace $\widehat{\boldsymbol{q}}_h$ is single-valued, and $\boldsymbol{q}_h \in H(\mathrm{div}, \Omega)$. Moreover, Theorem 2.1 asserts that the conservativity condition is equivalent to (2.9) with

$$a_h(\eta, \mu) = (c\,\mathbf{Q}\eta, \mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta, \,\mathcal{U}\mu)_{\mathcal{T}_h},$$
$$b_h(\mu) = \langle g_h\,, \mathbf{Q}\mu \cdot \boldsymbol{n}\rangle_{\partial\Omega} + (f, \,\mathcal{U}\mu)_{\mathcal{T}_h},$$

provided $g_h|_{\varepsilon_h^\circ} = 0$. This is, of course, a reasonable choice, since $g|_{\varepsilon_h^\circ} = 0$ and M_h is a space of discontinuous functions.

These results appeared earlier in [26, Theorem 2.1], where the hybridized RT method of arbitrary order was considered; the case of the lowest order RT method was previously considered in [21]. We can thus conclude that the original RT method is an RT-H method. In [41], bilinear form $a_h(\cdot, \cdot)$ was shown to be positive definite; this implies that λ_h is uniquely determined. Next, we apply our general approach to this method and verify Assumption 2.2 on the positive semidefiniteness of the local solvers and Assumption 2.3, the gluing condition. By Theorem 2.4, this ensures the existence and the uniqueness of λ_h and hence that of approximation $(\boldsymbol{q}_h, u_h)$.

PROPOSITION 3.1. *Assumption* 2.1 *on the existence and the uniqueness of the local solvers, and Assumption* 2.2 *on the positive semidefiniteness of the local solvers hold for the RT-H method. Assumption* 2.3, *the gluing condition, also holds with*

$$M(\partial K) = \{\mu : \mu|_e \in \mathcal{P}_k(e) \text{ for all faces } e \text{ of } \partial K\}.$$

Proof. Assumption 2.1 obviously holds. Let us prove Assumption 2.2. To do that, we first show that condition (2.13a) holds. By identity (2.14) with $\mu := \mathsf{m}$, we have that

$$-\left\langle \mathsf{m}\,, \widehat{\mathbf{Q}}\mathsf{m} \cdot \boldsymbol{n}\right\rangle_{\partial K} = (c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mathsf{m})_K + (d\,\mathcal{U}\mathsf{m}, \,\mathcal{U}\mathsf{m})_K,$$

by the definition of $\widehat{\mathbf{Q}}\mathsf{m}$. We thus see that condition (2.13a) is satisfied.

Now we verify condition (2.13b) with the given choice of $M(\partial K)$. If $\langle \mathsf{m}\,, \widehat{\mathbf{Q}}\mathsf{m} \cdot \boldsymbol{n}\rangle_{\partial K} = 0$, we immediately obtain $\mathbf{Q}\mathsf{m}|_K = 0$. This implies that (2.3a) can be rewritten

as

$$(3.1) \qquad (\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_K - \langle \mathcal{U}\mathsf{m} - \mathsf{m}\,, \boldsymbol{v}\cdot\boldsymbol{n} \rangle_{\partial K} = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K).$$

It is well known (see, for example, [12]) that for a given $\mathbf{grad}\,\mathcal{U}\mathsf{m}$ and $\mathcal{U}\mathsf{m} - \mathsf{m}$, there is a function $\boldsymbol{v} \in \boldsymbol{V}(K)$ such that

$$(3.2) \qquad (\boldsymbol{v}, \boldsymbol{p}_{k-1})_K = (\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{p}_{k-1})_K \qquad \text{for all } \boldsymbol{p}_{k-1} \in \mathcal{P}_{k-1}(K),$$

$$(3.3) \qquad \langle \boldsymbol{v}\cdot\boldsymbol{n}\,, p_k \rangle_e = -\langle \mathcal{U}\mathsf{m} - \mathsf{m}\,, p_k \rangle_e \qquad \text{for all } p_k \in \mathcal{P}_k(e)$$

for all faces e of K. Using this $\boldsymbol{v}$ in (3.1), we find that

$$(\mathbf{grad}\,\mathcal{U}\mathsf{m}, \mathbf{grad}\,\mathcal{U}\mathsf{m})_K + (\,\mathcal{U}\mathsf{m} - \mathsf{m}\,, \mathcal{U}\mathsf{m} - \mathsf{m})_{\partial K} = 0.$$

This implies that $\mathcal{U}\mathsf{m}$ is a constant on K, so m is constant on ∂K. This proves that condition (2.13b) is satisfied with $M(\partial K)$ as described.

It remains to verify Assumption 2.3. Since we are assuming that triangulation $\mathcal{T}_h$ is conforming, each interior face $e = \partial K^+ \cap \partial K^-$ is *also* a face of both K^+ and K^-. Hence, since $\mu|_e \in \mathcal{P}_k(e)$, we have that $P_{\partial K^+}\mu = \mu = P_{\partial K^-}\mu$ on e. This completes the proof. $\quad\square$

3.2. The BDM-H method. To obtain the BDM-H method, we use the BDM method to define the main three ingredients of the hybridization method:

1. For each $K \in \mathcal{T}_h$, we take

$$\widehat{\mathbf{Q}}\mathsf{m} = \mathbf{Q}\mathsf{m}, \quad \widehat{\mathbf{Q}}f = \mathbf{Q}f \qquad \text{on } \partial K;$$

2. The finite element spaces are defined as

$$\boldsymbol{V}(K) = \mathcal{P}_k(K)^n, \quad W(K) = \mathcal{P}_{k-1}(K), \quad k \geq 1;$$

3. The space of approximate traces is defined as $\quad \mathsf{M}_h = \mathcal{M}_{h,k}$.
This defines the BDM-H method.

Everything said about the RT-H method in the previous subsection applies to the BDM-H method. In particular, we have that the original BDM method is a BDM-H method; see [41].

3.3. The LDG-H methods. The LDG-H methods are obtained by using the LDG method to define the local solvers. The following specifications completely define the class of LDG-H methods:

1. The numerical traces

$$(3.4) \qquad \widehat{\mathbf{Q}}\mathsf{m} = \mathbf{Q}\mathsf{m} + \tau_K(\mathcal{U}\mathsf{m} - \mathsf{m})\boldsymbol{n}, \quad \widehat{\mathbf{Q}}f = \mathbf{Q}f + \tau_K(\mathcal{U}f)\,\boldsymbol{n} \quad \text{on } \partial K,$$

where τ_K is a function that can vary on ∂K.

2. The space $\boldsymbol{V}(K) \times W(K)$ as one of the following choices:

$$(3.5a) \qquad \mathcal{P}_k(K)^n \times \mathcal{P}_{k-1}(K), \ k \geq 1 \text{ and } \tau_K \geq 0 \text{ on } \partial K;$$

$$(3.5b) \qquad \mathcal{P}_k(K)^n \times \mathcal{P}_k(K), \quad k \geq 0 \text{ and } \tau_K > 0 \text{ on at least}$$
$$\text{one face of the simplex } K;$$

$$(3.5c) \qquad \mathcal{P}_{k-1}(K)^n \times \mathcal{P}_k(K), \ k \geq 1 \text{ and } \tau_K > 0 \text{ on } \partial K.$$

3. The space of approximate traces is

$$(3.6) \qquad\qquad \mathsf{M}_h = \mathcal{M}_{h,k}.$$

Typically, the stabilization parameter τ of the LDG methods is a nonnegative constant on each face in $\mathcal{E}_h$. Here, we allow τ to be double valued on $\mathcal{E}_h^\circ$, with two branches $\tau^- = \tau_{K-}$ and $\tau^+ = \tau_{K+}$ defined on the edge e shared by the finite elements K^- and K^+. Now the functions $(\mathbf{Q}\mathsf{m}, \mathcal{U}\mathsf{m})$ and $(\mathbf{Q}f, \mathcal{U}f)$ are the approximations given by the LDG method to exact solutions of (1.4) on each element, as claimed. As is well known (see [34, 17, 5]), the LDG method is uniquely solvable for $\tau_K > 0$. However, the above specifications define a wider class of LDG-H methods. We show that the existence and the uniqueness of the solution of the method can be guaranteed for each of choices (3.5).

PROPOSITION 3.2. *Assumption* 2.1 *on the existence and the uniqueness of the local solvers holds for the numerical traces given by* (3.4) *and with any of choices* (3.5) *for* $\mathbf{V}(K) \times W(K)$.

To prove this result for all the above-mentioned cases, we use the following auxiliary lemma.

LEMMA 3.1. *Let* $\tau_K \geq 0$. *With the choice of numerical traces in* (3.4)*, local problems* (2.3) *and* (2.4) *are uniquely solvable if* $\mathbf{V}(K) \times W(K)$ *defined by* (3.5) *is such that whenever* $w \in W(K)$ *satisfies*
  (i) $\tau_K w = 0$ *on* ∂K, *and*
  (ii) $(w, \mathrm{div}\, \mathbf{v})_K = 0$ *for all* $\mathbf{v} \in \mathbf{V}(K)$,
we have that $w = 0$.

Proof. Let us prove the result for first local solver $(\mathbf{Q}\mathsf{m}, \mathcal{U}\mathsf{m})$ defined by (2.3). The result for the other local mapping (2.4) is similar. It suffices to prove uniqueness, since this implies existence. To prove uniqueness, we must show that, when $\mathsf{m} = 0$, the only solution of (2.3) is the trivial one.

Taking $\mathbf{v} = \mathbf{Q}\mathsf{m}$ and $w = \mathcal{U}\mathsf{m}$ in (2.3) and adding the resulting equations, we get

$$(c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mathsf{m})_K + \left\langle \mathcal{U}\mathsf{m}, \left(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m}\right) \cdot \boldsymbol{n} \right\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mathsf{m})_K = 0.$$

Inserting the definition of the numerical trace $\widehat{\mathbf{Q}}\mathsf{m}$, we get

$$(c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mathsf{m})_K + \langle \mathcal{U}\mathsf{m}, \tau_K\,\mathcal{U}\mathsf{m}\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mathsf{m})_K = 0,$$

and since c is positive definite and symmetric, $d \geq 0$, and $\tau_K \geq 0$, we have that $\mathbf{Q}\mathsf{m} = \mathbf{0}$.

It remains to show that $\mathcal{U}\mathsf{m} = 0$. To do so, we note that the above equation implies that $(\tau\,\mathcal{U}\mathsf{m})_K = 0$ on ∂K. By (2.3a), we also have

$$(\mathcal{U}\mathsf{m}, \mathrm{div}\, \mathbf{v})_K = 0 \qquad \text{for all } \mathbf{v} \in \mathbf{V}(K).$$

By hypothesis (ii) of Lemma 3.1, this implies that $\mathcal{U}\mathsf{m} = 0$. This completes the proof. $\square$

We are now ready to prove Proposition 3.2.

Proof. By Lemma 3.1, we have only to show that, for each of three choices (3.5), if $w \in W(K)$ satisfies $\tau_K w = 0$ on ∂K and $(w, \mathrm{div}\, \mathbf{v})_K = 0$ for all $\mathbf{v}$ in $\mathbf{V}(K)$, then $w = 0$ on K.

Let us show that this is true for the spaces given by (3.5a). Since $\mathrm{div} : \mathbf{V}(K) \to W(K)$ is surjective, we know there is a $\mathbf{v}$ in $\mathbf{V}(K)$ such that $\mathrm{div}\, \mathbf{v} = w$. This implies that $(w, w)_K = 0$ and hence that $w = 0$ on K.

Next, let us consider choice (3.5b). Since w must vanish on the face F where $\tau_K > 0$, we immediately have that $w = 0$ if $k = 0$. If $k \geq 1$, it can be factored as $w = \ell_F \, p_{k-1}$, with $p_{k-1} \in \mathcal{P}_{k-1}(K)$ and ℓ_F equal to the barycentric coordinate function of K that vanishes on F. Then, choosing $\boldsymbol{v}$ in $\boldsymbol{V}(K) = \mathcal{P}_k(K)^n$ such that div $\boldsymbol{v} = p_{k-1}$, equation

$$0 = (\mathrm{div}\,\boldsymbol{v}, w)_K = (\mathrm{div}\,\boldsymbol{v}, \ell_F\,p_{k-1})_K = (p_{k-1}, \ell_F\,p_{k-1})_K$$

implies that p_{k-1} vanishes on K, so $w = 0$ on K.

Finally, let us consider choice (3.5c). Since $\tau_K > 0$ on ∂K, we have that $w = 0$ on ∂K, and a simple integration by parts gives that

$$(\mathbf{grad}\,w, \boldsymbol{v})_K = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K) = \mathcal{P}_{k-1}(K)^n.$$

Taking $\boldsymbol{v} = \mathbf{grad}\,w$ allows us to conclude that w is a constant on K and hence identically zero on K. This completes the proof. $\quad\square$

Note that choices (3.4) of the numerical traces, (3.5) of the finite elements spaces $\boldsymbol{V}(K) \times W(K)$, and (3.6) for approximate trace space M_h clearly imply that, for all these LDG-H methods, conservativity condition (2.6) is satisfied strongly. Moreover, by Theorem 2.1, the conservativity condition is equivalent to $a_h(\lambda_h, \mu) = b_h(\mu)$ for all $\mu \in M_h$, where

$$a_h(\eta, \mu) = (c\,\boldsymbol{\mathsf{Q}}\eta, \boldsymbol{\mathsf{Q}}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta, \mathcal{U}\mu)_{\mathcal{T}_h} + \langle 1,\ [\![(\mathcal{U}\mu - \mu)(\tau\,(\mathcal{U}\eta - \eta)\boldsymbol{n})]\!]\rangle_{\varepsilon_h},$$
$$b_h(\mu) = \langle g_h,\ \boldsymbol{\mathsf{Q}}\mu \cdot \boldsymbol{n} + \tau\,\mathcal{U}\mu\rangle_{\partial\Omega} + (f, \mathcal{U}\mu)_{\mathcal{T}_h},$$

provided $g_h|_{\varepsilon_h^\circ} = 0$.

