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The Hybridizable Discontinuous Galerkin Methods

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Abstract

In this paper, we present and discuss the so-called hybridizable discontinuous Galerkin (HDG) methods. The discontinuous Galerkin (DG) methods were originally devised for numerically solving linear and then nonlinear hyperbolic problems. Their success prompted their extension to the compressible Navier-Stokes equations – and hence to second-order elliptic equations. The clash between the DG methods and decades-old, well-established finite element methods resulted in the introduction of the HDG methods. The HDG methods can be implemented more efficiently and are more accurate than all previously known DG methods; they represent a competitive alternative to the well established finite element methods. Here we show how to devise and implement the HDG methods, argue why they work so well and prove optimal convergence properties in the framework of diffusion and incompressible flow problems. We end by briefly describing extensions to other continuum mechanics and fluid dynamics problems.

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1. Introduction

In this paper, we study a recently introduced family of methods for numerically solving partial differential equations called the hybridizable discontinuous Galerkin (HDG) method. To motivate the advent of these methods, let us place them into a historical context.

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DG methods for hyperbolic problems. The original discontinuous Galerkin (DG) method was introduced in [47] back in 1973 for numerically solving the neutron transport equation, a linear hyperbolic equation for a scalar variable. The importance of the method was soon recognized and its first theoretical analysis was carried out in 1974 in [35]. The method lay dormant until the 90's, where it was successfully extended to non-linear time-dependent hyperbolic systems of conservation laws in a series of papers [27, 26, 25, 23, 28].

DG methods for convection-dominated problems. In 1997, prompted by the success of the RKDG methods for purely convective problems, the method was successfully extended [2] to the compressible Navier-Stokes equations. Soon after, many DG methods appeared for discretizing second-order symmetric elliptic problems and in 2002 a unifying framework for all of them was proposed in [1].

The new DG methods for second-order symmetric elliptic problems were then compared with the well established finite element methods, namely, the mixed methods of Raviart-Thomas (RT) [46] and Brezzi-Douglas-Marini (BDM [4], and the continuous Galerkin (CG) method. A definite advantage of the DG methods was their ability to handle *adaptive* algorithms, as they are able to easily work with meshes with hanging nodes and approximations of varying polynomial degrees. However, when compared with the CG method, the new DG methods were criticized for having too many degrees of freedom and for not being as easy to implement. And when compared with the mixed methods, for providing less accurate approximations and also for not being as efficiently implementable.

The HDG methods. As a response to these criticisms, the HDG methods were introduced in [16] in the framework of diffusion problems. Therein, it was shown that the RT and BDM mixed methods could be obtained as particular cases of these new HDG methods. This suggested that HDG methods *close* to the RT and BDM methods could be implemented as efficiently and could even share their superior convergence properties while retaining the advantages typical of the DG methods. It was soon proven that this was the case in [11, 22, 18].

This breakthrough opened the possibility of a *new* systematic approach to devising HDG methods geared towards *efficient* implementation and towards achieving optimal order of convergence for *all* the unknowns as well as superconvergence of some of them.

Organization of the paper. In this paper, we show how this approach is being developed. Thus, in Section 2, we begin by revisiting the original DG method for transport in order to display the features that will also be those of the HDG methods for diffusion and incompressible flow problems. In Section 3,

we consider in detail the HDG methods for diffusion problems. Then in Section 4, we consider the HDG methods for the Stokes equations of incompressible flow. We end in Section 5 by briefly describing on the ongoing work on HDG methods for problems arising in continuum mechanics and fluid flow.

A short bibliographical note. The reader interested in a detailed history of the development of the DG methods up to 1999 is referred to [24]. More information about the DG methods can be found in the 2001 review [29], the 2003 short essay [7] and the 2004 article [8]. The subsequent work on DG methods is impossible to cover in a few references. However, the reader might want to see the paper on stabilization mechanisms of the DG methods [3], as well as the special issues on DG methods in [30] and [31]. Finally, a short overview of the HDG methods can be found in [40].

2. The Original DG Method for Transport

In this section, we revisit the original DG method [47] which was devised to numerically solve the neutron transport equation,

$$\sigma\, u + \nabla \cdot (\boldsymbol{a}\, u) = f \qquad \text{in } \Omega, \tag{2.1a}$$

$$u = u_D \qquad \text{on } \partial\Omega_-, \tag{2.1b}$$

where σ is a positive number, $\boldsymbol{a}$ a constant vector and $\partial\Omega_-$ the *inflow* boundary of $\Omega \subset \mathbb{R}^d$, that is, $\partial\Omega_- = \{\boldsymbol{x} \in \partial\Omega : \boldsymbol{a} \cdot \boldsymbol{n}(\boldsymbol{x}) < 0\}$. Here $\boldsymbol{n}(\boldsymbol{x})$ is the outward unit normal at $\boldsymbol{x}$. For simplicity, we assume that Ω is a bounded polyhedral domain.

Our intention is to present the features of the method which are also going to be present in the HDG methods we consider in the following sections.

2.1. The method.

Discretization of the domain. We begin by discretizing the domain Ω. We consider disjoint open sets K called *elements* such that $\overline{\Omega} = \cup_{K\in\Omega_h} \overline{K}$. Their outward unit normal will be denoted by $\boldsymbol{n}$. We denote by Ω_h the collection of all these elements and set $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$. We say that F is an *interior face* of the triangulation Ω_h if there are two elements K^+ and K^- in Ω_h such that $F = \partial K^+ \cap \partial K^-$. In this case, we denote by $\boldsymbol{n}^\pm$ the outward unit normal of $K^\pm$ at F. The collection of all interior faces is denoted by $\mathcal{E}_h^o$. We say that F is a *boundary face* of the triangulation Ω_h if there is an element K in Ω_h such that $F = \partial K \cap \partial\Omega$. The collection of all boundary faces is denoted by $\mathcal{E}_h^\partial$. The set $\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial$ is called the set of the faces of the triangulation Ω_h.

Rewriting the equations. For each $K \in \Omega_h$, the DG method is devised to provide an approximation u_h to the *restriction* to K of the exact solution u *as well as* an approximation $\boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{u}_h$ to the *trace* on ∂K of the normal component of the flux $\boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{u}$. Thus, to define these approximations, we need to rewrite the original problem (2.1) in terms of those functions.

We do this as follows. On each element $K \in \Omega_h$, we obtain u in terms of $\widehat{u}$ on ∂K_- by solving

$$\sigma\, u + \nabla\cdot(\boldsymbol{a}\, u) = f \qquad \text{in } K, \tag{2.2a}$$

$$u = \widehat{u} \qquad \text{on } \partial K_-. \tag{2.2b}$$

In turn, the function $\widehat{u}$ on ∂K is expressed in terms of u and u_D as follows:

$$\widehat{u}(\boldsymbol{x}) := \begin{cases} u_D(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega_-, \\ \lim_{\epsilon\downarrow 0} u(\boldsymbol{x} - \epsilon\boldsymbol{a}) & \text{otherwise.} \end{cases} \tag{2.2c}$$

Here $\partial K_- := \{\boldsymbol{x} \in \partial K :\ \boldsymbol{a}\cdot\boldsymbol{n}(\boldsymbol{x}) < 0\}$.

The Galerkin formulation and the numerical trace. Now we discretize the above equations by combining a Galerkin method with a suitable definition of the numerical trace of the flux. So, on the element K, we take the approximation u_h in the finite dimensional space $W(K)$ and determine it by requiring that it satisfy a formulation we describe next. If we multiply the equation (2.2a) by a smooth function w, integrate over K, integrate by parts and use equation (2.2b), we get that

$$\sigma\,(u,w)_K - (u, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}\widehat{u}, w\rangle_{\partial K} = (f,w)_K.$$

Here $(u,w)_K$ is the integral of $u\,w$ over K and $\langle w, v\rangle_{\partial K}$ the integral of $w\,v$ over ∂K. Thus, we require that

$$\sigma\,(u_h,w)_K - (u_h, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}\widehat{u}_h, w\rangle_{\partial K} = (f,w)_K, \tag{2.3a}$$

for all $w \in W(K)$. On ∂K, $\widehat{u}_h$ is expressed in terms of u_h and u_D by

$$\widehat{u}_h(\boldsymbol{x}) := \begin{cases} u_D(\boldsymbol{x}) & \text{for } \boldsymbol{x} \in \partial K \cap \partial\Omega_-, \\ \lim_{\epsilon\downarrow 0} u_h(\boldsymbol{x} - \epsilon\boldsymbol{a}) & \text{otherwise.} \end{cases} \tag{2.3b}$$

This completes the definition of the original DG method [47].

The discrete energy identity and the existence and uniqueness of the approximation. Let us show that the method is well defined. To do that,

we use the following *energy* argument. We begin by noting that if we multiply the equation (2.1a) by u, integrate over Ω and carry out some simple algebraic manipulations, we obtain the following *energy* identity:

$$\sigma\|u - f/2\sigma\|^2_{L^2(\Omega)} + \frac{1}{2}\langle|\boldsymbol{a}\cdot\boldsymbol{n}|u,u\rangle_{\partial\Omega\setminus\partial\Omega_-} = \Psi(f,u_d),$$

where $\Psi(f,u_d) := \frac{1}{4\sigma}\|f\|^2_{L^2(\Omega)} + \frac{1}{2}\langle|\boldsymbol{a}\cdot\boldsymbol{n}|u_D,u_D\rangle_{\partial\Omega_-}$. A discrete version of this identity can be obtained by taking $w := u_h$ in the equation (2.3a), adding over all the elements $K \in \Omega_h$ and performing similar manipulations to get

$$\sigma\|u_h - f/2\sigma\|^2_{L^2(\Omega)} + \frac{1}{2}\langle|\boldsymbol{a}\cdot\boldsymbol{n}|u_h,u_h\rangle_{\partial\Omega\setminus\partial\Omega_-} + \Theta_h(u_h-\widehat{u}_h) = \Psi(f,u_D), \quad (2.4)$$

where $\Theta_h(u_h-\widehat{u}_h) = \frac{1}{2}\sum_{K\in\Omega_h}\langle|\boldsymbol{a}\cdot\boldsymbol{n}|(u_h-\widehat{u}_h),u_h-\widehat{u}_h\rangle_{\partial K_-} \geq 0$.

Now, to prove that the DG method is well defined, we only have to show that if we set $f = 0$ and $u_D = 0$, we obtain the trivial solution. But, by the energy identity we immediately get that $u_h = 0$ since σ is a positive number. This proves that the approximate solution exists and is unique.

Implementation. Note that, by construction, we have that

$$(\sigma\, u_h, w)_K - (u_h, \boldsymbol{a}\cdot\nabla w)_K + \langle\boldsymbol{a}\cdot\boldsymbol{n}\, u_h, w\rangle_{\partial K\setminus\partial K_-} = (f,w)_K - \langle\boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{u}_h, w\rangle_{\partial K_-}.$$

Thus we see that if $\widehat{u}_h$ on ∂K_- is known, then u_h on K can be computed. This is a remarkable property as it allows for an efficient implementation of the method. One cannot praise enough the importance of a property like this in a numerical method.

