CS3230 – Design and Analysis of Algorithms (S2 AY2024/25)

Lecture 3a: Proof of Correctness

Correctness of an algorithm

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.
    - Iterative algorithms.
    - Recursive algorithms.

```
$\mathbf{Fib}(n)$
• If $n \leq 1$, return $n$.
• Else, return $\mathbf{Fib}(n-1) + \mathbf{Fib}(n-2)$.
```

```
$\mathbf{IFib}(n)$
• If $n \leq 1$
    • return $n$
• Else,
    • prev2 $=$ 0
    • prev1 $=$ 1
    • for $i$ $=$ 2 to $n$
        • temp $=$ prev1
        • prev1 $=$ prev1+prev2
        • prev2 $=$ temp
    • return prev1
```

```
While ( some condition is met )
    • Do { some work }
```

```
For ($i = 1, 2, \ldots$ up to $i = n$)
    • Do { some work }
```

Iterative algorithms

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.
- Analysis of an iterative algorithm:
  - **Loop invariant:**
    - Some desirable conditions that should be satisfied at the start of each iteration.

Iterative algorithms

```
**While** ( some condition is met )
  • **Do** { some work }
```

```
**For** ($i = 1, 2, \ldots$ up to $i = n$)
  • **Do** { some work }
```

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of an iterative algorithm:
  - **Loop invariant:**
    - Some desirable conditions that should be satisfied at the start of each iteration.
  - **Initialization:**
    - The loop invariant is true at the start of the first iteration.

Iterative algorithms

```
**While** ( some condition is met )
  • **Do** { some work }
```

```
**For** ($i = 1, 2, \ldots$ up to $i = n$)
  • **Do** { some work }
```

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of an iterative algorithm:
  - **Loop invariant:**
    - Some desirable conditions that should be satisfied at the start of each iteration.
  - **Initialization:**
    - The loop invariant is true at the start of the first iteration.
  - **Maintenance**:
    - If the loop invariant is satisfied at the start of the current iteration, then the loop invariant must be satisfied at the start of the next iteration.

Equivalently, the end of the current iteration (be careful of index value)

Iterative algorithms

```
**While** ( some condition is met )
• **Do** { some work }
```

```
**For** ($i = 1, 2, \dots$ up to $i = n$)
• **Do** { some work }
```

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of an <u>iterative algorithm</u>:
  - **Loop invariant:**
    - Some desirable conditions that should be satisfied at the start of each iteration.
  - **Initialization:**
    - The loop invariant is true at the start of the first iteration.
  - **Maintenance**:
    - If the loop invariant is satisfied at the start of the current iteration, then the loop invariant must be satisfied at <u>the start of the next iteration</u>.

Equivalently, the end of the current iteration (be careful of index value)

  - **Termination**:
    - Loop invariant at the end of the last iteration → The algorithm outputs a correct answer.

While (some condition is met)
- **Do** { some work }

For $(i = 1, 2, \ldots \text{ up to } i = n)$
- **Do** { some work }

Iterative algorithms

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of an <u>iterative algorithm</u>:
  - **Loop invariant:** Induction hypothesis
    - Some desirable conditions that should be satisfied at the start of each iteration.
  - **Initialization:** Base case
    - The loop invariant is true at the start of the first iteration.
  - **Maintenance**: Inductive step
    - If the loop invariant is satisfied at the start of the current iteration, then the loop invariant must be satisfied at the start of the next iteration.
  - **Termination**: The proof by induction implies the correctness of the algorithm
    - Loop invariant at the end of the last iteration → The algorithm outputs a correct answer.

Fibonacci numbers

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

```
IFib(n)
• If $n \le 1$
    • return $n$
• Else,
    • prev2 $= 0$
    • prev1 $= 1$
    • for $i = 2$ to $n$
        • temp $=$ prev1
        • prev1 $=$ prev1$+$prev2
        • prev2 $=$ temp
    • return prev1
```

If $n \le 1$, then the algorithm is correct.

- $\mathbf{Fib}(0) = 0$
- $\mathbf{Fib}(1) = 1$

Fibonacci numbers

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

```
IFib(n)
• If $n \leq 1$
    • return $n$
• Else,
    • prev2 $= 0$
    • prev1 $= 1$
    • for $i = 2$ to $n$
        • temp $=$ prev1
        • prev1 $=$ prev1$+$prev2
        • prev2 $=$ temp
    • return prev1
```

Now, consider the case of $n \geq 2$.

