CS3230 – Design and Analysis of Algorithms (S2 AY2024/25)

Lecture 2: Recurrences and Master Theorem

Analyzing the running time of an algorithm

- **Goal:** For a given algorithm $\mathcal{A}$, analyze the $\underbrace{\text{asymptotic running time}}_{\text{Time complexity}}$ $T(n)$ as a function of the input size n.

Unless otherwise stated, we consider the worst-case running time.

- $T(n)$ is the worst-case running time over all possible inputs of size n.

Analyzing the running time of an algorithm

recursive

- **Goal:** For a given algorithm $\mathcal{A}$, analyze the asymptotic running time $T(n)$ as a function of the input size n.
  - Step 1: Derive a recurrence.
  - Step 2: Solve the recurrence.

Analyzing the running time of an algorithm

recursive

- **Goal:** For a given algorithm $\mathcal{A}$, analyze the asymptotic running time $T(n)$ as a function of the input size n.
  - Step 1: Derive a recurrence.
  - Step 2: Solve the recurrence.

$\mathbf{Fib}(n)$
- If $n \leq 1$, return n.
- Else, return $\mathbf{Fib}(n-1) + \mathbf{Fib}(n-2)$.

Step 1

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ T(n-1) + T(n-2) + \Theta(1) & \text{if } n > 1 \end{cases}$$

Step 2

$$T(n) \in \Omega\left(2^{n/2}\right)$$

Merge sort

$T(n)$ **MergeSort**$(A[1..n])$

- If $n \geq 2$, do the following steps.
    - **MergeSort**$(A[1..\lceil n/2 \rceil])$.
    - **MergeSort**$(A[\lceil n/2 \rceil + 1..n])$.
    - "Merge" the two sorted arrays.

We omit the details.

Question: How to derive a recurrence for the running time $T(n)$ of Merge sort?

VisuAlgo (Merge sort): https://visualgo.net/en/sorting?mode=Merge

Merge sort

$T(n)$

MergeSort$(A[1..n])$
- If $n \geq 2$, do the following steps. $\leftarrow \Theta(1)$
  - **MergeSort**$(A[1..\lceil n/2 \rceil])$. $\leftarrow T(n/2)$
  - **MergeSort**$(A[\lfloor n/2 \rfloor..n])$. $\leftarrow T(n/2)$
  - "Merge" the two sorted arrays.

Here we approximate $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$ by $n/2$.
- This is slightly sloppy.
- As we will later discuss, removing floors and ceilings does not affect the asymptotic complexity in most of the cases.

VisuAlgo (Merge sort): https://visualgo.net/en/sorting?mode=Merge

Merge sort

$T(n)$ → **MergeSort**$(A[1..n])$

- If $n \geq 2$, do the following steps. ← $\Theta(1)$
  - **MergeSort**$(A[1..\lceil n/2 \rceil])$. ← $T(n/2)$
  - **MergeSort**$(A[\lceil n/2 \rceil..n])$. ← $T(n/2)$
  - "Merge" the two sorted arrays. ← $\Theta(n)$

$[\mathbf{1}, 5, 7]$,	$[\mathbf{2}, 4, 6]$,	$[\quad]$
$[1, \mathbf{5}, 7]$,	$[\mathbf{2}, 4, 6]$,	$[1 \quad]$
$[1, \mathbf{5}, 7]$,	$[2, \mathbf{4}, 6]$,	$[1, 2 \quad]$
$[1, \mathbf{5}, 7]$,	$[2, 4, \mathbf{6}]$,	$[1, 2, 4 \quad]$
$[1, 5, \mathbf{7}]$,	$[2, 4, \mathbf{6}]$,	$[1, 2, 4, 5 \quad]$
$[1, 5, \mathbf{7}]$,	$[2, 4, 6]$,	$[1, 2, 4, 5, 6 \quad]$
$[1, 5, 7]$,	$[2, 4, 6]$,	$[1, 2, 4, 5, 6, 7]$

VisuAlgo (Merge sort): https://visualgo.net/en/sorting?mode=Merge

Merge sort

$T(n) \rightarrow$ $\mathbf{MergeSort}(A[1..n])$

- If $n \geq 2$, do the following steps. $\leftarrow \Theta(1)$
  - $\mathbf{MergeSort}(A[1..\lceil n/2 \rceil])$. $\leftarrow T(n/2)$
  - $\mathbf{MergeSort}(A[\lceil n/2 \rceil..n])$. $\leftarrow T(n/2)$
  - "Merge" the two sorted arrays. $\leftarrow \Theta(n)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$

VisuAlgo (Merge sort): https://visualgo.net/en/sorting?mode=Merge

Merge sort

$T(n)$ → **MergeSort**$(A[1..n])$

- If $n \geq 2$, do the following steps. ← $\Theta(1)$
  - **MergeSort**$(A[1..\lceil n/2 \rceil])$. ← $T(n/2)$
  - **MergeSort**$(A[\lceil n/2 \rceil..n])$. ← $T(n/2)$
  - "Merge" the two sorted arrays. ← $\Theta(n)$

Base case

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Inductive step

Note: We often omit stating the base case because $T(n)$ is $\Theta(1)$ whenever $n \in O(1)$.

- The precise constant does not matter in most of the cases.

VisuAlgo (Merge sort): https://visualgo.net/en/sorting?mode=Merge

Solving a recurrence

- How to solve a given recurrence:
  - Merge sort: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

- **Four methods:**
  - Telescoping
  - Substitution
  - Recursion tree
  - Master theorem

Solving a recurrence

- How to solve a given recurrence:
  - Merge sort: $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$

Remarks:

If $f(n) \in O(n)$, then there exist two constant $c > 0$ and $n_0 > 0$ such that $f(n) \leq cn$ if $n \geq n_0$.

- For <u>upper bound</u> calculation, we can replace $\Theta(n)$ with cn.
  - $T(n) \leq 2T\left(\frac{n}{2}\right) + cn$ (if $n \geq n_0$).

If $f(n) \in \Omega(n)$, then there exist two constant $c > 0$ and $n_0 > 0$ such that $f(n) \geq cn$ if $n \geq n_0$.

- For <u>lower bound</u> calculation, we can replace $\Theta(n)$ with cn.
  - $T(n) \geq 2T\left(\frac{n}{2}\right) + cn$ (if $n \geq n_0$).

Telescoping series

- **An example:**

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Telescoping method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + {\color{red}n} & \text{if } n > 1 \end{cases}$

For the sake of simplicity, we omit $\Theta(\cdot)$ here and assume that $n = 2^k$ for some integer $k \geq 0$.

Telescoping method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + n & \text{if } n > 1 \end{cases}$

$$\log n \left\{ \begin{array}{l} \dfrac{T(n)}{n} = \dfrac{T\left(\frac{n}{2}\right)}{\frac{n}{2}} + 1 \\[2ex] \dfrac{T\left(\frac{n}{2}\right)}{\frac{n}{2}} = \dfrac{T\left(\frac{n}{4}\right)}{\frac{n}{4}} + 1 \\[2ex] \dfrac{T\left(\frac{n}{4}\right)}{\frac{n}{4}} = \dfrac{T\left(\frac{n}{8}\right)}{\frac{n}{8}} + 1 \\[2ex] \vdots \\[2ex] \dfrac{T(2)}{2} = \dfrac{T(1)}{1} + 1 \end{array} \right.$$

Telescoping method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + n & \text{if } n > 1 \end{cases}$

$$\log n \left\{ \begin{aligned} \frac{T(n)}{n} &= \frac{T\left(\frac{n}{2}\right)}{\frac{n}{2}} + 1 \\ \frac{T\left(\frac{n}{2}\right)}{\frac{n}{2}} &= \frac{T\left(\frac{n}{4}\right)}{\frac{n}{4}} + 1 \\ \frac{T\left(\frac{n}{4}\right)}{\frac{n}{4}} &= \frac{T\left(\frac{n}{8}\right)}{\frac{n}{8}} + 1 \\ &\vdots \\ \frac{T(2)}{2} &= \frac{T(1)}{1} + 1 \end{aligned} \right.$$

$$\frac{T(n)}{n} = \frac{T(1)}{1} + \log n$$

$$T(n) \in \Theta(n \log n)$$

Telescoping method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 4T\left(\frac{n}{2}\right) + n & \text{if } n > 1 \end{cases}$

$$\left.\begin{aligned} \frac{T(n)}{n^2} &= \frac{T\left(\frac{n}{2}\right)}{\left(\frac{n}{2}\right)^2} + \frac{1}{n} \\ \frac{T\left(\frac{n}{2}\right)}{\left(\frac{n}{2}\right)^2} &= \frac{T\left(\frac{n}{4}\right)}{\left(\frac{n}{4}\right)^2} + \frac{2}{n} \\ \frac{T\left(\frac{n}{4}\right)}{\left(\frac{n}{4}\right)^2} &= \frac{T\left(\frac{n}{8}\right)}{\left(\frac{n}{8}\right)^2} + \frac{4}{n} \\ &\vdots \\ \frac{T(2)}{2^2} &= \frac{T(1)}{1^2} + 1 \end{aligned}\right\} \log n$$

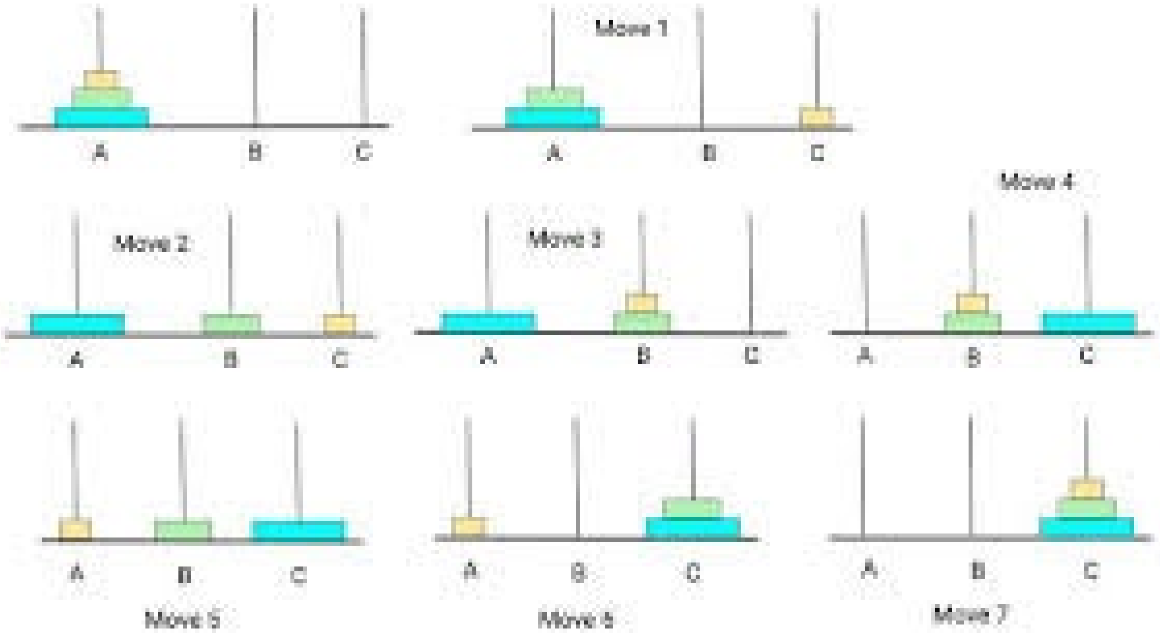
$$\frac{T(n)}{n^2} = \frac{T(1)}{1^2} + \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{n}\right)$$

$$T(n) \in \Theta(n^2)$$

Substitution method

- Step 1: Guess a solution.
- Step 2: Verify your solution by induction.