Form $a_h(\cdot, \cdot)$ is obviously symmetric. That it is also positive definite follows once Assumption 2.2 on the positive semidefiniteness of the local solvers is verified. Set

$$(3.7) \qquad \begin{aligned} M(\partial K) = \{\mu:\quad &\mu|_e \in \mathcal{P}_k(e) \text{ for all faces } e \text{ where } \tau_K = 0,\ \text{and} \\ &\mu|_e \in L^2(e) \text{ for all faces } e \text{ where } \tau_K > 0\}. \end{aligned}$$

PROPOSITION 3.3. *Let the numerical traces be set by* (3.4), *the local spaces be as in any of choices* (3.5), *and the space of approximate traces be set by* (3.6). *Then, Assumption* 2.2 *on the positive semidefiniteness of the local solvers and Assumption* 2.3, *the gluing condition, are satisfied with $M(\partial K)$ defined by* (3.7).

Proof. We begin by showing that condition (2.13a) holds. By identity (2.14) with $\mu := \mathsf{m}$ and the definition of $\widehat{\boldsymbol{\mathsf{Q}}}\mathsf{m}$, we have that

$$-\left\langle \mathsf{m},\ \widehat{\boldsymbol{\mathsf{Q}}}\mathsf{m} \cdot \boldsymbol{n} \right\rangle_{\partial K} = (c\,\boldsymbol{\mathsf{Q}}\mathsf{m}, \boldsymbol{\mathsf{Q}}\mathsf{m})_K + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mathsf{m})_K + \langle \tau_K(\mathcal{U}\mathsf{m} - \mathsf{m}),\ \mathcal{U}\mathsf{m} - \mathsf{m}\rangle_{\partial K}.$$

Since $\tau_K \geq 0$ in all three cases (3.5), we see that condition (2.13a) is satisfied.

Now, let us verify condition (2.13b). If we assume that $\langle \mathsf{m},\ \widehat{\boldsymbol{\mathsf{Q}}}\mathsf{m} \cdot \boldsymbol{n}\rangle_{\partial K} = 0$, we immediately obtain that $\boldsymbol{\mathsf{Q}}\mathsf{m}|_K = 0$ and $\tau\,(\mathcal{U}\mathsf{m} - \mathsf{m})|_{\partial K} = 0$. This implies that the first equation defining first local solver (2.3a) can be rewritten as

$$(3.8) \qquad (\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_K - \langle \mathcal{U}\mathsf{m} - \mathsf{m},\ \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial K} = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K).$$

We use this equation to show that in all three cases (3.5), condition (2.13b) is satisfied with $P_{\partial K}$ defined, on the face e of K, as the L^2-projection into $\mathcal{P}_k(e)$ if $\tau|_e = 0$ and as the identity if $\tau|_e > 0$:

(i) In case (3.5a), the result follows exactly as in the proof of Proposition 3.1.

(ii) In case (3.5b), we know (see [24]) that there is a function $\boldsymbol{v} \in \mathcal{P}_k(K)^n$ such that

$$(3.9) \qquad (\boldsymbol{v}, \boldsymbol{p}_{k-1})_K = (\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{p}_{k-1})_K \qquad \text{for all } \boldsymbol{p}_{k-1} \in \mathcal{P}_{k-1}(K)^n,$$

$$(3.10) \qquad \langle \boldsymbol{v} \cdot \boldsymbol{n}, p_k \rangle_e = -\langle \mathcal{U}\mathsf{m} - \mathsf{m}, p_k \rangle_e \qquad \text{for all } p_k \in \mathcal{P}_k(e)$$

for all the faces e of K *except* one, say, the face e' on which $\tau > 0$. Setting this $\boldsymbol{v}$ in (3.8) and using the fact that on e' we have that $\mathsf{m} = \mathcal{U}\mathsf{m}$, we obtain that $\mathcal{U}\mathsf{m}$ is a constant on K and that $\mathsf{m} = \mathcal{U}\mathsf{m}$ on the remaining faces of ∂K. Thus, m is constant on ∂K and condition (2.13b) is verified. Assumption 2.3, the gluing condition, is trivially satisfied by virtue of the definition of $M(\partial K)$ in (3.7).

(iii) In case (3.5c), we immediately see that $\mathsf{m} = \mathcal{U}\mathsf{m}$ on ∂K. Now we take $\boldsymbol{v} = \mathbf{grad}\,\mathcal{U}\mathsf{m}$ in (3.8) to get that $\mathcal{U}\mathsf{m}$ is a constant. This verifies Assumption 2.2 as in the previous case. Assumption 2.3 obviously holds from the definition of $M(\partial K)$ in (3.7). $\quad\square$

Our next result sheds light into the nature of numerical traces $\widehat{\boldsymbol{q}}_h$ and $\widehat{u}_h$ of the LDG-H schemes.

PROPOSITION 3.4 (characterization of LDG-H methods). *Let the numerical traces be set by* (3.4), *the local spaces be as in any of choices* (3.5), *the space of approximate traces be set by* (3.6), *and* $(\boldsymbol{q}_h, u_h)$ *be as defined in* (2.5). *Then conservativity condition* (2.6) *holds on* $\mathcal{E}_h^\circ$ *if and only if*

$$(3.11a) \qquad \lambda_h = \widehat{u}_h = \left(\frac{\tau^+}{\tau^- + \tau^+}\right)u_h^+ + \left(\frac{\tau^-}{\tau^- + \tau^+}\right)u_h^- + \left(\frac{1}{\tau^+ + \tau^-}\right)[\![\boldsymbol{q}_h]\!],$$

$$(3.11b) \qquad \widehat{\boldsymbol{q}}_h = \left(\frac{\tau^-}{\tau^- + \tau^+}\right)\boldsymbol{q}_h^+ + \left(\frac{\tau^+}{\tau^- + \tau^+}\right)\boldsymbol{q}_h^- + \left(\frac{\tau^+\tau^-}{\tau^- + \tau^+}\right)[\![u_h]\!].$$

Proof. Suppose the conservativity condition holds. We need to prove (3.11a) and (3.11b). By the definition of $\widehat{\boldsymbol{q}}_h$ (see (2.7)) we have

$$\begin{aligned}
\widehat{\boldsymbol{q}}_h &= \widehat{\boldsymbol{Q}}\lambda_h + \widehat{\boldsymbol{Q}}g_h + \widehat{\boldsymbol{Q}}f \\
&= (\boldsymbol{Q}\lambda_h + \boldsymbol{Q}g_h + \boldsymbol{Q}f) + \tau(\mathcal{U}\lambda_h + \mathcal{U}g_h + \mathcal{U}f - \lambda_h - g_h)\,\boldsymbol{n} \\
&= \boldsymbol{q_h} + \tau(u_h - \lambda_h - g_h)\,\boldsymbol{n}.
\end{aligned}$$

Inserting this expression into the conservativity condition and taking g_h equal to zero on $\mathcal{E}_h^\circ$, we obtain that, for any $\mu \in M_h$,

$$\langle \mu, [\![\widehat{\boldsymbol{q}}_h]\!]\rangle_{\mathcal{E}_h^\circ} = \langle \mu, [\![\boldsymbol{q}_h + \tau(u_h - \lambda_h)\,\boldsymbol{n}]\!]\rangle_{\mathcal{E}_h^\circ} = 0,$$

which implies, by our choice of spaces, that $[\![\widehat{\boldsymbol{q}}_h]\!] = 0$ on $\mathcal{E}_h^\circ$ or equivalently that

$$[\![\boldsymbol{q}_h]\!] + \left(\tau^+ u_h^+ + \tau^- u_h^-\right) - \left(\tau^+ + \tau^-\right)\lambda_h = 0 \text{ on } \mathcal{E}_h^\circ.$$

Solving for λ_h, we obtain (3.11a). To prove (3.11b), we simply insert the expression for λ_h into the identity

$$\widehat{\boldsymbol{q}}_h^+ \cdot \boldsymbol{n}^+ = \boldsymbol{q}_h^+ \cdot \boldsymbol{n}^+ + \tau^+\left(u_h^+ - \lambda_h\right)$$

and perform a few algebraic manipulations.

The converse asserted by the proposition is trivial: If identities (3.11) hold, then the normal component of $\widehat{\boldsymbol{q}}_h$ is single valued on $\mathcal{E}_h^\circ$ and the conservativity condition is satisfied. This completes the proof. $\square$

COROLLARY 3.2. *The LDG method is not an LDG-H method for any finite τ.*

Proof. On any interior face $e \in \mathcal{E}_h^\circ$, the LDG method has a numerical trace $\widehat{u}_h$ independent of $\boldsymbol{q}_h$; see [34, 17, 5]. On the other hand, by Proposition 3.4, the LDG-H methods have numerical traces $\widehat{u}_h$ that depend on $[\![\boldsymbol{q}_h]\!]$. Since this dependence cannot be removed for any finite value of τ, we see that no LDG method is an LDG-H method. This completes the proof. $\square$

As known from [34, p. 2445] and [17, p. 1681], the independence of numerical trace $\widehat{u}_h$ of the LDG methods of $\boldsymbol{q}_h$ on interior faces $\mathcal{E}_h^\circ$ allows us to eliminate the unknown $\boldsymbol{q}_h$ from the equations and to obtain a primal formulation involving only u_h. In contrast, in the LDG-H methods, $\widehat{u}_h$ *must* depend on $\boldsymbol{q}_h$ as well. Both approaches recover $\boldsymbol{q}_h$ locally but using different mechanisms. Since the LDG-H methods lead to a formulation involving only numerical trace λ_h, they have fewer globally coupled unknowns than the LDG method for high order polynomials.

The LDG-H methods considered in this subsection were studied in [17] where it was proven, in particular, that the method is well defined for $\tau > 0$ on $\mathcal{E}_h$. Methods with $\tau = 0$ do not fit in the framework proposed in [5]; they have been recently studied in [24].

3.4. A limiting case of LDG-H methods. Here we consider hybridizable Galerkin methods that can be obtained formally considering limiting values of the penalty parameter in LDG-H methods. The motivation for doing this arises from the previous corollary (Corollary 3.2), whereby we know that the only chance for showing that an LDG method can be hybridized lies in cases where τ is allowed to be not finite.

We first examine how numerical traces of the previous LDG-H method change as we *formally* pass to a limit in τ. By letting τ^+ go to infinity on the interior face $e = \partial K^+ \cap \partial K^-$ while maintaining a fixed finite τ^-, we find that the expressions for the numerical traces obtained in Proposition 3.4 become

$$(3.12) \qquad \widehat{u}_h = u_h^+ \quad \text{and} \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^- + \tau^- \, [\![u_h]\!].$$

Note that the above expression for primal numerical trace $\widehat{u}_h$ is independent of the fluxes, or, in other words, such traces will result in an LDG method. Indeed, the LDG method defined by these numerical traces have been thoroughly studied in the case $\tau^- > 0$; see [34, 17, 5].

In the special case $\tau^- = 0$, we get

$$\widehat{u}_h = u_h^+ \quad \text{and} \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h^-,$$

which also defines a previously studied LDG method. For this scheme, the discontinuities of the approximate solution across interior interelement boundaries do not introduce any dissipation. The dissipative effect of the discontinuities is concentrated on the boundary of the domain and hence reduced to a "minimum," which is the reason for its name, the *minimal dissipation* LDG method. Since this scheme does not fit the unified analysis in [5], it was studied in [20] and [24] for problems in one and several space dimensions, respectively.

The formal passage to limit solely in the expressions for numerical traces does not clarify if the limiting methods are hybridizable. In particular, we must explain

precisely what we mean by setting $\tau_K = \infty$ in the context of local solvers. To do so, let F_K be the union of one or more faces of the element K where we want to set the branch τ_K to ∞. Since

$$\widehat{\boldsymbol{Q}}\mathsf{m} = \boldsymbol{Q}\mathsf{m} + \tau_K(\mathcal{U}\mathsf{m} - \mathsf{m})\boldsymbol{n},$$

we expect that in the formal limit of $\tau_K = \infty$, we should have $\mathcal{U}\mathsf{m} - \mathsf{m} = 0$. Then the value of $\widehat{\boldsymbol{Q}}\mathsf{m}$ on F_K becomes an unknown because the last term above is an unknown formal product of 0 with ∞. Motivated by this, we now define the local solvers with $\widehat{\boldsymbol{Q}}\mathsf{m}$ and $\widehat{\boldsymbol{Q}}f$ as *new* unknowns. More precisely, setting

$$W(K) = \mathcal{P}_k(K), \quad \boldsymbol{V}(K) = \mathcal{P}_k(K)^n, \quad \boldsymbol{T}_K(F_K) = \{\boldsymbol{n}_K\, w|_{F_K} : w \in W(K)\},$$

we define local solution $(\boldsymbol{Q}\mathsf{m}, \mathcal{U}\mathsf{m}, (\widehat{\boldsymbol{Q}}\mathsf{m})_{F_K}) \in \boldsymbol{V}(K) \times W(K) \times \boldsymbol{T}_K(F_K)$ for any $\mathsf{m} \in \mathsf{M}_h$ by

(3.13a) $(c\,\boldsymbol{Q}\mathsf{m}, \boldsymbol{v})_K - (\mathcal{U}\mathsf{m}, \operatorname{div}\boldsymbol{v})_K = -\langle \mathsf{m}\,, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}$ for all $\boldsymbol{v} \in \boldsymbol{V}(K)$,

(3.13b) $-(\mathbf{grad}\,w, \boldsymbol{Q}\mathsf{m})_K + \left\langle w\,, \widehat{\boldsymbol{Q}}\mathsf{m}\cdot\boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, w)_K = 0$ for all $w \in W(K)$,

(3.13c) $\mathcal{U}\mathsf{m} = \mathsf{m}$ on F_K.

