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NUMERICAL APPROXIMATION OF THE ONE-DIMENSIONAL VLASOV–POISSON SYSTEM WITH PERIODIC BOUNDARY CONDITIONS*

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Abstract. A particle method of approximation is given for the one-dimensional Vlasov–Poisson system with periodic boundary conditions. The method is proved to converge without mollification of the electric field as the interparticle distance and time step are made small. A numerical study is carried out to verify the theory and further study convergence properties of the computed solution.

Key words. Vlasov–Poisson system, periodic boundary conditions, particle method, proof of convergence

AMS subject classifications. 35Q99, 65M12, 82D99

Introduction. A simple model problem in the kinetic theory of plasma is the one-dimensional Vlasov–Poisson system with periodic boundary conditions. We give a numerical method for solving this system of equations and prove the convergence of the algorithm. In addition, a computational study is carried out to demonstrate the validity of the theory and gain further insight into the accuracy of the computed solution. The numerical method is a type of particle method in which the continuous distribution function is replaced by a discrete set of points. The path of each of these particles in phase space is then computed.

A number of papers have previously been written on convergence of particle methods for the Vlasov–Poisson system. The first proofs for the one-dimensional periodic problem were given by Neunzert and Wick [12], [13]. In these papers, convergence of the method is proved in a measure theoretic sense that depends on an asymptotic distribution of initial data points. Later Cottet and Raviart gave a rather different theoretical development for the one-dimensional problem which is an extension of the theory of vortex methods in fluid dynamics [2], [3]. A good description of this work on vortex methods is given in [18]. Here the rate of convergence of the method depends on a ratio between a mollification parameter in the field ϵ and the interparticle distance β. The condition for convergence of the method is $\beta = O(\epsilon^{\alpha})$, $\alpha \geq 1$. Recently, a series of papers has been written by Victory et al., which develop a theory of convergence of particle methods for the Vlasov–Poisson system in higher dimensions. In higher dimensions one encounters more difficulties in carrying out the analysis as a result of the singularity in the fundamental solution of the Poisson equation. The papers [18]–[20] are an extension of the type of theory given in [2], [3], in which convergence depends on the β/ϵ ratio. In [21] the authors consider asymptotically distributed points as in [12], [13]. Also, a proof of convergence of a particle method for a three-dimensional spherically symmetric model is given by Schaeffer in [16]. Some additional references on particle methods are [6], [7], [14], [22].

In the present paper we show that the convergence of the particle method in one dimension depends primarily on the spacing between initial data points and the time step Δt. Thus the theory applies for a general nonuniform initial distribution of approximating points. Also, the convergence of the method does not depend on a mollification parameter in the electric field. In particular the proof shows that the method converges for equally spaced points and no mollification of the electric field, a case not covered by previous theorems. The present theory is an extension of work done in [24], [25] and the techniques used are similar to those in [16]. However, we note that the proof of [3, Thm. 2] employs similar methods.

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The results of computational work are given in §5 of the paper. We compare computations based on equally spaced initial data points and no mollification of the field with some other methods for doing particle computations. The methods we consider are equal weighting of points similar to what is used in [4], [18], asymptotically distributed data as in [13], and computations based on a mollifier as in [2]. Here the term "asymptotically distributed" refers to a low discrepancy point set of the type referred to in [13], [14]. The theory in the present paper shows that in general the rate of convergence of the method of point particles is the order of the interparticle distance. However, from the computational work we can draw some further conclusions about the relative accuracy of various methods. The least accurate method that we study is the elementary form of equally weighted points, and by comparison the computations using point particles either equally spaced or asymptotically distributed have a significantly better rate of convergence. This is not unexpected. For a modified version of the Vlasov–Poisson system the theory of Neunzert and Wick [14] shows that the computation based on equally weighted points is $O(N^{-1/2})$ whereas for a low discrepancy sequence of points the error is $O(\log(N)/N)$; N is the number of particles. Our computations indicate that one should expect a similar theoretical result to hold for the Vlasov–Poisson system without modification. Also, there is reason to assume that point particles with equal spacing have a rate of convergence better than $O(\beta)$. It has recently been shown that the point vortex method with equal spacing as applied to Euler's equation can be quadratically convergent, and in this work the problems are more singular than the current one-dimensional Vlasov–Poisson system [1], [8], [10]. Also, we find in computations that a mollifier of the type in [2] does not significantly improve the rate of convergence for equally spaced points as would be expected if the method without mollification is only $O(\beta)$.

Although point particles with equal spacing appears to be a higher-order method than indicated by the present theory, in studying a problem from plasma physics we obtained the best computed solution using asymptotically distributed points. The specific computational problem considered is a two-stream instability. For sufficiently small parameters the accuracy of the computation based on equally spaced points is comparable with that of asymptotically distributed points. However, asymptotically distributed data results in a more stable computation over a wider range of parameters. By comparison with the solution of two-stream instability given in [4] we are better able to reproduce the physics through the use of asymptotically distributed initial data. Also, the method of equally weighted points, which in its simplest form is not very accurate, can be significantly improved by a technique of staggering velocities as described in [4]. In this way the range of discrete velocities used in the computation is increased and effects particularly due to particles at high velocity are more accurately represented.

1. The one-dimensional periodic boundary value problem. The one-dimensional Vlasov–Poisson system with periodic boundary conditions is given as follows: given the region of phase space $A = \{(x, v) | 0 \le x \le L, -\infty < v < \infty\}$, $f(x, v, t)$ is the phase space distribution function defined on A. Then the function $f(x, v, t)$ satisfies the one-dimensional Vlasov–Poisson system with periodic boundary conditions in dimensionless variables as follows:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(x, t)\frac{\partial f}{\partial v} = 0, \tag{1.1a}$$

$$\frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f\, dv - 1, \tag{1.1b}$$

with boundary conditions

$$f(0, v, t) = f(L, v, t), \tag{1.1c}$$

(1.1d) $$E(0,t) = E(L,t), \quad t \geq 0,$$

and initial condition

(1.1e) $$f(x,v,0) = f_0(x,v).$$

The function

$$\rho(x,t) = \int_{-\infty}^{\infty} f\,dv - 1$$

is the charge density and condition (1.1d) is equivalent to

$$\int_0^L \rho(x,t)\,dx = \int_0^L \left[\int_{-\infty}^{\infty} f\,dv - 1\right] dx = 0,$$

which is the condition for total charge neutrality, i.e.,

$$\frac{1}{L}\int_0^L \int_{-\infty}^{\infty} f\,dv\,dx = 1.$$

In order to restate the above problem in terms of a potential function we add the zero mean electric field condition

(1.1f) $$\int_0^L E(x,t)\,dx = 0.$$

The periodic boundary value problem is thus (1.1a–f).

The above problem can be reformulated in terms of a potential function. Let $\phi(x,t)$ be the potential such that $\frac{\partial \phi}{\partial x} = E(x,t)$. Then from (1.1f) $\phi(x,t)$ satisfies $\phi(0,t) = \phi(L,t)$. Since there is an added constant of integration in the function $\phi(x,t)$, this constant can be chosen such that $\phi(0,t) = \phi(L,t) = 0$. For the potential function one therefore solves the boundary value problem

$$\frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{\infty} f\,dv - 1,$$

$$\phi(0,t) = \phi(L,t) = 0.$$

The solution to this problem is given in [2] as

$$\phi(x,t) = \int_0^L G(x,y)\left(\int_{-\infty}^{\infty} f\,dv - 1\right) dy,$$

$$G(x,y) = \begin{cases} (\frac{x}{L} - 1)y, & 0 \leq y \leq x, \\ (x(\frac{y}{L} - 1), & x \leq y \leq L, \end{cases}$$

and $E(x,t) = \frac{\partial \phi}{\partial x}$ is given as

$$E(x,t) = \int_0^L K(x,y)\left(\int_{-\infty}^{\infty} f\,dv - 1\right) dy,$$

$$K(x,y) = \begin{cases} \frac{y}{L}, & 0 \leq y \leq x, \\ \frac{y}{L} - 1, & x \leq y \leq L. \end{cases}$$

In terms of the potential function, the initial boundary value problem on the region A is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(x,t)\frac{\partial f}{\partial v} = 0, \tag{1.2a}$$

$$E(x,t) = \int_0^L K(x,y)\left(\int_{-\infty}^{\infty} f\,dv - 1\right) dy, \tag{1.2b}$$

$$f(x,v,0) = f_0(x,v), \tag{1.2c}$$

and where $f_0(x,v)$ satisfies

$$f_0(0,v) = f_0(L,v), \tag{1.2d}$$

$$\frac{1}{L}\int_0^L \int_{-\infty}^{\infty} f_0(x,v)\,dv\,dx = 1. \tag{1.2e}$$

This formulation of the problem therefore replaces (1.1a–f).

The solution to (1.2a–e) can be defined in terms of the characteristic system of differential equations. For a function $h(t)$ let

$$m(t) = \left[\left|\frac{h(t)}{L}\right|\right],$$

where $[x]$ is the integer part of x and

$$h^*(t) = \begin{cases} h(t) - m(t)L, & h(t) \geq 0, \\ h(t) + (m(t)+1)L, & h(t) < 0. \end{cases}$$

Then the range of $h^*(t)$ is $[0, L]$ and $h^*(t) = h(t)$ for $0 \leq h(t) \leq L$. The characteristic system for (1.2a–e) can be given as

$$\frac{dx}{dt} = v, \qquad x(0) = x_0, \tag{1.3a}$$

$$\frac{dv}{dt} = E(x^*(t),t), \qquad v(0) = v_0, \ (x_0, v_0) \in A. \tag{1.3b}$$

The solution to this system can be written as $x = x(x_0, v_0, t)$, $v = v(x_0, v_0, t)$, which are continuously differentiable functions of $(x_0, v_0) \in A$ and t. It can easily be seen that the transformation

$$(x_0, v_0) \to (x^*(x_0, v_0, t),\ v(x_0, v_0, t))$$

is a bijective and measure preserving transformation of A onto itself. The inverse of this transformation can be written as $(x, v) \to (x_0(x, v, t),\ v_0(x, v, t))$ for $(x, v) \in A$, and the solution to (1.2a–e) can then be written as

$$f(x,v,t) = f_0(x_0(x,v,t),\ v_0(x,v,t)). \tag{1.4}$$

Thus the periodic boundary value problem is equivalent to the system of equations (1.3a,b) combined with (1.4) and (1.2b–e). If $f_0(x,v)$ is extended periodically in x as a continuously differentiable function then the above solution to the Vlasov–Poisson system extends to each strip $iL \leq x \leq (i+1)L$, $-\infty < v < \infty$ as a continuously differentiable function of x, v, and t and gives a periodic solution to the Vlasov–Poisson system on all of the phase space.

To state a result on the existence and uniqueness of solutions to (1.2a–e) we make the following definitions: for $A = \{(x, v) | 0 \le x \le L,\ -\infty < v < \infty\}$, then

$C^1(A)$ is the class of continuous bounded functions defined on A with continuous, bounded, first partial derivatives. (Assume an appropriate one-sided derivative on the boundary of A.)

$C^1(A \times [0, T])$ is similarly defined.

Then it is assumed that $f_0(x, v) \in C^1(A)$ and has bounds

$$|f_0(x, v)| \le M, \tag{1.5a}$$

$$\left|\frac{\partial f_0}{\partial x}\right|, \left|\frac{\partial f_0}{\partial v}\right| \le D_0, \tag{1.5b}$$

$$\int_0^L \int_{-\infty}^{\infty} f_0(x, v)\, dv\, dx = L, \tag{1.5c}$$

$$\text{supp } f_0(x, v) \subset \{(x, v) | 0 \le x \le L,\ -R_0 < v < R_0\}. \tag{1.5d}$$

It is further assumed that $f_0(x, v)$ satisfies the condition

$$f_0(0, v) = f_0(L, v). \tag{1.6}$$

The existence and uniqueness of solutions to (1.2a–e) are given in Theorem 1.1.

THEOREM 1.1. *Let $f_0(x, v)$ be of class $C^1(A)$, have bounds* (1.5a–d), *and satisfy condition* (1.6); *then for any $T > 0$ the solution to* (1.2a–e) *exists and is unique as an element of $C^1(A \times [0, T])$. For t in the interval $[0, T]$ the solution f has the bounds*

$$|f(x, v, t)| \le M, \tag{1.7a}$$

$$\left|\frac{\partial f}{\partial x}\right|, \left|\frac{\partial f}{\partial v}\right| \le D, \tag{1.7b}$$

$$\int_0^L \int_{-\infty}^{\infty} f(x, v, t)\, dv\, dx = L, \tag{1.7c}$$

$$\text{supp } f(x, v, t) \subset \{(x, v) | 0 \le x \le L,\ -R \le v \le R\}. \tag{1.7d}$$

Proof. See [2], [23].

In the following sections a numerical method for approximating solutions to (1.2a–e) is defined. The method is a type of particle method in which approximations to the solutions of the characteristic system (1.3a,b) are obtained. A proof of convergence is given which shows that the approximate trajectories converge to the exact trajectories of the system. The theory is written in such a way as to apply to a general nonuniform distribution of approximating points.