Tower of Hanoi



- Hanoi(n, source, destination, temp)
  - If $n > 0$
    - Hanoi($n - 1$, source, temp, destination)
    - Move disk n from source to destination
    - Hanoi($n - 1$, temp, destination, source)

Analysis:

- $T(n) = 2T(n-1) + 1$
- $T(0) = 0$
- Prove: $T(n) = 2^n \ - 1$
- Base Cases: $T(0) = 0, T(1) = 1$
- Induction Step:
- $T(n) = 2T(n-1) + 1 = 2 \cdot \ (2^{n-1} - 1) + 1 = 2^n - 1$

Substitution method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \text{if } n > 1 \end{cases}$

Substitution method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} c & \text{if } n \le 1 \\ 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \text{if } n > 1 \end{cases}$
- **Induction hypothesis:** $T(n) \le (c+1)n^2 - n$.

Guessing an upper bound of $T(n)$.
If we can prove this for all $n \in \{1, 2, 3, \dots\}$, then $T(n) \in O(n^2)$.

Substitution method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \text{if } n > 1 \end{cases}$
- **Induction hypothesis:** $T(n) \leq (c+1)n^2 - n$.

Base case: $n = 1$.

- If $n = 1$, then $T(n) = c = (c+1)n^2 - n$.

Substitution method

- **Goal:** Solve the recurrence $T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \text{if } n > 1 \end{cases}$
- **Induction hypothesis:** $T(n) \leq (c+1)n^2 - n$.

Base case: $n = 1$.

- If $n = 1$, then $T(n) = c = (c+1)n^2 - n$.

Inductive step: $n \geq 2$.

$$\begin{aligned} T(n) &= 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \\ &\leq 4(c+1)\left\lfloor \frac{n}{2} \right\rfloor^2 - 4\left\lfloor \frac{n}{2} \right\rfloor + n \\ &\leq 4(c+1)\left(\frac{n}{2}\right)^2 - 4\left(\frac{n}{2}\right) + n \\ &= (c+1)n^2 - n \end{aligned}$$

Induction hypothesis: $T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \leq (c+1)\left\lfloor \frac{n}{2} \right\rfloor^2 - \left\lfloor \frac{n}{2} \right\rfloor$.

The function $(c+1)x^2 - x$ is increasing when $x \geq 1$.

Therefore, $T(n) \in O(n^2)$.

A common mistake

- **Goal:** Solve the recurrence $T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n & \text{if } n > 1 \end{cases}$
- **Induction hypothesis:** $T(n) \leq {\color{red}c}n^2$.

Base case: $n = 1$.

- If $n = 1$, then $T(n) = c = cn^2$.

Induction hypothesis: $T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \leq c \left\lfloor \frac{n}{2} \right\rfloor^2$.

Inductive step: $n \geq 2$.

$$\begin{aligned} T(n) &= 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n \\ &= 4c \left\lfloor \frac{n}{2} \right\rfloor^2 + n \\ &\in O(n^2) \end{aligned}$$

Incorrect proof!

You need to show that $T(n) \leq cn^2$.

Recursion tree

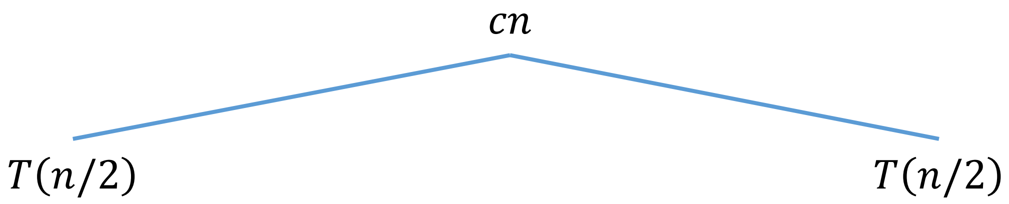
- **Goal:** Solve the recurrence $T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$

VisuAlgo (Recursion tree): https://visualgo.net/en/recursion

Recursion tree

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$



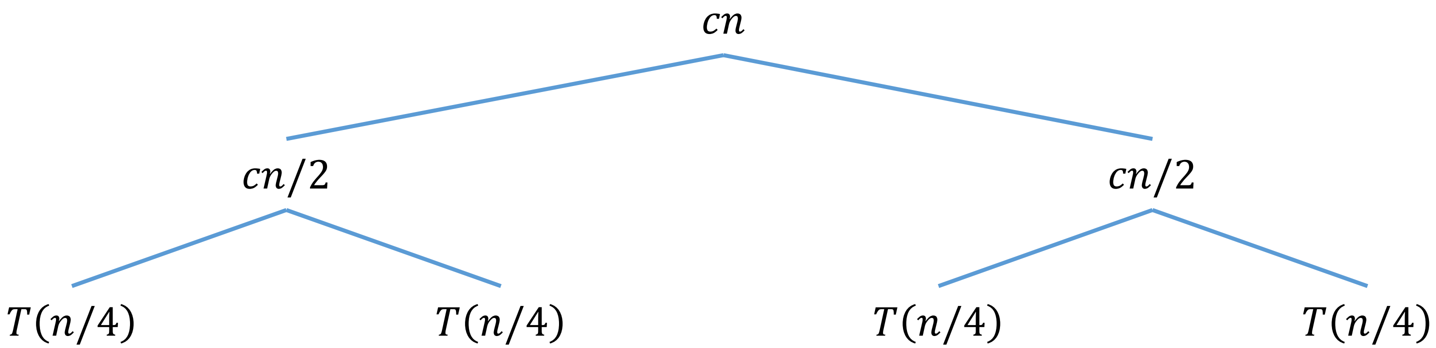
VisuAlgo (Recursion tree): https://visualgo.net/en/recursion