Loop invariant:

- At the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$

Fibonacci numbers

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

```
IFib(n)
• If $n \leq 1$
    • return $n$
• Else,
    • prev2 $= 0$
    • prev1 $= 1$
    • for $i = 2$ to $n$
        • temp $=$ prev1
        • prev1 $=$ prev1$+$prev2
        • prev2 $=$ temp
    • return prev1
```

Now, consider the case of $n \geq 2$.

Loop invariant:

- At the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$

Initialization:

- At the start of iteration $i = 2$,
  - prev2 $= \mathbf{Fib}(0) = 0$
  - prev1 $= \mathbf{Fib}(1) = 1$

Fibonacci numbers

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

```
IFib(n)
• If $n \leq 1$
    • return $n$
• Else,
    • prev2 $= 0$
    • prev1 $= 1$
    • for $i = 2$ to $n$
$\mathbf{Fib}(i-1)$    • temp = prev1    $\mathbf{Fib}(i-1)$
$\mathbf{Fib}(i)$      • prev1 = prev1+prev2    $\mathbf{Fib}(i-1) + \mathbf{Fib}(i-2)$
$\mathbf{Fib}(i-1)$    • prev2 = temp    $\mathbf{Fib}(i-1)$
    • return prev1
```

Now, consider the case of $n \geq 2$.

Loop invariant:
- At the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$

Initialization:
- At the start of iteration $i = 2$,
  - prev2 $= \mathbf{Fib}(0) = 0$
  - prev1 $= \mathbf{Fib}(1) = 1$

Maintenance:
- Suppose at the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$
- Then at the end of the iteration,
  - prev2 $= \mathbf{Fib}(i-1)$
  - prev1 $= \mathbf{Fib}(i)$

Fibonacci numbers

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

```
IFib(n)
• If n ≤ 1
    • return n
• Else,
    • prev2 = 0
    • prev1 = 1
    • for i = 2 to n
        • temp = prev1
        • prev1 = prev1+prev2
        • prev2 = temp
    • return prev1
```

Now, consider the case of $n \geq 2$.

Loop invariant:

- At the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$

Initialization:

- At the start of iteration $i = 2$,
  - prev2 $= \mathbf{Fib}(0) = 0$
  - prev1 $= \mathbf{Fib}(1) = 1$

Maintenance:

- Suppose at the start of iteration i,
  - prev2 $= \mathbf{Fib}(i-2)$
  - prev1 $= \mathbf{Fib}(i-1)$ $i = n$
- Then at the end of the iteration,
  - prev2 $= \mathbf{Fib}(i-1)$
  - prev1 $= \mathbf{Fib}(i)$

Termination:

- The algorithm returns $\mathbf{Fib}(n)$.

Question

- What is a suitable loop invariant to analyze this sorting algorithm?

  at the start of iteration j

  - A is sorted.
  - $A[1..j-1]$ is sorted.
  - $x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$.
  - ($A[1..j-1]$ is sorted) and ($x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$).

```
**SelectionSort**$(A[1..n])$
• For $j = 1$ to $n - 1$
    • Select $s \in \{j, j+1, \dots, n\}$ so that $A[s]$ is a smallest number in $A[j..n]$.
    • Swap $A[j]$ and $A[s]$.
```

VisuAlgo (Selection sort): https://visualgo.net/en/sorting?mode=Selection

Answer

- What is a suitable loop invariant to analyze this sorting algorithm?

  ✗ A is sorted. ← ✗ Initialization ← In general, A is not sorted at the start of the algorithm.
  ✓ Maintenance
  ✓ Termination

  - $A[1..j-1]$ is sorted.
  - $x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$.
  - ($A[1..j-1]$ is sorted) and ($x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$).

```
**SelectionSort**($A[1..n]$)
• For $j = 1$ to $n - 1$
    • Select $s \in \{j, j+1, \dots, n\}$ so that $A[s]$ is a smallest number in $A[j..n]$.
    • Swap $A[j]$ and $A[s]$.
```

VisuAlgo (Selection sort): https://visualgo.net/en/sorting?mode=Selection

Answer

- What is a suitable loop invariant to analyze this sorting algorithm?
  - ✗ A is sorted.
  - ✗ $A[1..j-1]$ is sorted. ← ✓ Initialization ✗ Maintenance ← Maintenance fails if $A[j-1]$ is larger than the smallest number in $A[j..n]$. ✓ Termination
  - $x \le y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$.
  - ($A[1..j-1]$ is sorted) and ($x \le y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$).

```
**SelectionSort**($A[1..n]$)
• For $j = 1$ to $n - 1$
    • Select $s \in \{j, j+1, \dots, n\}$ so that $A[s]$ is a smallest number in $A[j..n]$.
    • Swap $A[j]$ and $A[s]$.
```

VisuAlgo (Selection sort): https://visualgo.net/en/sorting?mode=Selection

Answer

- What is a suitable loop invariant to analyze this sorting algorithm?