2. The numerical method. The initial data $f_0(x, v) \in C^1(A)$ is assumed to satisfy (1.5a–d), (1.6), and for this initial data $f(x, v, t)$ is the solution to (1.2a–e) of class $C^1(A \times [0, T])$ which has bounds (1.7a–d). Let N_x, N_v be positive integers and for L and R_0, (1.5d), the interval $[0, L]$ is given the partition

$$0 = a_0 < a_1 < a_2 < \cdots < a_{N_x} = L,$$

where $\Delta x_i = a_i - a_{i-1}$ and $\|\Delta x\| = \max_i(\Delta x_i)$, and the interval $[-R_0, R_0]$ is given the partition

$$-R_0 = b_0 < b_1 < b_2 < \cdots < b_{N_v} = R_0,$$

where $\Delta v_j = b_j - b_{j-1}$ and $\|\Delta v\| = \max_j(\Delta v_j)$. The mesh points (x_i, v_j) are then specified such that

$$\begin{aligned} a_{i-1} < x_i < a_i, \qquad & i = 1, \ldots, N_x, \\ b_{j-1} < v_j < b_j, \qquad & j = 1, \ldots, N_v. \end{aligned}$$

Thus the region of phase space given by $0 \le x \le L$, $-R_0 \le v \le R_0$ is divided into $N_x N_v$ rectangles $\{(x, v) \mid a_{i-1} \le x \le a_i,\ b_{j-1} \le v \le b_j\}$ of area $\Delta x_i \Delta v_j$ with the points (x_i, v_j) interior to the rectangles. Let N_t be a positive integer and define $\Delta t = \frac{T}{N_t}$. The numerical approximation is then given as follows: let $\bar{x}_{i,j}(0) = x_i$, $\bar{v}_{i,j}(0) = v_j$, and $k = 1, 2, \ldots, N_t$. For $(k-1)\,\Delta t < t \le k\,\Delta t$, then

$$\begin{aligned} \bar{x}_{i,j}(t) &= \tilde{x}^k_{i,j}(t - (k-1)\Delta t), \\ \bar{v}_{i,j}(t) &= \tilde{v}^k_{i,j}(t - (k-1)\Delta t), \end{aligned}$$

where $\tilde{x}^k_{i,j}(s)$, $\tilde{v}^k_{i,j}(s)$ are solutions for $0 \le s \le \Delta t$ to

$$\begin{aligned} \tfrac{dx}{ds} &= v + \widetilde{E}_{k-1}(\bar{x}^*_{i,j}((k-1)\Delta t))s, \quad & x(0) &= \bar{x}_{i,j}((k-1)\Delta t), \\ \tfrac{dv}{ds} &= \widetilde{E}_{k-1}(\bar{x}^*_{i,j}((k-1)\Delta t)), & v(0) &= \bar{v}_{i,j}((k-1)\Delta t). \end{aligned}$$

Here

$$\widetilde{E}_{k-1}(x) = \sum_{i,j} \Delta x_i \Delta v_j \frac{f_0(x_i, v_j)}{\lambda} \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) - \int_0^L K(x, y)\, dy, \tag{2.1}$$

where $\lambda = \big(\sum_{i,j} \Delta x_i \Delta v_j\, f_0(x_i, v_j)\big)/L$ and

$$\gamma(x, y) = \begin{cases} \frac{y}{L}, & 0 \le y < \frac{L}{L+\delta} x, \\ \frac{x-y}{\delta}, & \frac{L}{L+\delta} x \le y \le \frac{L}{L+\delta}(x+\delta), \\ \frac{y}{L} - 1, & \frac{L(x+\delta)}{L+\delta} < y \le L. \end{cases}$$

The small parameter δ is introduced to make the approximate field Lipschitz continuous and it is assumed that $\delta \le \|\Delta x\|$. Thus

$$\begin{aligned} \tilde{x}^k_{i,j}(s) &= \bar{x}_{i,j}((k-1)\Delta t) + \bar{v}_{i,j}((k-1)\Delta t)s + \widetilde{E}_{k-1}(\bar{x}^*_{i,j}((k-1)\Delta t))s^2, \\ \tilde{v}^k_{i,j}(s) &= \bar{v}_{i,j}((k-1)\Delta t) + \widetilde{E}_{k-1}(\bar{x}^*_{i,j}((k-1)\Delta t))s. \end{aligned}$$

If we let

$$\begin{aligned} \overline{E}(x, t) &= \widetilde{E}_{k-1}(x), \quad (k-1)\Delta t < t \le k\,\Delta t, \quad 0 \le x \le L, \\ \overline{E}(x, 0) &= \widetilde{E}_0(x), \end{aligned}$$

and define $n(t) = [\frac{t}{\Delta t}]\,\Delta t$, then the trajectories $\bar{x}_{i,j}(t)$, $\bar{v}_{i,j}(t)$ are solutions to

$$\frac{dx}{dt} = v + \overline{E}(x^*(n(t)), t)(t - n(t)), \quad x(0) = x_i, \tag{2.2a}$$

$$\frac{dv}{dt} = \overline{E}(x^*(n(t)), t), \qquad v(0) = v_j, \tag{2.2b}$$

except at the finite number of points $t = k\,\Delta t$ where $\overline{E}(x, t)$ is discontinuous.

The system (2.2a,b) can be solved for arbitrary $(\xi, \eta) \in A$. That is, let $\bar{x}(\xi, \eta, 0) = \xi$, $\bar{v}(\xi, \eta, 0) = \eta$; then, for $(k-1)\Delta t < t \leq k\,\Delta t$,

$$\bar{x}(\xi, \eta, t) = \tilde{x}^k(t - (k-1)\Delta t),$$
$$\bar{v}(\xi, \eta, t) = \tilde{v}^k(t - (k-1)\Delta t),$$

where $\tilde{x}^k(s)$, $\tilde{v}^k(s)$ are solutions for $0 \leq s \leq \Delta t$ to

$$\frac{dx}{ds} = v + \tilde{E}_{k-1}(\bar{x}^*(\xi, \eta, (k-1)\Delta t))s, \qquad x(0) = \bar{x}(\xi, \eta, (k-1)\Delta t),$$
$$\frac{dv}{ds} = \tilde{E}_{k-1}(\bar{x}^*(\xi, \eta, (k-1)\Delta t)), \qquad v(0) = \bar{v}(\xi, \eta, (k-1)\Delta t).$$

For given $(\xi, \eta) \in A$, we consider the transformations of A given by

$$(\xi, \eta) \to (x^*(\xi, \eta, t),\ v(\xi, \eta, t)), \tag{2.3a}$$

$$(\xi, \eta) \to (\bar{x}^*(\xi, \eta, t)),\ \bar{v}(\xi, \eta, t)), \tag{2.3b}$$

where $(x(\xi, \eta, t),\ v(\xi, \eta, t))$ is the solution to (1.3a,b) and $(\bar{x}(\xi, \eta, t),\ \bar{v}(\xi, \eta, t))$ is the solution to (2.2a,b). That these transformations are measure preserving from A onto A is shown as in [13], [25]. We note that the transformation based on $\frac{dx}{dt} = v$ without the additional term in (2.2a) does not have a Jacobian of one. Let $S_{i,j}$ be the rectangle

$$\{(x, v) | a_{i-1} \leq x \leq a_i,\ b_{j-1} \leq v \leq b_j\}$$

with the point (x_i, v_j) in the interior. Then under (2.3a) the rectangle $S_{i,j}$ is transformed into the region $\Omega_{i,j}(t) \subset A$ and under (2.3b) into the region $\overline{\Omega}_{i,j}(t) \subset A$, $\Omega_{i,j}(t) \cap \Omega_{k,\ell}(t) = \emptyset$, $i, j \neq k, \ell$, $\mu(\Omega_{i,j}) = \Delta x_i\, \Delta v_j$, and similarly for $\overline{\Omega}_{i,j}(t)$. Furthermore, since $|\gamma(x, y)| \leq 1$ and $|K(x, y)| \leq 1$ then there is a constant N such that $|E(x, t)|$, $|\overline{E}(x, t)| \leq N$, and for $(\xi, \eta) \in S_{i,j}$ then

$$|v(\xi, \eta, t)| \leq R_0 + NT = R,$$
$$|\bar{v}(\xi, \eta, t)| \leq R_0 + NT = R.$$

Thus

$$\bigcup_{i,j} \Omega_{i,j}(t),\ \bigcup_{i,j} \overline{\Omega}_{i,j}(t) \subset \{(x, v) | 0 \leq x \leq L,\ -R \leq v \leq R\}. \tag{2.4}$$

Let us assume that for (ξ_1, η_1), (ξ_2, η_2) in $S_{i,j}$, and $t \in [0, T]$,

$$|x(\xi_1, \eta_1, t) - x(\xi_2, \eta_2, t)| < \frac{L}{2}$$

and

$$|\bar{x}(\xi_1, \eta_1, t) - \bar{x}(\xi_2, \eta_2, t)| < \frac{L}{2}.$$

A diameter of $\Omega_{i,j}(t)$, and similarly for $\overline{\Omega}_{i,j}(t)$, is defined as follows. If $\Omega_{i,j}(t)$ does not include the boundary of A then

$$\operatorname{diam}(\Omega_{i,j}(t)) = \sup(|x - y| + |v - w|)$$

for (x, v), (y, w) any two points in $\Omega_{i,j}(t)$. If $\Omega_{i,j}(t)$ includes the boundary of A let $\Omega_{i,j}(t) = B_1 \bigcup B_2$ where

$$B_1 = \left\{(x, v) \in \Omega_{i,j}(t) \mid x < \frac{L}{2}\right\},$$

$$B_2 = \left\{(x, v) \in \Omega_{i,j}(t) \mid x > \frac{L}{2}\right\}.$$

Then

$$\operatorname{diam}(\Omega_{i,j}(t)) = \max(\ell_1, \ell_2, \ell_3),$$

where

$$\ell_1 = \sup(|x - y| + |v - w|) \quad \text{for} \quad (x, v), (y, w) \text{ in } B_1,$$
$$\ell_2 = \sup(|x - y| + |v - w|) \quad \text{for} \quad (x, v), (y, w) \text{ in } B_2,$$
$$\ell_3 = \sup(|x - y + L| + |v - w|) \quad \text{for} \quad (x, v) \text{ in } B_1 \text{ and } (y, w) \text{ in } B_2.$$

It can be shown that

$$\operatorname{diam}(\Omega_{i,j}(t)) \le C(t)(\|\Delta x\| + \|\Delta v\|)$$
$$\text{and} \quad \operatorname{diam}(\overline{\Omega}_{i,j}(t)) \le C(t)(\|\Delta x\| + \|\Delta v\|). \tag{2.5}$$
$$C(t) \text{ is a nondecreasing function of } t.$$

The result for $\operatorname{diam}(\Omega_{i,j}(t))$ is straightforward. We give a short proof of the result for $\operatorname{diam}(\overline{\Omega}_{i,j}(t))$. Let $p_1 = (\xi_1, \eta_1)$, $p_2 = (\xi_2, \eta_2)$ be two points in $S_{i,j}$ and $(x_1(p_1, t), v_1(p_1, t))$, $(x_2(p_2, t), v_2(p_2, t))$ be the solutions to (2.2a,b) such that $(x_1(p_1, 0), v_1(p_1, 0)) = p_1$, $(x_2(p_2, 0), v_2(p_2, 0)) = p_2$; then

$$x_2(t) - x_1(t) = (\xi_2 - \xi_1) + \int_0^t \big[(v_2(s) - v_1(s)) + (\overline{E}(x_2^*(n(s), s)) - \overline{E}(x_1^*(n(s), s))\big](s - n(s))\, ds,$$

$$v_2(t) - v_1(t) = (\eta_2 - \eta_1) + \int_0^t (\overline{E}(x_2^*(n(s), s)) - \overline{E}(x_1^*(n(s), s)))\, ds.$$

Now for $(k - 1)\Delta t < t \le k\,\Delta t$,

$$\overline{E}(x_2^*(n(t)), t) - \overline{E}(x_1^*(n(t)), t) = \widetilde{E}_{k-1}(x_2^*(p_2, (k-1)\Delta t)) - \widetilde{E}_{k-1}(x_1^*(p_1, (k-1)\Delta t))$$

$$= \sum_{i,j} \Delta x_i \Delta v_j \frac{f_0(x_i, v_j)}{\lambda} (\gamma(x_2^*, \bar{x}_{i,j}^*((k-1)\Delta t)) - \gamma(x_1^*, \bar{x}_{i,j}^*((k-1)\Delta t))).$$

Let us define

$$g(t) = \sup_{i,j}\Big(\sup_{p_1, p_2 \in S_{i,j},\, \tau \in [0,t]} (|x_1(p_1, \tau) - x_2(p_2, \tau)| + |v_1(p_1, \tau) - v_2(p_2, \tau)|)\Big).$$

To obtain the relevant estimate we assume for simplicity that $0 < x_1^*(t) - \delta - g(t) < x_1^*(t) < x_2^*(t) < x_2^*(t) + \delta + g(t) < L$, $x_2^*(t) - x_1^*(t) = x_2(t) - x_1(t) < \frac{L}{2}$, and note that $x_1^*(t) - \delta < \frac{L}{L+\delta} x_1^*(t) < \frac{L}{L+\delta}(x_2^*(t) + \delta) < x_2^*(t) + \delta$. Let

$$\Omega_1 = \{(x, v) \mid x_1^* - \delta \le x \le x_2^* + \delta,\ -R \le v \le R\},$$
$$\Omega_1' = \{(x, v) \mid 0 \le x < x_1^* - \delta \ \text{ or } x_2^* + \delta < x \le L,\ -R \le v \le R\}.$$

If $\bar{x}^*_{i,j} < x^*_1 - \delta$ or $\bar{x}^*_{i,j} > x^*_2 + \delta$ then

$$\gamma(x^*_1, \bar{x}^*_{i,j}((k-1)\Delta t)) - \gamma(x^*_2, \bar{x}^*_{i,j}((k-1)\Delta t)) = 0$$

so if $S_1 = \{(i,j) \,|\, x^*_1 - \delta \le \bar{x}^*_{i,j} \le x^*_2 + \delta\}$ then

$$\begin{aligned}
&|\widetilde{E}_{k-1}(x^*_2(p_2, (k-1)\Delta t)) - \widetilde{E}_{k-1}(x^*_1(p_1, (k-1)\Delta t))| \\
&\le \sum_{i,j\in S_1} \left(\Delta x_i \Delta v_j \frac{f_0(x_i, v_j)}{\lambda}\right) |\gamma(x^*_2, \bar{x}^*_{i,j}((k-1)\Delta t)) - \gamma(x^*_1, \bar{x}^*_{i,j}((k-1)\Delta t))| \\
&\le \sum_{i,j\in S_1} \left(\Delta x_i \Delta v_j \frac{f_0(x_i, v_j)}{\lambda}\right) \le \overline{M} \sum_{i,j\in S_1} \mu(\overline{\Omega}_{i,j}((k-1)\Delta t)) \\
&= \overline{M} \sum_{i,j\in S_1} (\mu(\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega_1) + \mu(\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega'_1)).
\end{aligned}$$