Recursion tree

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{cn}{2}$$



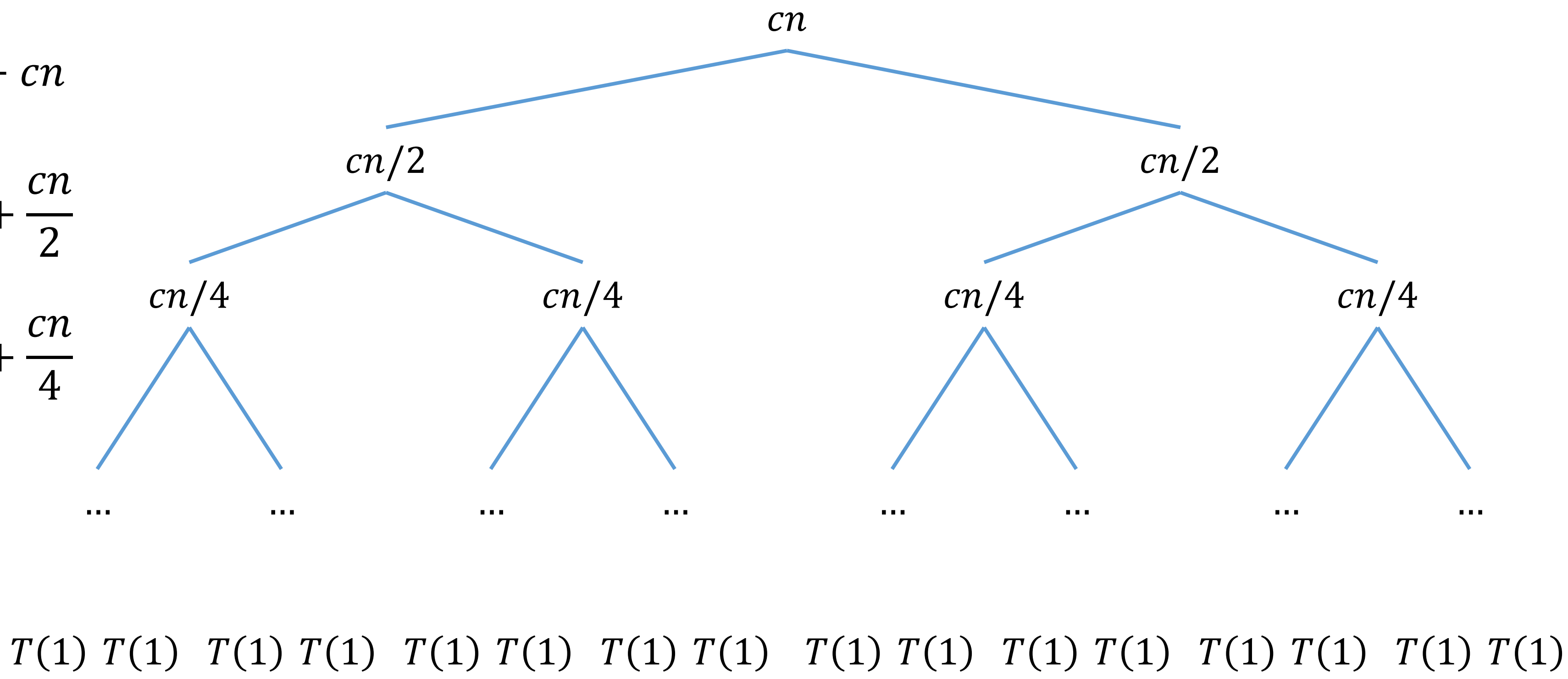
Recursion tree

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + cn$$

$$T\left(\frac{n}{2}\right) = 2T\left(\frac{n}{4}\right) + \frac{cn}{2}$$

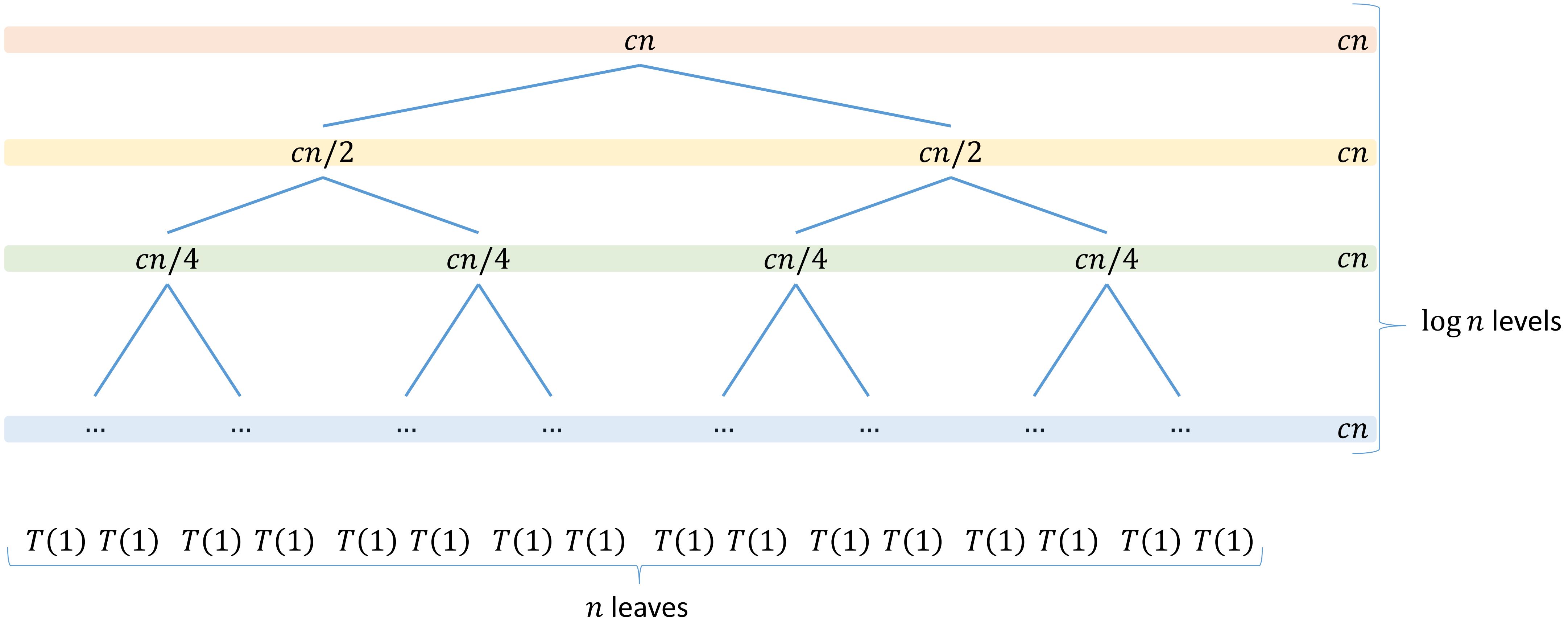
$$T\left(\frac{n}{4}\right) = 2T\left(\frac{n}{8}\right) + \frac{cn}{4}$$



VisuAlgo (Recursion tree): https://visualgo.net/en/recursion

Recursion tree

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$



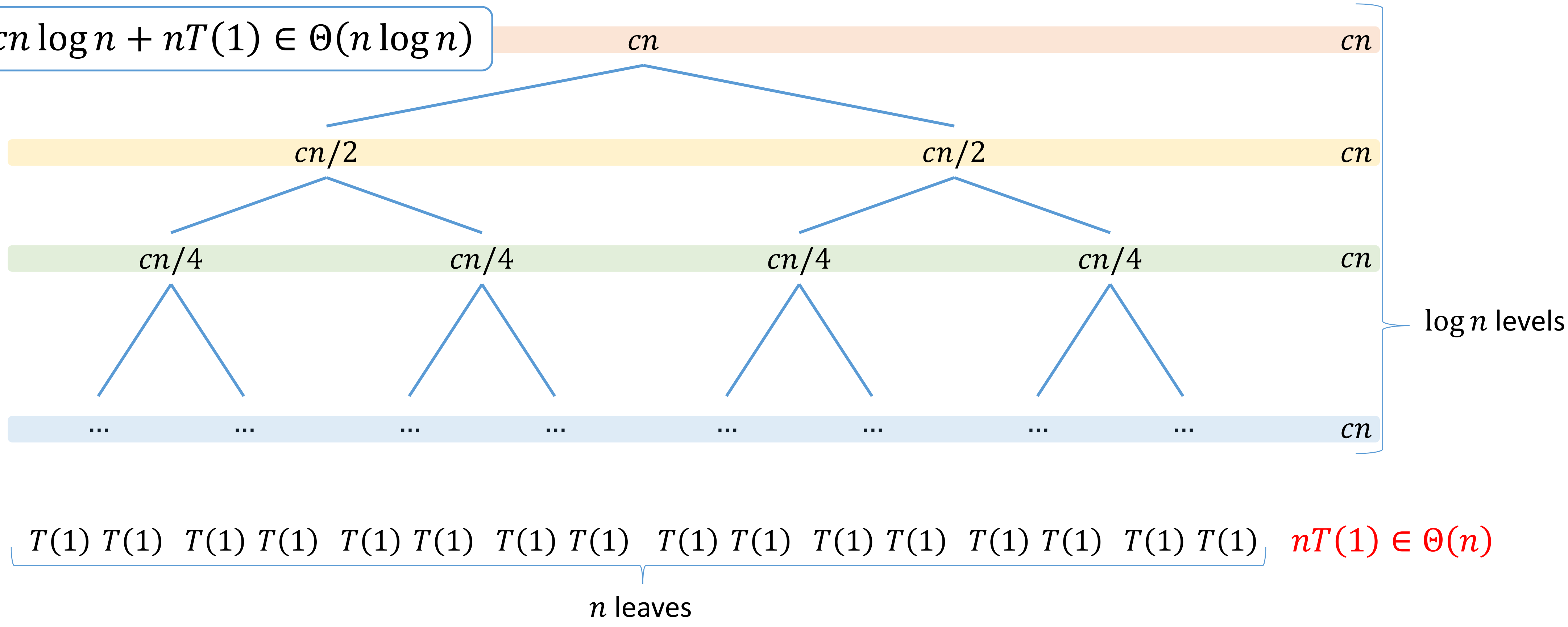
VisuAlgo (Recursion tree): https://visualgo.net/en/recursion

Recursion tree

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \leq 1 \\ 2T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \end{cases}$$

$cn \log n \in \Theta(n \log n)$

$\boxed{T(n) = cn \log n + nT(1) \in \Theta(n \log n)}$



$nT(1) \in \Theta(n)$

VisuAlgo (Recursion tree): https://visualgo.net/en/recursion

Question

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ T(n-1) + T(1) + cn & \text{if } n > 1 \end{cases}$$

Which of the following statements is **true**?

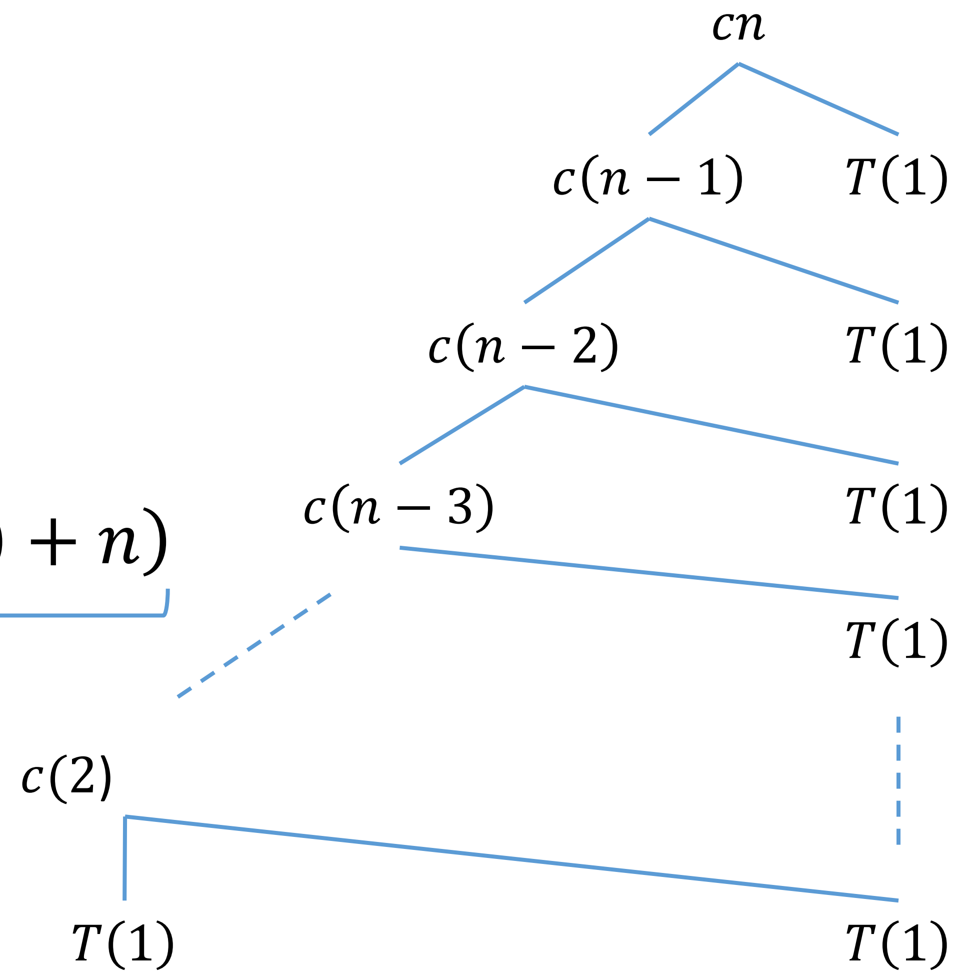
- $T(n) \in \Theta(n)$
- $T(n) \in \Theta(n \log n)$
- $T(n) \in \Theta(n^2)$
- $T(n) \in \Theta(n^3)$