Here, just as for the LDG-H methods, we set

$$\widehat{\boldsymbol{Q}}\mathsf{m} = \boldsymbol{Q}\mathsf{m} + \tau_K\,(\mathcal{U}\mathsf{m} - \mathsf{m})\,\boldsymbol{n} \quad \text{on } \partial K \setminus F_K.$$

Similarly, we define $(\boldsymbol{Q}f, \mathcal{U}f, (\widehat{\boldsymbol{Q}}f)_{F_K})$ as the element of $\boldsymbol{V}(K) \times W(K) \times \boldsymbol{T}_K(F_K)$ such that

(3.14a) $(c\,\boldsymbol{Q}f, \boldsymbol{v})_K - (\mathcal{U}f, \operatorname{div}\boldsymbol{v})_K = 0$ for all $\boldsymbol{v} \in \boldsymbol{V}(K)$,

(3.14b) $-(\mathbf{grad}\,w, \boldsymbol{Q}f)_K + \left\langle w, \widehat{\boldsymbol{Q}}f\cdot\boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}f, w)_K = (f, w)$ for all $w \in W(K)$,

(3.14c) $\mathcal{U}f = 0$ on F_K,

where

$$\widehat{\boldsymbol{Q}}f = \boldsymbol{Q}f + \tau_K\,(\mathcal{U}f)\,\boldsymbol{n} \quad \text{on } \partial K \setminus F_K.$$

We set the space of approximate traces by

(3.15) $\mathsf{M}_h = \{\mu \in \mathcal{M}_{h,k} : \mu|_{F_K} \text{ is continuous on } F_K \text{ for all } K \in \mathcal{T}_h\}$.

Note that the continuity condition in the above definition reflects the fact that the local solvers satisfy strong Dirichlet boundary conditions on F_K for all $K \in \mathcal{T}_h$; see (3.13c) and (3.14c). This completes the definition of the *limiting case of the LDG-H method* when $\tau_K = \infty$ on F_K. From now on, the above modification of the LDG local solvers is tacitly understood whenever we say that a branch of τ is infinity on a face. It is easy to check, by arguments similar to that in Proposition 3.2, that local problems (3.13) and (3.14) are uniquely solvable for every m in M_h and every $f \in L^2(\Omega)$ *provided*, for each element $K \in \mathcal{T}_h$, τ_K is not identically equal to zero on ∂K whenever F_K is the empty set.

Note that, although the local solvers have been modified, Theorem 2.1 continues to apply because its proof only relies on the form of the first two equations in the

local problems. Indeed, (3.13a) and (3.13b) are identical in form to (2.3a) and (2.3b), respectively; a similar remark applies to the equation of the second local solvers. Therefore, Theorem 2.1 also holds in this case. In particular, we have that

$$a_h(\eta, \mu) = (c\,\mathbf{Q}\eta, \mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta, \,\mathcal{U}\mu)_{\mathcal{T}_h} + \sum_{K \in \mathcal{T}_h} \langle \tau(\mathcal{U}\eta - \eta), (\mathcal{U}\mu - \mu) \rangle_{\partial K \backslash F_K}.$$

Finally, it is not difficult to see that Proposition 3.3 also holds. By Theorem 2.4, bilinear form $a_h(\cdot, \cdot)$ is positive definite, and we can immediately see that λ_h is uniquely determined.

Note that, unlike all previous examples, conservativity condition (2.6) for these methods is only imposed weakly. This is because while the jumps of $\widehat{\boldsymbol{q}}_h$ lie in $\mathcal{M}_{h,k}$, the approximate traces μ are in the space M_h, which is a strict subspace of $\mathcal{M}_{h,k}$. Since all LDG methods have single-valued numerical traces, this seems to suggest that no LDG method can be a limiting case of the LDG-H method. However, this is not the case, as we see next.

We consider the *one-sided limiting case of the LDG-H* method. This is the same as the above-defined limiting case of the LDG-H method but with the following additional assumption: For every interior face e in $\mathcal{E}_h^\circ$, one branch of τ is infinity, and the other branch is finite-valued.

COROLLARY 3.3. *The one-sided limiting case of the LDG-H method coincides with the LDG method whose numerical traces on the interior faces are given by* (3.12).

Proof. Let λ_h^∞ denote the solution of the one-sided limiting case of the LDG-H method, and let

$$\boldsymbol{q}_h^\infty = \mathbf{Q}\lambda_h^\infty + \mathbf{Q}g_h + \mathbf{Q}f, \qquad u_h^\infty = \mathcal{U}\lambda_h^\infty + \mathcal{U}g_h + \mathcal{U}f.$$

We will prove that $\boldsymbol{q}_h^\infty$ and u_h^∞ coincide with the corresponding solution components $\boldsymbol{q}_h^{\mathrm{LDG}}$ and u_h^{LDG}, respectively, of the LDG method with numerical traces set as in (3.12).

By the definition of the LDG method, $\boldsymbol{q}_h^{\mathrm{LDG}}$ and u^{LDG} satisfy (2.8a)–(2.8b) with the λ_h and $\widehat{\boldsymbol{q}}_h$ therein set, respectively, to $\widehat{u}_h$ and $\widehat{\boldsymbol{q}}_h$ of (3.12), which, for clarity, we will rewrite as $\widehat{u}_h^{\mathrm{LDG}}$ and $\widehat{\boldsymbol{q}}_h^{\mathrm{LDG}}$.

It suffices to show that $\boldsymbol{q}_h^\infty$ and u_h^∞ satisfy the same equations as $\boldsymbol{q}_h^{\mathrm{LDG}}$ and u_h^{LDG}. Adding local solver equations (3.13a) and (3.14a) over all elements, we find that $\boldsymbol{q}_h^\infty$ and u_h^∞ satisfy the first equation of the LDG method with λ_h^∞ in place of $\widehat{u}_h^{\mathrm{LDG}}$. But, since every interior edge has an infinite penalty branch and since

(3.16) $$\lambda_h^\infty|_{F_K} = (u_h^\infty)_{F_K} \qquad \text{for all elements } K,$$

we find that λ_h^∞ is in the same form as LDG numerical trace $\widehat{u}_h^{\mathrm{LDG}}$.

Also, summing local solver equations (3.13b) and (3.14b) over all elements, we find that $\boldsymbol{q}_h^\infty$ and u_h^∞ satisfy the second equation of the LDG methods, with $\widehat{\boldsymbol{q}}_h^\infty \equiv \widehat{\mathbf{Q}}\lambda_h^\infty + \widehat{\mathbf{Q}}g_h + \widehat{\mathbf{Q}}f$ in place of $\widehat{\boldsymbol{q}}_h^{\mathrm{LDG}}$. We will now show that the second equation, in fact, holds with the LDG flux. For this, we use the fact that

(3.17) $$\langle [\![\widehat{\boldsymbol{q}}_h^\infty]\!], \mu \rangle_{\mathcal{E}_h} = 0$$

for all μ in the subspace M_h of functions in M_h (defined by (3.15)), with $\mu|_{\partial\Omega} = 0$. Now, if w is any function in $W(K)$, then $w|_{F_K}$, extended by zero to $\mathcal{E}_h$, is in M_h. Therefore, (3.17) implies

$$\begin{aligned}
\langle \widehat{\boldsymbol{q}}_h^\infty \cdot \boldsymbol{n}, w \rangle_{F_K} &= -\langle (\widehat{\boldsymbol{q}}_h^\infty)_{K^c} \cdot (\boldsymbol{n})_{K^c}, w \rangle_{F_K} \\
&= -\langle (\widehat{\boldsymbol{q}}_h^\infty)_{K^c} + (\tau)_{K^c} \big((u_h^\infty)_{K^c} - \lambda_h^\infty \big)(\boldsymbol{n})_{K^c}, (\boldsymbol{n})_{K^c} w \rangle_{F_K}.
\end{aligned}$$

Here, for notational convenience, we have denoted the branch of a multivalued function f from outside K by $(f)_{K^c}$. By (3.16), we can rewrite the right-hand side as

$$\langle \widehat{\boldsymbol{q}}_h^\infty \cdot \boldsymbol{n} \,, w\rangle_{F_K} = -\langle\, (\widehat{\boldsymbol{q}}_h^\infty)_{K^c} + (\tau)_{K^c} [\![u_h^\infty]\!] \,,\ (\boldsymbol{n})_{K^c} w \,\rangle_{F_K}$$

and conclude that

$$(3.18) \qquad \sum_K \langle\widehat{\boldsymbol{q}}_h^\infty \cdot \boldsymbol{n}\,, w\rangle_{\partial K} = \sum_K \langle\widehat{\boldsymbol{q}}_h^{\mathrm{LDG}} \cdot \boldsymbol{n}\,, w\rangle_{\partial K}.$$

Thus, $\boldsymbol{q}_h^\infty$ and u_h^∞ satisfy the same equations as the LDG method with the same expressions for numerical traces as in the LDG case. □

Note that in the above proof, $\widehat{\boldsymbol{q}}_h^\infty$ and $\widehat{\boldsymbol{q}}_h^{\mathrm{LDG}}$ are not identical, in general, although (3.18) holds. This explains why the normal component of the limiting LDG-H numerical trace may not be single valued, although the numerical trace of its equivalent LDG method is single valued.

3.5. The CG-H method. The CG-H methods are obtained by using the CG method to define the local solvers. We are also going to see that they are also obtained from LDG-H methods by letting τ go to infinity everywhere.

Again, we need to specify the main ingredients of the local solvers. Similarly to the the limiting case of LDG-H methods, we need to give a new meaning of the local solvers since $\tau = \infty$. Since the numerical flux $\widehat{\boldsymbol{\mathcal{Q}}}\cdot$ will be unknown, we need an appropriate space for its approximation.

1. For any $k \geq 1$ and any $K \in \mathcal{T}_h$, we define the finite element spaces by

$$(3.19) \qquad \begin{aligned} &\boldsymbol{V}(K) = \mathcal{P}_{k-1}(K)^n, \quad W(K) = \mathcal{P}_k(K), \quad \text{and} \\ &\boldsymbol{T}(\partial K) := \{\boldsymbol{n}_K\, w|_{\partial K} :\ w \in W(K)\}. \end{aligned}$$

2. The numerical traces of fluxes $\widehat{\boldsymbol{\mathcal{Q}}}\cdot$ are unknown and will be determined by the modified local solvers as follows: $(\mathcal{Q}\mathsf{m}, \mathcal{U}\mathsf{m}, \widehat{\mathcal{Q}}\mathsf{m}) \in \boldsymbol{V}(K) \times W(K) \times \boldsymbol{T}(\partial K)$ is a solution to the problem

$$(3.20\mathrm{a}) \qquad (c\,\mathcal{Q}\mathsf{m}, \boldsymbol{v})_K - (\mathcal{U}\mathsf{m}, \operatorname{div}\boldsymbol{v})_K = -\langle\mathsf{m}\,, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K},$$

$$(3.20\mathrm{b}) \qquad -(\mathbf{grad}\,w, \mathcal{Q}\mathsf{m})_K + \left\langle w\,, \widehat{\mathcal{Q}}\mathsf{m}\cdot\boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, w)_K = 0,$$

$$(3.20\mathrm{c}) \qquad \mathcal{U}\mathsf{m} = \mathsf{m} \quad \text{on } \partial K.$$

for all $\boldsymbol{v} \in \boldsymbol{V}(K)$ and $w \in W(K)$. Similarly, $(\mathcal{Q}f, \mathcal{U}f, \widehat{\mathcal{Q}}f) \in \boldsymbol{V}(K) \times W(K) \times \boldsymbol{T}(\partial K)$ is defined by

$$(3.21\mathrm{a}) \qquad (c\,\mathcal{Q}f, \boldsymbol{v})_K - (\mathcal{U}f, \operatorname{div}\boldsymbol{v})_K = 0,$$

$$(3.21\mathrm{b}) \qquad -(\mathbf{grad}\,w, \mathcal{Q}f)_K + \left\langle w, \widehat{\mathcal{Q}}f\cdot\boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}f, w)_K = (f, w)_K,$$

$$(3.21\mathrm{c}) \qquad \mathcal{U}f = 0 \quad \text{on } \partial K.$$

for all $\boldsymbol{v} \in \boldsymbol{V}(K)$ and $\boldsymbol{v} \in W(K)$,

3. For the space of approximate traces, we take

$$(3.22) \qquad \mathsf{M}_h := \mathcal{M}_{h,k}^c.$$

We begin our discussion regarding the above CG-H method by verifying the assumptions required by Theorem 2.4.

PROPOSITION 3.5. *Assumption* 2.1 *on the existence and the uniqueness of the local solvers holds for the CG-H local solver. Assumption* 2.2 *on the positive semidefiniteness of the local solvers and Assumption* 2.3, *the gluing condition, hold with* $M(\partial K) = L^2(\partial K)$.

Proof. We prove the result for local solver $(\mathcal{Q}\mathsf{m}, \mathcal{U}\mathsf{m}, \widehat{\mathcal{Q}}\mathsf{m})$ defined by (3.20). The result for the local mapping defined by (3.21) is similar. Since the resulting system is square, we prove only uniqueness since this implies existence. Thus, we need to show that if $\mathsf{m} = 0$, then the only solution is the trivial one.

Taking $\boldsymbol{v} = \mathcal{Q}\mathsf{m}$ in (3.20a) and $w = \mathcal{U}\mathsf{m}$ in (3.20b) and adding the resulting equations, we get

$$(c\,\mathcal{Q}\mathsf{m}, \mathcal{Q}\mathsf{m})_K + \left\langle \mathcal{U}\mathsf{m}, \left(\widehat{\mathcal{Q}}\mathsf{m} - \mathcal{Q}\mathsf{m}\right)\cdot \boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mathsf{m})_K = 0.$$

Since, by (3.20c), $\mathcal{U}\mathsf{m} = 0$ on ∂K, we immediately obtain that $\mathcal{Q}\mathsf{m} = \boldsymbol{0}$. This implies that (3.20a) can be rewritten as follows:

$$(\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_K = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K),$$

which implies that $\mathcal{U}\mathsf{m} = 0$.

