assigment1

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Assignment 1: by ~~Hes~~ Hesen Cai

1. (a)

a.1> $2^{3n} = (2^3)^n = 8^n$ $\quad 3^{2n} = 9^n$

=> $\cancel{8^n < 9^n}$

$\lim_{n\to\infty} \frac{9^n}{8^n} = 0 \Rightarrow \cancel{8^n = o(9^n)}\ 9^n \in o(8^n) \Rightarrow 8^n < 9^n$

s.t. $2^{3n} < 3^{2n}$

a.2> $\because \lim_{n\to+\infty} \frac{n^{17}-n^{16}}{n^{17}} = 1 \quad \therefore n^{17}-n^{16} \in \Theta(n^{17})$

a.3> $\because 8^{\log_2 n} = 2^{3\log_2 n} = 2^{\log_2 n^3} = n^3 \quad \log_{10} 2^{(n^{18})} = \frac{\log_2 2^{n^{18}}}{\log_2 10} = \frac{n^{18}}{\log_2 10}$

and $\lim_{n\to+\infty} \frac{\frac{n^{18}}{\log_2 10}}{n^3} = +\infty \Rightarrow f(n) = \frac{n^{18}}{\log_2 10} = \omega(n^3) \Rightarrow n^3 < \frac{n^{18}}{\log_2 10} \Rightarrow 8^{\log_2 n} < \log_{10} 2^{n^{18}}$

a.4> apparently, $\lim_{n\to+\infty} \frac{n!}{\sqrt{n}} = +\infty \quad \therefore \sqrt{n} < n!$

Then, we have these orders as following.

$2^{3n} < 3^{2n} \quad n^{17}-n^{16} = n^{17} \quad 8^{\log_2 n} < \log_{10} 2^{n^{18}} \quad \sqrt{n} < n!$

a.5> $\lim_{n\to+\infty} \frac{n!}{3^{2n}} = \lim_{n\to+\infty} \frac{n!}{9^n} \geq \lim_{n\to+\infty} \frac{n \times \underbrace{9 \times 9 \cdots}_{n-9} \times 8\times7\times6\times5\times4\times3\times2\times1}{9 \times \underbrace{9 \times 9 \cdots}_{n-9} \times 9\times9\times9\times9\times9\times9\times9\times9} = +\infty$

$\Rightarrow \lim_{n\to+\infty} \frac{n!}{3^{2n}} = +\infty$

$\Rightarrow n! > 3^{2n} > 2^{3n}$

a.6> $\lim_{n\to+\infty} \frac{2^{3n}}{\log_{10} 2^{n^{18}}} = \lim_{n\to+\infty} \frac{8^n \cdot \log_2 10}{n^{18}} = \log_2 10 \lim_{n\to+\infty} \frac{8^n}{n^{18}} = \log_2 10 \cdot \frac{\ln 8}{18} \lim_{n\to+\infty} \frac{8^n}{n^{17}}$

$= \cdots = \log_2 10 \times \frac{\ln 8}{18} \times \frac{\ln 8}{17} \times \cdots \times \frac{\ln 8}{1} \lim_{n\to+\infty} \frac{8^n}{1} = +\infty$

$\therefore 2^{3n} > \log_{10} 2^{n^{18}} \Rightarrow n! > 3^{2n} > 2^{3n} > \log_{10} 2^{n^{18}}$

a.7> $\lim_{n\to+\infty} \frac{\log_{10} 2^{n^{18}}}{n^{17}-n^{16}} = \log_{10} 2 \lim_{n\to+\infty} \frac{n^{18}}{n^{17}-n^{16}} = +\infty \quad \therefore \log_{10} 2^{n^{18}} > n^{17}-n^{16}$

=> $n! > 3^{2n} > 2^{3n} > \log_{10} 2^{n^{18}} > n^{17} = n^{17} - n^{16}$ by Hesen Cai

Using the method above, we can easily prove: $n^{17}-n^{16} > 8^{\log_2 n} > \sqrt{n}$

Then, it's obvious that: $n! > 3^{2n} > 2^{3n} > \log_{10} 2^{n^{18}} > n^{17} \equiv n^{17}-n^{16} > 8^{\log_2 n} > \sqrt{n}$

