CS3230 – Design and Analysis of Algorithms (S2 AY2024/25)

Lecture 4a: Lower Bound for Comparison-Based Sorting

Sorting

- **Input:** an array $A = (a_1, a_2, \dots, a_n)$ of elements.
- **Goal:** Sort the elements in A in non-decreasing order.
  - A permutation $(a'_1, a'_2, \dots, a'_n)$ of A such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Examples:

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- Selection sort
- Merge sort
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Worst-case time complexity

Examples:

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We will restrict our attention to a certain class of algorithms.

Comparison-based sorting

- **Comparison-based** algorithms:
  - Elements can only be compared with each other:
    - $<$, $\leq$, $=$, $>$, $\geq$
  - No other information of the elements can be used.

All of them are comparison-based.

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Allowed

If $(A[i] < A[j])$, **then** { Do some work }

If $(A[i] = A[j])$, **then** { Do some work }

Not allowed

If $(A[i] + A[j] = A[k])$, **then** { Do some work }

If $(A[i] = k)$, **then** { Do some work }

If ($A[i]$ is odd), **then** { Do some work }

If (the jth bit of $A[i]$ is 1), **then** { Do some work }

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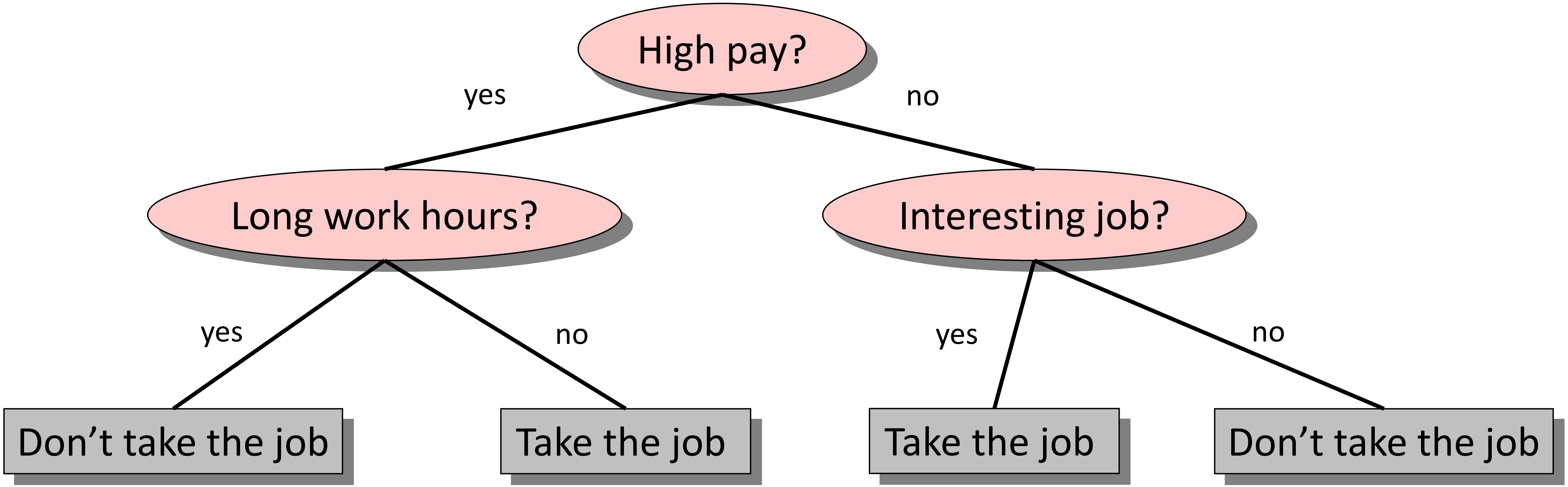
Merge sort and heap sort are **asymptotically optimal**!

Decision trees

- The proof of the theorem uses **decision trees**.