It remains to show that $\widehat{\mathcal{Q}}\mathsf{m} = 0$. To do that, we use (3.20b) rewritten as

$$\left\langle w, \widehat{\mathcal{Q}}\mathsf{m}\cdot \boldsymbol{n}\right\rangle_{\partial K} = 0 \qquad \text{for all } w \in W(K).$$

By the definition of space $\boldsymbol{T}(\partial K)$, we can find a function $w \in W(K)$ such that $\widehat{\mathcal{Q}}\mathsf{m} = w\,\boldsymbol{n}$. This readily implies that $\widehat{\mathcal{Q}}\mathsf{m} = 0$. This completes the verification of Assumption 2.1.

Inequality (2.13a) of Assumption 2.2 can easily be seen to hold. The second part of Assumption 2.2 also holds, since $M(\partial K) = L^2(\partial K)$. Finally, Assumption 2.3 trivially holds. ☐

Next, we discuss the conservativity condition. Flux approximation $\boldsymbol{q}_h$ of the CG-H method is, in general, not in $H(\mathrm{div}, \Omega)$. Nonetheless, it is interesting to observe that even the CG-H method has a weak conservativity property. This property holds for numerical flux trace $\widehat{\boldsymbol{q}}_h = \widehat{\mathcal{Q}}\lambda_h + \widehat{\mathcal{Q}}g_h + \widehat{\mathcal{Q}}f$, a quantity that is not present in the standard formulations of the CG methods but essential in our approach. Indeed, Theorem 2.1 asserts that $\widehat{\boldsymbol{q}}_h$ satisfies

$$\langle \mu, [\![\widehat{\boldsymbol{q}}_h]\!]\rangle_{\mathcal{E}_h^\circ} = 0 \quad \text{ for all } \mu \in \mathsf{M}_h,$$

which is a weak conservativity condition.

Observe that if a is a constant matrix on each element, by the definition of local solvers (3.20) and (3.21), we have that

(3.23) $$\mathcal{Q}\mathsf{m} = -a\mathbf{grad}\,\mathcal{U}\mathsf{m} \qquad \text{and} \qquad \mathcal{Q}f = -a\mathbf{grad}\,\mathcal{U}f.$$

Hence, $\boldsymbol{q}_h$ in (2.8a), being the sum of the local flux solutions, equals $-a\mathbf{grad}u_h$ on each element. Substituting this in (2.8b) and using the conservativity condition, we immediately see that u_h satisfies the standard CG equations. In addition, the boundary conditions defining local solvers (3.20c) and (3.21c) imply that u_h is continuous.

Thus, we conclude that this CG-H formulation coincides with the CG method whenever a is constant. In other words, the original CG method is a CG-H method when the matrix-valued function a is a constant on each element. In this case, we can also simplify the forms in (2.9) using (3.23) to

$$a_h(\eta, \mu) = (a\,\mathbf{grad}\,\mathcal{U}\eta, \mathbf{grad}\,\mathcal{U}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\, \mathcal{U}\mu)_{\mathcal{T}_h},$$

$$b_h(\mu) = \left\langle g_h\,,\ \left[\!\!\left[\widehat{\mathbf{Q}}\mu\right]\!\!\right]\right\rangle_{\mathcal{E}_h} + (f,\, \mathcal{U}\mu)_{\mathcal{T}_h}.$$

Note that in our case, we do not necessarily have that $g_h|_{\mathcal{E}_h^\circ} = 0$. Hence, the corresponding integral cannot be performed only on $\partial\Omega$ as in the previous cases.

Formulation (2.9) is nothing but the weak formulation for the CG method with static condensation of its interior degrees of freedom. This hybridization approach for the CG methods of degree k is explored in [31], where, in particular, a postprocessing technique providing locally conservative flux approximations competitive with that given by the RT methods of degree $k-1$ is introduced.

When the matrix-valued funcion a is not constant on each element, we cannot write (3.23) anymore. Instead, "a" has to be replaced by a function "a," which is, roughly speaking, the inverse of some local average of c, the inverse of a. In practice, however, we do not compute the matrix-valued function a; instead, we compute directly the functions $\mathbf{Q}\mathsf{m}$ and $\mathbf{Q}f$ by using the definition of the local solvers.

3.6. IP-H methods. The IP-H methods are obtained by using the numerical traces and the local solvers of the IP method. Thus,

1. the numerical traces are given by

(3.24)
$$\widehat{\mathbf{Q}}\mathsf{m} = -a\,\mathbf{grad}\,\mathcal{U}\mathsf{m} + \tau_K(\,\mathcal{U}\mathsf{m} - \mathsf{m})\,\boldsymbol{n},\,\widehat{\mathbf{Q}}f = -a\,\mathbf{grad}\,\mathcal{U}f + \tau_K(\,\mathcal{U}f)\,\boldsymbol{n}, \quad \text{on } \partial K;$$

2. the finite element space $\boldsymbol{V}(K) \times W(K)$ is defined for $k \geq 1$ as

(3.25) $$\boldsymbol{V}(K) = \mathcal{P}_k(K)^n, \quad W(K) = \mathcal{P}_k(K);$$

3. the space of approximate traces is chosen as

(3.26) $$\mathsf{M}_h := \mathcal{M}_{h,k}.$$

As before, τ is a double-valued function on $\mathcal{E}_h^\circ$, with two branches $\tau^- = \tau_{K^-}$ and $\tau^+ = \tau_{K^+}$ defined on the edge e shared by the finite elements K^- and K^+.

Note that IP methods can be defined by using a flux formulation, as the one employed here to define the local solvers or by means of a primal formulation; see [5]. These two IP methods, however, do coincide whenever the function a is a constant on each element $K \in \mathcal{T}_h$. For this reason, we are going to assume here that this is the case. All the results for this case, however, can be easily extended to the case in which a is not necessarily piecewise constant.

Next, we provide sufficient conditions for the IP-H method to be well defined. For simplicity, we assume that mesh $\mathcal{T}_h$ is shape regular, that is, that there is a constant $\gamma > 0$ such that $h_K/\rho_K \leq \gamma$ for all simplexes $K \in \mathcal{T}_h$, where h_K is the diameter of K and ρ_K the diameter of the largest ball contained in K.

PROPOSITION 3.6. *Let the numerical traces be given by* (3.24) *and the local spaces by* (3.25). *Suppose* $a(\boldsymbol{x})$ *is a constant matrix on each element* K. *Then Assumption* (2.1) *on the existence and the uniqueness of the local solvers holds provided* $\tau_K > c_0/h_K$ *for some constant* $c_0 > 0$ *depending on* γ *and* $a(\boldsymbol{x})$.

For a proof, see [6, 3]. Having established that the local solvers are well defined, we can apply Theorem 2.1. We find that the conservativity condition implies that λ_h solves (2.9), with

$$
\begin{aligned}
a_h(\eta,\mu) =&(c\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\,\mathcal{U}\mu)_{\mathcal{T}_h}\\
&+ \langle 1\,,\ [\![(\mu-\mathcal{U}\mu)(a\,\mathbf{grad}\,\mathcal{U}\eta+\mathbf{Q}\eta)]\!]\rangle_{\mathcal{E}_h},\\
&+ \langle 1\,,\ [\![(\mathcal{U}\mu-\mu)(\tau\,(\mathcal{U}\eta-\eta)\boldsymbol{n})]\!]\rangle_{\mathcal{E}_h},\\
b_h(\mu) =&\langle g_h\,,\ -a\mathbf{grad}\,\mathcal{U}\mu\cdot\boldsymbol{n}+\tau\,\mathcal{U}\mu\rangle_{\partial\Omega} + (f,\,\mathcal{U}\mu)_{\mathcal{T}_h},
\end{aligned}
$$

provided $g_h|_{\mathcal{E}_h^\circ}=0$. Using (2.12a) of Lemma 2.3 and the fact that $a(\boldsymbol{x})$ is constant on each K, we can simplify this expression as follows:

$$
\begin{aligned}
a_h(\eta,\mu) =&(c\,\mathbf{Q}\eta,\mathbf{Q}\mu)_{\mathcal{T}_h} - (c\,\mathbf{Q}\eta+\mathbf{grad}\,\mathcal{U}\eta,\mathbf{Q}\mu+a\mathbf{grad}\,\mathcal{U}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\,\mathcal{U}\mu)_{\mathcal{T}_h}\\
&+ \langle 1\,,\ [\![(\mathcal{U}\mu-\mu)(\tau\,(\mathcal{U}\eta-\eta)\boldsymbol{n})]\!]\rangle_{\mathcal{E}_h}\\
=&(a\mathbf{grad}\,\mathcal{U}\eta,\mathbf{grad}\,\mathcal{U}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\,\mathcal{U}\mu)_{\mathcal{T}_h}\\
&+ \langle 1\,,\ [\![((\mathcal{U}\eta-\eta)\,a\mathbf{grad}\,\mathcal{U}\mu+(\mathcal{U}\mu-\mu)\,a\mathbf{grad}\,\mathcal{U}\eta)]\!]\rangle_{\mathcal{E}_h}\\
&+ \langle 1\,,\ [\![(\mathcal{U}\mu-\mu)(\tau\,(\mathcal{U}\eta-\eta)\boldsymbol{n})]\!]\rangle_{\mathcal{E}_h}.
\end{aligned}
$$

The positive definiteness of the form $a_h(\cdot,\cdot)$ can be proven as in the case of LDG-H methods. Indeed, this fact is an immediate consequence of Theorem 2.4 and the following result.

PROPOSITION 3.7. *Let the numerical traces of the fluxes be set by* (3.24), *the local spaces be defined by* (3.25), *and the space of approximate traces be set by* (3.26). *Suppose $a(\boldsymbol{x})$ is a constant matrix on each element K. Then Assumption* 2.2 *on the positive semidefiniteness of the local solvers and Assumption* 2.3, *the gluing condition, are satisfied with $M(\partial K)=\{\mu:\mu|_e\in\mathcal{P}_k(e)$ for all faces $e\in\partial K\}$ whenever $\tau_K>c_0/h_K$ for some constant $c_0>0$ depending on γ and $a(\boldsymbol{x})$.*

The proof of this result is similar to that of Proposition 3.3.

Just as for LDG-H methods, we can give a characterization of the IP-H methods. It is given in the proposition below, which is an analog of Proposition 3.4 for the LDG-H methods. Since the proof is similar, we omit it.

PROPOSITION 3.8 (characterization of IP-H methods). *Let the numerical traces be set by* (3.24), *the spaces be as in* (3.25), *and $(\boldsymbol{q}_h,u_h)$ be as defined in* (2.5). *Then conservativity condition* (2.6) *holds if and only if on $\mathcal{E}_h^\circ$*

(3.27a)
$$
\lambda_h=\widehat{u}_h=\left(\frac{\tau^+}{\tau^-+\tau^+}\right)u_h^+ + \left(\frac{\tau^-}{\tau^-+\tau^+}\right)u_h^- - \left(\frac{1}{\tau^++\tau^-}\right)[\![a\mathbf{grad}\,u_h]\!],
$$

(3.27b)
$$
\widehat{\boldsymbol{q}}_h=-\left(\frac{\tau^-}{\tau^-+\tau^+}\right)a^+\mathbf{grad}\,u_h^+ - \left(\frac{\tau^+}{\tau^-+\tau^+}\right)a^-\mathbf{grad}\,u_h^- + \left(\frac{\tau^+\tau^-}{\tau^-+\tau^+}\right)[\![u_h]\!].
$$

We also have results analogous to Corollary 3.2.

COROLLARY 3.4. *The standard IP method is not an IP-H method for any finite τ.*

Proof. Comparing the numerical traces of the standard IP method (see [5, Table 3.1]), namely,

$$
\widehat{u}_h^{\mathrm{IP}}=\{\!\{u_h\}\!\}\ \text{ and }\ \widehat{\boldsymbol{q}}_h^{\mathrm{IP}}=-\{\!\{a\mathbf{grad}\,u_h\}\!\}+C\,[\![u_h]\!],
$$

with the expressions for the numerical traces in Proposition 3.8, we find that they cannot coincide for any value of τ. $\quad\square$

In spite of this negative result, a stabilized DG finite element method introduced in [38] and rewritten in [37] as an IP method, turns out to be an IP-H method. To describe this scheme in a simple setting, assume that $d = 0$ and $g = 0$. The method, as presented in [38], does not use the function λ_h approximating $u|_{\mathcal{E}_h}$. Instead, it uses approximate fluxes ℓ_h approximating the normal component of $a\mathbf{grad}u$. The space in which ℓ_h lies is the space of scalar double-valued functions defined by

$$L_h = \{q: \quad q|_e \in \mathcal{P}_k(e) \quad \text{for all } e \in \mathcal{E}_h \text{ and } q_{K^+} + q_{K^-} = 0 \ \text{ on } e = \partial K^+ \cap \partial K^-\}.$$

The DG method of [38] seeks $u_h \in W_h$, given by (2.1), with $W(K) = \mathcal{P}_k(K)$, and $\ell_h \in L_h$ such that

(3.28)
$$\sum_{K \in \mathcal{T}_h} \left\{ (a\mathbf{grad}u_h, \mathbf{grad}v)_K - \langle \ell_h, v\rangle_{\partial K} - \langle \eta, u_h\rangle_{\partial K} \right\}$$
$$- \alpha h \sum_{K \in \mathcal{T}_h} \langle \ell_h - a\mathbf{grad}u_h \cdot \boldsymbol{n}_K, \eta - a\mathbf{grad}v \cdot \boldsymbol{n}_K\rangle_{\partial K} = (f, v)$$

for all $v \in W_h$ and $\eta \in L_h$. Here, $\alpha > 0$ is a constant stabilization parameter, and $h = \max_{K \in \mathcal{T}_h} h_K$.