  ✗ A is sorted.

  ✗ $A[1..j-1]$ is sorted.

  ✗ $x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$. $\leftarrow$ ✓ Initialization ✓ Maintenance ✗ Termination $\leftarrow$ The final loop invariant is vacuously true.

  - ($A[1..j-1]$ is sorted) and ($x \leq y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$).

```
SelectionSort($A[1..n]$)
• For $j = 1$ to $n - 1$
    • Select $s \in \{j, j+1, \ldots, n\}$ so that $A[s]$ is a smallest number in $A[j..n]$.
    • Swap $A[j]$ and $A[s]$.
```

VisuAlgo (Selection sort): https://visualgo.net/en/sorting?mode=Selection

Answer

- What is a suitable loop invariant to analyze this sorting algorithm?

  ✗ A is sorted.

  ✗ $A[1..j-1]$ is sorted.

  ✗ $x \le y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$.

  ✓ ($A[1..j-1]$ is sorted) and ($x \le y$ for all $x \in A[1..j-1]$ and $y \in A[j..n]$). ←

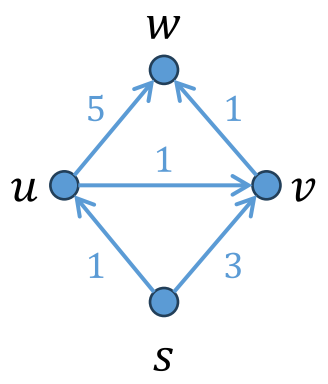
Exercise:
✓ Initialization
✓ Maintenance
✓ Termination

```
**SelectionSort**($A[1..n]$)
• For $j = 1$ to $n - 1$
    • Select $s \in \{j, j+1, \dots, n\}$ so that $A[s]$ is a smallest number in $A[j..n]$.
    • Swap $A[j]$ and $A[s]$.
```

VisuAlgo (Selection sort): https://visualgo.net/en/sorting?mode=Selection

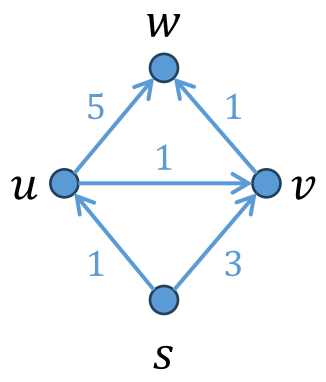
Weighted directed graphs

- Let $G = (V, E)$ be a directed graph.
- Each edge $e \in E$ has a positive weight $w(e)$.
    - If $(u, v) \in E$, then $w(u, v) = w(e)$, where $e = (u, v)$.
    - If $(u, v) \notin E$, then $w(u, v) = \infty$.



Single-source shortest paths

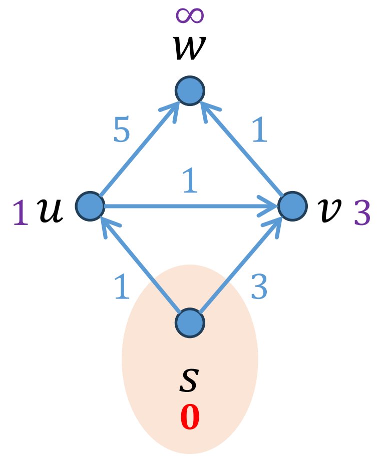
- Let $G = (V, E)$ be a directed graph.
- Each edge $e \in E$ has a positive weight $w(e)$.
  - If $(u, v) \in E$, then $w(u, v) = w(e)$, where $e = (u, v)$.
  - If $(u, v) \notin E$, then $w(u, v) = \infty$.
- **Goal:** Given a source $s \in V$, compute the shortest-path distance from s to all vertices.



