1. (a) $T(n) = 6T(n/3) + n^2$

Use Master theorem:

a.1> $a = 6$. $b = 3$. $d = \log_b a = \log_3 6 < \log_3 9 = 2$.

a.2> $f(n) = n^2$ suppose $\varepsilon_0 = \log_3 8 - \log_3 6$ $\Rightarrow d + \varepsilon_0 = \log_3 8 < \log_3 9$

Then we have $\exists \varepsilon_0 > 0$, s.t. $f(n) \in \Omega(n^{d+\varepsilon_0})$

a.3>. suppose $c = 1$ Then $af(n/b) = \frac{2}{3}n^2 < 1 \cdot n^2 = c \cdot f(n)$

With a.2> and a.3>, the conditions meet the requirements.

We have $T(n) \in \Theta(f(n)) \Rightarrow T(n) \in \Theta(n^2)$

(b) $T(n) = 9T(n/2) + 6n^3 + 4$

Use master theorem:

$a = 9$. $b = 2$. $f(n) = 6n^3 + 4$ $d = \log_b a = \log_2 9 > \log_2 8 = 3$

suppose $\varepsilon_0 = \log_2 9 - \log_2 8.5$, then $n^{d-\varepsilon_0} = n^{\log_2 8.5} > n^{\log_2 8} = n^3$

$\Rightarrow \exists \varepsilon_0 > 0$, s.t. $f(n) \in O(n^{d-\varepsilon_0})$ $\Rightarrow$ meets the case 1 requirement.

Then $T(n) \in \Theta(n^d) \Rightarrow T(n) \in \Theta(n^{\log_2 9})$

(c) $T(n) = T(n/7) + T(6n/7) + 5$

suppose $\forall n > 0$ $T(n) \le C_1 n - 5$

$T(n) = T(n/7) + T(\frac{6}{7}n) + 5$

$\le \frac{1}{7}C_1 n - 5 + \frac{6}{7}C_1 n - 5 + 5$

$= C_1 n - 5$ upper bound is shown. $T(n) \in O(n)$

suppose: $\forall n > 0$ $T(n) \ge C_2 n + 5$

$T(n) = T(n/7) + T(\frac{6}{7}n) + 5$

$\ge \frac{1}{7}C_2 n + 5 + \frac{6}{7}C_2 n + 5 + 5$

$= C_2 n + 15 \ge C_2 n + 5$ lower bound is shown. $T(n) \in \Omega(n)$

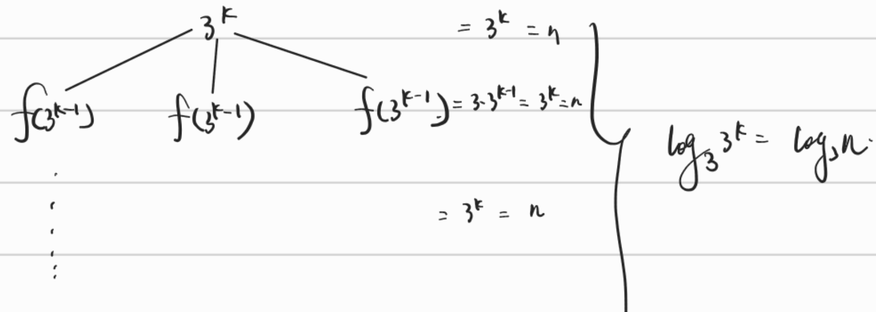
$\therefore$ $T(n) \in O(n)$ $T(n) \in \Omega(n)$

$\Rightarrow T(n) \in \Theta(n)$

2. $n = 3^k$ it's obvious: $f(3^k) = 3f(3^{k-1}) + 3^k$.

$f(3^k) = 3f(3^{k-1}) + 3^k$

$f(3^{k-1}) = 3f(3^{k-2}) + 3^{k-2}$.



$$f(3^1) \qquad \cdots\cdots \qquad f(3^1)$$

$$\underbrace{f(3^0)\,f(3^0)\,f(3^0) \cdots\cdots f(3^0)\,f(3^0)\,f(3^0)}_{3^k} = 0 \quad \text{(not including this line is our tree)}$$

the sum of every level $= 3^k = n$, The tree has $\log_3 n$ levels.

Then the total sum is $n \cdot \log_3 n$.

$\Rightarrow f(n) = n \log_3 n$.

3. assume computing "n mod m"'s Time is C_0

$T(m, n) = T(n \bmod m, m) + C$

suppose $n \geq m$. $\Rightarrow$ $n \bmod m < m$.

case 1: $m \leq n/2 \Rightarrow n \bmod m < m \leq n/2$. the bigger number m. in next level is smaller than $\frac{n}{2}$.

case 2: $m > n/2 \Rightarrow n \bmod m = n - m < n/2$, and the next time $m \bmod (n \bmod m) = m \bmod (n-m)$

$n - m < \frac{n}{2}$. the $m \bmod (n-m) < \frac{n}{2}$.

Then, we can comfirm. after two operations, the larger number is at least halved.

The sum of operations is at most $2 \cdot \log_2 n$.

The time is at most $C \cdot 2 \cdot \log_2 n$.

$\Rightarrow T(GCD(m, n)) = O(\log n)$