Here $\overline{M}$ is a constant such that $\sup \frac{f_0(x,v)}{\lambda} = \frac{M}{\lambda} \le \overline{M}$. It is assumed that $\delta \le \|\Delta x\| \le g(t)$. Now

$$\bigcup_{i,j\in S_1} (\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega_1) \subset \{(x,v) \,|\, x^*_1 - \delta \le x \le x^*_2 + \delta,\ -R \le v \le R\}$$

so

$$\begin{aligned}
&\mu\left(\bigcup_{i,j\in S_1} (\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega_1\right) \\
&= \sum_{i,j\in S_1} \mu(\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega_1) \\
&\le ((x^*_2(p_2, (k-1)\Delta t) - x^*_1(p_1, (k-1)\Delta t)) + 2\delta)(2R) \\
&\le (g((k-1)\Delta t) + 2\delta)(2R) \le 6Rg((k-1)\Delta t)
\end{aligned}$$

and

$$\begin{aligned}
\bigcup_{i,j\in S_1} (\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega'_1) \subset \{(x,v) \,|\, &x^*_1 - \delta - g((k-1)\Delta t) \le x \le x^*_1 - \delta, \\
&-R \le v \le R\} \\
\cup \{(x,v) \,|\, &x^*_2 + \delta \le x \le x^*_2 + g((k-1)\Delta t) + \delta, \\
&-R \le v \le R\}
\end{aligned}$$

so

$$\begin{aligned}
\mu\left(\bigcup_{i,j\in S_1} \overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega'_1\right) &= \sum_{i,j\in S_1} \mu(\overline{\Omega}_{i,j}((k-1)\Delta t) \cap \Omega'_1) \\
&\le 2g((k-1)\Delta t)(2R) \le 4R\, g((k-1)\Delta t).
\end{aligned}$$

Thus

$$|\widetilde{E}_{k-1}(x^*_2(p_2, (k-1)\Delta t)) - \widetilde{E}_{k-1}(x^*_1(p_1, (k-1)\Delta t))| \le 10\overline{M}R\, g((k-1)\Delta t) = 10\overline{M}R\, g(n(t)).$$

This estimate was derived by making some simplifying restrictions on the position of the trajectories $(x_2(p_2,t), v_2(p_2,t))$, $(x_1(p_1,t), v_1(p_1,t))$, but the result holds for any two trajectories with $p_1, p_2 \in S_{i,j}$. We therefore obtain the inequality

$$\begin{aligned}
&|x_2(t) - x_1(t)| + |v_2(t) - v_1(t)| \\
&\le |\xi_2 - \xi_1| + |\eta_2 - \eta_1| \\
&\quad + \int_0^t (|v_2(s) - v_1(s)| + 10\overline{M}R\, g(n(s))\Delta t + 10\overline{M}R\, g(n(s)))\, ds \\
&\le |\xi_2 - \xi_1| + |\eta_2 - \eta_1| + K \int_0^t g(n(s))\, ds,
\end{aligned}$$

where $K > 1 + 10(1 + \Delta t)\overline{M}R$. It follows that

$$\begin{aligned}
g(t) &= \sup_{i,j}\Big(\sup_{p_1,p_2\in S_{i,j},\,\tau\le t} |x_2(p_2,\tau) - x_1(p_1,\tau)| + |v_2(p_2,\tau) - v_1(p_1,\tau)|\Big) \\
&\le \|\Delta x\| + \|\Delta v\| + K \int_0^t g(n(s))\, ds \\
&\le \|\Delta x\| + \|\Delta v\| + K \int_0^t g(s)\, ds.
\end{aligned}$$

Thus

$$g(t) \le (\|\Delta x\| + \|\Delta v\|)\, e^{Kt} = C(t)\,(\|\Delta x\| + \|\Delta v\|),$$

which gives inequality (2.5). We note that for $\|\Delta x\| + \|\Delta v\|$ sufficiently small $g(t) < \frac{L}{2}$ which justifies the prior assumption.

3. Estimates for the electric field. Let $(x_{i,j}(t), v_{i,j}(t))$ be the solution to (1.3a,b) for the initial point (x_i, v_j). We want to prove that the solution to (2.2a,b), $(\bar{x}_{i,j}(t), \bar{v}_{i,j}(t))$ converges to $(x_{i,j}(t), v_{i,j}(t))$ as $\|\Delta x\|$, $\|\Delta v\|$, and Δt approach zero.

Thus we bound the solutions to

$$\begin{aligned}
x_{i,j}(t) - \bar{x}_{i,j}(t) &= \int_0^t \big[(v_{i,j}(s) - \bar{v}_{i,j}(s)) - \overline{E}(\bar{x}^*_{i,j}(n(s)), s)(s - n(s))\big]\, ds, \\
v_{i,j}(t) - \bar{v}_{i,j}(t) &= \int_0^t \big[E(x^*_{i,j}(s), s) - \overline{E}(\bar{x}^*_{i,j}(n(s)), s)\big]\, ds.
\end{aligned} \tag{3.1}$$

One needs to obtain an estimate for

$$\begin{aligned}
E(x^*_{i,j}(t), t) - \overline{E}(\bar{x}^*_{i,j}(n(t)), t) &= \big(E(x^*_{i,j}(t), t) - E(x^*_{i,j}(n(t), n(t))\big) \\
&\quad + \big(E(x^*_{i,j}(n(t)), n(t)) - E(\bar{x}^*_{i,j}(n(t)), n(t))\big) \\
&\quad + \big(E(\bar{x}^*_{i,j}(n(t)), n(t)\big) - \overline{E}(\bar{x}^*_{i,j}(n(t)), t)\big).
\end{aligned}$$

For $(k-1)\Delta t < t \le k\,\Delta t$, let us first consider

$$\begin{aligned}
E(x, n(t)) - \overline{E}(x, t) &= E(x, n(t)) - \widetilde{E}_{k-1}(x) \\
&= E(x, (k-1)\Delta t) - \widetilde{E}_{k-1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^L \int_{-\infty}^{+\infty} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy \\
&\quad - \sum_{i,j} \Delta x_i \Delta v_j \frac{f_0(x_i,v_j)}{\lambda}\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \\
&= \left(\int_0^L \int_{-\infty}^{+\infty} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy \right. \\
&\quad \left. - \sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i,v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \\
&\quad + \left(\sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i,v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right. \\
&\quad \left. - \sum_{i,j} \Delta x_i\, \Delta v_j \frac{f_0(x_i,v_j)}{\lambda}\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \\
&= \left(\int_0^L \int_{-\infty}^{+\infty} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy \right. \\
&\quad \left. - \sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i,v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \\
&\quad + (\lambda - 1) \sum_{i,j} \Delta x_i\, \Delta v_j \frac{f_0(x_i,v_j)}{\lambda}\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \Big).
\end{aligned}
$$

However

$$
\int_0^L \int_{-\infty}^{\infty} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy = \sum_{i,j} \int_{\Omega_{i,j}} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy,
$$

where $\Omega_{i,j} = \Omega_{i,j}((k-1)\Delta t)$. We bound

$$
\begin{aligned}
&\sum_{i,j} \left(\int_{\Omega_{i,j}} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy \right. \\
&\left. - \Delta x_i\, \Delta v_j\, f_0(x_i,v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \\
&= \sum_{i,j} \left(\int_{\Omega_{i,j}} K(x,y)\, f(y,v,(k-1)\Delta t)\, dv\, dy \right. \\
&\left. - \int_{\Omega_{i,j}} K(x,y)\, f(x^*_{i,j}((k-1)\Delta t),\, v_{i,j}((k-1)\Delta t),\, (k-1)\,\Delta t)\, dv\, dy \right) \\
&+ \sum_{i,j} \left(\int_{\Omega_{i,j}} K(x,y)\, f(x^*_{i,j}((k-1)\Delta t),\, v_{i,j}((k-1)\Delta t),\, (k-1)\Delta t)\, dv\, dy \right. \\
&\left. - \Delta x_i\, \Delta v_j\, f_0(x_i,v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right).
\end{aligned}
$$

From the bound (2.5) on $\operatorname{diam}(\Omega_{i,j}(t))$ it follows that

$$\left|\int_{\Omega_{i,j}} K(x,y)\big(f(y,v,(k-1)\Delta t)-f(x^*_{i,j}((k-1)\Delta t),v_{i,j}((k-1)\Delta t),(k-1)\Delta t)\big)\,dv\,dy\right|$$
$$\le DC((k-1)\Delta t)\,(\|\Delta x\|+\|\Delta v\|)\left|\int_{\Omega_{i,j}} K(x,y)\,dv\,dy\right|$$

and since $|\int_0^L\int_{-R}^R K(x,y)\,dv\,dy|\le 2RL$ then

$$\left|\sum_{i,j}\left(\int_{\Omega_{i,j}} K(x,y)(f(y,v,(k-1)\Delta t)-f(x^*_{i,j}((k-1)\Delta t),\ v_{i,j}((k-1)\Delta t),\right.\right.$$
$$\left.\left.(k-1)\Delta t))\,dv\,dy\right)\right|$$
$$\le DC((k-1)\Delta t)\,(\|\Delta x\|+\|\Delta v\|)\left|\sum_{i,j}\int_{\Omega_{i,j}} K(x,y)\,dv\,dy\right| \tag{3.2}$$
$$\le DC((k-1)\Delta t)\,(\|\Delta x\|+\|\Delta v\|)\,(2RL)$$
$$= C_1((k-1)\Delta t)\,(\|\Delta x\|+\|\Delta v\|),$$

with $C_1((k-1)\Delta t)=2RLDC((k-1)\Delta t)$.

Next let us consider

$$\int_{\Omega_{i,j}} K(x,y)\,f(x^*_{i,j}((k-1)\Delta t),\ v_{i,j}((k-1)\Delta t),\ (k-1)\Delta t)\,dv\,dy$$
$$-\Delta x_i\,\Delta v_j\,f_0(x_i,v_j)\,\gamma(x,\bar{x}^*_{i,j}((k-1)\Delta t))$$
$$=f_0(x_i,v_j)\textstyle\int_{\Omega_{i,j}}(K(x,y)-\gamma(x,\bar{x}^*_{i,j}((k-1)\Delta t)))\,dv\,dy.$$

Let $\Omega=\{(y,v)|0\le y\le L,\ -R\le v\le R\}$ and define

$$d(t)=\sup_{i,j}|x_{i,j}(t)-\bar{x}_{i,j}(t)|,$$
$$e(t)=d(t)+C(t)\,(\|\Delta x\|+\|\Delta v\|)+\delta.$$

It is assumed that $d(t)$, $e(t)$ are small; in particular,

$$d(t),\ e(t)<\frac{L}{2}.$$

For simplicity we will assume $0<x-2e(t)<x+2e(t)<L$. Let

$$A_1(x,t)=\big\{(y,v)|x-e(t)\le y\le x+e(t),\ -R\le v\le R\big\},$$

$A_2(x,t)=A_1'(x,t)$ (the complement of $A_1(x,t)$ in the set Ω). We have

$$S_1(x,t)=\big\{(i,j)|(x^*_{i,j}(t),\ v_{i,j}(t))\subset A_1(x,t)\big\},$$
$$S_2(x,t)=\big\{(i,j)|(x^*_{i,j}(t),\ v_{i,j}(t))\subset A_2(x,t)\big\}.$$

For $(k-1)\Delta t<t\le k\,\Delta t$ we first consider $(i,j)\in S_2(x,(k-1)\Delta t)$ and bound

$$\int_{\Omega_{i,j}}(K(x,y)-\gamma(x,\bar{x}^*_{i,j}((k-1)\Delta t)))\,dv\,dy,$$

where $\Omega_{i,j} = \Omega_{i,j}((k-1)\Delta t)$. For the type of estimate obtained for $i, j \in S_2$, let us assume that $x^*_{i,j}, \bar{x}^*_{i,j}$ are in the interval $0 \le y < \frac{L}{L+\delta} x$ so that

$$x^*_{i,j}((k-1)\Delta t) - \bar{x}^*_{i,j}((k-1)\Delta t) = x_{i,j}((k-1)\Delta t) - \bar{x}_{i,j}((k-1)\Delta t),$$

and for $(y, v) \subset \Omega_{i,j}$, $0 < y < \frac{L}{L+\delta} x$; then

$$|K(x,y) - \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t))| = \left| \frac{y}{L} - \frac{\bar{x}^*_{i,j}}{L} \right|$$

$$\le \frac{|y - x^*_{i,j}|}{L} + \frac{|x^*_{i,j} - \bar{x}^*_{i,j}|}{L}$$

$$\le \frac{1}{L}\big(C((k-1)\Delta t)\,(\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t)\big).$$

Near the boundaries this argument is modified by the periodicity of the solution but one obtains the same estimate. Thus

$$\left| \sum_{i,j \in S_2(x,(k-1)\Delta t)} \left(f_0(x_i, v_j) \int_{\Omega_{i,j}} \big(K(x,y) - \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t))\big)\, dv\, dy \right) \right|$$

$$\le \frac{M}{L}(C((k-1)\Delta t)\,(\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t)) \left(\sum_{i,j} \int_{\Omega_{i,j}} dv\, dy \right) \tag{3.3}$$

$$\le \frac{M}{L}(2RL)\,(C((k-1)\Delta t)\,(\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t))$$

$$\le C_2((k-1)\Delta t)\,\big((\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t)\big),$$

$$C_2((k-1)\Delta t) = 2MRC((k-1)\Delta t).$$

Let us now assume that $(i, j) \in S_1(x, (k-1)\Delta t)$. We make the simplifying assumption that $x - 2e((k-1)\Delta t) \ge 0$ and $x + 2e((k-1)\Delta t) \le L$. Near the boundary the argument is modified by periodicity. If $x - e((k-1)\Delta t) \le x^*_{i,j}((k-1)\Delta t) \le x + e((k-1)\Delta t)$ then

$$x - e((k-1)\Delta t) - d((k-1)\Delta t) \le \bar{x}^*_{i,j}((k-1)\Delta t) \le x + e((k-1)\Delta t) + d((k-1)\Delta t)$$

and $\Omega_{i,j}((k-1)\Delta t)$ and $\overline{\Omega}_{i,j}((k-1)\Delta t)$ are contained in the region

$$\Omega_1 = \big\{(y, v) | x - 2e((k-1)\Delta t) \le y \le x + 2e((k-1)\Delta t),\ -R \le v \le R\big\}$$

and

$$\mu(\Omega_1) = 4e((k-1)\Delta t)(2R) = 8Re((k-1)\Delta t).$$

Thus

$$\left| \sum_{i,j \in S_1} \left(\int_{\Omega_{i,j}} K(x,y)\, f_0(x_i, v_j)\, dv\, dy - \Delta x_i\, \Delta v_j\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \right|$$

$$\le M \sum_{i,j \in S_1} \int_{\Omega_{i,j}} |K(x,y)|\, dv\, dy$$

$$+ M \sum_{i,j \in S_1} |\gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t))|\, \mu(\overline{\Omega}_{i,j}).$$