2.2. The stabilization mechanism.

The relation between the residuals. Note that the DG method is actually defined by imposing a linear relation between the residual in the interior of the element K, $R_K := \sigma\, u_h + \nabla\cdot(\boldsymbol{a}\, u_h) - f$, and the residual on its boundary ∂K, $R_{\partial K} := \boldsymbol{a}\cdot\boldsymbol{n}\,(u_h - \widehat{u}_h)$.

Indeed, in terms of these residuals, the first equation defining the DG method (2.3a) reads

$$(R_K, w)_K = \langle R_{\partial K}, w\rangle_{\partial K} \qquad \forall\ w \in W(K).$$

Since this implies that $\|\mathsf{P}_{W(K)}R_K\|_{L^2(K)} \leq C\, h_K^{-1/2}\|R_{\partial K}\|_{L^2(\partial K)}$, where $\mathsf{P}_{W(K)}$ is the L^2-projection into $W(K)$, and since

$$\|R_K\|_{L^2(K)} \leq \|\mathsf{P}_{W(K)}R_K\|_{L^2(K)} + \|(\mathrm{Id} - \mathsf{P}_{W(K)})R_K\|_{L^2(K)},$$

we see that the size of the residual in the interior R_K is controlled by the residual on the boundary $R_{\partial K}$ and by the approximation properties of the L^2-projection

into $W(K)$. This means that the size of the residuals, and hence the quality of the approximation, depend only on the size of the jumps $|\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}(u_h-\widehat{u}_h)$ and on the approximation properties of the space $W(K)$.

Stabilization by the jumps. The discrete energy identity (2.4) suggests that, since $\Theta(u_h-\widehat{u}_h)\geq 0$, the size of the jumps $|\boldsymbol{a}\cdot\boldsymbol{n}|^{1/2}(u_h-\widehat{u}_h)$ remains bounded.

In fact,the quantity $\Theta_h(u_h-\widehat{u}_h)$ is a dissipative term that enhances the stability properties of the method. Indeed, note that there is more dissipation if the size of the jump $u_h-\widehat{u}_h$ is big. This happens, for example, whenever the exact solution is discontinuous and consequently the interior residual is big. Thus, the DG method has a built-in mechanism that transforms its potential inability to obtain a good approximation into *numerical dissipation* and into improved stability properties.

The counterpart of this mechanism in finite difference and finite volume methods for scalar hyperbolic conservation laws is the one induced by the so-called *artificial viscosity* term. One of the main problems for those methods is to define it in such a way that high-order accuracy can be attained. The stabilization mechanism of the DG method has such highly valued property. Indeed, next we show that optimal accuracy can be reached whenever the exact solution is smooth enough.

2.3. Convergence properties.

We end our review of the original DG method by showing that optimal convergence properties of u_h can be proven when Ω is a polyhedral domain and the triangulations Ω_h are made of shape-regular simplexes K satisfying two conditions: (i) Each simplex K has a unique $\boldsymbol{a}$-*outflow* face, F_K^+, and (ii) each interior face is the $\boldsymbol{a}$-*inflow* face of another simplex. We say that the face F is an $\boldsymbol{a}$-outflow (inflow) face when $\boldsymbol{a}\cdot\boldsymbol{n}>\ (<)\ 0$, where $\boldsymbol{n}$ is the outward unit normal at F.

The auxiliary projection. Indeed, in this case, and when $W(K)$ is the space of polynomials of degree k on K, $\mathcal{P}_k(K)$, for each $K\in\Omega_h$, we can find an auxiliary projection Π with which the error analysis is greatly simplified. It is defined as follows. On the element K, the projection of the function $u\in H^1(K)$, Πu, is defined as the element of $\mathcal{P}_k(K)$ satisfying

$$\begin{aligned}(\Pi u-u,w)_K&=0 \qquad \forall\ w\in\mathcal{P}_{k-1}(K),\\ \langle\Pi u-u,w\rangle_{F_K^+}&=0 \qquad \forall\ w\in\mathcal{P}_k(F_K^+).\end{aligned}$$

This projection is well defined and, for smooth functions u, it provides optimal approximation properties, that is,

$$\|\Pi u-u\|_{L^2(K)}\leq C\,|\,u\,|_{H^{k+1}(K)}\,h^{k+1}.$$

Estimate of the projection of the error. The main reason for considering the projection of the error, $\varepsilon_u := \Pi(u - u_h)$, is that the projection Π is tailored to the structure of the numerical trace of the method. Indeed, for $\boldsymbol{x} \in \mathcal{E}_h \setminus \partial\Omega_-$, we have, by construction, that

$$\begin{aligned} \boldsymbol{a}\cdot\boldsymbol{n}\ \mathsf{P}_{M_h} u(\boldsymbol{x}) &= \boldsymbol{a}\cdot\boldsymbol{n}\ \lim_{\epsilon\downarrow 0} \Pi u(\boldsymbol{x} - \epsilon\boldsymbol{a}) \\ \boldsymbol{a}\cdot\boldsymbol{n}\ \widehat{u}_h(\boldsymbol{x}) &= \boldsymbol{a}\cdot\boldsymbol{n}\ \lim_{\epsilon\downarrow 0} u_h(\boldsymbol{x} - \epsilon\boldsymbol{a}), \end{aligned}$$

and we see that $\boldsymbol{a}\cdot\boldsymbol{n}\ (\mathsf{P}_{M_h} u - \widehat{u}_h)(\boldsymbol{x}) = \boldsymbol{a}\cdot\boldsymbol{n}\ \widehat{\varepsilon_u}(\boldsymbol{x})$. As a consequence, ε_u is the solution of

$$\sigma\,(\varepsilon_u, w)_K - (\varepsilon_u, \boldsymbol{a}\cdot\nabla w)_K + \langle \boldsymbol{a}\cdot\boldsymbol{n}\,\widehat{\varepsilon_u}, w\rangle_{\partial K} = \sigma\,(\Pi u - u, w)_K \qquad \forall\ w \in W(K),$$

where $\widehat{\varepsilon_u} = 0$ on $\partial\Omega_-$. We immediately see that the projection of the error must depend *only* on $\Pi u - u$. In particular, by following the process used to obtain the discrete energy identity, we get

$$\sigma\|\varepsilon_u - \frac{1}{2}(\Pi u - u)\|^2_{L^2(\Omega)} + \Theta_h(\varepsilon_u - \widehat{\varepsilon_u}) = \frac{\sigma}{4}\|\Pi u - u\|^2_{L^2(\Omega)},$$

and we can deduce that,

$$\|u - u_h\|_{L^2(\Omega)} + \sigma^{-1/2}\Theta_h^{1/2}(\varepsilon_u - \widehat{\varepsilon_u}) \le C\,|\,u\,|_{H^{k+1}(\Omega_h)}\,h^{k+1}.$$

This result is optimal in both regularity and order of convergence; see more general results in [10, 9]. For arbitrary meshes, there is a loss in the order of convergence of 1/2; see [33]. Although in practice this loss is hard to observe, it has been proven to actually occur in [45] and [48].

Postprocessing. Finally, we show how to postprocess the approximate solution in order to get an optimally convergent approximation of $\partial_{\boldsymbol{a}} u := \boldsymbol{a}\cdot\nabla u$. We proceed as follows. First, for each simplex K, we define the approximation $\boldsymbol{q}_h$ of the flux $\boldsymbol{a}\,u$ as the element of $\boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x}\,\mathcal{P}_k(K)$ that is the solution of

$$\begin{aligned} (\boldsymbol{q}_h - \boldsymbol{a}u_h, \boldsymbol{v})_K &= 0, && \forall\ \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K) \\ \langle(\boldsymbol{q}_h - \boldsymbol{a}\widehat{u}_h)\cdot\boldsymbol{n}, w\rangle_F &= 0, && \forall\ w \in \mathcal{P}_k(F), \text{ for all faces } F \text{ of } K. \end{aligned}$$

Here $\boldsymbol{\mathcal{P}}_k(K) := [\mathcal{P}_k(K)]^d$. Then, if we set $\partial_{\boldsymbol{a}} u_h^\star := \nabla\cdot\boldsymbol{q}_h$, it is easy to show [10, 9] that $\partial_{\boldsymbol{a}} u_h^\star - \mathsf{P}_{W(K)}(\partial_{\boldsymbol{a}} u) = \sigma(\mathsf{P}_{W(K)} u - u_h)$ on each simplex $K \in \Omega_h$, and to conclude that

$$\|\partial_{\boldsymbol{a}} u_h^\star - \partial_{\boldsymbol{a}} u\|_{L^2(\Omega_h)} \le C(|u|_{H^{k+1}(\Omega_h)} + |\partial_{\boldsymbol{a}} u|_{H^{k+1}(\Omega_h)})\,h^{k+1}.$$

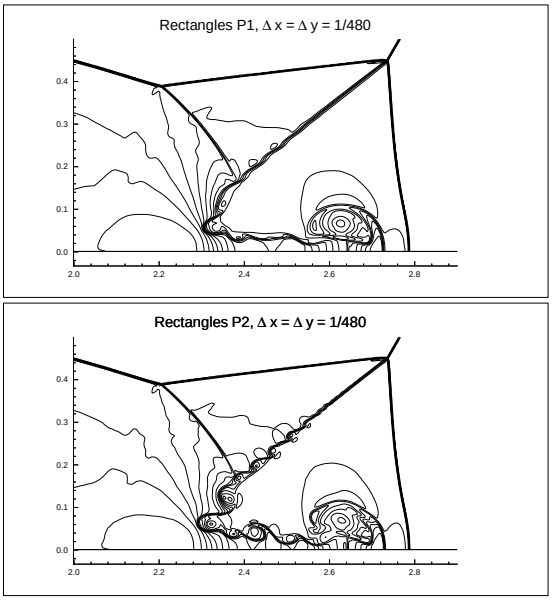

Figure 1. Euler equations of gas dynamics: Double Mach reflection problem. Isolines of the density around the double Mach stems. Linear polynomials on squares $\Delta x = \Delta y = \frac{1}{480}$ (top); and quadratic polynomials on squares $\Delta x = \Delta y = \frac{1}{480}$ (bottom).