Taking $v \equiv 0$ and using that $\{\!\{\eta\}\!\} = 0$ on $\mathcal{E}_h^\circ$, we get

(3.29)
$$\ell_h = \begin{cases} \{\!\{a\mathbf{grad}u_h\}\!\} \cdot \boldsymbol{n} - \frac{1}{2\alpha h} [\![u_h]\!] \cdot \boldsymbol{n} & \text{on } \mathcal{E}_h^\circ, \\ a\mathbf{grad}u_h \cdot \boldsymbol{n} - \frac{1}{\alpha h} u_h & \text{on } \mathcal{E}_h^\partial. \end{cases}$$

We see from the above equation that ℓ_h is indeed an approximation to the normal component of $a\mathbf{grad}u$. Next, taking $\eta \equiv 0$ in (3.28) and substituting therein the expression for ℓ_h from (3.29), we get that $u_h \in W_h$ satisfies

(3.30)
$$(a\mathbf{grad}u_h, \mathbf{grad}v)_{\mathcal{T}_h} - \langle \{\!\{a\mathbf{grad}v\}\!\}, [\![u_h]\!]\rangle_{\mathcal{E}_h}$$
$$- \left\langle \{\!\{a\mathbf{grad}u_h\}\!\} - \frac{1}{2\alpha h} [\![u_h]\!], [\![v]\!] \right\rangle_{\mathcal{E}_h^\circ}$$
$$- \left\langle a\mathbf{grad}u_h - \frac{1}{\alpha h} u_h \boldsymbol{n}, v\boldsymbol{n} \right\rangle_{\mathcal{E}_h^\partial}$$
$$- \left\langle \frac{\alpha h}{2} [\![a\mathbf{grad}u_h]\!], [\![a\mathbf{grad}v]\!] \right\rangle_{\mathcal{E}_h^\circ} = (f, v)$$

for all $v \in W_h$.

Now, we show that this is an IP-H method. Comparing the above formulation with the general primal formulation given by [5, equation (3.11)], we can easily verify that if we take

(3.31)
$$\widehat{u}_h = \{\!\{u_h\}\!\} - \frac{\alpha h}{2} [\![a\mathbf{grad}u_h]\!] \quad \text{on } \mathcal{E}_h^\circ,$$
$$\widehat{\boldsymbol{q}}_h = \begin{cases} -\{\!\{a\mathbf{grad}u_h\}\!\} + \frac{1}{2\alpha h} [\![u_h]\!] & \text{on } \mathcal{E}_h^\circ, \\ -a\mathbf{grad}u_h + \frac{1}{\alpha h} u_h \boldsymbol{n} & \text{on } \ \mathcal{E}_h^\partial, \end{cases}$$

we recover (3.30). Hence, the above numerical traces are exactly the numerical traces of the IP-H method given by Proposition 3.8 with $\tau^+ = \tau^- = (\alpha h)^{-1}$. This shows that the DG method proposed in [38] is an IP-H method. The correspondence between their flux approximation ℓ_h and our numerical flux trace follows immediately from (3.31) and (3.29):

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = -\ell_h.$$

It also follows from Proposition 3.6 that the IP method of (3.30) is well defined when $\alpha > 0$ is sufficiently small; a result already established in [38].

Let us end by pointing out that other IP–H-like methods can be obtained. For example, we could take $\boldsymbol{V}(K) = \mathcal{P}_{k-1}(K)^n$.

3.7. The NC-H methods. We now consider nonconforming hybridizable (NC-H) methods and show that methods like the P_1-nonconforming method introduced in [36] in the framework of the stationary Stokes equations, are, in fact, NC-H methods.

Again the main components of the NC-H method are defined as follows:

1. For any $k \geq 1$, set

$$\boldsymbol{V}(K) = \mathcal{P}_{k-1}(K)^n, \quad W(K) = \mathcal{P}_k(K),$$
(3.32) $$\qquad M(\partial K) = \{q : \ q|_e \in \mathcal{P}_{k-1}(e) \text{ for every face } e \text{ of } K\},$$
$$\boldsymbol{T}(\partial K) = \{q\boldsymbol{n}_K : \ q|_e \in \mathcal{P}_{k-1}(e) \text{ for every face } e \text{ of } K\}.$$

2. Define local solutions $(\boldsymbol{Q}\mathsf{m}, \mathcal{U}\mathsf{m}, (\widehat{\boldsymbol{Q}}\mathsf{m})_K)$ and $(\boldsymbol{Q}f, \mathcal{U}f, (\widehat{\boldsymbol{Q}}f)_K)$ as the elements of $\boldsymbol{V}(K) \times W(K) \times \boldsymbol{T}(\partial K)$ satisfying

(3.33a) $$\qquad (c\,\boldsymbol{Q}\mathsf{m}, \boldsymbol{v})_K - (\mathcal{U}\mathsf{m}, \operatorname{div}\boldsymbol{v})_K = -\langle \mathsf{m}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial K},$$

(3.33b) $$\qquad -(\mathbf{grad}w, \boldsymbol{Q}\mathsf{m})_K + \left\langle w, \widehat{\boldsymbol{Q}}\mathsf{m}\cdot \boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}\mathsf{m}, w)_K = 0,$$

(3.33c) $$\qquad \langle \mathcal{U}\mathsf{m}, \mu\rangle_{\partial K} = \langle \mathsf{m}, \mu\rangle_{\partial K},$$

for all $\boldsymbol{v} \in \boldsymbol{V}(K)$, $w \in W(K)$, and $\mu \in M(\partial K)$, and

(3.34a) $$\qquad (c\,\boldsymbol{Q}f, \boldsymbol{v})_K - (\mathcal{U}f, \operatorname{div}\boldsymbol{v})_K = 0,$$

(3.34b) $$\qquad -(\mathbf{grad}w, \boldsymbol{Q}f)_K + \left\langle w, \widehat{\boldsymbol{Q}}f\cdot \boldsymbol{n}\right\rangle_{\partial K} + (d\,\mathcal{U}f, w)_K = (f, w),$$

(3.34c) $$\qquad \langle \mathcal{U}f, \mu\rangle_{\partial K} = 0,$$

$\boldsymbol{v} \in \boldsymbol{V}(K)$, $w \in W(K)$, and $\mu \in M(\partial K)$.

3. The space of approximate traces is given by $\mathsf{M}_h = \mathcal{M}_{h,k-1}$.

Having completed the definition of the main ingredients of the method, we now verify the assumptions of Theorem 2.4.

Sufficient conditions under which Assumption 2.1 on the existence and the uniqueness of the local solvers hold are given next.

PROPOSITION 3.9. *For $k = 1$ and arbitrary n and for odd $k > 1$ and $n = 2$, local solvers (3.33) and (3.34) have unique solutions.*

Proof. We prove only the result for the first local solver, since the other can be proven in a similar way. Since (3.33) is a square system, it suffices to prove that if $\mathsf{m} = 0$, then $\boldsymbol{Q}\mathsf{m} = 0$, $\mathcal{U}\mathsf{m} = 0$, and $\widehat{\boldsymbol{Q}}\mathsf{m} = 0$. Choosing $\boldsymbol{v} = \boldsymbol{Q}\mathsf{m}$ and $w = \mathcal{U}\mathsf{m}$,

adding (3.33a) and (3.33b), and integrating by parts, we get

$$(c\,\mathbf{Q}\mathsf{m}, \mathbf{Q}\mathsf{m})_K + (d\,\mathcal{U}\mathsf{m}, \mathcal{U}\mathsf{m})_K + \left\langle \mathcal{U}\mathsf{m}, \left(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m}\right)\cdot\boldsymbol{n}\right\rangle_{\partial K} = 0.$$

If $\mathsf{m} = 0$, (3.33c) implies that $\langle \mathcal{U}\mathsf{m}, \mu\rangle_{\partial K} = 0$ for all $\mu \in M(\partial K)$. Since $(\widehat{\mathbf{Q}}\mathsf{m} - \mathbf{Q}\mathsf{m})\cdot\boldsymbol{n} \in M(\partial K)$, then the last term on the left-hand side above is zero, and hence, $\mathbf{Q}\mathsf{m} = 0$ and $d\,\mathcal{U}\mathsf{m} = 0$. Substituting this into (3.33a), we have

$$0 = (\mathcal{U}\mathsf{m}, \operatorname{div}\boldsymbol{v})_K = -(\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_K \quad \text{for all } \boldsymbol{v} \in \mathcal{P}_{k-1}(K)^n,$$

where, while integrating by parts, we have again used that $\langle \mathcal{U}\mathsf{m}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0$. Thus, $\mathbf{grad}\,\mathcal{U}\mathsf{m}$ vanishes, so $\mathcal{U}\mathsf{m}$ is a constant function, and $\langle \mathcal{U}\mathsf{m}, \mu\rangle_{\partial K} = 0$ implies that it vanishes identically.

It remains to show that $\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}$ also vanishes. Since both $\mathbf{Q}\mathsf{m}$ and $d\,\mathcal{U}\mathsf{m}$ vanish, (3.33b) implies that

$$(3.35) \qquad \left\langle w, \widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}\right\rangle_{\partial K} = 0 \quad \text{for all } w \in \mathcal{P}_k(K).$$

For $k = 1$, that is, for Crouzeix–Raviart nonconforming finite elements, the result follows easily for any dimension $n \geq 2$. Indeed, let $\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}|_{e_j} = a_j$ for some constants $a_j, j = 1,\dots,n+1$. Let $w \in \mathcal{P}_1(K)$ be a linear function on K which takes values a_j at the centroids of the faces e_j of K, $j = 1,\dots,n+1$. Then $0 = \langle w, \widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}\rangle_{\partial K} = \sum_{j=1}^{n+1} |e_j|a_j^2$ implies $a_j = 0$ for all faces, that is, $\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n} = 0$.

Finally, we show the same for k odd and $n = 2$. Let e_1, e_2, and e_3 denote the three edges of K, and let $L_i^{(j)}$ denote the ith Legendre polynomial mapped affinely to e_j from $[-1, 1]$. Assume that the first vertex of the edge e_j is mapped to the point -1, and that, as we go from its first to its second vertex, the triangle K is to our left. Since $\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}|_{e_j} \in \mathcal{P}_{k-1}(e_j)$, we can write

$$\left(\widehat{\mathbf{Q}}\mathsf{m}\right)_K \cdot \boldsymbol{n}_K|_{e_j} = \sum_{i=0}^{k-1} a_i^{(j)} L_i^{(j)}.$$

Note that when i is even, $L_i^{(j)}$ takes the same value at the endpoints of e_j. Therefore, for any even i, we can choose a w in (3.35) such that $w|_{e_1} = L_i^{(1)}$, $w|_{e_2} = -L_k^{(2)}$, and $w|_{e_3} = L_k^{(3)}$ (because with these choices $w|_{\partial K}$ is continuous). Then (3.35) implies that the coefficient $a_i^{(1)}$ vanishes. Repeating the argument for all edges, we find that $a_i^{(j)} = 0$ for all even i and $j = 1, 2, 3$. Next, for odd i, choose w such that $w|_{e_1} = L_i^{(1)}$, $w|_{e_2} = L_{i-1}^{(2)}$, and $w|_{e_3} = -L_k^{(3)}$. Since k is odd, these choices make $w|_{\partial K}$ continuous, so such a w can be found. With this w, (3.35) now gives that $a_i^{(1)} = 0$ for all odd i as well. Repeating this argument for other edges, we find all coefficients to be zero, so $\widehat{\mathbf{Q}}\mathsf{m}$ vanishes. □

Conservativity condition (2.6) with $\mathsf{M}_h = \mathcal{M}_{h,k-1}$ clearly implies strong conservativity. Using Theorem 2.1 and noting that the unknown fluxes $\widehat{\mathbf{Q}}\cdot$ cancel off in weak formulation (2.9), by boundary condition (3.33c) for the local solver, we have that the bilinear form is symmetric:

$$a_h(\eta, \mu) = (c\,\mathbf{Q}\eta, \mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta, \mathcal{U}\mu)_{\mathcal{T}_h}.$$

Its positive definiteness will follow from Theorem 2.4 once Assumptions 2.2 and 2.3 are verified, which we do next.

PROPOSITION 3.10. *Assumption* 2.2 *on the positive semidefiniteness of the local solvers and Assumption* 2.3, *the gluing condition, are satisfied with* $M(\partial K)$ *defined as in* (3.32).

Proof. First, we show that condition (2.13a) holds. Taking $\boldsymbol{v} = \mathbf{Q}\mathsf{m}$ in (3.33a), $w = \mathcal{U}\mathsf{m}$ in (3.33b), and adding the equations, we get, after a few simple algebraic manipulations, that

$$-\left\langle \mathsf{m}\,,\,\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}\right\rangle_{\partial K} = (c\,\mathbf{Q}\mathsf{m},\mathbf{Q}\mathsf{m})_K + (d\,\mathcal{U}\mathsf{m},\,\mathcal{U}\mathsf{m})_K + \left\langle\left(\widehat{\mathbf{Q}}\mathsf{m}-\mathbf{Q}\mathsf{m}\right)\cdot\boldsymbol{n}\,,\,\mathcal{U}\mathsf{m}-\mathsf{m}\right\rangle_{\partial K}$$
$$= (c\,\mathbf{Q}\mathsf{m},\mathbf{Q}\mathsf{m})_K + (d\,\mathcal{U}\mathsf{m},\,\mathcal{U}\mathsf{m})_K,$$

by boundary condition (3.33c) for the local solver. This implies that (2.13a) of Assumption 2.2 is satisfied.

