Tutorial 02

① Let $n \in \mathbb{Z}$.

n can be written as

$n = 5q + r$ ($\because$ division algorithm)

since $0 \leq r < 5$; n can be;

$n = 5k$

$n = 5k+1$ (suppose $q = k$)

$n = 5k+2$

$n = 5k+3$

$n = 5k+4$

considering the fourth power;

$n^4 = 5^4 k^4$

$n^4 = (5k+1)^4 = 5^4k^4 + 4 \cdot 5^3 k^3 + 6 \cdot 5^2 \cdot k^2 + 4 \cdot 5 \cdot k + 1$

$n^4 = (5k+2)^4 = 5^4k^4 + 4 \cdot 5^3 \cdot k^3 + 6 \cdot 5^2 \cdot k^2 + 4 \cdot 5 \cdot k + 16$

$n^4 = (5k+3)^4 = 5^4k^4 + 4 \cdot 5^3 \cdot k^3 + 6 \cdot 5^2 \cdot k^2 + 4 \cdot 5 \cdot k + 81$

$n^4 = (5k+4)^4 = 5^4k^4 + 4 \cdot 5^3 \cdot k^3 + 6 \cdot 5^2 \cdot k^2 + 4 \cdot 5 \cdot k + 256$

It is observable that the remainders are 1 in latter 4 cases.
Therefore; the fourth power is in the form $5k$ or $5k+1$. //

② n^{th} term $= \sum_{i=0}^{n} 10^i$

$S_n = 1 + 10 + 100 + 1000 + \dots + 10^n$

$= \dfrac{10^{n+1} - 1}{9}$

Suppose there exists a perfect square. Then for a general n^{th} term;

$S_n = k^2$ $(k \in \mathbb{Z})$

$\dfrac{10^{n+1} - 1}{9} = k^2$

$10^{n+1} - 1 = 9k^2$

when $n = 1$; $k^2 = \frac{10^2 - 1}{9} = 11$

when $n = 3$; $k^2 = \frac{10^4 - 1}{9} = \frac{10000 - 1}{9} = 1111$

These terms exist in the sequence. Therefore; it can contain perfect squares. //

(3) $a, b \in \mathbb{Z}$ $\quad x \in \mathbb{Z}$
$a, b \neq 0$

$(a,b) = (b,a) = (a,-b) = (a, b+ax)$

case 01 : $(a,b) = (b,a)$
GCD is commutative.
$\therefore$ By definition $\gcd(a,b) = \gcd(b,a)$

case 02 : $(a,b) = (a,-b)$
if d is a common divisor of a, b, it should also be a divisor of $-b$.
$\therefore$ case 02 holds true.

case 03 : $(a,b) = (a, b+ax)$
let $d = (a,b)$
$c = (a, b+ax)$

since d divides a and b, it divides $b+ax$ too. This implies that d is a common divisor of a and $b+ax$.

Since c is the gcd ; $c \geq d$.
however ; d is the gcd of a,b. Therefore $d \geq c$.
$\therefore$ $c = d$
$\therefore$ $(a,b) = (a, b+ax)$

$\therefore$ All three cases hold true. //

(4) $a \in \mathbb{Z} \setminus \{0\}$

$\gcd(a,0) = |a|$

when $a > 0$; $\gcd(a,0) = +a$

$a < 0$; $\gcd(a,0) = -a$

$\therefore \gcd(a,0) = \pm a$

$= |a|$ //

5 $a|c$ and $b|c$, $\gcd(a,b) = 1 \longrightarrow ab|c$

$c = m \cdot a \quad (m, n \in \mathbb{Z})$

$c = n \cdot b$

$\gcd(a,b) = 1$ implies that a and b are relatively prime.

According to extended euclidean algorithm ; $s \cdot a + tb = 1$

$s \cdot a + t \cdot b = 1$

$c \cdot sa + c \cdot tb = c$

since $a|c$ then $a|sc$ and $b|c \rightarrow b|tc$

6 p is a prime

$p|ab \longrightarrow p|a$ or $p|b$

By definition ; $ab = k \cdot p \ (k \in \mathbb{Z})$

Suppose $p \nmid a$ and $p \nmid b$.