2.4. The RKDG methods. Let us briefly point out that the extension of the orignal DG method to nonlinear hyperbolic conservation laws, called the Runge-Kutta discontinuous Galerkin (RKDG) methods, [27, 26, 25, 23, 28], shares with the original DG method many of the above-mentioned properties since it uses DG method to discretize the equations in space. To discretize the equations in time, a special type of *explicit* Runge-Kutta time marching methods is used. The distinctive feature of these Runge-Kutta methods is that their stability follows from the stability of a single Euler-forward step. A *crucial* component of the method is an operator (the so-called *slope limiter*) used to enforce the stability of the Euler-forward step and, as argued in [6], to ensure the convergence to the physically relevant solution. A rigorous convergence proof, however, remains a challenging open problem even for the scalar hyperbolic conservaton law. See [20] for rigorous error estimates for an *implicit*, *shock-capturing* DG method for that equation.

In practice, however, the methods turned out to be optimally convergent as well as able to capture very well the discontinuities of the solution. For example, consider the classical double-Mach reflection problem for the Euler equations of gas dynamics. In Fig. 1, obtained in [28], details of the approximation of the density are shown. The strong shocks are very well resolved by the RKDG solution using piecewise linear and piecewise quadratic polynomials defined on

squares. Note that there is a remarkable improvement in the approximation of the density near the contacts when going from linear to quadratic polynomials.

3. HDG Methods for Diffusion

In this section, we consider HDG methods to numerically solve the diffusion model problem

$$\begin{aligned} \mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega, && (3.1a)\\ \nabla \cdot \boldsymbol{q} &= f && \text{in } \Omega, && (3.1b)\\ u &= u_D && \text{on } \partial\Omega. && (3.1c) \end{aligned}$$

Here c is a matrix-valued function which is symmetric and uniformly positive definite on Ω.

We are going to show that despite the fact that the nature of this problem is radically different from the one just considered, we can device HDG methods by using a very similar approach. Most of the material for this section is contained in [16, 11, 22, 18].

3.1. The HDG methods.

Rewriting the equations. For each $K \in \Omega_h$, the methods provide an approximation to the restriction of $(\boldsymbol{q}, u)$ to K as well as an approximation to the traces $(\widehat{\boldsymbol{q}} \cdot \boldsymbol{n}, \widehat{u})$ on ∂K. We are thus going to rewrite the original equations in terms of those functions in order to be able to define the HDG methods by discretizing them.

Thus, if for each $K \in \Omega_h$, we assume that we know the trace $\widehat{u}$ on ∂K, we can obtain $(\boldsymbol{q}, u)$ inside K as the solution of

$$\begin{aligned} \mathrm{c}\,\boldsymbol{q} + \nabla u &= 0 && \text{in } K,\\ \nabla \cdot \boldsymbol{q} &= f && \text{in } K,\\ u &= \widehat{u} && \text{on } \partial K. \end{aligned}$$

The function $\widehat{u}$ can now be determined as the solution, on each $F \in \mathcal{E}_h$, of the equations

$$\begin{aligned} [\![\widehat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] &= 0 && \text{if } F \in \mathcal{E}_h^o,\\ \widehat{u} &= u_D && \text{if } F \in \mathcal{E}_h^\partial. \end{aligned}$$

Here we are using the notation $[\![\widehat{\boldsymbol{q}} \cdot \boldsymbol{n}]\!] := \widehat{\boldsymbol{q}}^+ \cdot \boldsymbol{n}^+ + \widehat{\boldsymbol{q}}^- \cdot \boldsymbol{n}^-$.

The Galerkin formulation and the numerical traces. Now we discretize these equations by using a Galerkin method together with a suitable approximation of the traces. First, we take $(\boldsymbol{q}_h, u_h)$ on the element $K \in \Omega_h$ in the finite dimensional space $\boldsymbol{V}(K) \times W(K)$ and $\widehat{u}_h$ on the face $F \in \mathcal{E}_h^o$ in the finite dimensional space $M(F)$.

On each element $K \in \Omega_h$, the function $(\boldsymbol{q}_h, u_h)$ is expressed in terms of $(\widehat{u}_h, f)$ by using a Galerkin method. Since

$$\begin{aligned}(c\,\boldsymbol{q}, \boldsymbol{v})_K - (u, \nabla \cdot \boldsymbol{v})_K + \langle \widehat{u}, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} &= 0,\\ -(\boldsymbol{q}, \nabla w)_K + \langle \widehat{\boldsymbol{q}} \cdot \boldsymbol{n}, w \rangle_{\partial K} &= (f, w)_K,\end{aligned}$$

for all sufficiently smooth functions $(\boldsymbol{v}, w)$, we determine $(\boldsymbol{q}_h, u_h)$ in terms of $(\widehat{u}_h, f)$ as the solution of

$$(c\,\boldsymbol{q}_h, \boldsymbol{v})_K - (u_h, \nabla \cdot \boldsymbol{v})_K + \langle \widehat{u}_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = 0, \tag{3.2a}$$

$$-(\boldsymbol{q}_h, \nabla w)_K + \langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, w \rangle_{\partial K} = (f, w)_K, \tag{3.2b}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$, where the numerical trace $\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}$ is *assumed* to have the following simple form:

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial K. \tag{3.2c}$$

Then the function $\widehat{u}_h$ is determined by enforcing weakly the single-valuedness of the normal component of the numerical trace $\widehat{\boldsymbol{q}}_h$ and by capturing the Dirichlet boundary condition. Thus, for each face $F \in \mathcal{E}_h$, we require that

$$\langle \mu, [\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] \rangle_F = 0 \qquad \forall\, \mu \in M(F), \tag{3.2d}$$

$$\widehat{u}_h = u_D \quad \text{ if } F \in \mathcal{E}_h^{\partial}. \tag{3.2e}$$

This completes the definition of the HDG methods. Note that equation (3.2d) is a condition on the single-valuedness of the normal component of $\widehat{\boldsymbol{q}}_h$ on $\mathcal{E}_h^o$. Indeed, if the restriction of $[\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!]$ to F lies in $M(F)$ for all $F \in \mathcal{E}_h^o$, we have that $[\![\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}]\!] = 0$ on $\mathcal{E}_h^o$ and the normal component of $\widehat{\boldsymbol{q}}_h$ is single valued. Studies of the importance of this property can be found in [22, 21].

The discrete energy identity and the existence and uniqueness of the approximation. The HDG methods are well defined under some very mild conditions, as we see in the next result.

Proposition 3.1. *The HDG method is well defined if (i) $\tau > 0$ on $\partial\Omega_h$, and if (ii) for any element $K \in \Omega_h$, $\nabla w \in \boldsymbol{V}(K)$ for all $w \in W(K)$.*

We can prove this result by using an energy argument. If we multiply the first two equations (3.1) defining the exact solution by $\boldsymbol{q}$ and u, respectively,

integrate over Ω and add the equations, we obtain the following *energy* identity

$$(c\,\boldsymbol{q},\boldsymbol{q})_\Omega = (f,u)_\Omega - \langle u_D, \boldsymbol{q}\cdot\boldsymbol{n}\rangle_{\partial\Omega}.$$

If we now apply a similar procedure to the HDG method, we get a *discrete energy* identity. So, taking $(\boldsymbol{v},w) := (\boldsymbol{q}_h,u_h)$ in the first two equations defining the HDG method, adding over all the elements $K \in \Omega_h$ and then adding the resulting equations, we get

$$(c\,\boldsymbol{q}_h,\boldsymbol{q}_h)_\Omega + \Theta_\tau(u_h-\widehat{u}_h) = (f,u_h)_\Omega - \langle u_D, \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\rangle_{\partial\Omega}. \tag{3.3}$$

where $\Theta_\tau(u_h-\widehat{u}_h) := \sum_{K\in\Omega_h}\langle\tau(u_h-\widehat{u}_h),u_h-\widehat{u}_h\rangle_{\partial K}$.

To prove Proposition 3.1, we only have to show that if we set $f=0$ and $u_D=0$, the only solution is the trivial one. But, by the discrete energy identity and condition (i) of Proposition 3.1, we have that $\boldsymbol{q}_h=\boldsymbol{0}$ on Ω_h and that $u_h=\widehat{u}_h$ on $\mathcal{E}_h$. Now, by equation (3.2a), this implies that $(\nabla u_h,\boldsymbol{v})_K=0 \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}(K)$, and by conditon (ii), we conclude that u_h is a constant on Ω. Since $u_h=\widehat{u}_h=0$ on $\partial\Omega$, we see that $u_h=0$ on Ω_h and hence that $\widehat{u}_h=0$ on $\mathcal{E}_h$. The proof of Proposition 3.1 is complete.

Implementation. To describe the implementation of the HDG methods, we need to introduce some notation. We denote by $(\boldsymbol{\mathcal{Q}}(\widehat{u}_h,f),\mathcal{U}(\widehat{u}_h,f))$ the linear mapping (see equations (3.2a), (3.2b) and (3.2c)) that associates $(\widehat{u}_h,f)$ to $(\boldsymbol{q}_h,u_h)$ and set

$$(\boldsymbol{\mathcal{Q}}^{\widehat{u}_h},\mathcal{U}^{\widehat{u}_h}) := (\boldsymbol{\mathcal{Q}}(\widehat{u}_h,0),\mathcal{U}(\widehat{u}_h,0)),$$
$$(\boldsymbol{\mathcal{Q}}^{f},\mathcal{U}^{f}) := (\boldsymbol{\mathcal{Q}}(0,f),\mathcal{U}(0,f)).$$

We also introduce the space $M_h := \{\mu\in L^2(\mathcal{E}_h):\ \mu|_F\in M(F)\ F\in\mathcal{E}_h^o\}$ and set $M_h(\zeta) := \{\mu\in M_h:\ \mu|_{\partial\Omega}=\zeta\}$.

With this notation, we can characterize the approximate solution given by the HDG method as follows.

Proposition 3.2. *We have that* $(\boldsymbol{q}_h,u_h) = (\boldsymbol{\mathcal{Q}}^{\widehat{u}_h},\mathcal{U}^{\widehat{u}_h})+(\boldsymbol{\mathcal{Q}}^{f},\mathcal{U}^{f})$, *where* $\widehat{u}_h\in M_h(u_D)$ *is the solution of*

$$a_h(\widehat{u}_h,\mu) = b_h(\mu) \qquad \forall\,\mu\in M_h(0).$$

Here, $a_h(\mu,\eta) := \sum_{K\in\Omega_h}((\mathrm{c}\boldsymbol{\mathcal{Q}}^{\mu},\boldsymbol{\mathcal{Q}}^{\eta})_K + \langle\tau(\mathcal{U}^{\mu}-\mu),\mathcal{U}^{\eta}-\eta\rangle_{\partial K})$, *and* $b_h(\mu) := (f,\mathcal{U}^{\mu})_\Omega$, *for any* $\mu,\eta\in M_h$.

Note that the formulation characterizing $\widehat{u}_h$ is a rewriting of equations (3.2d) and (3.2e).

We can thus see that the HDG method can be implemented as a typical finite element method. Once the function $\widehat{u}_h$ is computed, we can readily compute $(\boldsymbol{q}_h, u_h)$. For details of the implementation, see [34], where the HDG methods are shown to be *more* efficient than the CG method for high-degree polynomial approximations.