Since $|K(x, y)|,\ |\gamma(x, y)| \leq 1$,

$$\sum_{i,j \in S_1} \int_{\Omega_{i,j}} dv\, dy \leq \int_{\Omega_1} dv\, dy = \mu(\Omega_1) = 8Re((k-1)\Delta t)$$

and

$$\sum_{i,j \in S_1} \mu(\overline{\Omega}_{i,j}) \leq \mu(\Omega_1) = 8Re((k-1)\Delta t).$$

Then

$$\left| \sum_{i,j \in S_1} \left(\int_{\Omega_{i,j}} K(x, y)\, f_0(x_i, v_j)\, dv\, dy - \Delta x_i\, \Delta v_j\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right) \right|$$
$$\leq 16MRe((k-1)\Delta t). \tag{3.4}$$

Thus from (3.2), (3.3), (3.4),

$$\left| \int_0^L \int_{-\infty}^{\infty} K(x, y)\, f(y, v, (k-1)\Delta t)\, dv\, dy - \sum_{i,j} \Delta x_i \Delta v_j\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right|$$
$$\leq C_1((k-1)\Delta t)(\|\Delta x\| + \|\Delta v\|)$$
$$+C_2((k-1)\Delta t)((\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t)) + 16MRe((k-1)\Delta t). \tag{3.5}$$

The quantity $|\lambda - 1|$ is bounded as follows:

$$\left| \int_0^L \int_{-R_0}^{R_0} f_0(x, v)\, dv\, dx - \sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i, v_j) \right|$$
$$= \left| \sum_{i,j} \int_{S_{i,j}} \left(f_0(x, v) - f_0(x_i, v_j) \right) dv\, dx \right|$$
$$\leq \frac{D_0}{2} (\|\Delta x\| + \|\Delta v\|) \sum_{i,j} \int_{S_{i,j}} dv\, dx$$
$$\leq \frac{D_0}{2} (\|\Delta x\| + \|\Delta v\|)\, (2R_0 L) = D_0 R_0 L (\|\Delta x\| + \|\Delta v\|).$$

Thus

$$|\lambda - 1| = \left(\left| \int_0^L \int_{-R_0}^{R_0} f_0(x, v)\, dv\, dx - \sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i, v_j) \right| \right) / L$$
$$\leq D_0 R_0 L (\|\Delta x\| + \|\Delta v\|)/L$$

and

$$\left| (\lambda - 1) \sum_{i,j} \Delta x_i\, \Delta v_j \frac{f_0(x_i, v_j)}{\lambda}\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right|$$
$$\leq |\lambda - 1| \sum_{i,j} \Delta x_i\, \Delta v_j \frac{f_0(x_i, v_j)}{\lambda} \tag{3.6}$$
$$\leq \frac{D_0 R_0 L}{L} (\|\Delta x\| + \|\Delta v\|) L = D_0 R_0 L (\|\Delta x\| + \|\Delta v\|).$$

Combining estimates (3.5), (3.6) it follows that for $(k-1)\Delta t < t \le k\,\Delta t \le T$,

$$\begin{aligned}
&|E(x,n(t)) - \overline{E}(x,t)| = |E(x,(k-1)\Delta t) - \widetilde{E}_{k-1}(x)| \\
&\quad \le C_1((k-1)\Delta t)\,(\|\Delta x\| + \|\Delta v\|) + C_2((k-1)\Delta t)\,((\|\Delta x\| + \|\Delta v\|) + d((k-1)\Delta t)) \\
&\qquad + 16MR\,e((k-1)\Delta t) + D_0R_0L(\|\Delta x\| + \|\Delta v\|) \\
&\quad \le C_3(T)(\|\Delta x\| + \|\Delta v\|) + C_4(T)\,e((k-1)\,\Delta t).
\end{aligned} \tag{3.7}$$

For $0 \le t \le T$, it can be assumed that

$$\left|\frac{\partial E}{\partial x}\right|, \left|\frac{\partial E}{\partial t}\right| \le \overline{D}.$$

Then, for $(k-1)\Delta t < t \le k\,\Delta t$,

$$\left|E(x^*_{i,j}(t),t) - E(x^*_{i,j}(n(t),\,n(t)))\right| \le \overline{D}\left(|x_{i,j}(t) - x_{i,j}(n(t))| + |t - n(t)|\right)$$

and since

$$|x_{i,j}(t) - x_{i,j}(n(t))| \le \int_{n(t)}^{t} |v_{i,j}(s)|\,ds \le R\,\Delta t$$

then

$$\begin{aligned}
\left|E(x^*_{i,j}(t),t) - E(x^*_{i,j}(n(t)),n(t))\right| &\le \overline{D}(R\,\Delta t + \Delta t) \\
&\le 2R\overline{D}\,\Delta t.
\end{aligned} \tag{3.8}$$

Also,

$$\begin{aligned}
\left|E(x^*_{i,j}(n(t)),n(t)) - E(\bar{x}^*_{i,j}(n(t),\,n(t)))\right| &\le \overline{D}\left|x_{i,j}(n(t)) - \bar{x}_{i,j}(n(t))\right| \\
&\le \overline{D}\,d((k-1)\,\Delta t).
\end{aligned} \tag{3.9}$$

Finally from estimates (3.7)–(3.9) it follows that

$$\begin{aligned}
\left|E(x^*_{i,j}(t),t) - \overline{E}(\bar{x}^*_{i,j}(n(t)),t)\right| &\le \left|E(x^*_{i,j}(t),t) - E(x^*_{i,j}(n(t)),n(t))\right| \\
&\quad + \left|E(x^*_{i,j}(n(t)),n(t)) - E(\bar{x}^*_{i,j}(n(t)),n(t))\right| \\
&\quad + \left|E(\bar{x}^*_{i,j}(n(t)),n(t)) - \overline{E}(\bar{x}^*_{i,j}(n(t)),t)\right| \\
&\le 2R\overline{D}\,\Delta t + \overline{D}\,d((k-1)\Delta t) \\
&\quad + C_3(T)(\|\Delta x\| + \|\Delta v\|) + C_4(T)\,e((k-1)\Delta t) \\
&\le C_5(T)(\|\Delta x\| + \|\Delta v\| + \Delta t) + C_6(T)\,e((k-1)\Delta t) \\
&= C_5(T)(\|\Delta x\| + \|\Delta v\| + \Delta t) + C_6(T)\,e(n(t)).
\end{aligned} \tag{3.10}$$

4. Proof of convergence. From the expression (3.1), using estimate (3.10), and that $|\overline{E}| \le N$ one obtains the inequality

$$\begin{aligned}
&|x_{i,j}(t) - \bar{x}_{i,j}(t)| + |v_{i,j}(t) - \bar{v}_{i,j}(t)| \\
&\quad \le \int_0^t \left(|v_{i,j}(s) - \bar{v}_{i,j}(s)| + N\,\Delta t + C_5(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t) + C_6(T)\,e(n(s))\right) ds.
\end{aligned} \tag{4.1}$$

However, for $(k-1)\Delta t < t \le k\,\Delta t$,

$$\begin{aligned} e(n(t)) &= e((k-1)\Delta t) \\ &= \sup_{i,j} |x_{i,j}((k-1)\Delta t) - \bar{x}_{i,j}((k-1)\Delta t)| + C((k-1)\Delta t)\,(\|\Delta x\| + \|\Delta v\|) + \delta. \end{aligned}$$

Now

$$\begin{aligned} |x_{i,j}((k-1)\Delta t) - \bar{x}_{i,j}((k-1)\Delta t)| &\le |x_{i,j}((k-1)\Delta t) - x_{i,j}(t)| \\ &\quad + |x_{i,j}(t) - \bar{x}_{i,j}(t)| + |\bar{x}_{i,j}(t) - \bar{x}_{i,j}((k-1)\Delta t)| \end{aligned}$$

and

$$\begin{aligned} |x_{i,j}((k-1)\Delta t) - x_{i,j}(t)| &\le R\,\Delta t, \\ |\bar{x}_{i,j}((k-1)\Delta t) - \bar{x}_{i,j}(t)| &\le (R + N\,\Delta t)\,\Delta t \le 2R\,\Delta t. \end{aligned}$$

Thus

$$|x_{i,j}((k-1)\Delta t) - \bar{x}_{i,j}((k-1)\Delta t)| \le |x_{i,j}(t) - \bar{x}_{i,j}(t)| + 3R\,\Delta t.$$

Since $C((k-1)\Delta t) \le C(t)$, $\delta \le \|\Delta x\|$ then

$$\begin{aligned} e(n(t)) = e((k-1)\Delta t) &\le \sup_{i,j} |x_{i,j}(t) - \bar{x}_{i,j}(t)| + 3R\,\Delta t + (C(t)+1)\,(\|\Delta x\| + \|\Delta v\|) \\ &\le \sup_{i,j} |x_{i,j}(t) - \bar{x}_{i,j}(t)| + C_7(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t). \end{aligned}$$

(4.2)

Let

$$h(t) = \sup_{i,j}(|x_{i,j}(t) - \bar{x}_{i,j}(t)| + |v_{i,j}(t) - \bar{v}_{i,j}(t)|).$$

From (4.1) and (4.2) it follows that $h(t)$ satisfies an inequality of the form

$$h(t) \le B_1(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t) + B_2(T)\int_0^t h(s)\,ds.$$

$B_1(T)$, $B_2(T)$ are constants depending on bounds (1.7a–d) and T. The solution is

$$\begin{aligned} h(t) &\le B_1(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t)\,e^{B_2(T)t} \\ &\le K(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t) \quad \text{for } 0 \le t \le T. \end{aligned}$$

This result is summarized in Theorem 4.1.

THEOREM 4.1. *There is a constant $K(T)$ which depends on bounds* (1.7a–d) *and T such that*

$$\sup_{i,j}(|x_{i,j}(t) - \bar{x}_{i,j}(t)| + |v_{i,j}(t) - \bar{v}_{i,j}(t)|) \le K(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t) \tag{4.3}$$

for $0 \le t \le T$.

We note that the assumptions that go into the proof such as $e(t) < \frac{L}{2}$ will be satisfied on an interval $[0, T]$ if $\|\Delta x\| + \|\Delta v\| + \Delta t$ is sufficiently small. The theorem applies when the mesh points are equally spaced so that

$$a_i - a_{i-1} = \Delta x = \frac{L}{N_x},$$

$$b_j - b_{j-1} = \Delta v = \frac{2R_0}{N_v},$$

and where

$$x_i = \frac{a_{i-1} + a_i}{2},$$

$$v_j = \frac{b_{j-1} + b_j}{2}.$$

In this case one obtains the estimate

$$\sup_{i,j}(|x_{i,j}(t) - \bar{x}_{i,j}(t)| + |v_{i,j}(t) - \bar{v}_{i,j}(t)|) \le K(T)(\Delta x + \Delta v + \Delta t).$$

We can modify the previous theory so that it is more applicable to equally weighted points. This is done by giving a different definition to the field $\overline{E}(x, t)$ as given in system (2.2a,b). Let

$$q_{i,j} = \int\int_{S_{i,j}} f_0(x, v)\, dv\, dx.$$

It will be assumed that $q_{i,j}$ is known exactly from the initial data in that it can be approximated initially to any degree of accuracy. Then let

$$\overline{E}(x, t) = \widehat{E}_{k-1}(x), \quad (k-1)\Delta t < t \le k\,\Delta t, \quad 0 \le x \le L, \tag{4.4}$$

and

$$\widehat{E}_{k-1}(x) = \sum_{i,j} q_{i,j}\gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) - \int_0^L K(x, y)\, dy.$$

Here it is assumed that $(\sum_{i,j} q_{i,j})/L = 1$ so that the constant λ of the previous section has the value one.

The proof for the present method differs in the estimate for

$$E(\bar{x}^*_{i,j}(n(t)), n(t)) - \overline{E}(\bar{x}^*_{i,j}(n(t)), t)$$

in the expression following (3.1). Since

$$q_{i,j} = f_0(x_i, v_j)\,\Delta x_i \Delta v_j + r_{i,j} + s_{i,j},$$

where

$$r_{i,j} = \int\int_{S_{i,j}} \left(\frac{f_0(x, v) - f_0(x_i, v)}{x - x_i}\right)(x - x_i)\, dv\, dx,$$

$$s_{i,j} = \int\int_{S_{i,j}} \left(\frac{f_0(x_i, v) - f_0(x_i, v_j)}{v - v_j}\right)(v - v_j)\, dv\, dx,$$

then

$$\begin{aligned} E(x, (k-1)\Delta t) - \widehat{E}_{k-1}(x) = &\left(\int_0^L \int_{-\infty}^{\infty} K(x, y)\, f(y, v, (k-1)\Delta t)\, dv\, dy \right. \\ &\left. - \sum_{i,j} \Delta x_i \Delta v_j\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t))\right) \\ &+ \sum_{i,j}(r_{i,j} + s_{i,j})\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)). \end{aligned}$$

However

$$\left| \int_0^L \int_{-\infty}^{\infty} K(x, y)\, f(y, v, (k-1)\Delta t)\, dv\, dy - \sum_{i,j} \Delta x_i\, \Delta v_j\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right|$$

is bounded according to (3.5) and it can be shown that

$$\left| \sum (r_{i,j} + s_{i,j})\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) \right| \leq D_0 R_0 L(\|\Delta x\| + \|\Delta v\|),$$

which is the same as estimate (3.6). Thus the bound for $|E(x, n(t)) - \overline{E}(x, t)|$ is precisely the estimate (3.7). Otherwise, the convergence proof is unchanged. This result is summarized as follows.