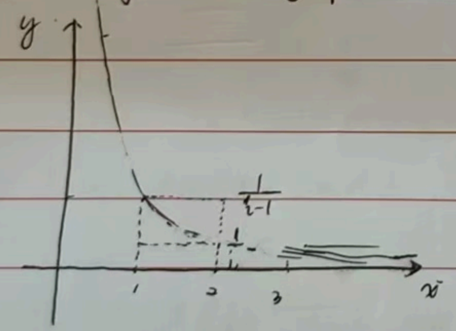
The final order is:

$$\sqrt{n} < 8^{\log_2 n} < n^{17}-n^{16} \equiv n^{17} < \log_{10} 2^{n^{18}} < 2^{3n} < 3^{2n} < n!$$

b> $\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\ln n)$

proof: $\int_{x=1}^{x=n} \frac{dx}{x} = \ln n$

Looking at the graph below:



We can find: In $[i, i+1]$, $\frac{1}{i} > \int_{x=i}^{x=i+1} \frac{1}{x} dx > \frac{1}{i+1}$ for any $i \geq 1$

Hence: sum up i from $i=1$ to $i=n$

$\therefore \sum_{i=1}^{n} \frac{1}{i} > \int_{x=1}^{x=n} \frac{1}{x} dx > \sum_{i=1}^{n} \frac{1}{i+1}$

$\sum_{i=1}^{n} \frac{1}{i+1} = \sum_{i=2}^{n+1} \frac{1}{i} = \sum_{i=1}^{n} \frac{1}{i} - 1 + \frac{1}{n+1} = \sum_{i=1}^{n} \frac{1}{i} - \frac{n}{n+1}$ $\quad \lim_{n\to+\infty}\left(\sum_{i=1}^{n} \frac{1}{i} + \frac{n}{n+1}\right) = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{i} - 1$

$\therefore \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{i} > \lim_{n\to+\infty} \int_{x=1}^{x=n} \frac{1}{x} dx > \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{i+1} = \lim_{n\to+\infty} \sum_{i=1}^{n} \frac{1}{i} - 1$

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$\because \lim\limits_{n\to+\infty} \dfrac{\int_{x=1}^{x=n} \frac{1}{x}dx}{\sum_{i=1}^{n} \frac{1}{i}} > \lim\limits_{n\to+\infty} \dfrac{\sum_{i=1}^{n} \frac{1}{i+1}}{\sum_{i=1}^{n} \frac{1}{i}} = \lim\limits_{n\to+\infty} \dfrac{\sum_{i=1}^{n} \frac{1}{i} - \frac{n}{n+1}}{\sum_{i=1}^{n} \frac{1}{i}}$

$\because \sum_{i=1}^{n} \frac{1}{i} > \int_{x=1}^{x=n} \frac{1}{x}dx = \ln n > \frac{n}{n+1}$ $\therefore \lim\limits_{n\to+\infty} \dfrac{\int_{x=1}^{x=n} \frac{1}{x}dx}{\sum_{i=1}^{n} \frac{1}{i}} > 0$

$\because \lim\limits_{n\to+\infty} \dfrac{\int_{x=1}^{x=n} \frac{1}{x}dx}{\sum_{i=1}^{n} \frac{1}{i}} < \lim\limits_{n\to+\infty} \dfrac{\sum_{i=1}^{n} \frac{1}{i}}{\sum_{i=1}^{n} \frac{1}{i}} = 1$

$\therefore 0 < \lim\limits_{n\to+\infty} \dfrac{\int_{x=1}^{x=n} \frac{1}{x}dx}{\sum_{i=1}^{n} \frac{1}{i}} < 1$

$\Rightarrow \sum_{i=1}^{n} \frac{1}{i} \in \Theta(\ln n)$

2. ① $m=1$ $m=F_1$ or F_2 meets the requirement.

② $m=2$. $m=F_3$ also meets the requirement.

③ assume when $m=k$. meets the requirement.

$m = F_{i_1} + F_{i_2} + \cdots F_{i_k}$

Then, consider $m+1 = F_1 + F_{i_1} + \cdots + F_{i_k}$ for any $1 \le j < k$

or $= F_2 + F_{i_1} + \cdots + F_{i_k}$. $i_j + 1 < i_{j+1}$

if $F_{i_1} \ge F_5$, then $m+1 = F_1 + F_{i_1} + \cdots F_{i_k}$ meets the requirement.

When $F_{i_1} \le F_4$ if $\exists j$ s.t. $i_j + 2 < i_{j+1}$

$m+1 = F_{i_j+1} + F_{i_{j+1}} + \cdots + F_{i_k}$. meets. all requirement

if $\nexists j$. s.t. $i_j + 2 < i_{j+1}$. $\Rightarrow m+1 = F_{i_k+1}$ also meets requirements,

Then. considering ① ② ③, we can ~~conclude~~ prove the 2.

3. No. assume $f(n) = n$ $g(n) = 2n$ $\Rightarrow n \in O(2n)$

but $2^n \notin O(2^{2n}) = O(4^n)$