3.2. The stabilization mechanism.

The relation between the residuals. Note that, also for diffusion problems, the HDG method is defined by imposing a linear relation between the residuals in the interior of the element K, $\boldsymbol{R}_K^u := \mathrm{c}\boldsymbol{q}_h + \nabla u_h$ and $R_K^{\boldsymbol{q}} := \nabla \cdot \boldsymbol{q}_h - f$, and the residuals on its boundary ∂K, $R_{\partial K}^u := u_h - \widehat{u}_h$ and $R_{\partial K}^{\boldsymbol{q}} := (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n} = -\tau\,(u_h - \widehat{u}_h)$.

Indeed, in terms of these residuals, the first two equations defining the HDG method read

$$\begin{aligned} (\boldsymbol{R}_K^u, \boldsymbol{v})_K &= \langle R_{\partial K}^u, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} \quad \forall\ \boldsymbol{v} \in \boldsymbol{V}(K), \\ (R_K^{\boldsymbol{q}}, w)_K &= \langle R_{\partial K}^{\boldsymbol{q}}, w \rangle_{\partial K} \qquad \forall\ w \in W(K). \end{aligned}$$

Since this implies that

$$\begin{aligned} \|\mathsf{P}_{\boldsymbol{V}(K)} \boldsymbol{R}_K^u\|_{L^2(K)} &\le C\, h_K^{-1/2} \|R_{\partial K}^u\|_{L^2(\partial K)}, \\ \|\mathsf{P}_{W(K)} R_K^{\boldsymbol{q}}\|_{L^2(K)} &\le C\, h_K^{-1/2} \|R_{\partial K}^{\boldsymbol{q}}\|_{L^2(\partial K)}, \end{aligned}$$

where $\mathsf{P}_{\boldsymbol{V}(K)}$ and $\mathsf{P}_{W(K)}$ are the L^2-projections into $\boldsymbol{V}(K)$ and $W(K)$, respectively, we see that the quality of the approximation depends only on $u_h - \widehat{u}_h$, τ and on the approximation properties of the spaces $\boldsymbol{V}(K)$ and $W(K)$.

Stabilization by the jumps. The discrete energy identity (3.3) indicates that we can control the jumps $u_h - \widehat{u}_h$ if we take τ to be strictly positive. In this case, $\Theta_\tau(u_h - \widehat{u}_h)$ becomes a dissipative term which enhances the stability of the numerical method, just as for the original DG method. That this stabilization does not affect in a negative manner the accuracy of the method is shown next.

3.3. Convergence properties. Here we discuss the convergence properties of the method when Ω is a polyhedral domain, the triangulations Ω_h are made of shape-regular simplexes K, and when we take $\boldsymbol{V}(K) \times W(K) := \boldsymbol{P}_k(K) \times P_k(K)$. For simplicity, we assume that the stabilization function τ is *constant* on each ∂K.

The auxiliary projection. To do this, we follow what was done for the original DG method and define an auxiliary projection. On any simplex K, the

projection of the function $(\boldsymbol{q}, u) \in \boldsymbol{H}^1(K) \times H^1(K)$, $\Pi(\boldsymbol{q}, u) := (\boldsymbol{\Pi q}, \Pi u)$ is the element of $\boldsymbol{P}_k(K) \times \mathcal{P}_k(K)$ which solves the equations

$$\begin{aligned}
(\boldsymbol{\Pi q}, \boldsymbol{v})_K &= (\boldsymbol{q}, \boldsymbol{v})_K && \forall\, \boldsymbol{v} \in \boldsymbol{P}_{k-1}(K),\\
(\Pi u, w)_K &= (u, w)_K && \forall\, w \in P_{k-1}(K),\\
\langle \boldsymbol{\Pi q} \cdot \boldsymbol{n} + \tau \Pi u, \mu\rangle_F &= \langle \boldsymbol{q} \cdot \boldsymbol{n} + \tau u, \mu\rangle_F && \forall\, \mu \in P_k(F),
\end{aligned}$$

for all faces F of the simplex K. This projection is well defined, as we see in the next result.

Theorem 3.3. *Suppose that $\tau_K := \tau|_K$ is positive. Then $(\boldsymbol{\Pi q}, \Pi u)$ is well defined for any $k \geq 0$. Furthermore, there is a constant C independent of K and τ_K such that*

$$\begin{aligned}
\|\boldsymbol{\Pi q} - \boldsymbol{q}\|_K &\leq C\, h_K^{k+1}\, |\boldsymbol{q}|_{\boldsymbol{H}^{k+1}(K)} + C\, h_K^{k+1}\, \tau_K\, |u|_{H^{k+1}(K)},\\
\|\Pi u - u\|_K &\leq C\, h_K^{k+1}\, |u|_{H^{k+1}(K)} + C\, \frac{h_K^{k+1}}{\tau_K}\, |\nabla \cdot \boldsymbol{q}|_{H^k(K)}.
\end{aligned}$$

Estimate of the projection of the errors. This projection is fitted to the structure of the numerical trace $\widehat{\boldsymbol{q}}_h$ because, if we consider the projection of the errors $(\varepsilon_{\boldsymbol{q}}, \varepsilon_u) := (\boldsymbol{\Pi q} - \boldsymbol{q}_h, \Pi u - u_h)$, we see that we have, by the last property of the projection,

$$P_{M_h}(\boldsymbol{q} \cdot \boldsymbol{n}) = \boldsymbol{\Pi q} \cdot \boldsymbol{n} + \tau(\Pi u - P_{M_h} u) \quad \text{ on } \partial K.$$

Comparing this expression with the definition of the numerical trace $\widehat{\boldsymbol{q}}_h$, (3.2c),

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau(u_h - \widehat{u}_h) \qquad \text{on } \partial K,$$

we obtain that $\boldsymbol{\varepsilon}_{\widehat{\boldsymbol{q}}} \cdot \boldsymbol{n} := \varepsilon_{\boldsymbol{q}} \cdot \boldsymbol{n} + \tau(\varepsilon_u - \varepsilon_{\widehat{u}})$ on ∂K, provided we set $\varepsilon_{\widehat{u}} := P_{M_h} u - \widehat{u}_h$ and $\boldsymbol{\varepsilon}_{\widehat{\boldsymbol{q}}} \cdot \boldsymbol{n} := P_{M_h}(\boldsymbol{q}_h \cdot \boldsymbol{n}) - \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}$, where P_{M_h} is the L^2-projection into the space M_h. This implies that the equations satisfied by the projection of the errors are the following. For each simplex $K \in \Omega_h$,

$$\begin{aligned}
(c\, \varepsilon_{\boldsymbol{q}}, \boldsymbol{v})_K - (\varepsilon_u, \nabla \cdot \boldsymbol{v})_K + \langle \varepsilon_{\widehat{u}}, \boldsymbol{v} \cdot \boldsymbol{n}\rangle_{\partial K} &= (c\, (\boldsymbol{\Pi q} - \boldsymbol{q}), \boldsymbol{v})_K,\\
-(\varepsilon_{\boldsymbol{q}}, \nabla w)_K + \langle \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{q}}} \cdot \boldsymbol{n}, w\rangle_{\partial K} &= 0,
\end{aligned}$$

for all $(\boldsymbol{v}, w) \in \boldsymbol{V}(K) \times W(K)$. Moreover, for each face $F \in \mathcal{E}_h$,

$$\begin{aligned}
\langle \mu,\, [\![\boldsymbol{\varepsilon}_{\widehat{\boldsymbol{q}}} \cdot \boldsymbol{n}]\!]\rangle_F &= 0 \qquad \forall\, \mu \in M(F),\\
\varepsilon_{\widehat{u}} &= 0 \qquad \text{if } F \in \mathcal{E}_h^{\partial}.
\end{aligned}$$

We thus see that the projections of the error solely depend on $\boldsymbol{\Pi q} - \boldsymbol{q}$. In

particular, the discrete energy identity for these equations is

$$(c\,\varepsilon_{\boldsymbol{q}},\varepsilon_{\boldsymbol{q}})_\Omega+\Theta_\tau(\varepsilon_u-\varepsilon_{\widehat{u}})=(c\,(\boldsymbol{\Pi q}-\boldsymbol{q}),\varepsilon_{\boldsymbol{q}})_\Omega,$$

and we can estimate the projection of error in the flux and in the jumps. In fact, we have the following result.

Theorem 3.4. *For $k\geq 0$, we have that*

$$\|\varepsilon_{\boldsymbol{q}}\|_{L^2(\Omega)}+\Theta_\tau^{1/2}(\varepsilon_u-\varepsilon_{\widehat{u}})\;\leq\; C\,\|\boldsymbol{\Pi q}-\boldsymbol{q}\|_{L^2(\Omega)}.$$

Moreover, if the elliptic regularity estimate $\|u\|_{H^2(\Omega)}\leq C\,\|\nabla\cdot(\mathrm{c}\nabla u)\|_{L^2(\Omega)}$ holds when $u=0$ on $\partial\Omega$, we have that

$$\|\varepsilon_u\|_{L^2(\Omega)}\leq C\,C_\tau\,h^{\min\{k,1\}}\,\|\boldsymbol{\Pi q}-\boldsymbol{q}\,\|_{L^2(\Omega)},$$

where $C_\tau=\max_{K\in\Omega_h}\{1,h_K\tau_K\}$.

We can now conclude that, whenever the exact solution $(\boldsymbol{q},u)$ is smooth enough, and the stabilization function τ is of order one, we have that

$$\|\boldsymbol{\varepsilon_q}\|_{L^2(\Omega)}\leq C\,h^{k+1}\quad\text{ and }\quad\|\varepsilon_u\|_{L^2(\Omega)}\leq C\,h^{k+1+\min\{k,1\}},$$

and so,

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega)}\leq C\,h^{k+1}\quad\text{ and }\quad\|u-u_h\|_{L^2(\Omega)}\leq C\,h^{k+1}.$$

Postprocessing. We can take advantage of the superconvergence of the projection of the error ε_u for $k\geq 1$ to define a better approximation $u_h^\star$ to u. The approximation $u_h^\star$ is defined on the simplex $K\in\Omega_h$ as the unique function in $\mathcal{P}_{k+1}(K)$ satisfying

$$\begin{aligned}(\nabla u_h^\star,\nabla w)_K=&-(\mathrm{c}\boldsymbol{q}_h,\nabla w)_K\qquad&&\text{for all } w\in\mathcal{W}_{k+1}(K),\\(u_h^\star,w)_K=&(u_h,w)_K&&\text{for all } w\in\mathcal{P}_{k-1}(K).\end{aligned}$$

Here $\mathcal{W}_{k+1}(K)$ denotes the $L^2(K)$-orthogonal complement of $\mathcal{P}_{k-1}(K)$ in $\mathcal{P}_{k+1}(K)$. This projection is a modification of the projection proposed in [51, 32, 52].