THEOREM 4.2. *For $\bar{x}_{i,j}(t)$, $\bar{v}_{i,j}(t)$, the solutions to* (2.2a,b) *where $\overline{E}(x, t)$ is defined by* (4.4), *there is a constant $K(T)$ which depends on bounds* (1.7a–d) *and T such that*

$$\begin{aligned} &\sup_{i,j}(|x_{i,j}(t) - \bar{x}_{i,j}(t)| + |v_{i,j}(t) - \bar{v}_{i,j}(t)|) \\ &\leq K(T)\,(\|\Delta x\| + \|\Delta v\| + \Delta t) \quad \textit{for} \quad 0 \leq t \leq T. \end{aligned} \tag{4.5}$$

As an application of Theorem 4.2 we consider the case of equally weighted initial data points. Let us assume that the initial function $f_0(x, v)$ satisfies (1.5d) and is of the form

$$f_0(x, v) = g(x)\, h(v),$$

where $\frac{1}{L}\int_0^L g(x)\, dx = 1$, $\int_{-R_0}^{R_0} h(v)\, dv = 1$. Then for the positive integer N_x the partition of $[0, L]$ is obtained by solving

$$\frac{i}{N_x} = \frac{1}{L} \int_0^{a_i} g(x)\, dx, \qquad i = 0, 1, \ldots, N_x$$

for the values

$$0 = a_0 < a_1 < a_2 < \cdots < a_{N_x} = L.$$

For the integer N_v the partition of $[-R_0, R_0]$ is obtained by solving

$$\frac{j}{N_v} = \int_{-R_0}^{b_j} h(v)\, dv, \qquad j = 0, 1, \ldots, N_v$$

for the values

$$-R_0 = b_0 < b_1 < b_2 < \cdots < b_{N_v} = R_0.$$

With this the region of phase space given by $0 \leq x \leq L$, $-R_0 \leq v \leq R_0$ is divided into the $N = N_x N_v$ rectangles $S_{i,j}$ such that

$$\int\int_{S_{i,j}} f_0(x, v)\, dv\, dx = \frac{L}{N}.$$

Mesh points (x_i, v_j) are located at the center of charge of $S_{i,j}$.

5. Numerical results. In the expressions for the approximate field (2.1), (4.4), it is assumed that $\delta \leq \|\Delta x\|$ and the final result on convergence (4.3), (4.5) does not depend on this parameter. So for the purposes of computation we let $\delta \to 0$ and in this limit the function $\gamma(x, y)$ is given as

$$\gamma(x, y) = \begin{cases} \frac{y}{L}, & 0 \leq y < x, \\ \frac{y}{L} - 1, & x < y \leq L, \\ \frac{x}{L} - \frac{1}{2}, & y = x. \end{cases} \tag{5.1}$$

The computation carried out with $\gamma(x, y)$ given by (5.1) is referred to as the unmollified problem to be compared subsequently with the case where a mollifier of the type used in [2] is introduced into the computation. Three sets of computations are carried out for the unmollified case, first where initial data points are equally spaced with unequal charge at each point, then when the points are equally weighted as in [4], [18], and finally when the points are asymptotically distributed as in [13], [14]. We are using the term asymptotically distributed to refer to a low discrepancy point set. We begin the discussion with the equally spaced problem.

The numerical method of Theorem 4.1 is applied in which the initial data points (x_i, v_j) are distributed as follows: $\Delta x = L/N_x$, $\Delta v = (v_{max} - v_{min})/N_v$,

$$x_i = \left(i - \frac{1}{2}\right) \Delta x, \qquad i = 1, \ldots, N_x,$$

$$v_j = \left(j - \frac{1}{2}\right) \Delta v, \qquad j = 1, \ldots, N_v.$$

The initial data is

$$f_0(x, v) = C\left(1 + 2\epsilon \cos \frac{2\pi x}{L}\right) h(v),$$

where

$$h(v) = \left[\frac{1}{\sqrt{2\pi}\, v_{th}^3}\right] v^2 e^{-v^2/2v_{th}^2}, \qquad v_{min} \leq v \leq v_{max}.$$

This choice of initial data results in a two-stream instability. We let $L = 1$, $v_{max} = 3.4v_{th}$, $v_{min} = -3.4v_{th}$, $\epsilon = 0.025$, $v_{th} = \frac{0.3}{\pi}$. The values of ϵ and v_{th} are taken from [4]. The value $3.4\, v_{th}$ is slightly larger than the maximal velocity on the mesh when the points are equally weighted with 100 discrete velocities as in [4]. The constant C is chosen so that

$$\int_0^L \int_{-\infty}^{\infty} f_0(x, v) dv dx = 1.$$

In the discussion that follows it is assumed that $f_0(x, v)$ has compact support in velocity space; i.e., $f_0(x, v) = 0$ for $v < v_{min}$ or $v > v_{max}$.

In this section we refer to the computed electric field as E_c; that is,

$$E_c(x, (k-1)\Delta x) = \frac{1}{\lambda} \sum_{i,j} (\Delta x\, \Delta v)\, f_0(x_i, v_j)\, \gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) - \int_0^L K(x, y)\, dy,$$

$$\lambda = \frac{\sum_{i,j} f_0(x_i, v_j) \Delta x\, \Delta v}{L}. \tag{5.2}$$

E_c is then given as

$$E_c = E_1 - E_2,$$

where

$$E_1(x) = \sum_{i,j} (\Delta x\, \Delta v) \frac{f_0(x_i, v_j)}{\lambda} \gamma(x, \bar{x}^*_{i,j}),$$

$$E_2(x) = \int_0^L K(x, y)\, dy,$$

which is computed explicitly as

$$E_2(x) = \frac{x}{L} - \frac{1}{2}.$$

To compute E_1 particles are ordered by the x-coordinate so that

$$0 < x_1 \leq x_2 \ldots \leq x_N < L, \quad N = N_x \times N_v.$$

Each x_i represents the x-coordinate of a particle. Except for the initial distribution we assume that no two particles have the exact same location in position space. If the ℓ, mth trajectory is at position x_i we then let

$$f_i = \frac{f_0(x_\ell, v_m)}{\lambda} \Delta x\, \Delta v \quad \text{and} \quad v_i = \bar{v}_{\ell,m}((k-1)\Delta t).$$

Then to compute the field at the particle position x_j at time $(k-1)\Delta t$, $k > 1$,

$$E_1(x_j) = \sum_{0 \leq x_i < x_j} \frac{f_i x_i}{L} + \sum_{x_j < x_i \leq L} f_i \left(\frac{x_i}{L} - 1\right) + f_j \left(\frac{x_j}{L} - \frac{1}{2}\right)$$

$$= \frac{1}{L} \sum_{i=1}^{N} f_i x_i - \sum_{x_i > x_j} f_i - \frac{1}{2} f_j$$

$$= \frac{1}{L} \sum_{i=1}^{N} f_i x_i - \sum_{i=1}^{N} f_i + \sum_{x_i \leq x_j} f_i - \frac{1}{2} f_j$$

$$E_c(x_j, (k-1)\Delta t) = E_1(x_j) - E_2(x_j).$$

We note that $C_1 = \frac{1}{L} \sum_{i=1}^{N} f_i x_i$ and $C_2 = \sum_{i=1}^{N} f_i$ are constants of the motion. The constant C_2 is given at the outset from the initial data and has the value one. The constant C_1 is computed at each time step and monitors the accuracy of the computation. At time $(k-1)\Delta t$ the particles are put into sequence by the x-coordinate and the sum for C_1 is computed. Then

$$E_1(x_1) = C_1 - C_2 + \frac{1}{2} f_1,$$

$$E_c(x_1, (k-1)\Delta t) = E_1(x_1) - E_2(x_1),$$

and

$$E_1(x_{j+1}) = E_1(x_j) + \frac{1}{2} f_j + \frac{1}{2} f_{j+1}, \quad j = 1, \ldots, N-1,$$

$$E_c(x_{j+1}, (k-1)\Delta t) = E_1(x_{j+1}) - E_2(x_{j+1}).$$

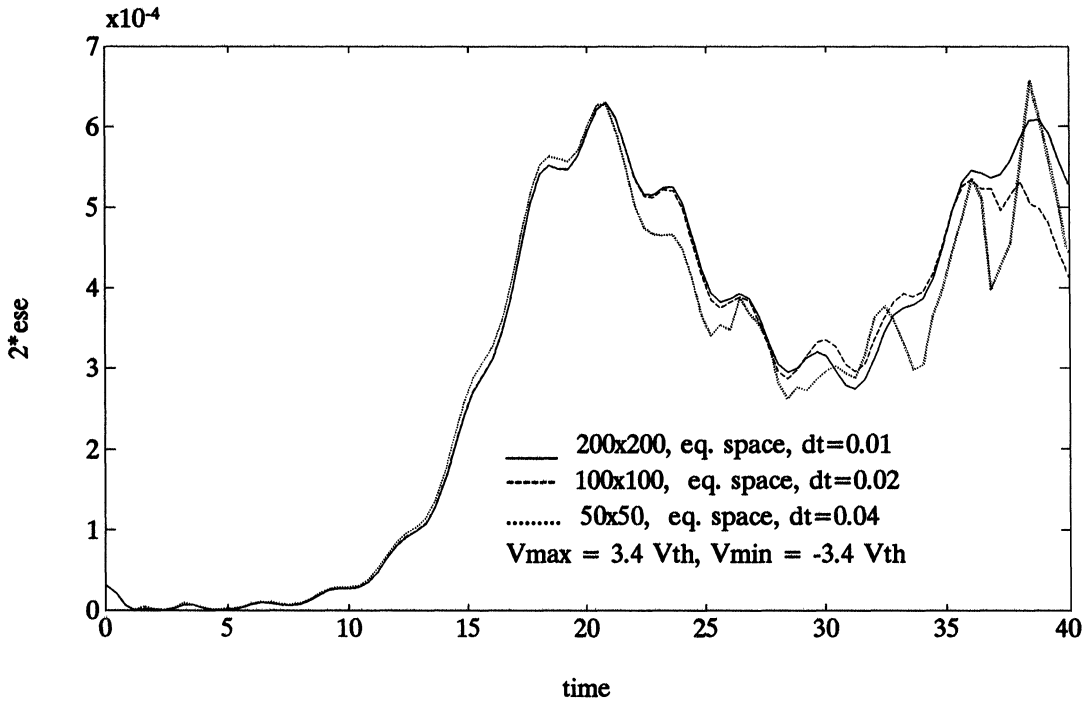

FIG. 1.

Having computed the field at all x_i positions, $i = 1, N$, we then advance the particle trajectories according to

$$x_{new} = x_i + v_i \Delta t + E_c(x_i, (k-1)\Delta t)(\Delta t)^2,$$
$$v_{new} = v_i + E_c(x_i, (k-1)\Delta t)(\Delta t).$$

(x_{new}, v_{new}) is the new position in (x, v)-space of the particle previously at (x_i, v_i). The particle distribution at time $k\,\Delta t$ is then sequenced with x-coordinates in ascending order. The particle previously at position i in the sequence is now at i' so that

$$x_{i'} = \bar{x}_{\ell,m}(k\,\Delta t), \quad v_{i'} = \bar{v}_{\ell,m}(k\,\Delta t), \quad f_{i'} = \frac{f_0(x_\ell, v_m)}{\lambda}.$$

To show the convergence of the algorithm just described computations were carried out on the IBM-370 to time $T = 40$. The graphs of electrostatic energy (*ese*) as a function of time are in Figure 1. The computation of *ese* is given as

$$ese = \frac{1}{2}\sum_{i=1}^{N-1}(E_c(x_i, (k-1)\Delta t))^2(x_{i+1} - x_i).$$

The solid curve was obtained with $N_x \times N_v = 200 \times 200$, $\Delta t = 0.01$ (40,000 particles). Figure 1 compares the *ese* computed with $N_x \times N_v = 50 \times 50$, $\Delta t = 0.04$ and $N_x \times N_v = 100 \times 100$, $\Delta t = 0.02$ with the 200×200 case. We see a clear convergence of the algorithm with the 40,000 particle computation giving the most accurate approximation to the solution to (1.2a–e). To give a point of reference the graph given by the solid line in Figure 2 is the

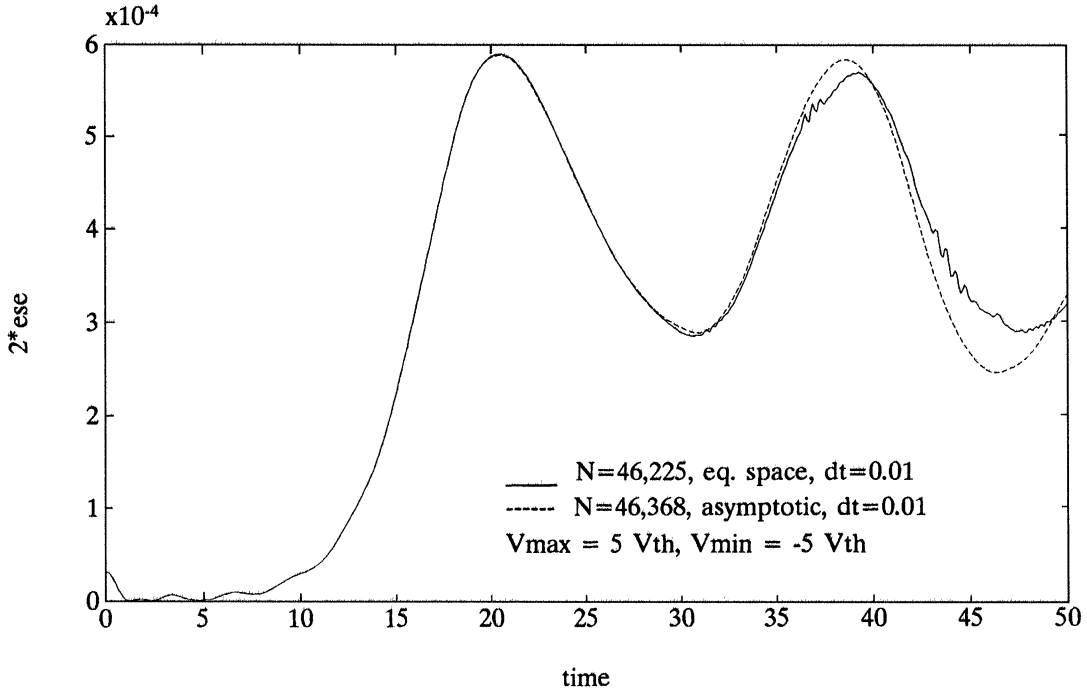

FIG. 2.

ese computed with 215 x 215 = 46,225 particles in steps of 0.01 and where $v_{max} = 5v_{\text{th}}$, $v_{min} = -5v_{th}$. With these parameters we get a better comparison with the graph in Figure 3 of [4]. The reason for restricting the domain in velocity space to $3.4v_{th}$ is to reduce Δv without increasing the number of data points which adds to the roundoff error. This improves the convergence of the computation so that we can better compare the different methods.