Answer

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ T(n-1) + T(1) + cn & \text{if } n > 1 \end{cases}$$

$$T(n) = \underbrace{n \cdot T(1)}_{\Theta(n)} + \underbrace{c(2 + 3 + \cdots + (n-1) + n)}_{\Theta(n^2)}$$

$$T(n) \in \Theta(n^2)$$



Question

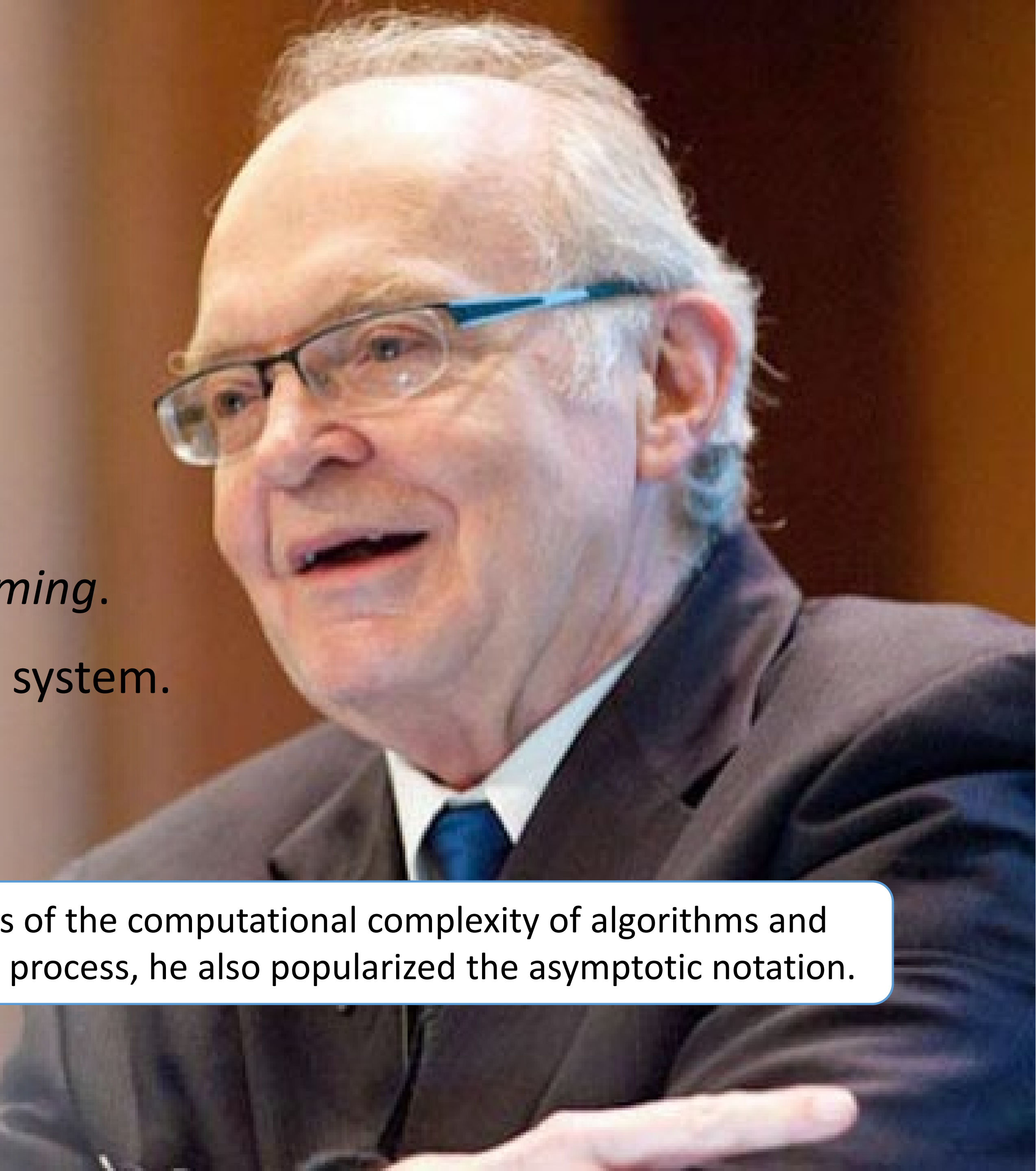
Who is the **Master of Algorithms** pictured below?

- Robert Floyd
- Richard Karp
- Donald Knuth
- Alan Turing

Answer

Donald Knuth

- 1974 Turing Award (at the age of 36).
- Author of *The Art of Computer Programming*.
- Creator of the TeX computer typesetting system.
- The father of the analysis of algorithms.



He contributed to the development of the rigorous analysis of the computational complexity of algorithms and systematized formal mathematical techniques for it. In the process, he also popularized the asymptotic notation.

Source: https://en.wikipedia.org/wiki/Donald_Knuth

Solving a recurrence of the generic form

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.
  - $a \geq 1$
  - $b \geq 1$
  - $f(n) \in \Omega(1)$

- **Goal:** Solve $T(n)$.

Solving a recurrence of the generic form

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.
  - $a \geq 1$
  - $b \geq 1$
  - $f(n) \in \Omega(1)$

- **Goal:** Solve $T(n)$.

- **Main idea:** Classify the work into two types and compare their costs.
  - Splitting/combining.
  - Solving the base cases.

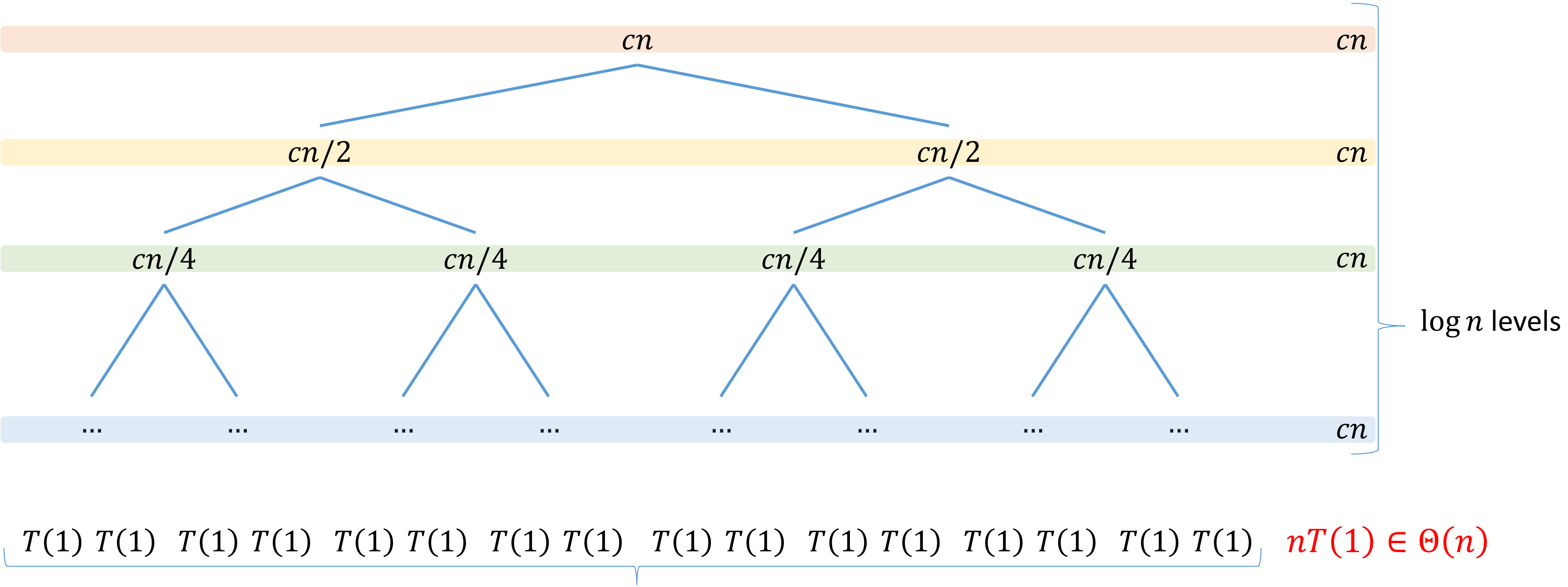
Two types of work

$$\boxed{T(n) = cn \log n + nT(1) \in \Theta(n \log n)}$$

Splitting/combining:

- Split a problem into sub-problems.
- Combine the solutions of subproblems.

$cn \log n \in \Theta(n \log n)$



n leaves

Solving the base cases:

- The cost is linear in the number of leaves.

Two types of work

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Solving the base cases **Splitting/combining**

The number of leaves $= n^d$, where $d = \log_b a$ is the critical exponent.

Recursion tree:
- Tree height: $\log_b n$
- The number of children of a node: a
- The number of leaves: $a^{\log_b n} = n^{\log_b a} = n^d$

Master theorem

Two types of work:

- Solving the base cases: n^d ← **Dominant term**
  - $d = \log_b a$
- Splitting/combining: $f(n)$

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Case 1:

- $f(n) \in O(n^{d-\epsilon})$ for some constant $\epsilon > 0$.

▷ $T(n) \in \Theta(n^d)$

Master theorem

Two types of work:

- Solving the base cases: n^d
  - $d = \log_b a$
- Splitting/combining: $f(n)$

Comparable

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Case 1:

- $f(n) \in O\left(n^{d-\epsilon}\right)$ for some constant $\epsilon > 0$.

▷ $T(n) \in \Theta\left(n^d\right)$

Case 2:

- $f(n) \in \Theta\left(n^d \log^k n\right)$ for some constant $k \geq 0$.

▷ $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Master theorem

Two types of work:

- Solving the base cases: n^d
  - $d = \log_b a$
- Splitting/combining: $f(n)$ ← **Dominant term**

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Case 1:

- $f(n) \in O(n^{d-\epsilon})$ for some constant $\epsilon > 0$.

▷ $T(n) \in \Theta(n^d)$

Case 2:

- $f(n) \in \Theta(n^d \log^k n)$ for some constant $k \geq 0$.

▷ $T(n) \in \Theta(n^d \log^{k+1} n)$

Case 3:

- $f(n) \in \Omega(n^{d+\epsilon})$ for some constant $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}$ for some constant $c < 1$.

▷ $T(n) \in \Theta(f(n))$

A **regularity condition** ensuring that the splitting/combining cost $f(n)$ at the top level of recursion is the dominant term.

Examples

Two types of work:
- Solving the base cases: n^d ← **Dominant term**
  - $d = \log_b a$
- Splitting/combining: $f(n)$

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Critical exponent: $d = \log_b a$

Case 1:
- $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

▷ $T(n) \in \Theta\left(n^d\right)$

Case 2:
- $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

▷ $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Case 3:
- $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}_{\text{Regularity condition}}$ for some $c < 1$.

▷ $T(n) \in \Theta\left(f(n)\right)$

Solve: $T(n) = 4T(n/2) + n$.
- $a = 4$
- $b = 2$
- $d = \log_b a = 2$
- $f(n) = n \in O\left(n^{d-\epsilon}\right)$ for $\epsilon = 1$
- **Case 1** $\rightarrow T(n) \in \Theta(n^2)$

VisuAlgo (Master theorem): https://visualgo.net/en/recursion?example=MT1L2 – you can change $n, a, b, f(n)$

Examples

Two types of work:

- Solving the base cases: n^d
  - $d = \log_b a$
- Splitting/combining: $f(n)$

Comparable

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Critical exponent: $d = \log_b a$

Case 1:

- $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

$\triangleright$ $T(n) \in \Theta\left(n^d\right)$

Case 2:

- $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

$\triangleright$ $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Case 3:

- $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}_{\text{Regularity condition}}$ for some $c < 1$.