Theorem 3.5. *We have that*

$$\|\,u-u_h^\star\,\|_{L^2(\Omega)}\leq C\,C_\tau\,h^{\min\{k,1\}}\|\,\boldsymbol{\Pi q}-\boldsymbol{q}\,\|_{L^2(\Omega)}+C\,h^{k+2}\,|\,u\,|_{H^{\ell+2}(\Omega_h)},$$

for any $k\geq 0$ where $C_\tau=\max_{K\in\Omega_h}\{1,h_K\,\tau_K\}$.

We thus conclude that, when the exact solution $(\boldsymbol{q},u)$ is smooth enough, the stabilization function τ is of order one and $k\geq 1$, we have that

$$\|\,u-u_h^\star\,\|_{L^2(\Omega)}\leq C\,h^{k+2}.$$

3.4. Comparison with other finite element methods.

Finite element methods fitting the HDG formulation. Many finite element methods fit the formulation (3.2); the main examples are displayed in Table 1. In fact, the RT and BDM mixed methods can be viewed as particular cases of HDG methods and the CG method can be considered as a limiting case. This suggests that we could consider that the HDG methods are *between* the RT and BDM mixed methods and the CG method.

Table 1. Methods fitting the formulation (3.2) for triangulations Ω_h of simplexes.

Method	$\boldsymbol{V}(K)$	$W(K)$	$M(F)$	τ
RT	$\boldsymbol{\mathcal{P}}_k(K) + \boldsymbol{x}\,\mathcal{P}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	0
BDM	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_{k-1}(K)$	$\mathcal{P}_k(F)$	0
HDG	$\boldsymbol{\mathcal{P}}_k(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	$(0,\infty)$
CG	$\boldsymbol{\mathcal{P}}_{k-1}(K)$	$\mathcal{P}_k(K)$	$\mathcal{P}_k(F)$	∞

Boundary residuals and accuracy. To further elaborate this idea, we compare in Table 2 how these methods deal with the residuals at the boundary of the elements and how this is reflected in their convergence properties.

Table 2. Residuals, stabilization and order of accuracy (for $k \geq 1$).

Method	$R^u_{\partial K}$	$R^{\boldsymbol{q}}_{\partial K}$	τ	$\boldsymbol{q}_h$	u_h	$u_h^\star$
RT	–	0	0	$k+1$	$k+1$	$k+2$
BDM	–	0	0	$k+1$	k	$k+2$
HDG	–	–	$\mathcal{O}(1)$	$k+1$	$k+1$	$k+2$
HDG	–	–	$\mathcal{O}(1/h)$	k	$k+1$	$k+1$
CG	0	–	∞	k	$k+1$	$k+1$

We see, on the one hand, that the RT and BDM methods force the residual $R^{\boldsymbol{q}}_{\partial K} := (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}$ to be equal to zero and obtain the orders of convergence of $k+1$ and $k+2$ for $\boldsymbol{q}_h$ and $u_h^\star$, respectively. On the other hand, the CG method forces the residual $R^u_{\partial K} := u_h - \widehat{u}_h$ to be equal to zero and obtain the orders of convergence of only k and $k+1$ for $\boldsymbol{q}_h$ and $u_h^\star$, respectively. (For a comparison between the RT and CG methods, see [19].) However, unlike these methods, the HDG method does not force any of these two residuals to be zero. Instead, it plays with the stabilization function τ to *balance* their relative sizes so that the approximation error is optimal. As we see in Table 2, this happens when both residuals have a similar weight, that is, when the stabilization function τ is of order one. Note that we would expect, from Table 1, that taking τ *small* or *big* enough would guarantee that the convergence properties of the HDG

method are closer to those of the RT and BDM methods or to the CG method, respectively. The results displayed in Table 2 actually confirm this. A rigorous explanation of this fact can be found in [22, 18].

4. HDG Methods for Incompressible Fluid Flow

In this section, we extend the HDG methods to the more involved Stokes equations of incompressible fluid flow,

$$-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{on } \Omega, \tag{4.1a}$$

$$\nabla\cdot\,\boldsymbol{u} = 0 \qquad \text{on } \Omega, \tag{4.1b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \partial\Omega, \tag{4.1c}$$

$$(p,1)_\Omega = 0, \tag{4.1d}$$

where $\langle \boldsymbol{u}_D\cdot\boldsymbol{n}, 1\rangle_{\partial\Omega} = 0$. Most of the material for this section is contained in [43, 17].

4.1. The HDG methods.

Rewriting the equations. For each $K \in \Omega_h$, the method provides an approximation to the restriction of $(\mathrm{L}, \boldsymbol{u}, p)$ to K, where L is the gradient of the velocity $\boldsymbol{u}$, as well as to the traces $(-\nu\widehat{\mathrm{L}}\boldsymbol{n} + \widehat{p}\,\boldsymbol{n}, \widehat{\boldsymbol{u}})$ on ∂K. So, we first rewrite the Stokes equations in a manner that will be suitable to defining the HDG methods.

If we assume that we know the trace of the velocity on ∂K, $\widehat{\boldsymbol{u}}$, as well as the average of the pressure on K, $\overline{p}$, we can obtain $(\mathrm{L}, \boldsymbol{u}, p)$ inside K as the solution of

$$\begin{aligned}
\mathrm{L} - \nabla\boldsymbol{u} &= 0 && \text{in } K,\\
-\nu\,\nabla\cdot\mathrm{L} + \nabla p &= \boldsymbol{f} && \text{in } K,\\
\nabla\cdot\,\boldsymbol{u} &= \frac{1}{|K|}\langle\widehat{\boldsymbol{u}}\cdot\boldsymbol{n}, 1\rangle_{\partial K} && \text{in } K,\\
\boldsymbol{u} &= \widehat{\boldsymbol{u}} && \text{on } \partial K,\\
\frac{1}{|K|}(p,1)_K &= \overline{p}.
\end{aligned}$$

The functions $\widehat{\boldsymbol{u}}$ and $\overline{p}$ can now be obtained as the solution of

$$\begin{aligned}
[\![-\nu\widehat{\mathrm{L}}\boldsymbol{n} + \widehat{p}\,\boldsymbol{n}]\!] &= 0 && \text{for all } F \in \mathcal{E}_h^o,\\
\langle\widehat{\boldsymbol{u}}\cdot\boldsymbol{n}, 1\rangle_{\partial K} &= 0 && \text{for all } K \in \Omega_h,\\
\widehat{\boldsymbol{u}} &= \boldsymbol{u}_D && \text{on } \partial\Omega,\\
(\overline{p}, 1)_\Omega &= 0.
\end{aligned}$$

The Galerkin method and the numerical traces. We now discretize the equations by using a Galerkin method together with a suitable approximation of the traces. On the element $K \in \Omega_h$, we take $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ in the space $\mathrm{G}(K) \times \boldsymbol{V}(K) \times Q(K)$, and on the face $F \in \mathcal{E}_h^o$, we take $\widehat{u}_h$ in the space $\boldsymbol{M}(F)$.

On each element $K \in \Omega_h$, the function $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ is expressed in terms of $(\widehat{\boldsymbol{u}}_h, \overline{p}_h, f)$ as follows. Since

$$
\begin{aligned}
(\mathrm{L}, \mathrm{G})_K + (\boldsymbol{u}, \nabla \cdot \mathrm{G})_K - \langle \widehat{\boldsymbol{u}}, \mathrm{G}\boldsymbol{n} \rangle_{\partial K} &= 0, \\
(\nu \mathrm{L}, \nabla \boldsymbol{v})_K - (p_h, \nabla \cdot \boldsymbol{v})_K - \langle \nu \widehat{\mathrm{L}} \boldsymbol{n} - \widehat{p} \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial K} &= (\boldsymbol{f}, \boldsymbol{v})_K, \\
-(\boldsymbol{u}, \nabla q)_K + \langle \widehat{\boldsymbol{u}} \cdot \boldsymbol{n}, q - \overline{q} \rangle_{\partial K} &= 0,
\end{aligned}
$$

for smooth emough $(\mathrm{G}, \boldsymbol{v}, q)$, we determine $(\mathrm{L}_h, \boldsymbol{u}_h, p_h)$ in terms of $(\widehat{\boldsymbol{u}}_h, \overline{p}_h, f)$ as the solution of

$$
(\mathrm{L}_h, \mathrm{G})_K + (\boldsymbol{u}_h, \nabla \cdot \mathrm{G})_K - \langle \widehat{\boldsymbol{u}}_h, \mathrm{G}\boldsymbol{n} \rangle_{\partial K} = 0, \tag{4.2a}
$$

$$
(\nu \mathrm{L}_h, \nabla \boldsymbol{v})_K - (p_h, \nabla \cdot \boldsymbol{v})_K - \langle \nu \widehat{\mathrm{L}}_h \boldsymbol{n} - \widehat{p}_h \boldsymbol{n}, \boldsymbol{v} \rangle_{\partial K} = (\boldsymbol{f}, \boldsymbol{v})_K, \tag{4.2b}
$$

$$
-(\boldsymbol{u}_h, \nabla q)_{\Omega_h} + \langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, q - \overline{q} \rangle_{\partial K} = 0, \tag{4.2c}
$$

$$
\frac{1}{|K|}(p_h, 1)_K = \overline{p}_h, \tag{4.2d}
$$

for all $(\mathrm{G}, \boldsymbol{v}, q) \in \mathrm{G}(K) \times \boldsymbol{V}(K) \times P_h(K)$, where the numerical trace $-\nu \widehat{\mathrm{L}}_h \boldsymbol{n} + \widehat{p}_h \boldsymbol{n}$ is *assumed* to be given by

$$
-\nu \widehat{\mathrm{L}}_h \boldsymbol{n} + \widehat{p}_h \boldsymbol{n} = -\nu \mathrm{L}_h \boldsymbol{n} + p_h \boldsymbol{n} + \nu\, \tau\, (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) \quad \text{ on } \partial K. \tag{4.2e}
$$

Then we determine $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ by enforcing the remaining equations, that is, by requiring that

$$
\langle [\![-\nu \widehat{\mathrm{L}}_h \boldsymbol{n} + \widehat{p}_h\, \boldsymbol{n}]\!], \boldsymbol{\mu} \rangle_F = 0 \qquad \forall \boldsymbol{\mu} \in \boldsymbol{M}(F)\ \ \forall\, F \in \mathcal{E}_h^o, \tag{4.2f}
$$

$$
\langle \widehat{\boldsymbol{u}}_h \cdot \boldsymbol{n}, 1 \rangle_{\partial K} = 0 \qquad \forall\, K \in \Omega_h, \tag{4.2g}
$$

$$
\widehat{\boldsymbol{u}}_h = \boldsymbol{u}_D \qquad \text{on } \partial\Omega, \tag{4.2h}
$$

$$
(\overline{p}_h, 1)_\Omega = 0. \tag{4.2i}
$$

This completes the definition of the HDG methods.