We now consider the case where the initial data points are equally weighted and compare the results with the previous computation for initial data points equally spaced. The convergence of the method is guaranteed by Theorem 4.2 and also by the theory on asymptotic distributions in [13]. Let

$$g(x) = \big(1 - 2\epsilon \cos(2\pi x)\big),$$

$$h(v) = \frac{1}{C\sqrt{2\pi}}\left(\frac{v^2}{v_{th}^3} e^{-v^2/2v_{th}^2}\right), \qquad v_{min} \le v \le v_{max},$$

where

$$C = \frac{1}{\sqrt{2\pi}} \int_{v_{min}}^{v_{max}} \frac{v^2}{v_{th}^3} e^{-v^2/2v_{th}^2}\, dv$$

$$= \left[\frac{1}{2} - \frac{v}{\sqrt{2\pi}\, v_{th}} e^{-v^2/2v_{th}^2} + \frac{1}{2} erf\left(\frac{v}{\sqrt{2}v_{th}}\right)\right]_{v_{min}}^{v_{max}}.$$

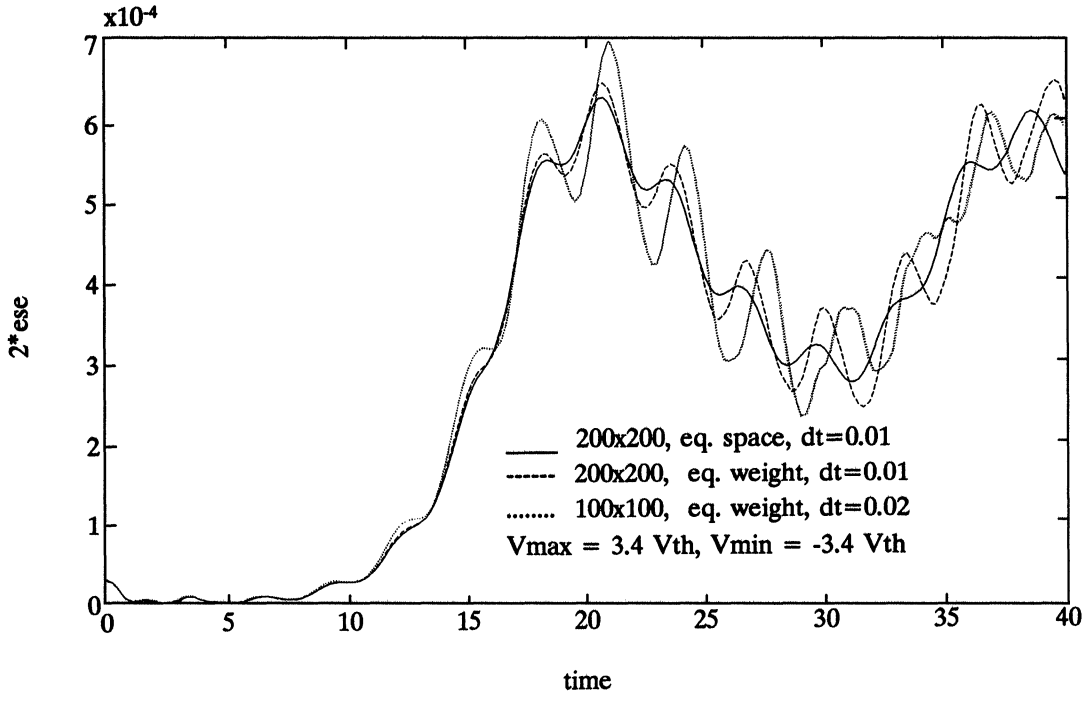

FIG. 3.

Here $erf(x)$ is the error function. The mesh points (x_i, v_j) are obtained by solving the nonlinear equations

$$\frac{i-1/2}{N_x} = \int_0^{x_i} g(x)\,dx, \quad \frac{j-1/2}{N_v} = \int_{v_{min}}^{v_j} h(v)\,dv,$$
$$i = 1, \ldots, N_x, \quad j = 1, \ldots, N_v.$$

The values x_i, v_j are computed to within an absolute error of 10^{-14}. If $(\bar{x}_{i,j}((k-1)\Delta t), \bar{v}_{i,j}((k-1)\Delta t))$ is the position in phase space of the particles initially at (x_i, v_j) then the computed electric field at time $(k-1)\Delta t$ is

$$E_c(x, (k-1)\Delta t) = \sum_{i,j} q_{i,j}\,\gamma(x, \bar{x}^*_{i,j}((k-1)\Delta t)) - \int_0^L K(x,y)\,dy,$$

where $q_{i,j} = \frac{1}{N}$ and $\gamma(x,y)$ is given by (5.1). Thus for $E_c = E_1 - E_2$, E_2 is computed explicitly as before and

$$E_1 = \sum_{i,j} \frac{1}{N}\gamma(x, \bar{x}^*_{i,j}).$$

Thus the algorithm for computing particle trajectories for equally weighted initial data is precisely the same as previously described for equally spaced data points except the quantity $\frac{\Delta x\,\Delta v}{\lambda} f_0(x_i, v_j)$ in (5.2) is replaced by $q_{i,j} = \frac{1}{N}$.

Figure 3 gives the graphs of *ese* as a function of time for initial data equally weighted and compares these graphs with the computation for equally spaced points. The solid curve is the graph previously obtained using 200 × 200 equally spaced data points and a time step

of 0.01. The dotted line and broken line correspond to the *ese* obtained using 100×100 data points equally weighted and a time step of 0.02 and 200×200 initial data points equally weighted with a time step of 0.01. The graphs of *ese* for equally weighted points appear to be converging to the graph for the equally spaced data points which still seems to be the most accurate approximation to the solution of (1.2a–e).

The computation done with equally weighted points is an example of the use of asymptotically distributed initial data in the sense of Neunzert and Wick [13]. However, one can make better use of this theory. To do so we have to introduce some concepts from number theory involving low discrepancy sequences of points. In the theory developed by Neunzert and Wick [12], [13], the initial distribution function is approximated in a measure theoretic sense by a sequence of equally weighted points. The particle method based on this initial distribution then gives a good measure theoretic approximation to the phase space distribution function which is the solution to the Vlasov–Poisson system. In this theory the error in the approximation is measured in terms of the "discrepancy" (or in terms of the "bounded Lipschitz distance"). For a definition of discrepancy and discussion of various concepts used in this theory we refer to [11], [14]. To obtain the initial particle distribution in phase space we start with a sequence of N points in the unit square $(e_{1,i}, e_{2,i})$, $i = 1, \ldots, N$ which as measured by the discrepancy is a good approximation to Lebesgue measure. The distribution of points in phase space is obtained from a transformation based on the initial distribution function. The theoretical development of this procedure is given in [9]. In the present problem when the distribution function factors as $f(x, v) = g(x)h(v)$ we obtain $(x_i, v_i) \in R_2$ from $(e_{1,i}, e_{2,i})$ by solving the equations

$$e_{1,i} = \int_0^{x_i} g(x)\,dx, \qquad e_{2,i} = \int_{v_{min}}^{v_i} h(v)\,dv. \tag{5.3}$$

In our previous computation using equally weighted points the sequence $(e_{1,i}, e_{2,i})$ is therefore given as

$$\begin{aligned} e_{1,i} &= \left(2\left[\frac{i-1}{N_v}\right] + 1\right)/2N_x, \\ e_{2,i} &= (2 \bmod(i-1, N_v) + 1)/2N_v, \end{aligned} \tag{5.4}$$

where $[a]$ is the integer part of a. However, as measured by the discrepancy this sequence gives an $O(N^{-1/2})$ approximation to Lebesgue measure in the unit square [17, p. 122]. For optimal low discrepancy sequences in the unit square the error is $O(\log(N))/N$. Neunzert and Wick give an example of such a low discrepancy sequence based on Fibonacci numbers [14]. The theoretical justification for the use of Fibonacci numbers in the construction of low discrepancy sequences is given in [26]. Here let α_k be the kth Fibonacci number $\alpha_0 = \alpha_1 = 1$, $\alpha_{k+1} = \alpha_k + \alpha_{k-1}$. Then let $N = \alpha_k$ and for the sequence $(e_{1,i}, e_{2,i})$ let

$$\begin{aligned} e_{1,i} &= \frac{(2i-1)}{2\alpha_k}, \qquad i = 1, 2, \ldots, N, \\ e_{2,i} &= \left\{\frac{2(i-1)\alpha_{k-1} + 1}{2\alpha_k}\right\}, \end{aligned} \tag{5.5}$$

where $\{a\}$ refers to the fractional part of a. Given the sequence $(e_{1,i}, e_{2,i})$ in the unit square the distribution of points in phase space (x_i, v_i), $i = 1, \ldots, N$ is then obtained from (5.3), giving a low discrepancy approximation to the initial distribution function.

Our procedure for computing with low discrepancy point sets is essentially the same as that previously employed for equally weighted points. What has changed is our initial

TABLE 1
Approximate rms error in the electrostatic energy up to time $T = 30$.

Equal space		Asymptotic		Equal weight	
N	e_{rms}	N	e_{rms}	N	e_{rms}
2,500	.20069435 D-4	2,584	.17555609 D-4		
4,900	.97498878 D-5	4,181	.42172450 D-5	4,900	.57499389 D-4
10,000	.34060607 D-5			10,000	.41033675 D-4
11,025	.29902862 D-5	10,946	.35277492 D-5	11,025	.41857985 D-4
17,689	.10398880 D-5	17,711	.22669199 D-5	17,689	.29088867 D-4
28,900	.42533723 D-6	28,657	.18269698 D-5	28,900	.17991787 D-4
40,000	.33682445 D-6			40,000	.13421366 D-4
46,225	.23055021 D-6	46,368	.58132383 D-6	46,225	.12541912 D-4
		75,025	.54364697 D-6		

TABLE 2
Approximate rms error in the electrostatic energy up to time $T = 40$.

Equal space		Asymptotic		Equal weight	
N	e_{rms}	N	e_{rms}	N	e_{rms}
2,500	.68027609 D-4	2,584	.39219445 D-4		
4,900	.39326597 D-4	4,181	.10756664 D-4	4,900	.55495285 D-4
10,000	.31670553 D-4			10,000	.43679096 D-4
11,025	.12017573 D-4	10,946	.94821164 D-5	11,025	.45934283 D-4
17,689	.16795783 D-4	17,711	.37287982 D-5	17,689	.36110570 D-4
28,900	.94279639 D-5	28,657	.59359742 D-5	28,900	.29677687 D-4
40,000	.36557252 D-5			40,000	.26705527 D-4
46,225	.22498863 D-5	46,368	.52764443 D-5	46,225	.25417208 D-4
		75,025	.24493154 D-5		

sequence $(e_{1,i}, e_{2,i})$ in the unit square, and we are also describing each particle by only the single index $i = 1, \ldots, N$. Thus $(\bar{x}_i(t), \bar{v}_i(t))$ is the ith trajectory associated with the initial point (x_i, v_i). The field at time $0 < (k-1)\Delta t \leq T$ is given as

$$E_c(x, (k-1)\Delta t) = \sum_i q_i \gamma(x, \bar{x}_i((k-1)\Delta t)) - \int_0^L K(x, y)\, dy,$$

where $q_i = \frac{1}{N}$. The discrete method based on (2.2a,b) is now exactly that of Neunzert and Wick [13]. As we expect from our previous discussion, using a low discrepancy initial sequence of points improves the rate of convergence for equally weighted particles. Improvements in computations based on low discrepancy sequences were also observed by Denavit and Walsh in [5].

Tables 1 and 2 show the results of computations done with low discrepancy initial point sets in comparison with computations done with a comparable number of equally spaced and equally weighted points. We approximate the rms error in the graph of electrostatic energy as a function of time in which we take as our "exact" solution the result of the computation done with $250 \times 250 = 62,500$ points equally spaced with time step $\Delta t = 0.01$. At these parameters a further refinement of the mesh in either space or time did not significantly improve the accuracy of the computation on the time intervals under consideration. This lack of improvement can be due to roundoff error. Specifically we compute

$$e_{rms} = \left(\frac{1}{N_\tau + 1} \sum_{k=1}^{N_\tau + 1} (e((k-1)\Delta\tau) - \bar{e}((k-1)\Delta\tau))^2 \right)^{1/2},$$

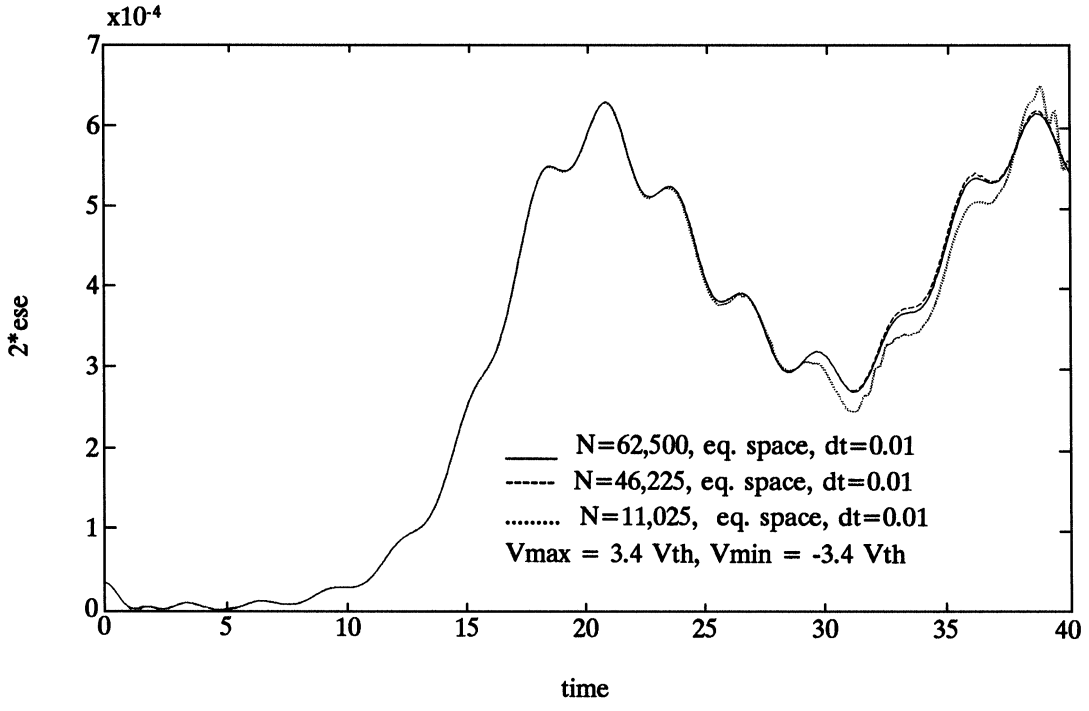

FIG. 4.

where $e((k-1)\Delta\tau) = 2 \times ese$ for the "exact" solution at time $(k-1)\Delta\tau$, with $\Delta\tau = 0.1$, $N_\tau = \frac{T}{\Delta\tau}$, and $\bar{e}((k-1)\Delta\tau) = 2 \times ese$ for the approximate solution. That is we use the values of $e((k-1)\Delta\tau)$, $\bar{e}((k-1)\Delta\tau)$ at every tenth time step of the computation. In all computations the time step is $\Delta t = 0.01$. Tables 1 and 2 give the results for computations up to time $T = 30$ and 40 in which N is the number of particles in the approximate solution.