Now, we prove condition (2.13b). If $\langle \mathsf{m}\,,\,\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n}\rangle_{\partial K} = 0$, then $\mathbf{Q}\mathsf{m}|_K = 0$ and (3.33a) becomes

$$(\mathbf{grad}\,\mathcal{U}\mathsf{m}, \boldsymbol{v})_K = \langle\,\mathcal{U}\mathsf{m} - \mathsf{m}\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K} = 0 \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{V}(K).$$

This implies that that $\mathcal{U}\mathsf{m}$ is a constant. This shows that condition (2.13b) of Assumption 2.2 is satisfied.

Assumption 2.3 is trivially satisfied, and this completes the proof. □

In Tables 3.1 and 3.2, we give the simplified weak formulation of the NC-H method under the further assumption that $c(\boldsymbol{x})$ is a constant matrix on each K in $\mathcal{T}_h$. In this case, we can show that the original NC method is an NC-H method. To see why, first observe that by summing up the last equation of the local solvers, we find that $u_h = \mathcal{U}\lambda_h + \mathcal{U}g_h + \mathcal{U}f$ satisfies

$$\langle\,[\![u_h]\!]\,,\,\mu\rangle_e = 0 \quad \text{for all } \mu \in \mathcal{P}_{k-1}(e)$$

for all interior faces e, so the weak continuity constraints of the discontinuous method are satisfied. Now, (2.12a) and (2.12c) become $(c\,\mathbf{Q}\lambda_h + \mathbf{grad}\,\mathcal{U}\lambda_h, \boldsymbol{v})_{\mathcal{T}_h} = 0$ and $(c\,\mathbf{Q}f + \mathbf{grad}\,\mathcal{U}f, \boldsymbol{v})_{\mathcal{T}_h} = 0$, which gives

$$\boldsymbol{q}_h = \mathbf{Q}\lambda_h + \mathbf{Q}g_h + \mathbf{Q}f = -a\mathbf{grad}(\mathcal{U}\lambda_h + \mathcal{U}g + \mathcal{U}f) = -a\mathbf{grad}u_h.$$

Then (2.8a) implies

$$(a\mathbf{grad}u_h, \mathbf{grad}v_h)_{\mathcal{T}_h} + (du_h, v_h)_{\mathcal{T}_h} = (f, v_h)$$

for all $v_h \in \{w : w \in W_h, \langle\,[\![w]\!]\,,\,\mu\rangle_{\mathcal{E}_h^\circ} = 0$ for all $\mu \in M_h$ and $\langle w\,,\,\mathsf{m}\rangle_{\partial\Omega} = 0$ for all $\mathsf{m} \in \mathsf{M}_h\}$, which is the familiar primal form of this nonconforming method. Note that although the information in g_h disappears from the right-hand side above, it is contained in u_h as $u_h = \mathcal{U}\lambda_h + \mathcal{U}g_h + \mathcal{U}f$.

Let us end this subsection by pointing out that, in the case of lowest order polynomials $k = 1$ and for the case in which $d = 0$ and both c and f are constant on each simplex K of triangulation $\mathcal{T}_h$, our hybridization framework allows us to recover a well-known relationship between the RT method of lowest degree and the nonconforming method [4, 47]. Let us sketch how to obtain it. In this case, we can easily show that local solver $\mathbf{Q}\mathsf{m}$ is the *same* for both this nonconforming method and that of the RT method of lowest degree; see the computation of the RT method in [26]. Since we also have that $\widehat{\mathbf{Q}}\mathsf{m}\cdot\boldsymbol{n} = \mathbf{Q}\mathsf{m}\cdot\boldsymbol{n}$, we can conclude that the stiffness matrix associated with bilinear form $a_h(\cdot,\cdot)$ of both methods is also the same—if the degrees

of freedom for the numerical traces are the barycenters of the faces. Moreover, since the average on each simplex of the local solver $\mathcal{U}\mathsf{m}$ coincides with the local solver $\mathcal{U}\mathsf{m}$ of the RT method under consideration, the matrix associated with linear form $b_h(\cdot)$ is also the same for both methods. Of course, in both cases, we take g_h at the barycenter of each face $e \in \mathcal{E}_h^\partial$ to be the average of g on the face e. By Theorem 2.1, the degrees of freedom of the approximate traces are the same for both methods. The above-mentioned relation between the two methods now easily follows from the definition of approximate solutions (2.5).

4. Other novel methods. In this section, we build on the work done in the previous section and construct what are perhaps the three most important examples of methods of the unifying framework. The first is a class of methods employing different local solvers in different parts of the domain, which can easily deal with nonconforming meshes. The second is an RT method that can handle hanging nodes. The third is the family of EDG methods; they are constructed from already known hybridized methods in this unified framework in order to reduce their computational complexity. As for the examples of the previous section, we assume that the mesh is simplicial; however, we do not assume it to be necessarily conforming.

4.1. A class of hybridizable methods well suited for adaptivity. We introduce here a class of hybridizable methods able to use different local solvers in different elements and to easily handle nonconforming meshes. They are thus ideal to use with adaptive strategies. After introducing the methods, we prove that they are all well defined. We then discuss their main advantages and give several examples.

To define the methods, we need to specify the numerical fluxes, the local finite element spaces, and the space of approximate traces:

1. For any simplex $K \in \mathcal{T}_h$, we take

$$(4.1) \qquad \widehat{\mathbf{Q}}\mathsf{m} = \mathbf{Q}\mathsf{m} + \tau_K(\mathcal{U}\mathsf{m} - \mathsf{m})\boldsymbol{n}, \quad \widehat{\mathbf{Q}}f = \mathbf{Q}f + \tau_K(\mathcal{U}f)\boldsymbol{n} \quad \text{on } \partial K;$$

the function τ_K is allowed to change on ∂K.

2. The local space $\boldsymbol{V}(K) \times W(K)$ can be any of the following:

$$(4.2a) \qquad \left(\mathcal{P}_{k(K)}(K)^n + \boldsymbol{x}\,\mathcal{P}_{k(K)}(K)\right) \times \mathcal{P}_{k(K)}(K),$$
$$\text{where } k(K) \geq 0 \text{ and } \tau_K \geq 0 \text{ on } \partial K,$$

$$(4.2b) \qquad \mathcal{P}_{k(K)}(K)^n \times \mathcal{P}_{k(K)-1}(K),$$
$$\text{where } k(K) \geq 1 \text{ and } \tau_K \geq 0 \text{ on } \partial K,$$

$$(4.2c) \qquad \mathcal{P}_{k(K)}(K)^n \times \mathcal{P}_{k(K)}(K),$$
$$\text{where } k(K) \geq 0 \text{ and } \tau_K > 0 \text{ on at least one face } e \text{ of } K,$$

$$(4.2d) \qquad \mathcal{P}_{k(K)-1}(K)^n \times \mathcal{P}_{k(K)}(K),$$
$$\text{where } k(K) \geq 1 \text{ and } \tau_K > 0 \text{ on } \partial K.$$

3. The space of approximate traces is

$$(4.3a) \qquad \mathsf{M}_h = \mathfrak{M}_h \cap \left\{\mu : \mu|_{\partial K} \in \mathcal{C}(\overline{\{\boldsymbol{x} \in \partial K : \tau_K(\boldsymbol{x}) = \infty\}}) \quad \forall K \in \mathcal{T}_h\right\},$$

where

$$(4.3b) \qquad \mathfrak{M}_h := \{\mu \in L^2(\mathcal{E}_h) : \ \mu|_e \in \mathcal{P}_{k(e)}(e) \text{ for all } e \in \mathcal{E}_h^\circ\}.$$
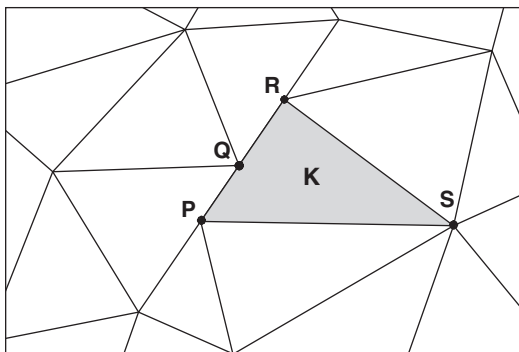
FIG. 4.1. *The interior edges $e = \mathbf{PQ}$ and $e = \mathbf{QR}$ are contained in the face $\mathbf{PR}$ of the element K. Assumption* 4.1 *is satisfied for this element if $\tau_K|_{\partial K} \in [0, \infty]$ and if $\tau_K|_{\mathbf{PQ}}$ and $\tau_K|_{\mathbf{QR}}$ are taken in $(0, \infty)$.*

Here, if $e = \partial K^+ \cap \partial K^-$, we set

$$(4.3c) \qquad k(e) := \begin{cases} \max\{k(K^+), k(K^-)\} & \text{if } \tau^+ < \infty \text{ and } \tau^- < \infty, \\ k(K^+) & \text{if } \tau^+ = \infty \text{ and } \tau^- < \infty, \\ k(K^-) & \text{if } \tau^+ < \infty \text{ and } \tau^- = \infty, \\ \min\{k(K^+), k(K^-)\} & \text{if } \tau^+ = \infty \text{ and } \tau^- = \infty, \end{cases}$$

and take $g_h = I_h g$ for some interpolation operator I_h into M_h.

Note that the choice $\tau = \infty$ on some interior faces $e \in \mathcal{E}_h^\circ$ is allowed. We follow the convention that in this case, the definition of the local solvers has to be modified as for the limiting cases of LDG-H methods; see subsection 3.4. The definition of the methods is competed with the following assumption on the values of the stabilization parameter τ.

Assumption 4.1. For each element $K \in \mathfrak{T}_h$ and each interior face $e \in \mathcal{E}_h^\circ$ on ∂K, $\tau_K|_e \in [0, \infty]$ and

$$(4.4) \qquad \qquad \tau_K|_e \in (0, \infty) \text{ if } e \text{ is not a face of } K.$$

Let us briefly discuss this assumption. First, let us recall the difference between an interior face $e \in \mathcal{E}_h^\circ$ and the faces of the simplexes of the triangulation. Each simplex K in the partition $\mathfrak{T}_h$ has $n + 1$ faces determined by its vertices. On the other hand, if e is an interior face, we have that $e = \partial K^+ \cap \partial K^-$ for some elements K^+ and K^- in $\mathfrak{T}_h$. We thus see that, for nonconforming meshes, although each interior face e is contained in a face of K^+ and a face of K^-, it is not necessarily a face of K^+ or K^-. See an example in Figure 4.1. The main motivation of the above assumption can now be easily seen. Indeed, take any $K \in \mathfrak{T}_h$. If $e \subset \partial K$ is a face in $\mathcal{E}_h^\circ$ which is *not a face* of K, then the above assumption forces us to take the numerical trace corresponding to an LDG-H method; in this way, the nonconformity of the mesh can be dealt with in a very natural way. If, on the contrary, e is actually a face, the assumption allows us to take either $\tau_K = 0$, $\tau_K \in (0, \infty)$, or even $\tau_K = \infty$. In this way, the verification of Assumptions 2.1, 2.2, and 2.3 becomes extremely easy, as we are going to see next.

Next, we show that the approximate solution $(\boldsymbol{q}_h, u_h)$, (2.5), provided by this method is well defined.

PROPOSITION 4.1. *Consider the method defined by* (4.1), (4.2), *and* (4.3), *and let Assumption* 4.1 *hold. Then Assumption* 2.1 *on the existence and the uniqueness of*

the local solvers, Assumption 2.2 *on the positive semidefiniteness of the local solvers, and Assumption* 2.3, *the gluing condition, hold with*

$$
(4.5) \quad M(\partial K) = \{\mu : \quad \text{on any face } e \in \mathcal{E}_h^\circ \text{ on } \partial K, \quad \mu|_e \in \mathcal{P}_{k(K)}(e) \text{ if } \tau_K|_e = 0,
$$
$$
\text{and} \quad \mu|_e \in L^2(e) \text{ if } \tau_K|_e > 0\}.
$$

Proof. Thanks to Theorem 2.4, we have only to satisfy Assumptions 2.1, 2.2, and 2.3. We begin by verifying Assumption 2.1 on the existence and the uniqueness of the local solvers. Let K be an arbitrary simplex of triangulation $\mathcal{T}_h$. Then, as discussed above, by condition (4.4), we have either $\tau_K = 0$, $\tau_K \in (0, \infty)$, or $\tau_K = \infty$ on each of the faces of each simplex K of triangulation $\mathcal{T}_h$. As a consequence, the fact that the local solvers are well defined can be easily obtained by a straightforward modification of the proofs of similar results for the LDG-H methods, Proposition 3.5, and the CG-H method, Proposition 3.5. For this reason, we do not present here the proof. However, let us note that whenever $\tau_K|_e = \infty$, we strongly impose a Dirichlet boundary condition, and so the space of approximate traces restricted to ∂K and local space $W(K)$ must satisfy the following compatibility condition:

$$
\{\mu|_S : \quad \mu \in \mathsf{M}_h\} \subset \{w|_S : \quad w \in W(K)\}, \quad \text{where} \quad S := \overline{\{\boldsymbol{x} \in \partial K : \tau_K(\boldsymbol{x}) = \infty\}}.
$$

This condition can be easily verified by noting that, if $\tau = \infty$ on the interior face $e \in \mathcal{E}_h^\circ$, then e must be a face of K by the conditions on the stabilization parameters (4.4), and since, by the definition of $k(e)$, (4.3c), we have that $k(e) \leq k(K)$.

Next, let us prove that Assumption 2.2 on the positive semidefiniteness of the local solvers is satisfied with $M(\partial K)$ as in (4.5). For choice (4.2a), it is easy to see that it follows from Proposition 3.1 and from the definition of $k(e)$, (4.3c). For the remaining choices, the result follows from Proposition 3.3 and the definition of $k(e)$, (4.3c).