$\triangleright$ $T(n) \in \Theta\left(f(n)\right)$

Solve: $T(n) = 2T(n/2) + n$.

- $a = 2$
- $b = 2$
- $d = \log_b a = 1$
- $f(n) = n \in \Theta\left(n^d \log^k n\right)$ for $k = 0$
- **Case 2** $\rightarrow T(n) \in \Theta(n \log n)$

VisuAlgo (Master theorem): https://visualgo.net/en/recursion?example=MT1L2 – you can change $n, a, b, f(n)$

Question

Question: $T(n) = 4T\left(\frac{n}{2}\right) + n^3$ satisfies which case of the master theorem?

$T(n) = aT(n/b) + f(n)$.
Critical exponent: $d = \log_b a$

Case 1:
- $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

$\triangleright$ $T(n) \in \Theta\left(n^d\right)$

Case 2:
- $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

$\triangleright$ $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Case 3:
- $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}_{\text{Regularity condition}}$ for some $c < 1$.

$\triangleright$ $T(n) \in \Theta\left(f(n)\right)$

- Case 1.
- Case 2.
- Case 3.
- None of the above.

VisuAlgo (Master theorem): https://visualgo.net/en/recursion?example=MT1L2 – you can change $n, a, b, f(n)$

Answer

Two types of work:

- Solving the base cases: n^d
  - $d = \log_b a$
- Splitting/combining: $f(n)$ ← **Dominant term**

- Consider a recurrence of the generic form: $T(n) = aT(n/b) + f(n)$.

Critical exponent: $d = \log_b a$

Case 1:
- $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

▷ $T(n) \in \Theta\left(n^d\right)$

Case 2:
- $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

▷ $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Case 3:
- $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}_{\text{Regularity condition}}$ for some $c < 1$.

▷ $T(n) \in \Theta\left(f(n)\right)$

Solve: $T(n) = 4T(n/2) + n^3$.
- $a = 4$
- $b = 2$
- $d = \log_b a = 2$
- $f(n) = \Omega\left(n^{d+\epsilon}\right)$ for $\epsilon = 1$
- **Regularity condition:**

  $af(n/b) = \frac{n^3}{2} \leq cn^3 = cf(n)$ for $c = \frac{1}{2}$
- **Case 3** → $T(n) \in \Theta(n^3)$

VisuAlgo (Master theorem): https://visualgo.net/en/recursion?example=MT1L2 – you can change $n, a, b, f(n)$

Remarks

- The master theorem does not cover all recurrences of the generic form:
  - $T(n) = aT(n/b) + f(n)$.

- **Example:**
  - $T(n) = T(n/2) + 2^{\sqrt{\log n}}$.

- Critical exponent: $d = \log_b a = 0$.
- $f(n) = 2^{\sqrt{\log n}} \notin O(n^{0-\varepsilon}) \rightarrow$ Not Case 1.
- $f(n) = 2^{\sqrt{\log n}} \notin \Theta\left(n^0 \log^k n\right)$ for any $k \geq 0$ $\rightarrow$ Not Case 2.
- $f(n) = 2^{\sqrt{\log n}} \notin \Omega(n^{0+\varepsilon}) \rightarrow$ Not Case 3.

Remarks

- The master theorem does not cover all recurrences of the generic form:
  - $T(n) = aT(n/b) + f(n)$.

- **Example:**
  - $T(n) = T(n/2) + 2^{\sqrt{\log n}}$.

- Critical exponent: $d = \log_b a = 0$.
- $f(n) = 2^{\sqrt{\log n}} \notin O(n^{0-\varepsilon})$ → Not Case 1.
- $f(n) = 2^{\sqrt{\log n}} \notin \Theta\left(n^0 \log^k n\right)$ for any $k \geq 0$ → Not Case 2.
- $f(n) = 2^{\sqrt{\log n}} \notin \Omega(n^{0+\varepsilon})$ → Not Case 3.

- **Exercise:**
  - $T(n) \in \Theta\left(2^{\sqrt{\log n}} \cdot \sqrt{\log n}\right)$ ▷ Indeed, all three cases are not applicable.

Remarks

- The condition $f(n) \in \Omega(n^{d+\epsilon})$ is redundant in Case 3.

Critical exponent: $d = \log_b a$

Case 3:
- $f(n) \in \Omega(n^{d+\epsilon})$ for some $\epsilon > 0$.
- $\underbrace{af(n/b) \leq cf(n)}_{\text{Regularity condition}}$ for some $c < 1$.

Regularity condition

$$af(n/b) \leq cf(n)$$

$$\Downarrow$$

$$\frac{a}{c} f(n/b) \leq f(n)$$

$$\Downarrow$$

$$\left(\frac{a}{c}\right)^i f(1) \leq f(b^i)$$

$$\Downarrow$$

$$\Omega(n^{d+\epsilon}) \ni n^{\log_b\left(\frac{a}{c}\right)} f(1) = \left(\frac{a}{c}\right)^{\log_b n} f(1) \leq f(n)$$

$$\epsilon = \log_b\left(\frac{a}{c}\right) - \log_b a > 0 \qquad n = b^i$$

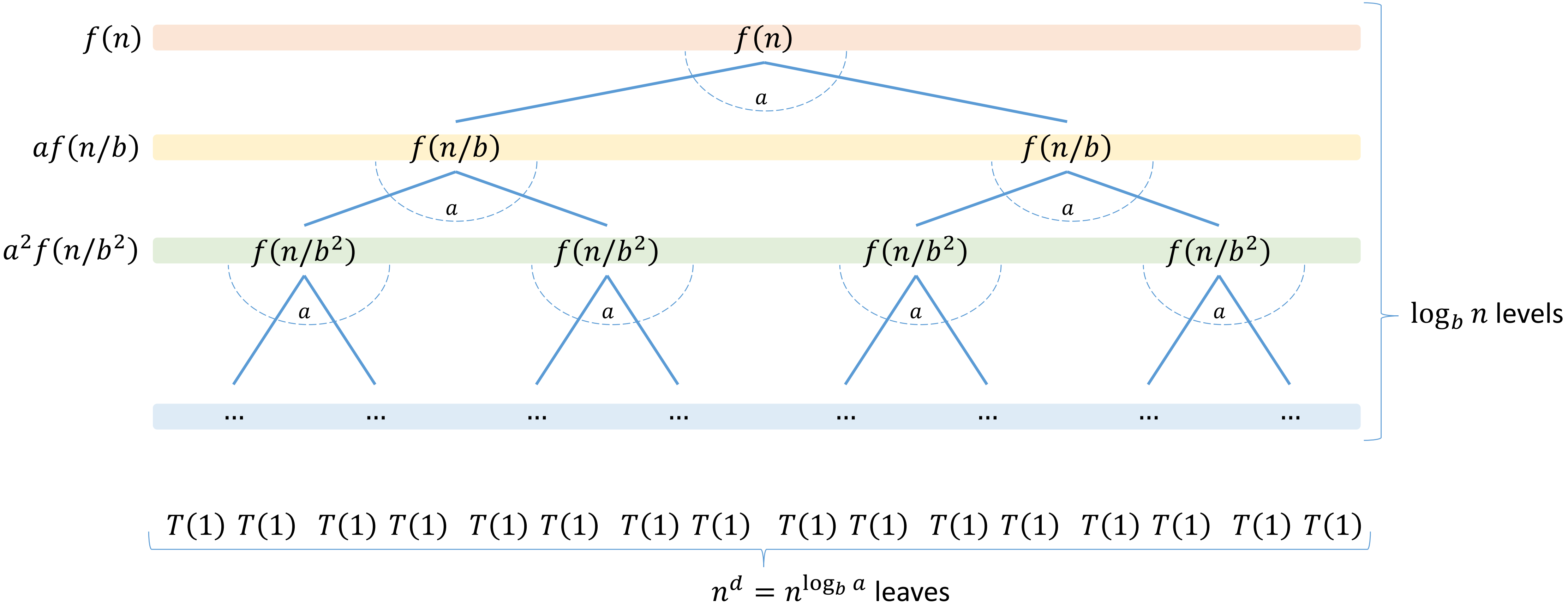
Critical exponent: $d = \log_b a$

Case 3: $f(n) \in \Omega(n^{d+\epsilon})$ for some $\epsilon > 0$.

- $af(n/b) \le cf(n)$ for some $c < 1$.

Goal: $T(n) \in \Theta(f(n))$

Proof of the master theorem



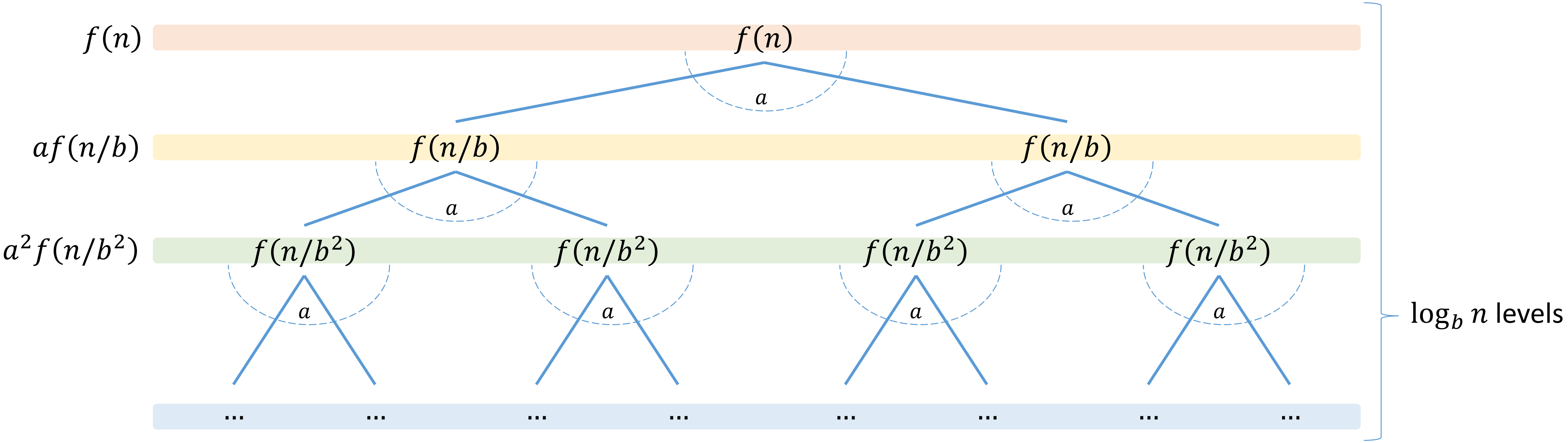
Critical exponent: $d = \log_b a$

Case 3: $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $af(n/b) \leq cf(n)$ for some $c < 1$.