It is clear that the rate of convergence for asymptotically distributed initial data is significantly better than for equally weighted initial data. This is consistent with the theory on asymptotic distributions. The theory of Neunzert and Wick [14] shows that for particles of a fixed finite size equal weighting is an $O(N^{-1/2})$ method whereas for asymptotically distributed data the method is $O(\log(N)/N)$. Here, the convergence is measured in terms of the bounded Lipschitz distance. As far as we know, there is no proof of this result for point particles. The rate of convergence for equal spaced initial data is also much better than the equally weighted case and seems to be comparable with that for asymptotically distributed data. The rate of convergence of this method appears to be closer to $O(\beta^2)$ rather than the $O(\beta)$ rate given by the theory in this paper. Figures 5 and 6 show the graphs of electrostatic energy for asymptotically distributed data for $N = 10,946$, $N = 46,369$, and $N = 75,025$, and Figure 4 shows the graphs of equally spaced data for $N = 11,025$ and $N = 46,225$. The solid curve in all figures is the electrostatic energy for 250×250 equally spaced initial points with $\Delta t = 0.01$ in all the computations. Figure 2 shows the result of increasing the support in velocity space to $v_{min} = -5v_{th}$, $v_{max} = 5v_{th}$. The broken line represents the *ese* for asymptotically distributed points and the solid line represents equally spaced points. With the increase in velocity support, numerical instabilities appear in the graph for equally spaced points but do not appear in the graph for asymptotically distributed points. Also the result obtained with asymptotically distributed initial data compares well with the solution of two-stream instability as shown in [4, Fig. 3].

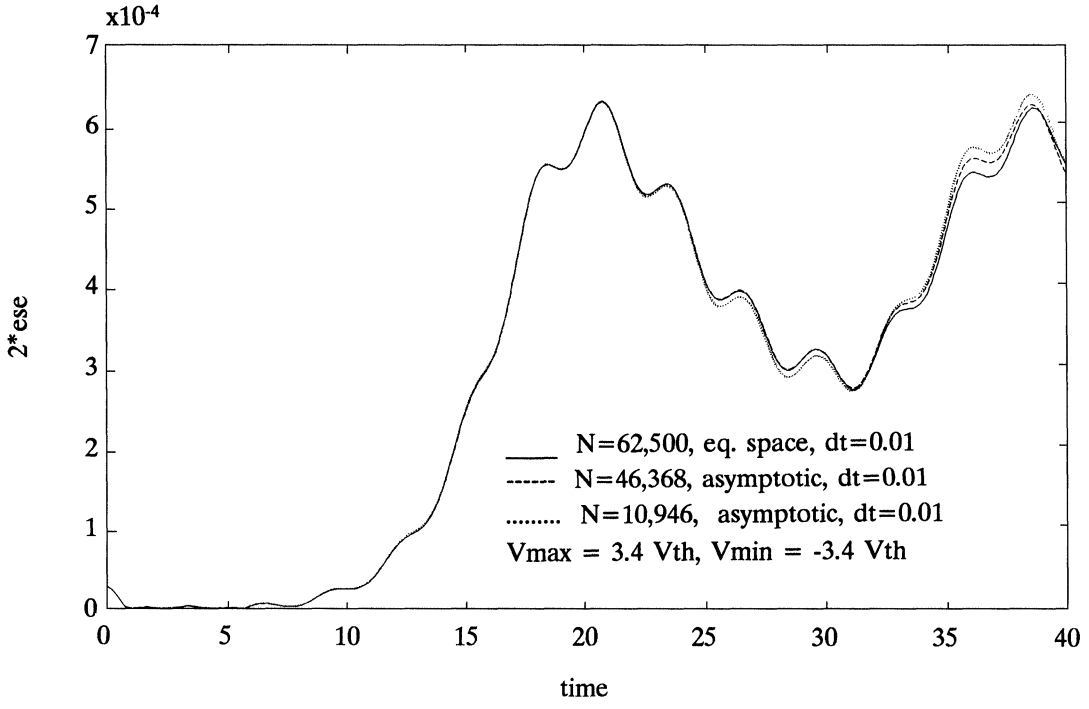

FIG. 5.

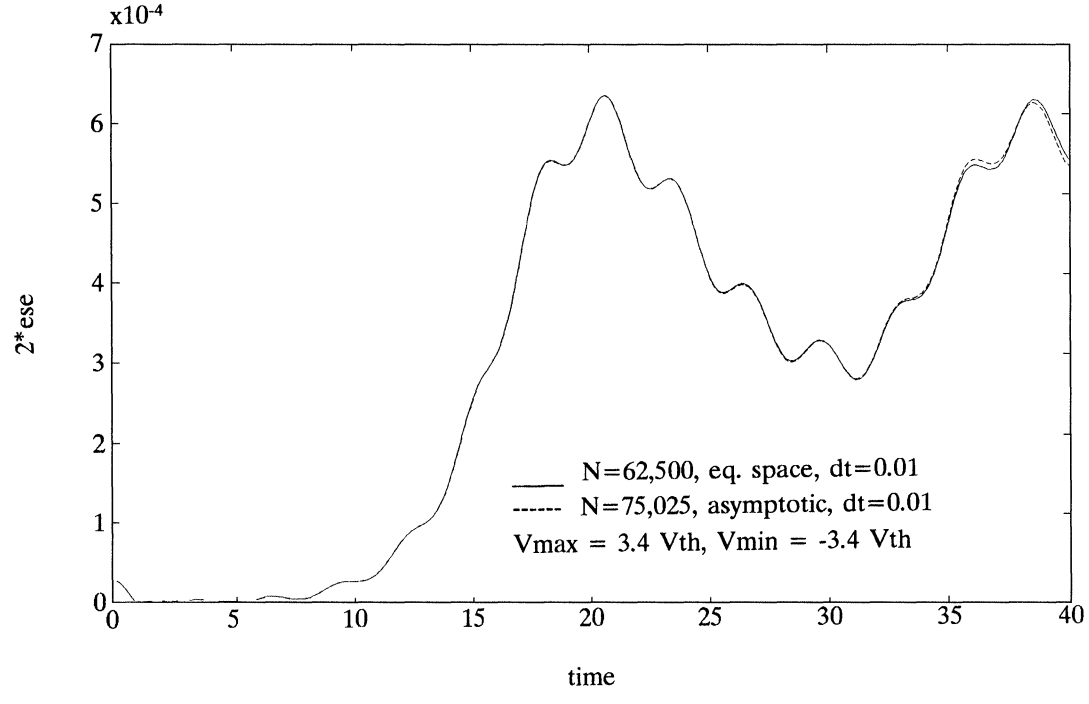

FIG. 6.

The estimate of the rate of convergence for equally spaced points is made more precise. The kinetic energy is a quantity which is several orders of magnitude larger than the electro-

static energy, and as the mesh is refined its computation retains a few more significant digits. Therefore, to estimate the rate of convergence we consider the rms error in the graph of kinetic energy as a function of time. At time $(k-1)\Delta t$ the kinetic energy for the approximate solution is

$$ke = \sum_{i,j} (\bar{v}_{i,j}((k-1)\Delta t))^2 \frac{f_0(x_i, v_j)}{\lambda} \Delta x \Delta v.$$

Let g represent the graph of kinetic energy for the exact solution and $\bar{g}$ the graph of kinetic energy for the approximate solution for $0 \leq \tau \leq t$. Here, t, τ take on discrete values $t = (k-1)\Delta t, k = 1, 2, \ldots$ and $\tau = (l-1)\Delta t, l = 1, 2, \ldots, k$. Then

$$e_{rms}(t) = rms(g - \bar{g}).$$

We assume for fixed t a form for the error

$$e_{rms} = C\beta^r + o(\beta^r),$$

where $\beta = \sqrt{(\Delta x)^2 + (\Delta v)^2}$. Here the effect of time discretization is being ignored. The rate of convergence r is estimated as follows: let $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$ be the graphs of kinetic energy for $0 \leq \tau \leq t$ associated with $\beta_1 = \beta$, $\beta_2 = \frac{\beta}{2}$, $\beta_3 = \frac{\beta}{4}$, and $D_1 = rms(\bar{g}_1 - \bar{g}_2)$, $D_2 = rms(\bar{g}_2 - \bar{g}_3)$. Then

$$r = \frac{\log(D_1/D_2)}{\log(\beta_1/\beta_2)}. \tag{5.6}$$

The quantity r as computed by (5.6) depends on time, t; however, in the limit of small parameters r approaches a constant in time and gives the order of accuracy in the spatial parameters of the numerical method. The rate of convergence is obtained similarly in [15]. We note that in computing r data at every time step is used whereas Tables 1 and 2 are based on data at every tenth time step. Figure 7 shows the result of a computation of rate of convergence as a function of time in which $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$ are obtained with $N_x \times N_v = 50 \times 50$, 100×100, 200×200, $\Delta t = 0.01$, $v_{max}, |v_{min}| = 3.4 v_{th}$. It appears from this estimate that the order of accuracy for equally spaced points is around 2.

We now show that the computation of two-stream instability using equally weighted points can be improved by a technique of staggering the position of points in velocity space as described in [4]. This refinement of the equal weighting method is obtained in the present context by a modification of the sequence in the unit square given by (5.4). The distribution of points in phase space is computed as before according to (5.3). Let K, N_x, N_v be positive integers such that N_x, N_v are multiples of K, $N = N_x N_v$. For $i = 1, \ldots, N_x$, $j = 1, \ldots, N_v$, and $n = (i-1)N_v + j$, then

$$\begin{aligned} e_{1,n} &= \left(i - \frac{1}{2}\right)/N_x, \\ e_{2,n} &= \left[(j-1)K + Mod(i-1, K) + \frac{1}{2}\right]/(KN_v), \end{aligned} \tag{5.7}$$

where $n = 1, \ldots, N$. If $K = 1$ then the sequence (5.7) is precisely (5.4) which gives an initial distribution having N_v discrete velocities and $N = N_x N_v$ data points. If $K > 1$, the sequence (5.7) gives an initial distribution with $N = N_x N_v$ points but KN_v discrete velocities.

Figure 8 shows the result of a computation based on (5.7) with staggered velocities. The broken line gives the graph of ese obtained with $N_x = 216$, $N_v = 216$, $N = 46,656$, $K = 8$, $v_{max}, |v_{min}| = 5 v_{th}$. The dotted line gives the graph of ese for $N_x = 216$, $N_v = 216$, $K = 1$,

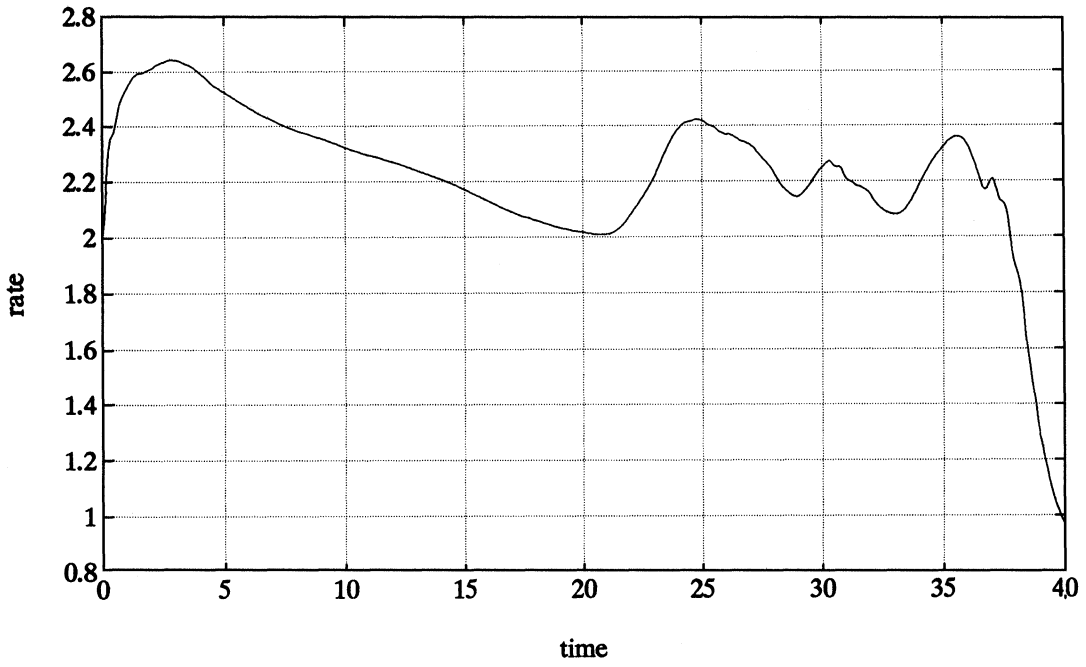

FIG. 7.