The discrete energy identity and the existence and uniqueness of the approximation. These methods are well defined under very mild conditions, as we see in the next result.

Proposition 4.1. *The HDG method is well defined if (i) the stabilization function τ is strictly positive on $\partial\Omega_h$, (ii) $\nabla \boldsymbol{v} \in \mathrm{G(K)}$ for any $\boldsymbol{v} \in \boldsymbol{V}(K)$, and if (iii) $\nabla q \in \boldsymbol{V}(K)$ for all $q \in Q(K)$.*

To prove this result, we begin by establishing an energy identity. Note that for the exact solution we have

$$(\mathrm{L},\mathrm{L})_\Omega = (\boldsymbol{f},\boldsymbol{u})_\Omega + \langle -\nu \mathrm{L}\boldsymbol{n} + p\,\boldsymbol{n}, \boldsymbol{u}_D\rangle_{\partial\Omega},$$

and so we should have a similar energy identity for the solution of the HDG method. Indeed, it is not difficult to obtain

$$(\mathrm{L}_h,\mathrm{L}_h)_\Omega + \Theta_\tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) = (\boldsymbol{f},\boldsymbol{u}_h)_\Omega + \langle(-\nu\widehat{\mathrm{L}}_h + \widehat{p}_h\,\mathrm{I})\boldsymbol{n}, \boldsymbol{u}_D\rangle_{\partial\Omega}, \tag{4.3}$$

where $\Theta_\tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h) := \sum_{K\in\Omega_h}\langle\tau(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h), \boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h\rangle_{\partial K}$.

Once again, to prove Proposition 4.1, we only have to show that when if we set $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{u}_D = \boldsymbol{0}$, the only solution is the trivial one. By the discrete energy identity, we see that in this case we have that $\mathrm{L}_h = 0$ on Ω_h and that $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h$ on $\mathcal{E}_h$. By equation (4.2a), this implies that

$$(\nabla\boldsymbol{u},\mathrm{G})_K = 0 \quad \forall\, \mathrm{G}\in\mathrm{G}(K),$$

and by condition (ii), we conclude that $\boldsymbol{u}_h$ is constant on Ω. Since $\boldsymbol{u}_h = \widehat{\boldsymbol{u}}_h = 0$ on $\partial\Omega$, we see that $\boldsymbol{u}_h = \boldsymbol{0}$ on Ω and, as a consequence, that $\widehat{u}_h = \boldsymbol{0}$ on $\mathcal{E}_h$. Finally, by equation (4.2b),

$$(\nabla p_h, \boldsymbol{v})_K = 0 \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}(K),$$

and by condition (iii), we have that p_h is a constant on Ω. By equations (4.2d) and (4.2i), we conclude that $p_h = \overline{p}_h = 0$ on Ω_h. This completes the proof of Proposition 4.1.

Implementation. To describe the implementation of the HDG methods, we need to introduce some notation. We denote by $(\mathcal{L},\boldsymbol{\mathcal{U}},\mathcal{P})$ the linear mapping (given by equations (4.2a) to (4.2e)) that associates $(\widehat{\boldsymbol{u}}_h,\overline{p}_h,f)$ to $(\mathrm{L}_h,\boldsymbol{u}_h,p_h)$, and set

$$\begin{aligned}
(\mathcal{L}^{\widehat{\boldsymbol{u}}_h},\boldsymbol{\mathcal{U}}^{\widehat{\boldsymbol{u}}_h},\mathcal{P}^{\widehat{\boldsymbol{u}}_h}) &:= (\mathcal{L},\boldsymbol{\mathcal{U}},\mathcal{P})(\widehat{\boldsymbol{u}}_h,0,0),\\
(\mathcal{L}^{\overline{p}_h},\boldsymbol{\mathcal{U}}^{\overline{p}_h},\mathcal{P}^{\overline{p}_h}) &:= (\mathcal{L},\boldsymbol{\mathcal{U}},\mathcal{P})(0,\overline{p}_h,0),\\
(\mathcal{L}^{f},\boldsymbol{\mathcal{U}}^{f},\mathcal{P}^{f}) &:= (\mathcal{L},\boldsymbol{\mathcal{U}},\mathcal{P})(0,0,f).
\end{aligned}$$

We also introduce the spaces

$$\begin{aligned}
\boldsymbol{M}_h &:= \{\boldsymbol{\mu}\in\boldsymbol{L}^2(\mathcal{E}_h): \quad \boldsymbol{\mu}|_F\in\boldsymbol{M}(F) \quad \forall\, F\in\mathcal{E}_h^o\},\\
\overline{P}_h &:= \{\overline{q}_h\in L^2(\Omega): \quad \overline{q}_h\in\mathcal{P}_0(K) \quad \forall\, K\in\Omega_h\},
\end{aligned}$$

and set $\boldsymbol{M}_h(\boldsymbol{\zeta}) := \{\boldsymbol{\mu}\in\boldsymbol{M}_h: \quad \boldsymbol{\mu}|_{\partial\Omega} = \boldsymbol{\zeta}\}$.

With this notation, we can characterize the approximate solution given by the HDG method as follows.

Theorem 4.2. *We have that*

$$(\mathrm{L}_h, \boldsymbol{u}_h, p_h) = (\mathcal{L}^{\widehat{\boldsymbol{u}}_h}, \boldsymbol{\mathcal{U}}^{\widehat{\boldsymbol{u}}_h}, \mathcal{P}^{\widehat{\boldsymbol{u}}_h}) + (\mathcal{L}^{\overline{p}_h}, \boldsymbol{\mathcal{U}}^{\overline{p}_h}, \mathcal{P}^{\overline{p}_h}) + (\mathcal{L}^{f}, \boldsymbol{\mathcal{U}}^{f}, \mathcal{P}^{f}),$$

where $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ *is the only element in* $\boldsymbol{M}_h(\boldsymbol{u}_D) \times \overline{P}_h$ *such that*

$$\begin{aligned} a_h(\widehat{\boldsymbol{u}}_h, \boldsymbol{\mu}) + b_h(\overline{p}_h, \boldsymbol{\mu}) &= \ell_h(\boldsymbol{\mu}), \quad && \forall\, \boldsymbol{\mu} \in \boldsymbol{M}_h(\boldsymbol{0}), \\ b_h(\overline{q}, \widehat{\boldsymbol{u}}_h) &= 0, && \forall\, \overline{q} \in \overline{P}_h, \\ (\overline{p}_h, 1)_\Omega &= 0. \end{aligned}$$

Here the forms are given by

$$\begin{aligned} a_h(\boldsymbol{\eta}, \boldsymbol{\mu}) &= \sum_{K \in \Omega_h} \left((\nu \mathcal{L}^{\boldsymbol{\eta}}, \mathcal{L}^{\boldsymbol{\mu}})_K + \langle \nu\tau(\boldsymbol{\mathcal{U}}^{\boldsymbol{\eta}} - \boldsymbol{\eta}), (\boldsymbol{\mathcal{U}}^{\boldsymbol{\mu}} - \boldsymbol{\mu}) \rangle_{\partial K} \right), \\ b_h(\overline{q}, \boldsymbol{\mu}) &= -\sum_{K \in \Omega_h} \langle \overline{q}, \boldsymbol{\mu} \cdot \boldsymbol{n} \rangle_{\partial K}, \\ \ell_h(\boldsymbol{\mu}) &= (\boldsymbol{f}, \boldsymbol{\mathcal{U}}^{\boldsymbol{\mu}})_\Omega, \end{aligned}$$

for all $\boldsymbol{\eta} \in \boldsymbol{M}_h, \boldsymbol{\mu} \in \boldsymbol{M}_h$, *and* $\overline{q} \in \overline{P}_h$.

Note that the first equation of the formulation characterizing $(\widehat{\boldsymbol{u}}_h, \overline{p}_h)$ is a rewriting of equation (4.2f) whereas the second is a rewriting of equation (4.2g).

We see that the HDG method can be implemented as a typical mixed method. In fact, an augmented lagrangian algorithm can be used to further improve its implementation; see [43].

4.2. The stabilization mechanism.

The relation between the residuals. Note that also for The Stokes problem, the HDG method is defined by imposing a linear relation between the residuals in the interior of the element K, $\mathrm{R}_K^{\boldsymbol{u}} := \mathrm{L}_h - \nabla \boldsymbol{u}_h$, $\boldsymbol{R}_K^{\mathrm{L},p} := \nabla \cdot (-\nu \mathrm{L}_h + p_h\, \mathrm{I}) - \boldsymbol{f}$, and $R_K^{\nabla\cdot\boldsymbol{u}} := \nabla \cdot \boldsymbol{u}_h$ and the residuals on its boundary ∂K, $\mathrm{R}_{\partial K}^{\boldsymbol{u}} := (\widehat{\boldsymbol{u}}_h - \boldsymbol{u}_h) \otimes \boldsymbol{n}$ and $\boldsymbol{R}_{\partial K}^{\mathrm{L},p} := (-\nu \mathrm{L}_h \boldsymbol{n} + p_h\, \boldsymbol{n}) - (-\nu \widehat{\mathrm{L}}_h \boldsymbol{n} + \widehat{p}_h\, \boldsymbol{n}) = -\nu\, \tau\, (\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h)$.

Indeed, the equations of the Galerkin method defining the HDG method can be rewritten as follows:

$$\begin{aligned} (\mathrm{R}_K^{\boldsymbol{u}}, \mathrm{G})_K &= \langle \mathrm{R}_{\partial K}^{\boldsymbol{u}}, \mathrm{G} \rangle_{\partial K} \\ (\boldsymbol{R}_K^{\mathrm{L},p}, \boldsymbol{v})_K &= \langle R_{\partial K}^{\mathrm{L},p}, \boldsymbol{v} \rangle_{\partial K}, \\ (R_K^{\nabla\cdot\boldsymbol{u}}, q)_K &= \langle tr \mathrm{R}_{\partial K}^{\boldsymbol{u}}, q \rangle_{\partial K}. \end{aligned}$$

Since this implies that

$$\begin{aligned} \|\mathsf{P}_{\mathrm{G}(K)} \mathrm{R}_K^{\boldsymbol{u}}\|_{L^2(K)} &\le C\, h_K^{-1/2} \|\mathrm{R}_{\partial K}^{\boldsymbol{u}}\|_{L^2(\partial K)}, \\ \|\mathsf{P}_{\boldsymbol{V}(K)} \boldsymbol{R}_K^{\mathrm{L},p}\|_{L^2(K)} &\le C\, h_K^{-1/2} \|\boldsymbol{R}_{\partial K}^{\mathrm{L},p}\|_{L^2(\partial K)}, \\ \|\mathsf{P}_{Q(K)} R_K^{\nabla\cdot\boldsymbol{u}}\|_{L^2(K)} &\le C\, h_K^{-1/2} \|tr \mathrm{R}_{\partial K}^{\boldsymbol{u}}\|_{L^2(\partial K)}, \end{aligned}$$

we see that the quality of the approximation depends only on $\boldsymbol{u} - \widehat{\boldsymbol{u}}_h$, τ and on the approximation properties of the spaces $\mathrm{G}(K)$, $\boldsymbol{V}(K)$ and $Q(K)$.