If $p \nmid a$; p should divide b as $p|ab$. $\therefore p \nmid b$ becomes false.

If $p \nmid b$; p should divide a as $p|ab$. $\therefore p \nmid a$ becomes false.

In either case ; our assumption is contradicted.

$\therefore p|ab \rightarrow p|a$ or $p|b$. //

⑦ $\{2x+4y : x,y \in \mathbb{Z}\} = \{2018x + 18y ; x,y \in \mathbb{Z}\}$

⑨ $\gcd(a, a+n) \mid n$

when $n = 1$; $\gcd(a, a+1) \mid 1$.
The only possible integers that divide 1 is +1 or −1. Therefore; $\gcd(a, a+1) = \pm 1$. However $\gcd > 0$. $\therefore \gcd(a, a+1) = +1$.
Therefore from theorom, a, $a+1$ are relatively prime. //

⑩ $L = \text{lcm}(1001, 10101)$

From theorom; $ab = \text{lcm}(a,b) \cdot \gcd(a,b)$ $(a,b \in \mathbb{Z}^+)$

a	b	r
10101	1001	91
1001	91	0

$\therefore \gcd(1001, 10101) = 91$

$\therefore 1001 \times 10101 = \text{lcm}(1001, 10101) \times 91$
$\text{lcm}(1001, 10101) = 111\,111$
$L = 111\,111$

Sum of the digits of $L = 1+1+1+1+1+1$
$= 6$ //

⑪ $100 = 2^2 \times 5^2$

number of factors of 5 in the prime factorization of 100!
$\left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor + \left\lfloor \frac{100}{125} \right\rfloor = 20 + 4 + 0 = 24$

⑫ Assume $\log_2 3$ is a rational number.
$\log_2 3 = \frac{a}{b}$ $(a,b \in \mathbb{Z})$

$2^{a/b} = 3$

$2^a = 3^b$

This implies that 3^b is a power of 2. This is false. Therefore our assumption is incorrect.

$\therefore \log_2 3$ is an irrational number. //

(13)

a. divisors of 6 $= \{1, 2, 3, 6\}$

sum of divisors except 6 itself $= 1+2+3$

$= 6$

6 is "perfect" positive integer. //

divisors of 28 $= \{1, 2, 4, 7, 14, 28\}$

sum $= 1+2+4+7+14 = 28$

$\therefore$ 28 is "perfect" positive. //

(b.) $2^{p-1}(2^p-1)$

if 2^p-1 is prime, the only factors of 2^p-1 is 1 and itself. Therefore, the divisors of (2^p-1) are those factors.

Divisors of 2^{p-1} are powers of 2 ranging from 2^0 to 2^{p-1}.

$$\text{Sum} = \underbrace{2^0+2^1+2^2+\dots+2^{p-1}} + 1$$

$$= 2^p-1+1$$

$$= 2^p$$

(14)

a. $n \in \mathbb{N}$

n is prime $\longleftrightarrow \phi(n) = n-1$

if n is a prime, all the numbers that are lesser than n are relatively

prime to n.
Therefore $\phi(n) = n-1$

if $\phi(n) = n-1$, it means $n-1$ number of integers are relatively prime to n. let $a_1, a_2, .. a_{n-1}$ be those numbers.
then $\gcd(n, a_i) = 1$

if p is not prime, there should be a divisor p that divides both n and a_i. However, it is not true.
Therefore p is prime. //

b. $\phi(p^k)$
p is a prime
k is a positive integer.

$$\phi(p^k) = p^k - p^{k-1}$$
$$= p^k(1 - 1/p)$$

15 $n \in \mathbb{N}$
$n^2 - 79n + 1601$

counter example : $n = 81$
$$81^2 - 79 \times 81 + 1601 = 1681$$
$$= 41 \times 41$$