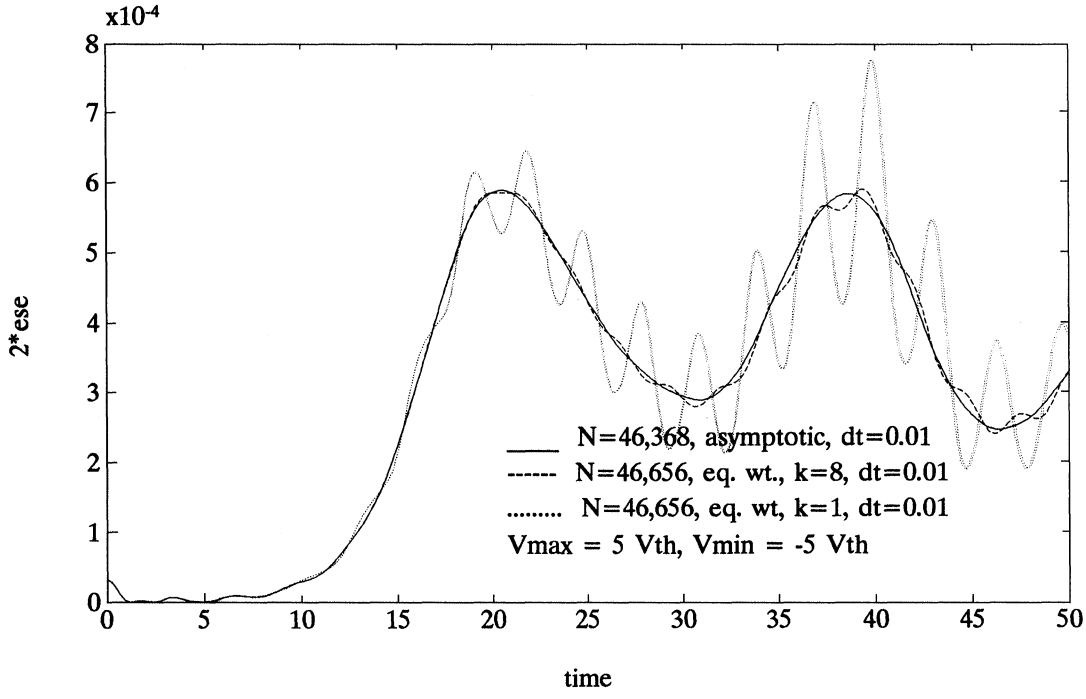

FIG. 8.

v_{max}, $|v_{min}| = 5v_{th}$ (unstaggered velocities as in (5.4)). The graphs are shown in reference to the *ese* for asymptotically distributed points represented by the solid line. The time step is

$\Delta t = .01$. Clearly staggering the velocities improves the computation with equally weighted points; furthermore, the gain in accuracy obtained with $K = 8$ may not be altogether explained on the basis of lowering the discrepancy as in the theory of Neunzert and Wick. Using 46,656 points and $K = 1$ the extreme discrepancy of sequence (5.7) is 4.62×10^{-3} (approximately $1/\sqrt{N}$) and with $K = 8$ it is 2.64×10^{-3}. The computation of extreme discrepancy is done according to the theorem of Niederreiter [27]. By contrast we estimate the extreme discrepancy of the sequence (5.5) with 46,368 points to be approximately 9×10^{-5}. Thus the discrepancy of the sequence (5.7) with $K = 8$ is somewhat lower than with $K = 1$ but still of the same order of magnitude. The sequence obtained from (5.5) clearly has much lower discrepancy. To further explain this gain in accuracy obtained from (5.7) with $K = 8$ we observe that increasing the array of discrete velocities used in the computation improves the approximation of initial kinetic energy particularly that due to particles at high velocity. We note that initial distributions based on either equally spaced or asymptotically distributed points also give a significantly better approximation to initial kinetic energy than that obtained on the basis of (5.7) with $K = 1$.

We now consider the question of whether the accuracy of the computation can be improved by introducing into the field computation a mollifier of the type used by Cottet and Raviart in [2]. In this theory initial data points are equally spaced. Let

$$\sigma(x) = \begin{cases} x + 1, & -1 \le x < 0, \\ 1 - x, & 0 \le x \le 1, \end{cases}$$

and

$$\sigma_\delta(x) = \frac{1}{\delta}\sigma\left(\frac{x}{\delta}\right).$$

Then in the computation of the electric field the kernel $\gamma(x, y)$ given by (5.1) is replaced with a mollified kernel

$$\gamma_\delta(x, y) = \int_{-\infty}^{\infty} K(x, y - \eta)\, \sigma_\delta(\eta)\, d\eta.$$

The function $\gamma_\delta(x, y)$ can be computed explicitly. We give the expression for $0 \le x - \delta \le x + \delta \le L$:

$$\gamma_\delta(x, y) = \begin{cases} \frac{y}{L}, & 0 \le y \le x - \delta, \\ \frac{y}{L} - \frac{(y-x)^2}{2\delta^2} - \frac{(y-x)}{\delta} - \frac{1}{2}, & x - \delta \le y \le x, \\ \frac{y}{L} + \frac{(y-x)^2}{2\delta^2} - \frac{(y-x)}{\delta} - \frac{1}{2}, & x \le y \le x + \delta, \\ \frac{y}{L} - 1, & x + \delta \le y \le L. \end{cases}$$

When $x - \delta < 0$ or $x + \delta > L$, this formula is modified based on the periodicity of the solution. Thus the field $E_c(x, (k-1)\Delta t)$ is now given as

$$E_c(x, (k-1)\Delta t) = \frac{1}{\lambda}\sum_{i,j}(\Delta x\, \Delta v)\, f_0(x_i, v_j)\, \gamma_\delta(x, \bar{x}^*_{i,j}) - \int_0^L K(x, y)\, dy. \tag{5.8}$$

The function $\sigma(x)$ is of class $W^{1,\infty}$ in the notation of [2] and, in addition,

$$\int_{-\infty}^{\infty} x\, \sigma(x)\, dx = 0.$$

Thus in [2, Thm. 3] we let $m = 1$, $k = 2$. If $\beta = \sqrt{(\Delta x)^2 + (\Delta v)^2}$ then the result of this theorem predicts a rate of convergence in Δx, Δv parameters of

$$O\left(\delta^2 + \frac{\beta^2}{\delta}\right).$$

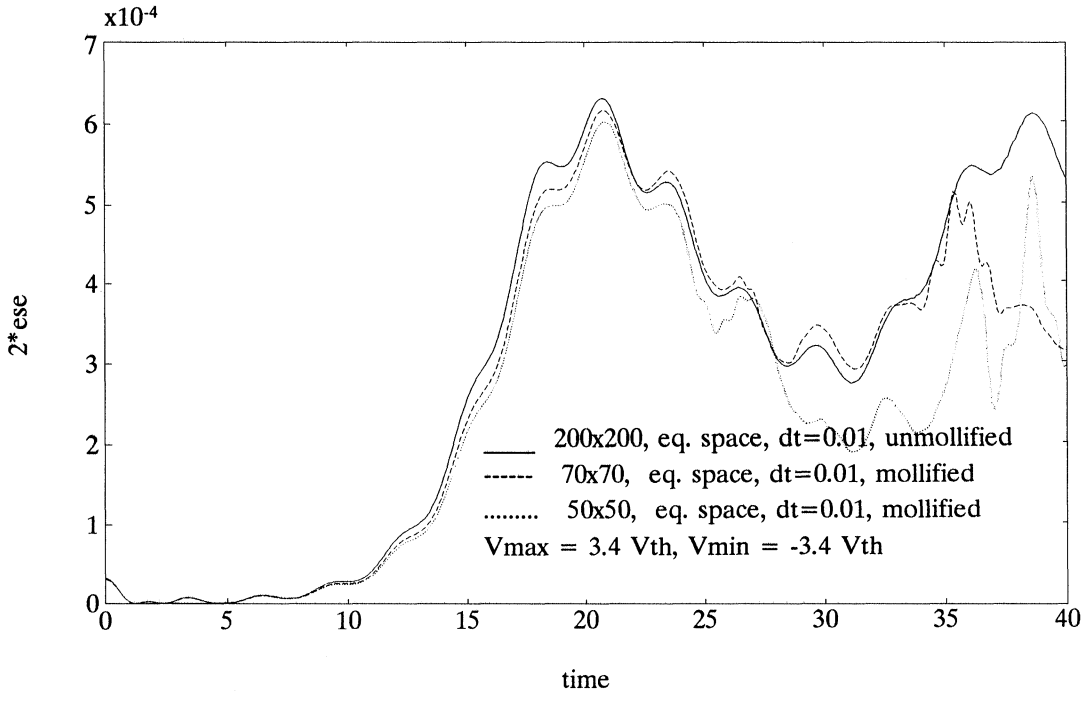

FIG. 9.

This expression is optimized in δ for $\delta = \beta^{2/3}$ in which case we get a rate of convergence $O(\beta^{4/3})$.

Figure 9 shows the result of computations done with expression (5.8) for the field E_c and where the time stepping and computation of *ese* is done the same as for the unmollified case. The solid line represents the *ese* previously obtained with $N_x \times N_v = 200 \times 200$ data points, $\Delta t = 0.01$, and no mollification of E_c. The dotted line gives the graph of *ese* obtained for the mollified case with $N_x \times N_v = 50 \times 50$ initial data points equally spaced and the broken line corresponds to the mollified case with $N_x \times N_v = 70 \times 70$ initial data points. The time step for the computations with the mollifier is $\Delta t = 0.01$ and $\delta = \beta^{2/3}$ in both computations. We see a convergence of the result with the mollifier to the *ese* curve obtained without the mollifier and with 40,000 points. Figure 10 shows the *ese* graph computed with $N_x \times N_v = 70 \times 70$ and no mollification of the field (in comparison with the 200×200 case). In comparing Figures 9 and 10 we cannot conclude that there is a significant gain in accuracy when the mollifier is used. However, because of time constraints in computing the electric field we did not further increase the number of particles when computing with a mollifier. Nonetheless, this computation gives some further indication that the order of accuracy for point particles equally spaced is better than $O(\beta)$. Otherwise, the mollifier should significantly improve the rate of convergence of this method. To better understand the use of the mollifier in improving convergence the problem needs further study. This has been done more thoroughly from a computational standpoint for vortex methods as in [15]. Also in regard to the Vlasov–Poisson system the effectiveness of the mollifier can perhaps be better studied in the context of the particle-in-cell method in which the electric field is obtained by solving the Poisson equation on a fixed grid. The width of the fixed grid takes the role of the mollification parameter and according to the results in [3] adjusting this width relative to the interparticle distance may improve the rate of convergence of the method.

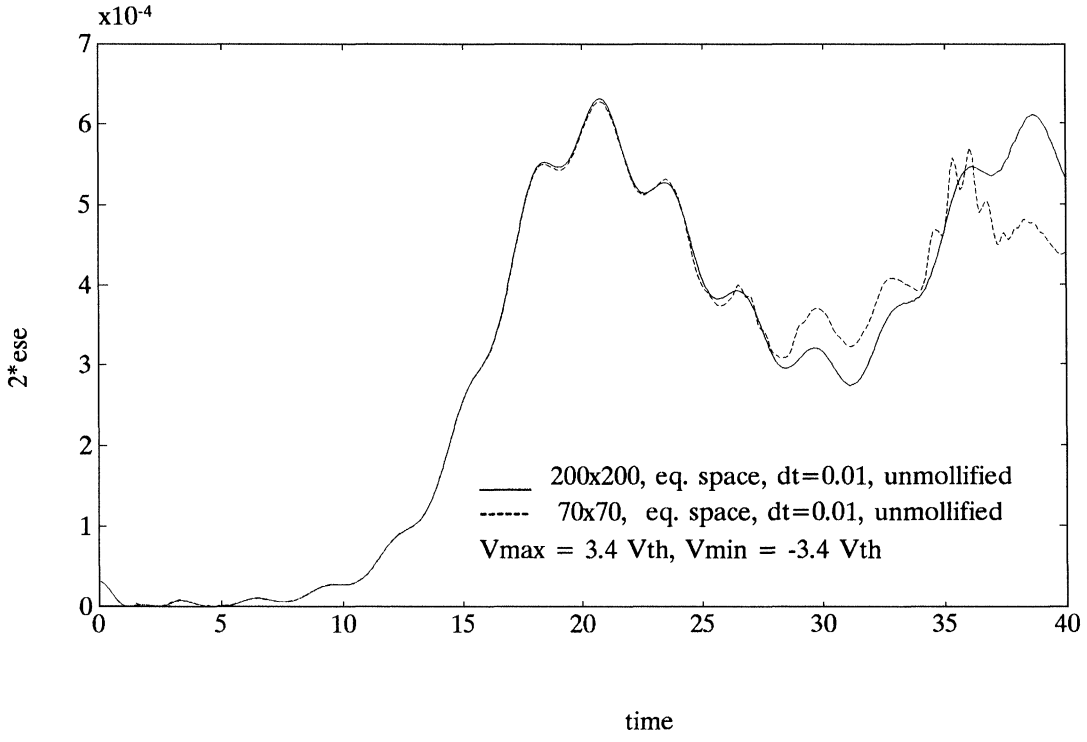

FIG. 10.

Conclusion. A conclusion we can draw from this work is that the convergence of the particle method in one dimension depends mainly on the spacing between initial data points and the time step. Including a mollifier in the electric field computation is not necessary for convergence. In general the rate of convergence in the spatial parameters for the point particle method is on the order of the interparticle distance. However, the accuracy of the method is improved for specific particle distributions. From the computational work it appears that we get a higher order of accuracy using point particles with equal spacing of initial data. Including a simple mollifier does not significantly improve the rate of convergence of this method. Our best results are obtained through the use of asymptotically distributed points. In computing the solution to two-stream instability the rate of convergence of this method is comparable to that for equally spaced data. In addition the method is stable over a wider range of parameters and better simulates the physics. Finally, with equally weighted points we get a good computed solution to the problem of two-stream instability by staggering the velocities. Without staggering this method is seen to be relatively inaccurate. Theory suggests that equal weighting is a lower-order method; however, its effectiveness is increased by widening the array of discrete velocities used in the computation.

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