Decision trees

- A **decision tree** is a rooted tree.
  - Start from the root.
  - At every vertex, a question is asked.
  - Depending on the answer, a child is chosen.
  - At a leaf, a decision is taken.

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- A **decision tree** is a rooted tree.
  - Start from the root.
  - At every vertex, a question is asked.
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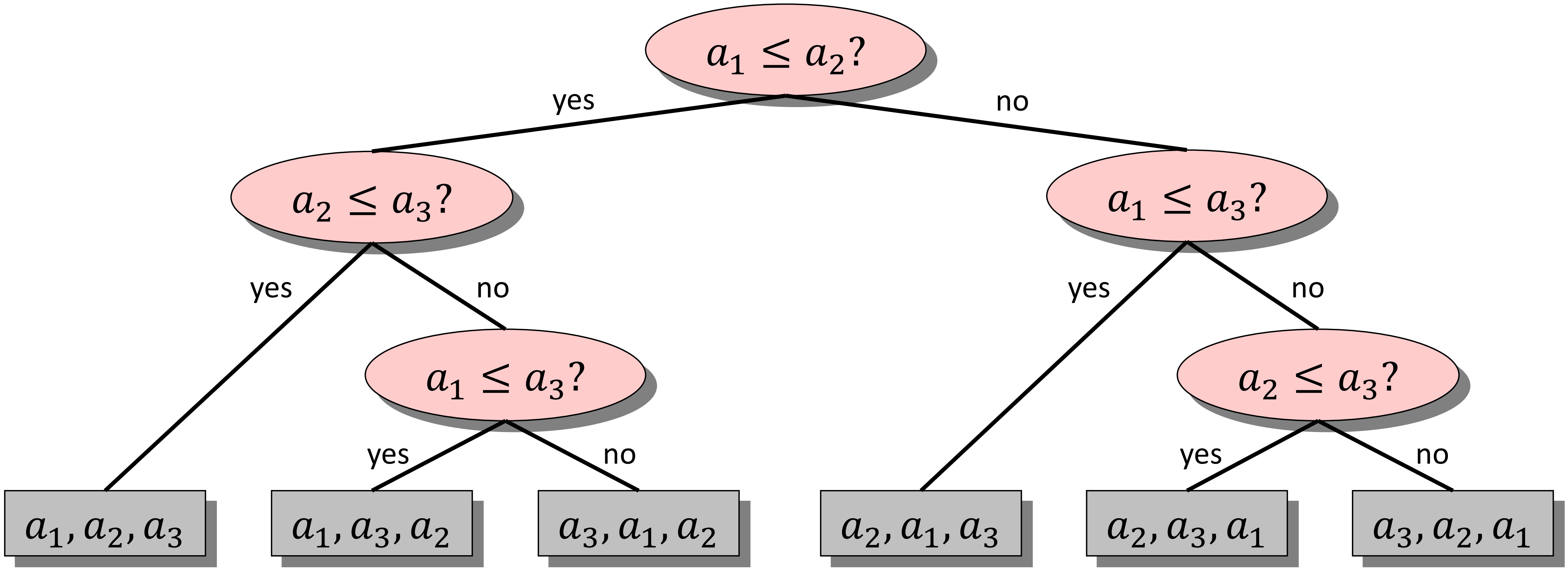
  - At a leaf, a decision is taken.
- Any comparison-based algorithm can be modeled using a decision tree:
  - A comparison $\longleftrightarrow$ A question asked at a node.
  - Program state depends on the result of the comparison $\longleftrightarrow$ Chosen child depends on the answer to the question.
  - $\underbrace{\text{Output of the algorithm}}_{\text{A permutation } (a'_1, a'_2, \ldots, a'_n) \text{ of } A}$ $\longleftrightarrow$ Decision at a leaf.