Stabilization by the jumps. By the energy identity (4.3), we see that we can control the jumps $\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h$ if we require the stabilization function τ to be positive on $\partial\Omega_h$. Next, we show that this stabilization mechanism does not spoil the accuracy of the method.

4.3. Convergence properties. Here we discuss the convergence properties of the method when Ω is a polyhedral domain, the triangulations Ω_h are made of shape-regular simplexes K, and when we take

$$\mathrm{G}(K) \times \boldsymbol{V}(K) \times W(K) := \mathrm{P}_K(K) \times \boldsymbol{P}_k(K) \times P_k(K).$$

Here $\mathrm{P}_k(K)$ is the space of matrix-valued functions whose components belong to $P_k(K)$. Once again, for simplicity, we assume that the stabilization function τ is *constant* on each ∂K.

The auxiliary projection. Given a simplex $K \in \Omega_h$ and a function $(\mathrm{L}, \boldsymbol{u}, p)$ in $\mathrm{H}^1(K) \times \boldsymbol{H}^1(K) \times H^1(K)$, we define its projection $\mathit{\Pi}(\mathrm{L}, \boldsymbol{u}, p) := (\Pi\,\mathrm{L}, \boldsymbol{\Pi u}, \mathit{\Pi} p)$ as the element of $\mathrm{G}_h \times \boldsymbol{V}_h \times P_h$ that solves the equations

$$\begin{aligned}
(\Pi\,\mathrm{L}, \mathrm{G})_K &= (\mathrm{L}, \mathrm{G})_K && \forall\ \mathrm{G} \in \mathrm{P}_{k-1}(K),\\
(\boldsymbol{\Pi u}, \boldsymbol{v})_K &= (\boldsymbol{u}, \boldsymbol{v})_K && \forall\ \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k-1}(K),\\
(\mathit{\Pi} p, q)_K &= (p, q)_K && \forall\ q \in \mathcal{P}_{k-1}(K),\\
(\mathrm{tr}\ \Pi\,\mathrm{L}, q)_K &= (\mathrm{tr}\ \mathrm{L}, q)_K && \forall\ q \in \mathcal{P}_k(K),\\
\langle \nu\Pi\,\mathrm{L}\boldsymbol{n} - \mathit{\Pi} p\,\boldsymbol{n} - \nu\,\tau\boldsymbol{\Pi u}, \boldsymbol{\mu}\rangle_F &= \langle \nu\mathrm{L}\boldsymbol{n} - p\,\boldsymbol{n} - \nu\,\tau\boldsymbol{u}, \boldsymbol{\mu}\rangle_F && \forall\ \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}_k(F),
\end{aligned}$$

for all faces F of the simplex K. This projection is actually well defined.

Theorem 4.3. *Suppose that $\tau_K := \tau|_K$ is a positivie constant on ∂K. Then the projection $\mathit{\Pi}$ is well defined. Moreover, on each element $K \in \Omega_h$, we have that*

$$\begin{aligned}
\|\boldsymbol{\Pi u} - \boldsymbol{u}\|_K &\le C\,h^{k+1}\left(|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)} + {\tau_K}^{-1}|\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(K)}\right),\\
\|\nu\Pi\,\mathrm{L} - \nu\mathrm{L}\|_K &\le C\,h^{k+1}\nu\ \left(|\boldsymbol{u}|_{\boldsymbol{H}^{k+2}(K)} + \tau_K|\boldsymbol{u}|_{\boldsymbol{H}^{k+1}(K)}\right),\\
\|\mathit{\Pi} p - p\|_K &\le C\,h_K^{k+1}\,|p|_{H^{k+1}(K)} + C\,\|\nu\Pi\,\mathrm{L} - \nu\mathrm{L}\|_K.
\end{aligned}$$

We have assumed that $\mathrm{tr}\ \mathrm{L} = 0$ *for the last two inequalities and that* $\nabla \cdot \boldsymbol{u} = 0$ *in the last one.*

Estimates of the projection of the errors. This projection is fitted to the structure of the numerical trace $-\nu\widehat{\mathrm{L}}\boldsymbol{n} + \widehat{p}\,\boldsymbol{n}$ in the following sense. Consider

the projection of the errors $(\mathrm{E}_{\mathrm{L}}, \boldsymbol{\varepsilon}_{\boldsymbol{u}}, \varepsilon_p) := (\Pi\,\mathrm{L} - \mathrm{L}_h, \boldsymbol{\Pi}\boldsymbol{u} - \boldsymbol{u}_h, \Pi p - p_h)$. Then, by the last equation defining this projection, we have that

$$\mathbf{P}_M(-\nu\mathrm{L}\boldsymbol{n} + p\boldsymbol{n}) = -\nu\Pi\,\mathrm{L}\boldsymbol{n} + \Pi p\,\boldsymbol{n} + \nu\,\tau(\boldsymbol{\Pi}\boldsymbol{u} - \mathbf{P}_M\boldsymbol{u}),$$

where $\mathbf{P}_M$ is the L^2-projection into $\boldsymbol{M}_h$. Comparing this with the definition of the numerical trace

$$-\nu\widehat{\mathrm{L}}_h\boldsymbol{n} + \widehat{p}_h\boldsymbol{n} = -\nu\mathrm{L}_h\boldsymbol{n} + p_h\boldsymbol{n} + \nu\,\tau\,(\boldsymbol{u}_h - \widehat{\boldsymbol{u}}_h),$$

we get that $-\nu\varepsilon_{\widehat{\mathrm{L}}}\boldsymbol{n} + \varepsilon_{\widehat{p}}\boldsymbol{n} = -\nu\mathrm{E}_{\mathrm{L}}\boldsymbol{n} + \varepsilon_p\,\boldsymbol{n} + \tau(\boldsymbol{\varepsilon}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}})$ provided $-\nu\varepsilon_{\widehat{\mathrm{L}}}\boldsymbol{n} + \varepsilon_{\widehat{p}}\boldsymbol{n} := \mathbf{P}_M(-\nu\mathrm{L}\boldsymbol{n} + p\boldsymbol{n}) - (-\nu\widehat{\mathrm{L}}_h\boldsymbol{n} + \widehat{p}_h\boldsymbol{n})$ and $\boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}} := \mathbf{P}_M(\boldsymbol{u} - \widehat{\boldsymbol{u}})$.

The equations satisfied by the projection of the errors are then the following. For each simplex $K \in \Omega_h$,

$$\begin{aligned}(\mathrm{E}_{\mathrm{L}}, \mathrm{G})_K + (\boldsymbol{\varepsilon}_{\boldsymbol{u}}, \boldsymbol{\nabla}\cdot \mathrm{G})_K - \langle \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}}, \mathrm{G}\boldsymbol{n}\rangle_{\partial K} &= (\Pi\ \mathrm{L} - \mathrm{L}, \mathrm{G})_K,\\ -(\nabla\cdot(\nu\mathrm{E}_{\mathrm{L}}), \boldsymbol{v})_K + (\nabla\varepsilon_p, \boldsymbol{v})_K + \langle \nu\tau\,(\boldsymbol{\varepsilon}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}}), \boldsymbol{v}\rangle_{\partial K} &= 0,\\ -(\boldsymbol{\varepsilon}_{\boldsymbol{u}}, \nabla q)_K + \langle \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}}, q\,\boldsymbol{n}\rangle_{\partial K} &= 0,\end{aligned}$$

for all $(\mathrm{G}, \boldsymbol{v}, q)$ in $\mathrm{G}(\mathrm{K}) \times \boldsymbol{V}(K) \times Q(K)$. Moreover,

$$\begin{aligned}\langle -\nu\mathrm{E}_{\mathrm{L}}\boldsymbol{n} + \varepsilon_p\boldsymbol{n} + \nu\tau\,(\boldsymbol{\varepsilon}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}}), \boldsymbol{\mu}\rangle_F &= 0 \quad \forall\,\boldsymbol{\mu} \in \boldsymbol{M}(F)\ \forall F \in \mathcal{E}_h^o,\\ \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}} &= 0 \quad \text{on } \partial\Omega,\\ (\varepsilon_p, 1)_\Omega &= (\Pi p - p, 1)_\Omega.\end{aligned}$$

We thus see that the projection of the errors only depend on $\Pi\mathrm{L} - \mathrm{L}$ and on $(\Pi p - p, 1)_\Omega$, the latter quantity being equal to zero for $k \geq 1$.

In particular, the discrete energe identity for the equations is

$$(\mathrm{E}_{\mathrm{L}}, \mathrm{E}_{\mathrm{L}})_\Omega + \Theta_\tau(\boldsymbol{\varepsilon}_{\boldsymbol{u}} - \boldsymbol{\varepsilon}_{\widehat{\boldsymbol{u}}}) = (\Pi\,\mathrm{L} - \mathrm{L}, \mathrm{E}_{\mathrm{L}})_\Omega,$$

and we immediately obtain an estimate of the projection of the error in the gradient and in the jumps of the velocity. In fact, we can prove the following result.

Theorem 4.4. *We have*

$$\begin{aligned}\|\,\mathrm{E}_{\mathrm{L}}\,\|_{L^2(\Omega)} + \Theta_\tau^{1/2}(\boldsymbol{u} - \widehat{\boldsymbol{u}}_h) &\leq C\,\|\,\Pi\,\mathrm{L} - \mathrm{L}\,\|_{L^2(\Omega)},\\ \|\varepsilon_p\|_{L^2(\Omega)} &\leq |(\Pi p - p, 1)_\Omega|\,|\Omega|^{-1/2} + C\,\sqrt{C_\tau}\,\nu\,\|\Pi\ \mathrm{L} - \mathrm{L}\|_{L^2(\Omega)},\end{aligned}$$

where $C_\tau := \max_{K\in\Omega_h}\{1, \tau_K\,h_K\}$. *Moreover, if the elliptic regularity estimate* $\nu\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)} \leq C\|-\nu\Delta\boldsymbol{u} + \nabla p\|_{\boldsymbol{L}^2(\Omega)}$ *holds whenever* $\boldsymbol{u} = \boldsymbol{0}$ *on* $\partial\Omega$, *we have that*

$$\|\,\boldsymbol{\varepsilon}_{\boldsymbol{u}}\,\|_{L^2(\Omega)} \leq C\,C_\tau\,h^{\min\{k,1\}}\,\|\,\Pi\,\mathrm{L} - \mathrm{L}\,\|_{L^2(\Omega)}.$$

We can now conclude that, whenever the exact solution $(\boldsymbol{q}, u)$ is smooth enough, and the stabilization function τ is of order one, we have that

$$\nu\|\mathrm{E}_{\mathrm{L}}\|_{\mathrm{L}^2(\Omega)} + \|\varepsilon_p\|_{L^2(\Omega)} \leq C\, h^{k+1} \quad \text{and} \quad \|\boldsymbol{\varepsilon}_u\|_{L^2(\Omega)} \leq C\, h^{k+1+\min\{k,1\}},$$

and so,

$$\nu\|\mathrm{L} - \mathrm{L}_h\|_{\mathrm{L}^2(\Omega)} + \|p - p_h\|_{L^2(\Omega)} \leq C\, h^{k+1} \quad \text{and} \quad \|u - u_h\|_{L^2(\Omega)} \leq C\, h^{k+1}.$$

Postprocessing. Here we show how to obtain a new approximate velocity which is exactly divergence-free, $\boldsymbol{H}(\mathrm{div})$-conforming, and converges with an additional order for $k \geq 1$. We only describe the three dimensional case, as the two dimensional case is much simpler. We quote [17] almost *verbatim*.