∇

Goal: $T(n) \in \Theta\big(f(n)\big)$

Proof of the master theorem



$$\underbrace{T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)}_{n^d = n^{\log_b a} \text{ leaves}}$$

Solving the base cases: The cost is $n^d T(1) \in o(f(n))$.

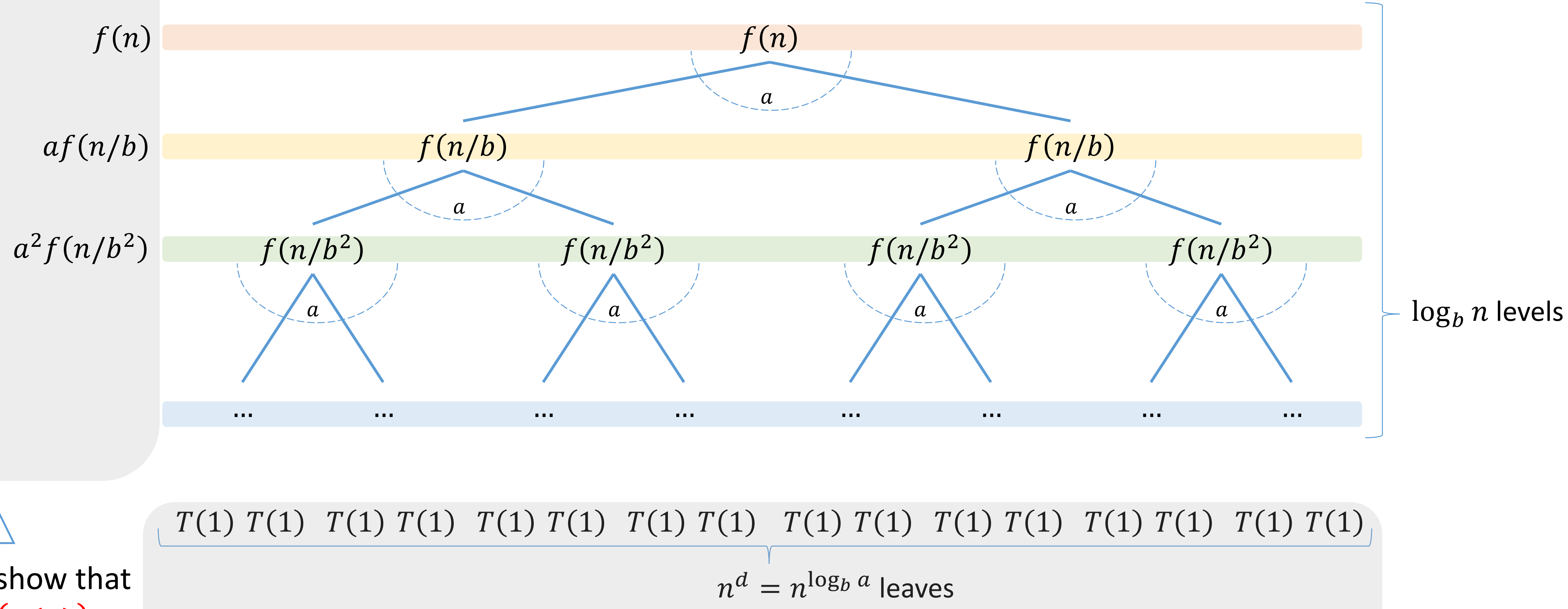
Critical exponent: $d = \log_b a$

Case 3: $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $af(n/b) \leq cf(n)$ for some $c < 1$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(f(n)\right)$

Splitting/combining:



Just need to show that this part is $\Theta\left(f(n)\right)$.

Solving the base cases: The cost is $n^d T(1) \in o\left(f(n)\right)$.

Critical exponent: $d = \log_b a$

Case 3: $f(n) \in \Omega\left(n^{d+\epsilon}\right)$ for some $\epsilon > 0$.
- $af(n/b) \le cf(n)$ for some $c < 1$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(f(n)\right)$

Splitting/combining:

$\log_b n$ levels

$f(n)$

$af(n/b) \quad \le cf(n)$

$a^2 f(n/b^2) \quad \le c^2 f(n)$

⋮

…

$$(1 + c + c^2 + \cdots) < \frac{1}{1-c}$$

Just need to show that this part is $\Theta\left(f(n)\right)$.

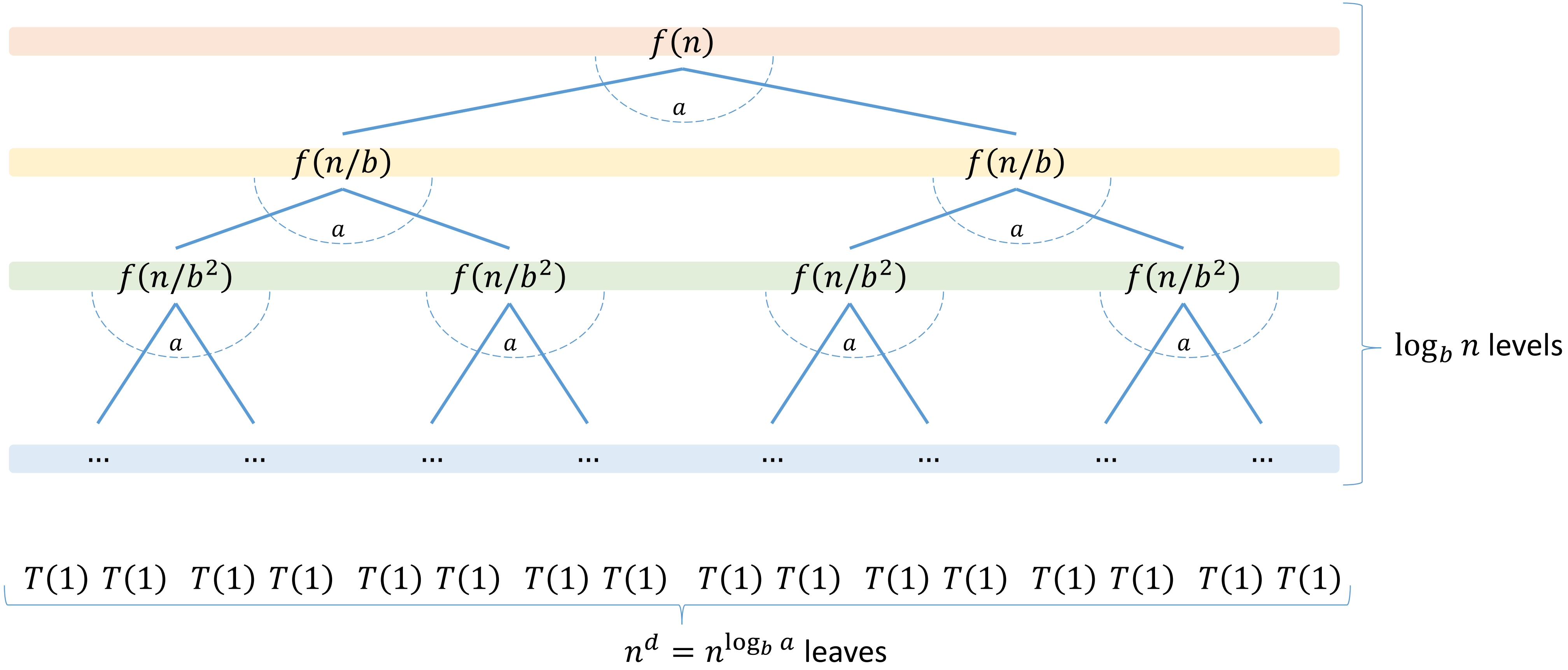
$$\Omega\left(f(n)\right) \ni f(n) \le \textbf{overall cost} \le f(n) \cdot \underbrace{(1 + c + c^2 + \cdots)}_{\log_b n \text{ terms}} \in O\left(f(n)\right)$$

Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Goal: $T(n) \in \Theta\left(n^d\right)$

Proof of the master theorem

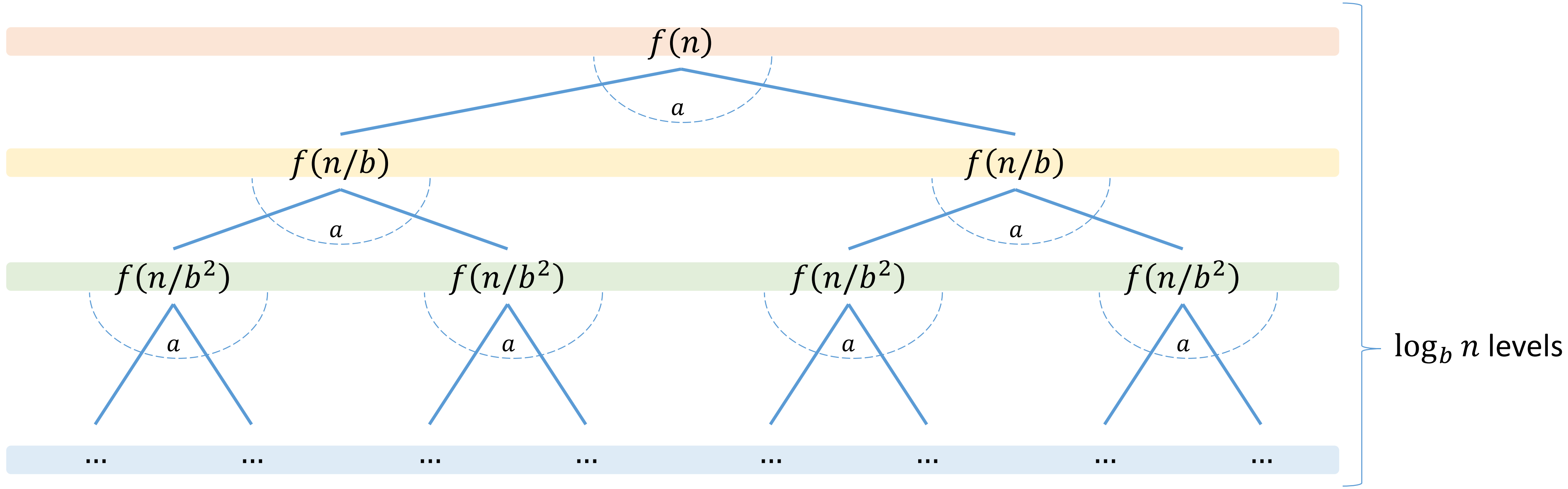


Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Goal: $T(n) \in \Theta\left(n^d\right)$

Proof of the master theorem



$\underbrace{T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)\ \ T(1)\ T(1)}_{n^d \,=\, n^{\log_b a} \text{ leaves}}$

Solving the base cases: The cost is already $n^d T(1) \in \Theta\left(n^d\right)$.