Assumption 2.3, the gluing condition, also follows by using the arguments of the previous section. Indeed, for an interior face $e = \partial K^+ \cap \partial K^-$, if τ^+ or τ^- is positive, the result trivially follows from condition (4.3c) and the fact that on e, one of the projections $P_{\partial K^+}$ or $P_{\partial K^-}$ becomes the identity by the definition of $k(e)$, (4.3c). It remains to consider the case $\tau^+ = \tau^- = 0$. By (4.3c), either $k(K^+)$ or $k(K^-)$ equals $k(e)$, say, $k(e) = k(K^+)$. Then we immediately have that $P_{\partial K^+}\mu = \mu$. This completes the proof. $\quad\square$

Next, let us discuss the main features of these methods.

(i) **Variable degree approximation spaces on conforming meshes**. The RT-H, BDM-H, and LDG-H methods considered in the previous section used a single local solver in each of the elements K of the conforming triangulation $\mathcal{T}_h$. A variable-degree version of each of these methods is a particular case of the class of methods presented here. Note that the case of the variable degree RT method, introduced and analyzed in [27], is exactly the variable-degree version of the method using the RT method as local solvers.

(ii) **Automatic coupling of different methods on conforming meshes**. The methods presented here allow for the use of different local solvers in different elements K of $\mathcal{T}_h$, which are then automatically coupled. For example, if we are working with the RT, LDG, and CG local solvers, the conservativity condition implicitly imposes

the following expressions for the numerical traces:

$$\widehat{u}_h = u_h|_{\Omega_{\mathrm{LDG}}} + \frac{1}{\tau_{\mathrm{LDG}}}\, [\![\boldsymbol{q}_h]\!], \ \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h|_{\Omega_{\mathrm{RT}}} \qquad \text{(coupling RT and LDG)},$$

$$\widehat{u}_h = u_h|_{\Omega_{\mathrm{CG}}}, \qquad\qquad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h|_{\Omega_{\mathrm{LDG}}} + \tau_{\mathrm{LDG}}\, [\![u_h]\!] \ \text{(coupling of LDG and CG)},$$

$$\widehat{u}_h = u_h|_{\Omega_{\mathrm{CG}}}, \qquad\qquad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h|_{\Omega_{\mathrm{RT}}}. \qquad\qquad \text{(coupling of CG and RT)}.$$

Note that this coupling holds even for nonconforming meshes.

It is interesting to compare the above couplings with other couplings in the available literature, namely,

$$\widehat{u}_h = u_h|_{\Omega_{\mathrm{LDG}}}, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h|_{\Omega_{\mathrm{RT}}} + C_{11}\, [\![u_h]\!] \qquad \text{(coupling of RT and LDG in [23])},$$

$$\widehat{u}_h = u_h|_{\Omega_{\mathrm{CG}}}, \quad \widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h|_{\Omega_{\mathrm{LDG}}} + \tau_{\mathrm{LDG}}\, [\![u_h]\!] \quad \text{(coupling of LDG and CG in [48])}.$$

(iii) **Mortaring capabilities (for nonconforming meshes)**. One of the advantageous features of DG methods is their ability to handle nonconforming meshes; see [52] for an application to structural mechanics. The methods under consideration incorporate this *mortaring* ability thanks to the very form that the numerical trace of the flux on ∂K takes on an interior face $e \in \mathcal{E}_h^\circ$ which is not a face of K, and thanks to the definition of the stabilization parameter τ therein. Let us give two examples.

If we have a conforming mesh, we can take the first choice of local spaces (4.2a) and set $\tau \equiv 0$. The resulting method, as we have seen, is nothing but the RT-H method. We can easily modify this method to handle nonconforming meshes by simply taking $\tau_K \in (0, \infty)$ on every interior face $e \in \mathcal{E}_h^\circ$ which is not a face of K, and otherwise, taking $\tau_K = 0$. Thus, the resulting method can be considered as a variation of the RT method, which is capable of handling nonconforming meshes.

We can do something similar with the CG method. Indeed, if the mesh is conforming, we can take the last choice of local spaces (4.2d) and set $\tau \equiv \infty$ to obtain the hybridized CG method. For nonconforming meshes, we can slightly modify the method by simply taking $\tau_K \in (0, \infty)$ on every interior face $e \in \mathcal{E}_h^\circ$ which is not a face of K, and otherwise, taking $\tau_K = \infty$. The resulting method is thus a variation of the CG method capable of handling nonconforming meshes. It constitutes an alternative to the coupling of the CG and the LDG methods proposed in [48] to deal with nonmatching meshes.

(iv) **The conservativity condition**. Let us end by noting that the stiffness matrix associated to the approximate trace λ_h is always symmetric and positive definite. Moreover, on the interior faces on which $\tau < \infty$, the conservativity condition is enforced strongly.

4.2. The RT method on meshes with hanging nodes. Consider the case of the variable degree RT-H method. The method is obtained by taking the numerical traces as in (4.1) with $\tau \equiv 0$, the local space (4.2a), and the multiplier space as in (4.3). This method does not belong to the family of methods described in the previous subsection because our choice of stabilization parameter does not satisfy condition (4.4). Thus, to ensure the existence and the uniqueness of the approximate solution, we have to impose special conditions on the meshes and *link* the definition of $k(K)$ to the structure of the mesh.

Let us illustrate how to do this in the two-dimensional case. The meshes $\mathcal{T}_h$ we consider are constructed as follows. First, construct a conforming triangulation of Ω, $\mathcal{T}_h^{(0)} := \{K^{(0)}\}$. Then, take a subset of that triangulation $\mathcal{T}_h^{(0,1)}$ and divide each of its triangles into four congruent triangles; the set of those triangles is denoted by
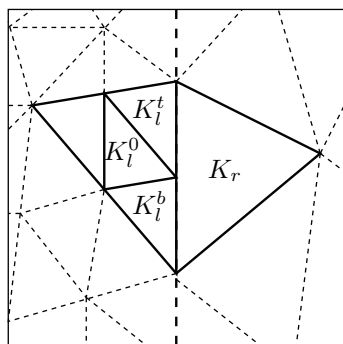
FIG. 4.2. *In the presence of hanging nodes like the above, an RT-H method with the spaces on edges chosen to have the maximum degree from either side is well defined if $k(K_l^t) \geq k(K_r)$ and $k(K_l^b) \geq k(K_r)$. The degree $k(K_l^0)$ can be arbitrarily chosen.*

$\mathcal{T}_h^{(1)}$. Next, for $j = 2, \ldots, \ell$, given the set $\mathcal{T}_h^{(j-1)}$, pick the subset $\mathcal{T}_h^{(j-1,j)}$ and create the set of smaller triangles $\mathcal{T}_h^{(j)}$. The simple case $\ell = 1$ is illustrated in Figure 4.2. Finally, we establish a link between the mesh and the definition of the polynomial degree of the RT method on the triangle K, $k(K)$, as follows. If $e = \partial K^+ \cap \partial K^-$, we take $k(e) := \max\{k(K^+), k(K^-)\}$ and require that

$$(4.6) \qquad \text{if } e \text{ is not an edge of } K^-, \text{ then } k(K^+) \geq k(K^-).$$

Next, we show that the method is well defined.

PROPOSITION 4.2.   *The variable-degree RT-H method on meshes with hanging nodes as described above is uniquely solvable.*

Proof. If we proceed exactly as in Proposition 3.1, we can see that Assumption 2.1 on the existence and the uniqueness of the local solvers is verified and that Assumption 2.2 on the positive semidefiniteness of the local solvers is also verified *provided* we change the definition of the set $M(\partial K)$ to

$$M(\partial K) = \{\mu : \mu|_e \in \mathcal{P}_{k(e)}(e) \text{ for all edges } e \text{ of } \partial K\}.$$

The result follows if we prove that there is only one solution $\lambda_h \in M_h$ of weak formulation (1.7).

To do that, we proceed exactly as in the proof of Theorem 2.4. First, since Assumption 2.1 holds, we have that system (2.9) is well defined. Next, we show that $a_h(\mu, \mu) = 0$ for $\mu \in M_h$ implies $\mu = 0$. By Assumption 2.2, we readily obtain that, for any given $K \in \mathcal{T}_h$, we have that, on ∂K,

$$C_K = P_{\partial K}\mu,$$

where $P_{\partial K}$ is the L²-projection into $M(\partial K)$ as defined above. It remains to show that this implies that μ is a constant on $\mathcal{E}_h$. To do that, we use the structure of the meshes and the definition of $k(K)$ for all $K \in \mathcal{T}_h$.

We proceed as follows. We claim that, for $j = \ell, \ell - 1, \ldots, 0$, we have that $\mu|_{\partial K}$ is a constant for all $K \in \mathcal{T}_h^{(j)}$. This immediately implies that μ is a constant on $\mathcal{E}_h$, and since $\mu|_{\partial \Omega} = 0$, that $\mu = 0$ on $\mathcal{E}_h$.

It remains to prove the claim. We proceed by induction on j. Let us prove the inductive hypothesis for $j = \ell$. Let K be any triangle in $\mathcal{T}_h^{(\ell)}$ and pick any of its edges

e. If the edge e lies on $\partial\Omega$, we immediately have that $\mu = C_K = 0$. If $e = \partial K \cap \partial K'$ for some triangle $K' \in \mathcal{T}_h^{(\ell)}$, we proceed as in the proof of Theorem 2.4 to conclude that

$$\mu = C_K = C_{K'} \qquad \text{on } e.$$

The only other remaining possibility, by construction of triangulation $\mathcal{T}_h^{(\ell)}$, is that $e = \partial K \cap \partial K'$ for some triangle $K' \in \mathcal{T}_h^{(\ell')}$, with $\ell' < \ell$. In this case, e is not an edge of K', and, by condition (4.6) on $k(K)$, we have that $k(K) \geq k(K')$ and hence $k(e) := \max\{k(K), k(K')\} = k(K)$. This implies that

$$\mu = C_K \qquad \text{on } e.$$

Since edge e was picked arbitrarily, we conclude that $\mu|_{\partial K}$ is a constant, as wanted.

Now, let us assume that the inductive hypothesis holds for $j = J$ and let us prove it also holds for $j = J - 1$. Let K be any triangle in $\mathcal{T}_h^{(J-1)}$ and pick any of its edges e. Since, by the inductive hypothesis, $\mu|_{\partial K}$ is a constant for all $K \in \mathcal{T}_h^{(J)}$, we have that $\mu|_{\partial K}$ is a constant for all $K \in \mathcal{T}_h^{(J-1,J)}$, since, by construction, each of the triangles in $\mathcal{T}_h^{(J-1,J)}$ is subdivided in four congruent triangles in $K \in \mathcal{T}_h^{(J)}$. Hence, $\mu = C_K$ on e if the edge e lies in the border of any triangle in $K \in \mathcal{T}_h^{(J-1,J)}$. To finish the proof, we need only to prove the same result in the remaining three cases: (i) if the edge e lies on $\partial\Omega$, (ii) if $e = \partial K \cap \partial K'$ for some triangle $K' \in \mathcal{T}_h^{(J-1)} \setminus \mathcal{T}_h^{(J-1,J)}$, and (iii) if $e = \partial K \cap \partial K'$ for some triangle $K' \in \mathcal{T}_h^{(J'-1)}$, with $J' < J$. This can be done exactly as in the previous step. $\square$

4.3. The EDG methods. Now we show that new methods [33] can be immediately generated from already existing hybridized methods by simply reducing the space of their approximate traces. The main interest of these EDG methods, introduced in the setting of shells problems in [43], stems from the further reduction in globally coupled unknowns achieved by reducing the approximate trace space M_h.

To construct such methods, we begin with selecting any method defined by uniquely solvable local problems (2.3), (2.4), and conservativity condition (2.6), yielding a unique approximate trace λ_h. Then, by Theorem 2.1, $\lambda_h \in M_h$ is the only solution of the weak formulation

$$(4.7) \qquad a_h(\lambda_h, \mu) = b_h(I_h g; \mu) \qquad \text{for all } \mu \in M_h,$$

where we are writing $b_h(I_h g; \mu)$ instead of just $b_h(\mu)$ in order to stress its dependency on $I_h g \in \mathsf{M}_h$. We now define an EDG method by replacing the original approximate trace space M_h by a subspace $\widetilde{\mathsf{M}}_h$. This forces us to replace $I_h g \in \mathsf{M}_h$ by $\widetilde{I}_h g \in \widetilde{\mathsf{M}}_h$ and to change the conservativity condition, but the local solvers remain the same. Now, define the operator $\mathcal{J}_h : \widetilde{\mathsf{M}}_h \to \mathsf{M}_h$ as the identity operator representing the natural *embedding* of $\widetilde{\mathsf{M}}_h$ into M_h, hence, the name of these methods, and set

$$\widetilde{M}_h := \left\{ \widetilde{\mu} \in \widetilde{\mathsf{M}}_h : \widetilde{\mu} = 0 \text{ on } \partial\Omega \right\}.$$

Then by Theorem 2.1, the new conservativity condition is equivalent to

$$(4.8) \qquad a_h\left(\mathcal{J}_h \widetilde{\lambda}_h, \mathcal{J}_h \widetilde{\mu}\right) = b_h\left(\mathcal{J}_h \widetilde{I}_h g; \mathcal{J}_h \widetilde{\mu}\right) \qquad \text{for all } \widetilde{\mu} \in \widetilde{M}_h,$$

where $\widetilde{\lambda}_h \in \widetilde{M}_h$ is the new approximate trace. Note that we have that $\mathcal{J}_h \widetilde{\mu}|_{\partial\Omega} = 0$ for all $\widetilde{\mu} \in \widetilde{M}_h$.

To show that this EDG method is well defined, it suffices to prove that homogeneous equation (4.8) has only a trivial solution. For simplicity, let us assume that $a_h(\cdot, \cdot)$ is symmetric and positive definite. Thus, taking $\widetilde{\mu} = \widetilde{\lambda}_h$, we get $a_h(\mathcal{J}_h \widetilde{\lambda}_h, \mathcal{J}_h \widetilde{\lambda}_h) = 0$. By the positive definiteness of $a_h(\cdot, \cdot)$, we have $\mathcal{J}_h \widetilde{\lambda}_h = 0$, which implies $\widetilde{\lambda}_h = 0$. Hence, (4.8) is uniquely solvable.