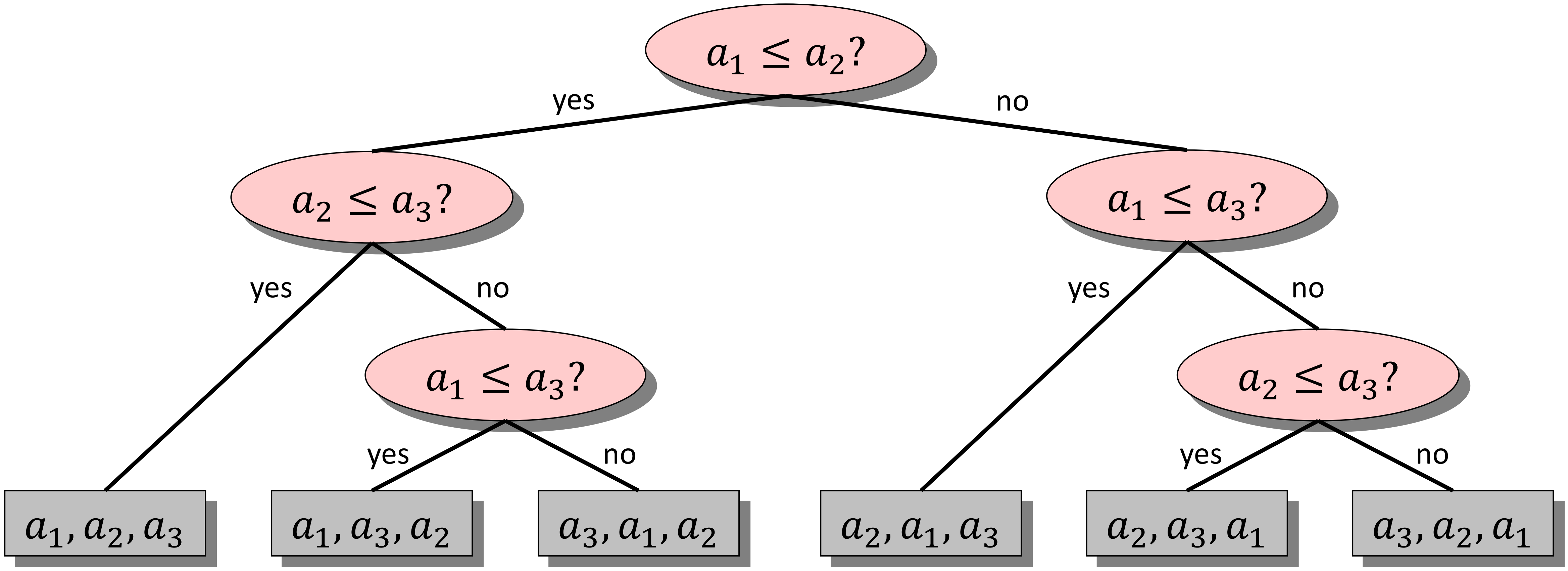
An example

- A comparison-based algorithm for sorting $A = (a_1, a_2, a_3)$.



An example

Worst-case running time $\geq$ worst-case number of comparisons $=$ height of the tree



Proof of the theorem

Theorem: The worst-case time complexity of any comparison-based sorting algorithm is $\Omega(n \log n)$.

Proof:

- Model the algorithm as a decision tree, which is a binary tree with at least $n!$ leaves:
  - Each permutation is a possible answer.
- The height of the binary tree is at least $\log(n!)$.

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- The height of the binary tree is at least $\log(n!)$.

$$\log(n!) \in n \log n - n \log e + O(\log n) \subseteq \Theta(n \log n)$$

Stirling's approximation
https://en.wikipedia.org/wiki/Stirling%27s_approximation

Question

- Is the following claim **true** or **false**?

> There exists a comparison-based sorting algorithm that can sort any 5-element array using at most 6 comparisons.

Answer

- Is the following claim **true** or **false**?

There exists a comparison-based sorting algorithm that can sort any 5-element array using at most 6 comparisons.