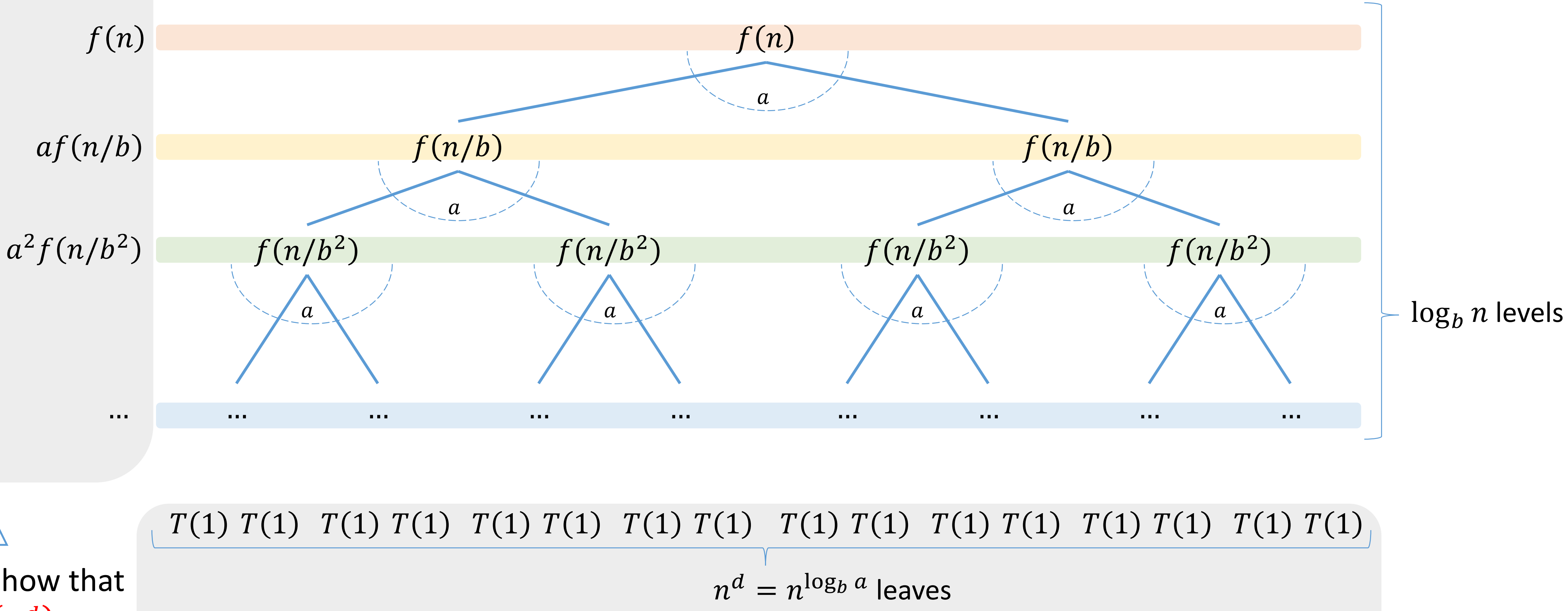
Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(n^d\right)$

Splitting/combining:



Just need to show that this part is $O\left(n^d\right)$.

Solving the base cases: The cost is already $n^d T(1) \in \Theta\left(n^d\right)$.

Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(n^d\right)$

Splitting/combining:

$\log_b n$ levels

$$\left\{\begin{array}{rl} f(n) & \leq cn^{d-\epsilon} \\ af(n/b) & \leq ac\left(\frac{n}{b}\right)^{d-\epsilon} \\ a^2 f(n/b^2) & \leq a^2 c\left(\frac{n}{b^2}\right)^{d-\epsilon} \\ \ldots & \end{array}\right.$$

$\exists n_0 > 0$ and $\exists c > 0$
such that $f(n) \leq cn^{d-\epsilon}$

Just need to show that
this part is $O\left(n^d\right)$.

Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(n^d\right)$

Splitting/combining:

$\log_b n$ levels

$$
\left\{
\begin{array}{rll}
f(n) & \leq cn^{d-\epsilon} & \\
af(n/b) & \leq ac\left(\frac{n}{b}\right)^{d-\epsilon} & = cn^{d-\epsilon} \cdot ab^{\epsilon-d} \\
a^2 f(n/b^2) & \leq a^2 c\left(\frac{n}{b^2}\right)^{d-\epsilon} & = cn^{d-\epsilon} \cdot \left(ab^{\epsilon-d}\right)^2 \\
\ldots & &
\end{array}
\right.
$$

$\exists n_0 > 0$ and $\exists c > 0$
such that $f(n) \leq cn^{d-\epsilon}$

Just need to show that
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Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(n^d\right)$

Splitting/combining:

$\log_b n$ levels

$$
\left\{
\begin{array}{rllll}
f(n) & \leq cn^{d-\epsilon} & & & \\
af(n/b) & \leq ac\left(\frac{n}{b}\right)^{d-\epsilon} & = cn^{d-\epsilon} \cdot ab^{\epsilon-d} & & = cn^{d-\epsilon} \cdot b^{\epsilon} \\
a^2 f(n/b^2) & \leq a^2 c\left(\frac{n}{b^2}\right)^{d-\epsilon} & = cn^{d-\epsilon} \cdot \left(ab^{\epsilon-d}\right)^2 & & = cn^{d-\epsilon} \cdot b^{2\epsilon} \\
\ldots & & & &
\end{array}
\right.
$$

$\exists n_0 > 0$ and $\exists c > 0$
such that $f(n) \leq cn^{d-\epsilon}$

$b^d = a$

Just need to show that this part is $O\left(n^d\right)$.

Critical exponent: $d = \log_b a$

Case 1: $f(n) \in O\left(n^{d-\epsilon}\right)$ for some $\epsilon > 0$.

$\bigtriangledown$

Goal: $T(n) \in \Theta\left(n^d\right)$

Proof of the master theorem

Splitting/combining:

$\log_b n$ levels

$$f(n) \le cn^{d-\epsilon}$$

$$af(n/b) \le ac\left(\frac{n}{b}\right)^{d-\epsilon} = cn^{d-\epsilon} \cdot ab^{\epsilon-d} = cn^{d-\epsilon} \cdot b^{\epsilon}$$

$$a^2 f(n/b^2) \le a^2 c\left(\frac{n}{b^2}\right)^{d-\epsilon} = cn^{d-\epsilon} \cdot \left(ab^{\epsilon-d}\right)^2 = cn^{d-\epsilon} \cdot b^{2\epsilon}$$

...

$\exists n_0 > 0$ and $\exists c > 0$ such that $f(n) \le cn^{d-\epsilon}$

$b^d = a$

Just need to show that this part is $O\left(n^d\right)$.

Overall cost:

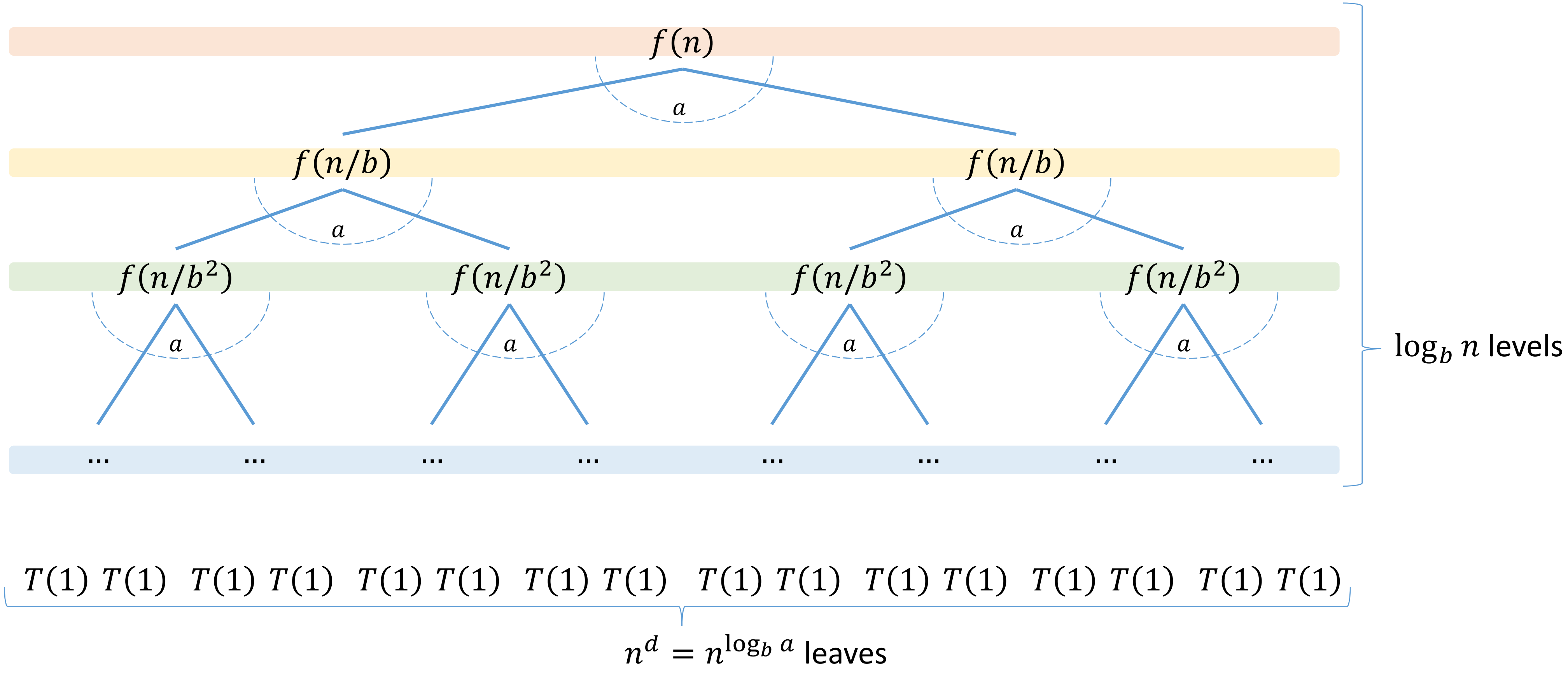
$$O\left(n^{d-\epsilon}\right) \cdot \underbrace{(1 + b^{\epsilon} + b^{2\epsilon} + \cdots)}_{\log_b n \text{ terms}} = O\left(n^{d-\epsilon}\right) \cdot O\left(b^{\epsilon \log_b n}\right) = O\left(n^{d-\epsilon}\right) \cdot O\left(n^{\epsilon}\right) = O\left(n^d\right)$$