Now, let us show that it is very easy to obtain the equations for the EDG method once those of the original method have been obtained. Denote by $[\lambda_h]$ the vector of the degrees of freedom of the function λ_h with respect to some basis in M_h. Similarly, denote by $[\widetilde{\lambda}_h]$ the vector of degrees of freedom vector of the function $\widetilde{\lambda}_h$ in $\widetilde{M}_h$. Equation (4.7) can be written in a matrix form as $A[\lambda_h] = b(I_h g)$, and if $[\mathcal{J}_h \widetilde{\lambda}_h] = T[\widetilde{\lambda}_h]$, then the equation for $[\widetilde{\lambda}_h]$ is $T^t A T[\widetilde{\lambda}_h] = T^t b(\mathcal{J}_h \widetilde{I}_h g)$. Here T is the rectangular matrix representing the basis of $\widetilde{M}_h$ with respect to basis of M_h. Since $\widetilde{M}_h \subset M_h$, if we use the Lagrange basis functions, T is nothing but a connectivity matrix whose entries are zeroes and ones, so it is extremely easy to compute.

Note that the above considerations continue to hold if $\mathcal{J}_h$ is any injective operator from $\widetilde{M}_h$ into M_h such that $\mathcal{J}_h \widetilde{\mu}|_{\partial\Omega} = 0$ for all $\widetilde{\mu} \in \widetilde{M}_h$. Thus, new methods can also be created by using spaces $\widetilde{M}_h$ that are not necessarily subspaces of M_h. The main task here would be to find the matrix T which represents the basis of $\widetilde{M}_h$ with respect to basis of M_h.

Let us give some examples of EDG methods. The first example of an EDG method was proposed in [43]: It is obtained from an LDG-H method using approximations of degree k in each variable by forcing the continuity of the traces. Thus, whereas the functions in the space of approximate traces for the LDG-H method M_h are discontinuous on the borders of the elements, the functions of $\widetilde{M}_h$ are continuous therein. This allows the method to be immediately incorporated into commercial codes. On the other hand, this also results in the degradation of the conservativity properties of the EDG method, which hold only weakly. In some cases, this induces a *degradation* in the approximating properties of the method as recently proven in [32].

Indeed, in that paper, it was shown that when the stabilization parameter τ is taken to be of order one, the EDG method converges with order k for $\boldsymbol{q}$ and order $k+1$ for u. This has to be contrasted with the fact that the original LDG-H method converges with order $k+1$ in *both* variables; see [33]. Moreover, in this case, the LDG-H has superconvergence properties that allow us to compute, in an element–by-element fashion, a new approximation to u converging with order $k+2$; see also [33]. Such property does *not* hold for the corresponding EDG method. Even more, numerical experiments show that the computational advantage of the EDG method does not compensate for its loss of accuracy. On the other hand, if the stabilization parameter τ is taken to be of order h^{-1}, both the EDG and the LDG-H methods converge with the same orders, namely, k in $\boldsymbol{q}$ and order $k+1$ in u.

The second example is associated with the constructions of subspaces $\widetilde{M}_h$ of M_h that could be required to be very smooth. For example, we could ask that they be not only continuous on $\mathcal{E}_h^\circ$ but $\mathcal{C}^1$-continuous. This might be reasonable to do if the solution is very smooth and varies slowly in Ω.

The third and last example is associated to methods for nonmatching grids. Suppose that Ω is divided into two domains Ω_1 and Ω_2 independently meshed, and that we are using the variation of the CG method to handle nonconforming methods described in the first subsection. Then, all the interior faces e lying on the interface $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ must be computed and used to define the space of traces M_h. (This can be done, although it is a computational geometry tour de force, especially in

three dimensions; see [52].) To reduce the size of the interface space on Γ, we can alternately find a subspace of M_h of functions which are polynomials on the interior faces determined by, say, the vertices of the triangulation of Ω_1 lying on Γ.

5. Extensions and generalizations. Although in all examples we have given, only simplicial elements have been considered, this is not essential. Obviously, quadrilateral, prismatic, and other elements could be handled easily by DG methods. Furthermore, our framework is applicable for mixed methods using other types of elements; see [12].

Note also that the considered DG, mixed, CG, and nonconforming finite element methods used to define the local solvers are not the only choices. Stabilized, Petrov–Galerkin methods, boundary element, and even (if possible) the exact solution can be used as local solvers. For example, the hybridization of the discontinuous Petrov–Galerkin method can be found in [18, 19].

In what follows, we sketch how to extend our results to include Neumann boundary conditions and interface transmission conditions. We also extend them to DG methods using other stabilization mechanisms.

5.1. Other boundary and transmission conditions. The hybridization method proposed here can be easily extended to other types of boundary and transmission conditions.

Neumann boundary condition. For example, the case when on part $\partial\Omega_N$ of the boundary $\partial\Omega$ the Neumann boundary condition $\boldsymbol{q}\cdot\boldsymbol{n} = q_N$ is specified can be incorporated easily in the hybridization procedure. We simply require that the approximate trace λ_h belongs to

$$M_h = \{\mu \in \mathsf{M}_h : \quad \mu = 0 \text{ on } \partial\Omega_D\},$$

where $\partial\Omega_D := \partial\Omega \backslash \partial\Omega_N$ is the Dirichlet boundary and replace conservativity condition (2.6) by

$$\left\langle \mu, \; \left[\!\left[\widehat{\boldsymbol{\mathsf{Q}}}\lambda_h + \widehat{\boldsymbol{\mathsf{Q}}}g_h + \widehat{\boldsymbol{\mathsf{Q}}}f \right]\!\right] \right\rangle_{\mathcal{E}_h} = \langle \mu, \; q_N \rangle_{\partial\Omega_N} \quad \text{for all } \mu \in M_h.$$

Transmission condition. To handle transmission condition $[\![\boldsymbol{q}]\!] = t$ on the $(n-1)$-dimensional surface Γ_t, we simply have to write

$$\left\langle \mu, \; \left[\!\left[\widehat{\boldsymbol{\mathsf{Q}}}\lambda_h + \widehat{\boldsymbol{\mathsf{Q}}}g_h + \widehat{\boldsymbol{\mathsf{Q}}}f \right]\!\right] \right\rangle_{\mathcal{E}_h} = \langle \mu, \; q_N \rangle_{\partial\Omega_N} + \langle \mu, \; t \rangle_{\Gamma_t} \quad \text{for all } \mu \in M_h,$$

where we are assuming that $\Gamma_t \subset \mathcal{E}_h$. This case is equivalent to having a right-hand side that is a δ-function with a support on Γ_t.

Jump condition. Now, we can add jump condition $[\![u]\!] = \boldsymbol{j}$ on the $(n-1)$-dimensional surface Γ_j, where $\boldsymbol{j}\cdot\boldsymbol{n}$ is given. Then we take triangulation $\mathcal{T}_h$ such that $\Gamma_j \subset \mathcal{E}_h$ and proceed as follows. Since the exact solution is double valued on Γ_j, that is, since its traces on Γ_j are $u^\pm := \{\!\{u\}\!\} + 1/2\,\boldsymbol{n}^\pm \cdot \boldsymbol{j}$, we take the approximation to these traces to be $\lambda_h + 1/2\,\boldsymbol{n}^\pm \cdot \boldsymbol{j}$ on Γ_j and define the function $(\boldsymbol{\mathsf{Q}}\mathsf{m}_{\boldsymbol{j}}, \mathcal{U}\mathsf{m}_{\boldsymbol{j}})$ as the solution of local solver (2.3), with $\mathsf{m}_{\boldsymbol{j}}$ given by

$$\mathsf{m}_{\boldsymbol{j}} = \begin{cases} \frac{1}{2}\,\boldsymbol{n}_K \cdot \boldsymbol{j} & \text{on } \partial K \cap \Gamma_j, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, we simply rewrite the conservativity condition as

$$\left\langle \mu, \; \left[\!\left[\widehat{\boldsymbol{\mathsf{Q}}}\lambda_h + \widehat{\boldsymbol{\mathsf{Q}}}\mathsf{m}_{\boldsymbol{j}} + \widehat{\boldsymbol{\mathsf{Q}}}g_h + \widehat{\boldsymbol{\mathsf{Q}}}f \right]\!\right] \right\rangle_{\mathcal{E}_h} = \langle \mu, \; q_N \rangle_{\partial\Omega_N} + \langle \mu, \; t \rangle_{\Gamma_t} \text{ for all } \mu \in M_h.$$

We see that the global system for λ_h has the same matrix and a right-hand side that incorporates the data related to the boundary and interface conditions. This particular example shows the ease with which the hybridizable methods can handle various types of boundary and transmission conditions for the differential equation.

5.2. Hybridizable DG methods with other stabilization mechanisms. For each finite element $K \in \mathcal{T}_h$, the LDG-H method uses on ∂K the numerical trace $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \tau(u_h - \lambda_h)\boldsymbol{n}$ and the IP-H uses the numerical trace $\widehat{\boldsymbol{q}}_h = -a\mathbf{grad}u_h + \tau(u_h - \lambda_h)\boldsymbol{n}$. However, these are not the only choices for numerical traces we could use to generate stabilization through the difference between u_h and λ_h. Indeed, in the unified analysis of DG methods [5], we see that we can also take $\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \boldsymbol{\alpha}_r((u_h - \lambda_h)\,\boldsymbol{n})$ for the Brezzi–Manzini–Marini–Pietra–Russo (BMMPR) method [13] and $\widehat{\boldsymbol{q}}_h = -a\mathbf{grad}u_h + \boldsymbol{\alpha}_r((u_h - \lambda_h))$ for the Bassi–Rebay–Mariotti–Pedinotti–Savini (BRMPS) method [7]. Here, for any $\boldsymbol{\varphi} \in \boldsymbol{L}^2(\partial K)$, the vector $\boldsymbol{\alpha}_r(\boldsymbol{\varphi})$ is the element of $\boldsymbol{V}(K)$ such that

$$\boldsymbol{\alpha}_r(\boldsymbol{\varphi}) = -\tau\,\boldsymbol{r}_{e,K}(\boldsymbol{\varphi}) \text{ on each face } e \text{ of } K,$$
$$(\boldsymbol{r}_{e,K}(\boldsymbol{\varphi}), \boldsymbol{v})_K = -\langle \boldsymbol{\varphi}\,,\,\boldsymbol{v}\rangle_e \text{ for all } \boldsymbol{v} \in \boldsymbol{V}(K).$$

It is not difficult to verify that results similar to those obtained for the LDG-H and IP-H methods can also be obtained for similar BMMPR-H and BRMPS-H methods, respectively. Let us briefly comment on a couple of interesting details. To fix ideas, we consider the BMMPR-H methods. For these methods, Theorem 2.1 holds with

$$a_h(\eta, \mu) = (c\,\mathbf{Q}\eta, \mathbf{Q}\mu)_{\mathcal{T}_h} + (d\,\mathcal{U}\eta,\,\mathcal{U}\mu)_{\mathcal{T}_h} + \langle 1\,,\,[\![(\mathcal{U}\mu - \mu)(\boldsymbol{\alpha}_r((\mathcal{U}\eta - \eta)\,\boldsymbol{n}))]\!]\rangle_{\mathcal{E}_h},$$
$$b_h(\mu) = \langle g_h\,,\,(\mathbf{Q}\mu + \boldsymbol{\alpha}_r(\mathcal{U}\mu\,\boldsymbol{n})) \cdot \boldsymbol{n}\rangle_{\partial\Omega} + (f,\,\mathcal{U}\mu)_{\mathcal{T}_h},$$

provided $g_h|_{\mathcal{E}_h^\circ} = 0$. It is not difficult to see that bilinear form $a_h(\cdot, \cdot)$ is symmetric. Indeed, we have that

$$\langle (\mathcal{U}\mu - \mu)\,\boldsymbol{n}\,,\,\boldsymbol{\alpha}_r((\mathcal{U}\eta - \eta)\,\boldsymbol{n})\rangle_{\partial K}$$
$$= -\sum_{e \text{ face of } K} \tau_K|_e\,\langle (\mathcal{U}\mu - \mu)\,\boldsymbol{n}\,,\,\boldsymbol{r}_{e,K}((\mathcal{U}\eta - \eta)\,\boldsymbol{n})\rangle_e$$
$$= +\sum_{e \text{ face of } K} \tau_K|_e\,(\boldsymbol{r}_{e,K}((\mathcal{U}\mu - \mu)\,\boldsymbol{n}), \boldsymbol{r}_{e,K}((\mathcal{U}\eta - \eta)\,\boldsymbol{n}))_K.$$

The fact that bilinear form $a_h(\cdot, \cdot)$ is positive definite follows from Theorem 2.4 and a slight modification of Proposition 3.3; in it, we take $M(\partial K) = \{v : v|_e \in \mathcal{P}_k(e), e \in \partial K\}$. Note that Assumption 2.2 is then satisfied, since $\boldsymbol{r}_{e,K}(\boldsymbol{\varphi}) = \boldsymbol{0}$, if and only if the L^2-projection of $\boldsymbol{\varphi}|_e$ into $\mathcal{P}_k(e)$ is zero.

Finally, note that the conservativity condition is enforced strongly. In this case, however, we do not have an explicit expression of the approximate trace λ_h in terms of $(\boldsymbol{q}_h, u_h)$ as we have for the LDG-H methods in Proposition 3.4. Instead, we have only the relation

$$[\![\boldsymbol{\alpha}_r(\lambda_h\,\boldsymbol{n})]\!] = [\![\boldsymbol{\alpha}_r(u_h\,\boldsymbol{n})]\!] + [\![\boldsymbol{q}_h]\!] \quad \text{on } \mathcal{E}_h^\circ.$$

Let us end by noting that extensions of this work to other problems arising in continuum mechanics, fluid dynamics, and electromagnetism constitutes the subject of ongoing work.

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