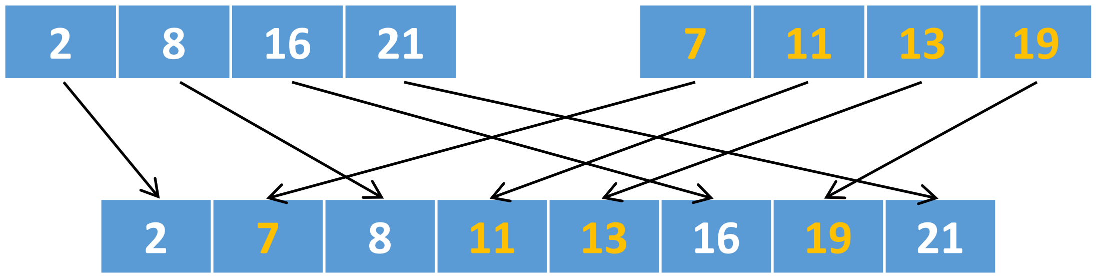
False:

- There are $5! = 120$ permutations of 5 elements.
- In any binary tree of height at most 6, the number of leaves is at most $2^6 = 64$.
- $120 > 64$, so 6 comparisons are not enough.

Question

Input: k sorted arrays $A_1[1..n], A_2[1..n], \ldots, A_k[1..n]$.

Goal: Merge the k sorted arrays into one sorted array of length kn.



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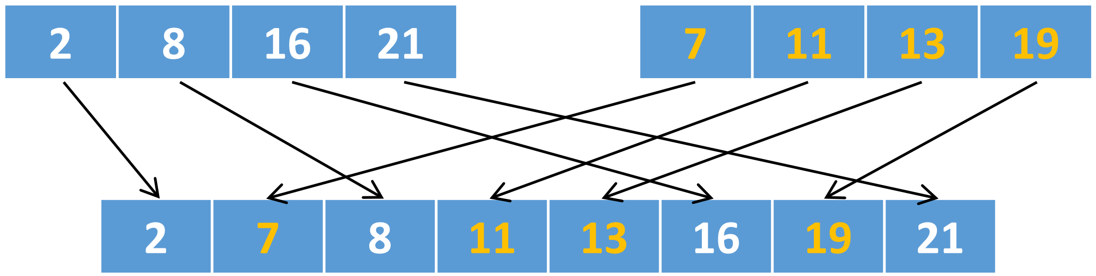
Goal: Merge the k sorted arrays into one sorted array of length kn.

Question: What is a tight lower bound of the worst-case running time for comparison-based algorithms for this task?

- $\Omega(kn)$
- $\Omega(kn \log k)$
- $\Omega(kn \log n)$
- $\Omega(k^2 n)$



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- As the k arrays are sorted, the number of possible ways to combine them is $\frac{(kn)!}{(n!)^k}$.
- Therefore, any comparison-based algorithm requires at least $\log \frac{(kn)!}{(n!)^k}$ comparisons.

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$$\log\left(\frac{(kn)!}{(n!)^k}\right) = \log\big((kn)!\big) - k\log(n!)$$
$$\in \big(kn\log(kn) - kn\log e + O(\log kn)\big) - k\big(n\log n - n\log e + O(\log n)\big) \leq kn\log k + O(\log kn)$$
$$\subseteq \Theta(kn\log k)$$

$$\log(n!) \in n\log n - n\log e + O(\log n)$$

Non-comparison sorts

Question: Can we bypass the $\Omega(n \log n)$ lower bound by an algorithm that is not comparison-based?

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Suppose each element in the array A belongs to the range $\{1,2,\dots,k\}$.

$\mathbf{CountingSort}(A)$

- For all $i \in \{1,2,\dots,k\}$, compute $\mathbf{count}_i =$ the number of appearances of i in A.
- Set the initial $\mathbf{count}_1$ entries of A to be 1.
- Set the next $\mathbf{count}_2$ entries of A to be 2.
- Set the next $\mathbf{count}_3$ entries of A to be 3.
- …

Exercise: Show that the algorithm can be implemented to finish in $O(n+k)$ time.

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