Critical exponent: $d = \log_b a$

Case 2: $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

Goal: $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Proof of the master theorem



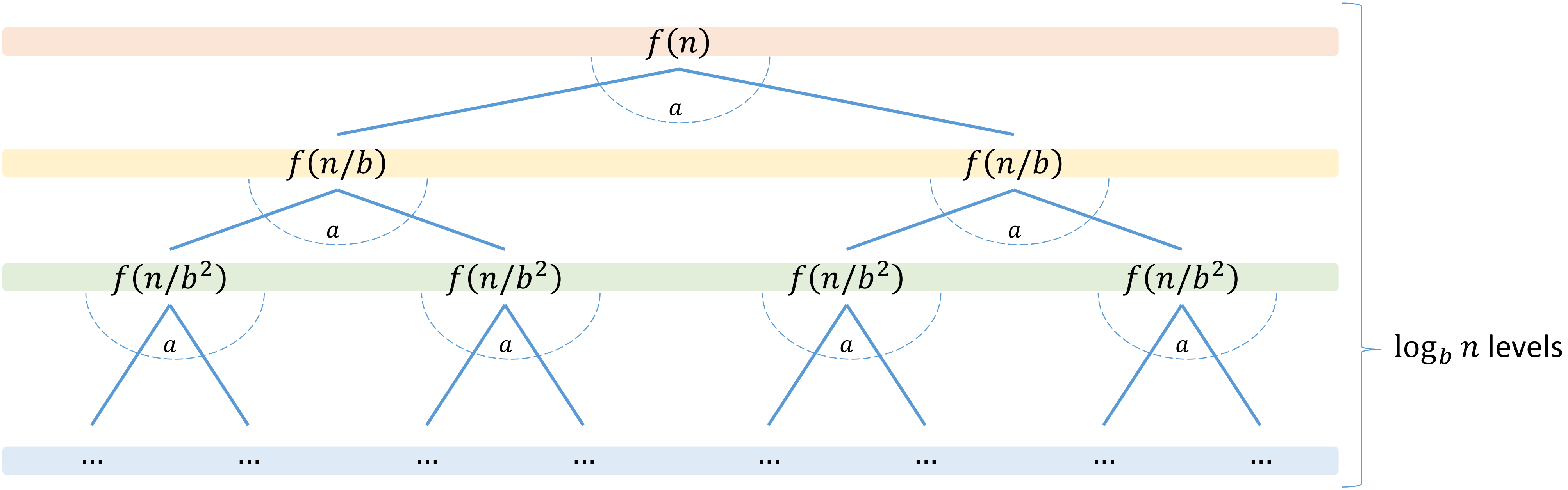
Critical exponent: $d = \log_b a$

Case 2: $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

$\bigtriangledown$

Goal: $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Proof of the master theorem



$\underbrace{T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)}_{n^d = n^{\log_b a} \text{ leaves}}$

Solving the base cases: The cost is $n^d T(1) \in o\left(n^d \log^{k+1} n\right)$.

Critical exponent: $d = \log_b a$

Case 2: $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

Proof of the master theorem

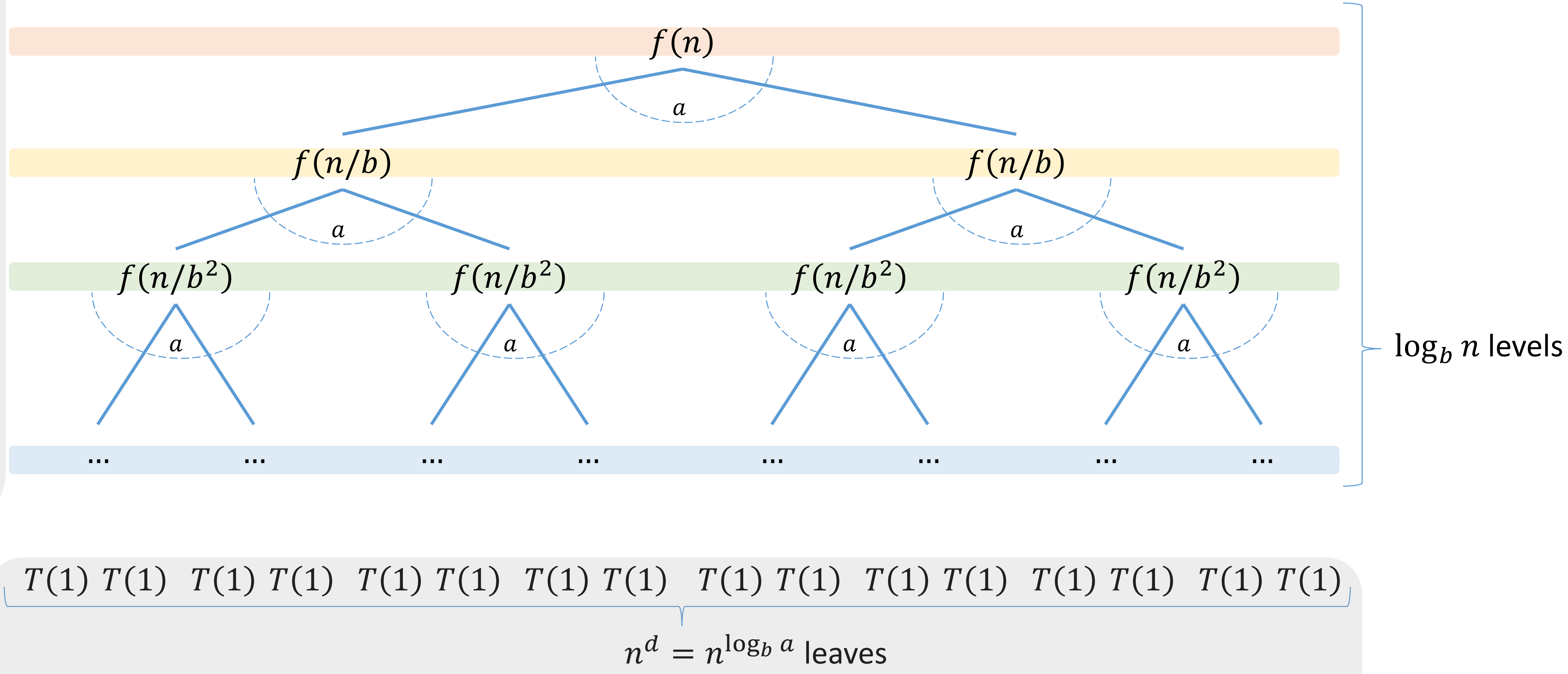
Goal: $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Splitting/combining:

$f(n)$

$af(n/b)$

$a^2 f(n/b^2)$



Need to show that this part is $\Theta\left(n^d \log^{k+1} n\right)$.

Solving the base cases: The cost is $n^d T(1) \in o\left(n^d \log^{k+1} n\right)$.

Critical exponent: $d = \log_b a$

Case 2: $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

Goal: $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Proof of the master theorem

Splitting/combining:

$$\log_b n \text{ levels} \begin{cases} f(n) & \in \Theta(n^{\log_b a} \log^k n) = \Theta\left(n^d \log^k n\right) \\ af(n/b) & \in a \cdot \Theta\left(\left(\frac{n}{b}\right)^{\log_b a} \log^k \frac{n}{b}\right) = \Theta\left(n^{\log_b a} \log^k \frac{n}{b}\right) = \Theta\left(n^d \log^k \frac{n}{b}\right) \\ a^2 f(n/b^2) & \in a^2 \cdot \Theta\left(\left(\frac{n}{b^2}\right)^{\log_b a} \log^k \frac{n}{b^2}\right) = \Theta\left(n^{\log_b a} \log^k \frac{n}{b^2}\right) = \Theta\left(n^d \log^k \frac{n}{b^2}\right) \\ & \vdots \\ \ldots & \end{cases}$$

Need to show that this part is $\Theta\left(n^d \log^{k+1} n\right)$.

Critical exponent: $d = \log_b a$

Case 2: $f(n) \in \Theta\left(n^d \log^k n\right)$ for some $k \geq 0$.

Proof of the master theorem

Goal: $T(n) \in \Theta\left(n^d \log^{k+1} n\right)$

Splitting/combining:

$\log_b n$ levels

$$f(n) \in \Theta(n^{\log_b a} \log^k n) = \Theta\left(n^d \log^k n\right)$$

$$af(n/b) \in a \cdot \Theta\left(\left(\frac{n}{b}\right)^{\log_b a} \log^k \frac{n}{b}\right) = \Theta\left(n^{\log_b a} \log^k \frac{n}{b}\right) = \Theta\left(n^d \log^k \frac{n}{b}\right)$$

$$a^2 f(n/b^2) \in a^2 \cdot \Theta\left(\left(\frac{n}{b^2}\right)^{\log_b a} \log^k \frac{n}{b^2}\right) = \Theta\left(n^{\log_b a} \log^k \frac{n}{b^2}\right) = \Theta\left(n^d \log^k \frac{n}{b^2}\right)$$

$\vdots$

...

$$\log^k n + \log^k \frac{n}{b} + \log^k \frac{n}{b^2} + \cdots = (\log n)^k + (\log n - \log b)^k + (\log n - 2\log b)^k + \cdots \in \Theta\left((\log n)^{k+1}\right)$$

Need to show that this part is $\Theta\left(n^d \log^{k+1} n\right)$.

$$\textbf{Overall cost: } \Theta\left(n^d\right) \cdot \underbrace{\left(\log^k n + \log^k \frac{n}{b} + \log^k \frac{n}{b^2} + \cdots\right)}_{\log_b n \text{ terms, half of them being at least } \left(\frac{1}{2}\right)^k \log^k n} = \Theta(n^d \log^{k+1} n)$$

Floors and ceilings

- In the master theorem, floors and ceilings within the recursive subproblem sizes do not affect the asymptotic growth of the function.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ 2T\left(\frac{n}{2}\right) + \Theta(n) & \text{if } n > 1 \end{cases} \qquad T(n) = \begin{cases} \Theta(1) & \text{if } n \le 1 \\ T\left(\left\lceil\frac{n}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + \Theta(n) & \text{if } n > 1 \end{cases}$$

- **Optional readings:**
  - Section 4.7 of CLRS 4e "Akra–Bazzi recurrences."
  - William Kuszmaul and Charles E. Leiserson. "Floors and Ceilings in Divide-and-Conquer Recurrences." Symposium on Simplicity in Algorithms (SOSA 2021). https://epubs.siam.org/doi/10.1137/1.9781611976496.15

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  - Wing-Kin Sung