In the *three dimensional case* we define the postprocessed approximate velocity $\boldsymbol{u}_h^\star$ on the tetrahedron $K \in \Omega_h$ as the element of $\boldsymbol{\mathcal{P}}_{k+1}(K)$ such that

$$\begin{aligned} \langle (\boldsymbol{u}_h^\star - \widehat{\boldsymbol{u}}_h)\cdot \boldsymbol{n}, \mu\rangle_F = 0 \quad \forall\, \mu \in \mathcal{P}_k(F), \\ \langle (\boldsymbol{n}\times\nabla)(\boldsymbol{u}_h^\star\cdot\boldsymbol{n}) - \boldsymbol{n}\times(\{\!\{\mathrm{L}_h^t\}\!\}\boldsymbol{n}), (\boldsymbol{n}\times\nabla)\mu\rangle_F = 0 \quad \forall\, \mu \in \mathcal{P}_{k+1}(F)^\perp, \end{aligned}$$

for all faces F of K, and such that

$$\begin{aligned} (\boldsymbol{u}_h^\star - \boldsymbol{u}_h, \nabla w)_K = 0 \quad \forall\, w \in \boldsymbol{\mathcal{P}}_k(K), \\ (\boldsymbol{\nabla}\times \boldsymbol{u}_h^\star - \mathbf{w}_h, (\boldsymbol{\nabla}\times \boldsymbol{v})\,\mathrm{B}_K)_K = 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_k(K). \end{aligned}$$

Here $\mathcal{P}_{k+1}(F)^\perp := \{\mu \in \mathcal{P}_{k+1}(F) \,:\, \langle \mu, \widetilde{\mu}\rangle_F = 0, \quad \forall \widetilde{\mu} \in \mathcal{P}_k(F)\}$, $\mathbf{n}\times\nabla$ is the tangential gradient rotated $\pi/2$ in the positive sense (from the point of view of the normal vector) and the function $\{\!\{\mathrm{L}_h^t\}\!\}$ is the single-valued function on $\mathcal{E}_h$ equal to $((\mathrm{L}_h^t)^+ + (\mathrm{L}_h^t)^-)/2$ on the set $\mathcal{E}_h \setminus \partial\Omega$ and equal to L_h^t on $\partial\Omega$. In the last equation, we have that $\mathbf{w}_h := (\mathrm{L}_{32}^h - \mathrm{L}_{23}^h, \mathrm{L}_{13}^h - \mathrm{L}_{31}^h, \mathrm{L}_{21}^h - \mathrm{L}_{12}^h)$ is the approximation to the vorticity and $\mathrm{B}_K := \sum_{\ell=0}^3 \lambda_{\ell-3}\lambda_{\ell-2}\lambda_{\ell-1}\nabla\lambda_\ell \otimes \nabla\lambda_\ell$ is the so-called *symmetric bubble matrix* introduced in [15]. Here the $\lambda_i's$ are the barycentric coordinates associated with the tetrahedron K, the subindices being counted modulo 4. Finally, to define $\boldsymbol{\mathcal{S}}_k(K)$, recall the Nédélec space of the first kind [36], defined by $\boldsymbol{N}_k = \boldsymbol{\mathcal{P}}_{k-1}(K) \oplus \boldsymbol{S}_k$, where $\boldsymbol{S}_\ell$ is the space of vector-valued homogeneous polynomials $\boldsymbol{v}$ of degree ℓ such that $\boldsymbol{v}\cdot\boldsymbol{x} = 0$. Then, define $\boldsymbol{\mathcal{S}}_k(K) := \{\boldsymbol{p} \in \boldsymbol{N}_k : (\boldsymbol{p}, \nabla\phi)_K = 0 \text{ for all } \phi \in \mathcal{P}_{k+1}(K)\}$.

Theorem 4.5. *We have that* $\boldsymbol{u}_h^\star \in \boldsymbol{H}(div, \Omega)$ *and that* $\nabla\cdot\boldsymbol{u}_h^\star = 0$ *on* Ω. *Moreover,*

$$\|\boldsymbol{u}_h^\star - \boldsymbol{u}\|_{L^2(\Omega)} \leq C\, h^{k+2} |\,\boldsymbol{u}\,|_{\boldsymbol{H}^{\ell_{\boldsymbol{u}}+2}(\Omega)} + C\, C_\tau\, h^{\min\{k,1\}}\, \|\Pi\,\mathrm{L} - \mathrm{L}\|_{L^2(\Omega)}.$$

We thus conclude that, when the exact solution $(\mathrm{L}, \boldsymbol{u}, p)$ is smooth enough, the stabilization function τ is of order one and $k \geq 1$, we have that

$$\|\,\boldsymbol{u} - \boldsymbol{u}_h^\star\,\|_{\boldsymbol{L}^2(\Omega)} \leq C\, h^{k+2}.$$

5. Conclusion and Ongoing Work

The described approach to devise HDG methods has proven to be very powerful for the model problems considered in the previous sections. We believe that it can be used in a systematic manner to obtain efficiently implementable and accurate HDG methods for a wide variety of problems of practical interest. In fact, many HDG methods have already been defined and numerically tested on a variety of problems; their analyses contitute the subject of ongoing research. To end this paper, we describe them and briefly discuss their main convergence properties.

HDG methods have been devised for linear, steady-state convection-diffusion problems in [13], and for time-dependent linear and nonlinear convection-diffusion problems in [41] and [42], respectively. The convergence properties for HDG methods for the purely diffusive case seem to carry over to all these problems in the diffusion-dominated regime.

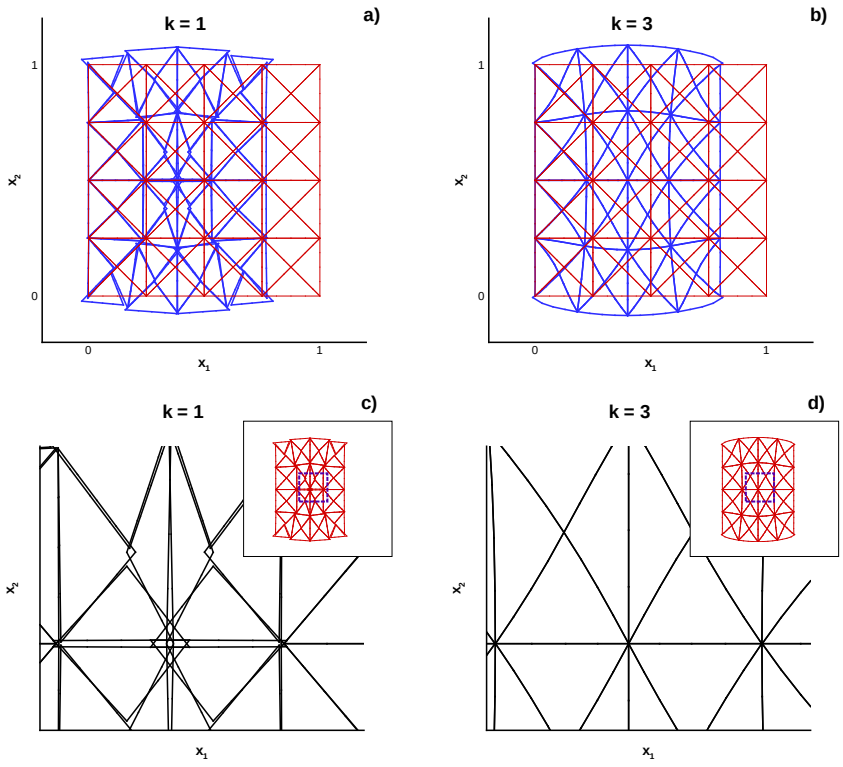

Figure 2. a) deformed shape using $\mathcal{P}^1$, b) deformed shape using $\mathcal{P}^3$, c) closeup view of Figure a), d) closeup view of Figure b).

HDG methods for linear and nonlinear elasticity have been devised in [50] and [49] with very good results. Indeed, by using polynomial approximations of degree k for all the components of the stress and displacement, the order of convergence of $k+1$ for $k \geq 0$ in all variables is obtained. Moreover, by means of a local postprocessing, a new approximation of the displacement can be computed which converges with order $k+2$ for $k \geq 2$. In Fig. 2, we show the approximate displacement of the borders of the elements for a nonlinear elasticity problem; see [49] for a detailed description. The approximation with polynomials of degree one is not as good as the approximation using polynomials of degree three. In full agreement with the properties of the HDG methods, this is reflected in the fact that the jumps for the former are highly visible whereas those of the latter are not.

HDG methods for Timoshenko beams have been developed in [5] with optimal convergence results. Indeed, if polynomials of degree k are used to approximate the displacement, rotation angle, bending moment and shear stress, numerical experiments suggest that all of these variables converge with order $k+1$ for $k \geq 0$. For biharmonic problems [12], the HDG methods provide the optimal order of convergence of $k+1$ for the scalar variable and its gradient. However, the approximation of the laplacian is only of order $k+1/2$ and that of its gradient of only order $k-1/2$. On the other hand, on strict subdomains, the order is the optimal $k+1$ for all these variables. Further analysis of this phenomenon is required to obtain an optimally convergent HDG method on the the whole domain.

HDG methods for vorticity-velocity-pressure formulations of the Stokes equations have been proposed in [14] and later numerically compared with other HDG methods in [38]. The results indicate that the HDG method considered in the previous section performs better. Extensions of this HDG method for the incompressible Navier-Stokes has been recently proposed in [39, 37]. Once again, all the convergence properties of the HDG methods seem to carry over to these equations.

Finally, we would like to report that the HDG methods for both the Euler equations of gas dynamics and the compressible Navier-Stokes equations been devised in [44] seem to provide approximations with optimally convergent properties.

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