$\mathbf{dist}(s, s) = 0$
$\mathbf{dist}(s, u) = 1$
$\mathbf{dist}(s, v) = 2$
$\mathbf{dist}(s, w) = 3$

Dijkstra’s algorithm

```
**Dijkstra**$(G = (V, E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}(d(u) + w(u, v))$
    • $d(v) = \min_{u \in R}(d(u) + w(u, v))$
    • Add $v$ to $R$
```



$\mathbf{dist}(s, s) = 0$
$\mathbf{dist}(s, u) = 1$
$\mathbf{dist}(s, v) = 2$
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VisuAlgo (Dijkstra): https://visualgo.net/en/sssp?slide=7

Dijkstra’s algorithm

```
**Dijkstra**$(G = (V, E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u, v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u, v)\big)$
    • Add $v$ to $R$
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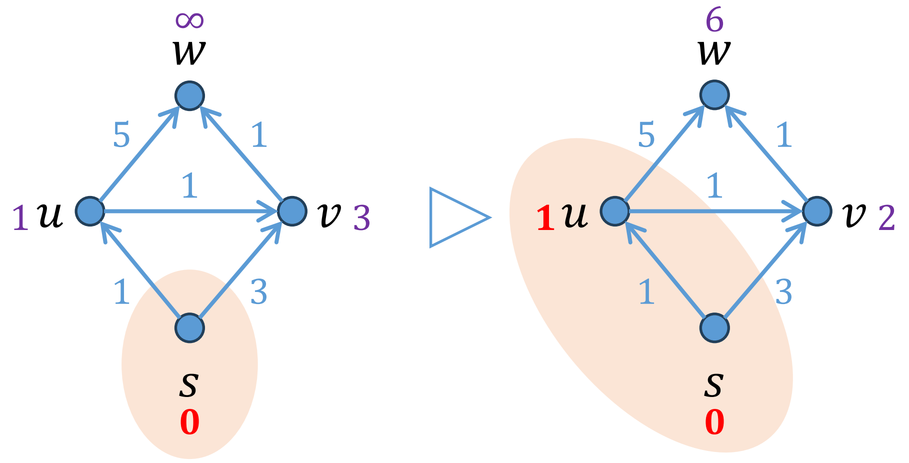

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VisuAlgo (Dijkstra): https://visualgo.net/en/sssp?slide=7

Dijkstra’s algorithm

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• $d(s) = 0$
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• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u, v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u, v)\big)$
    • Add $v$ to $R$
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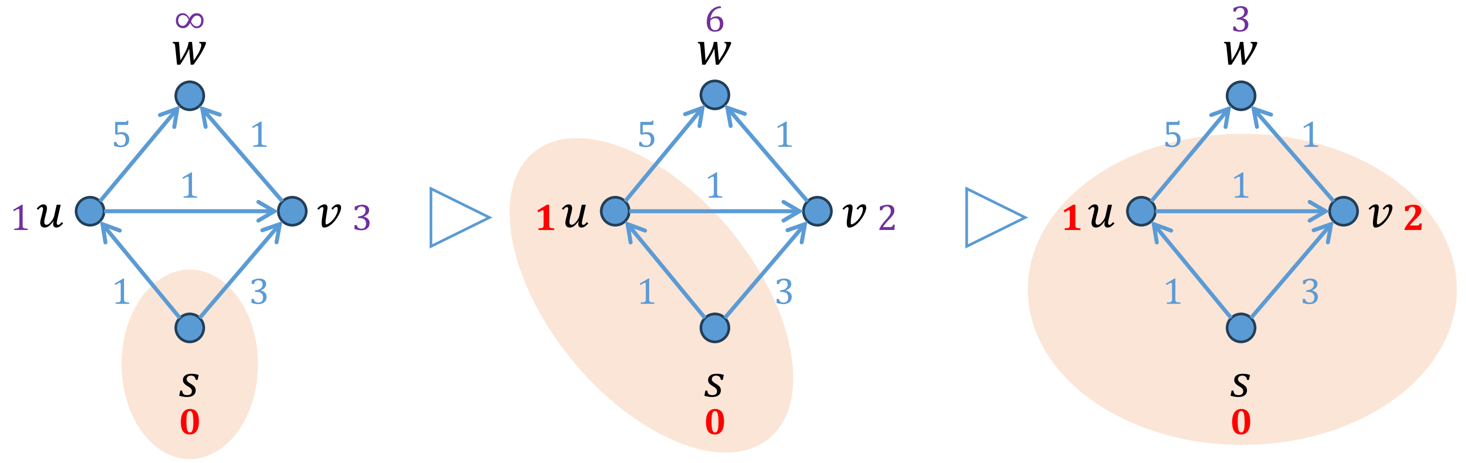

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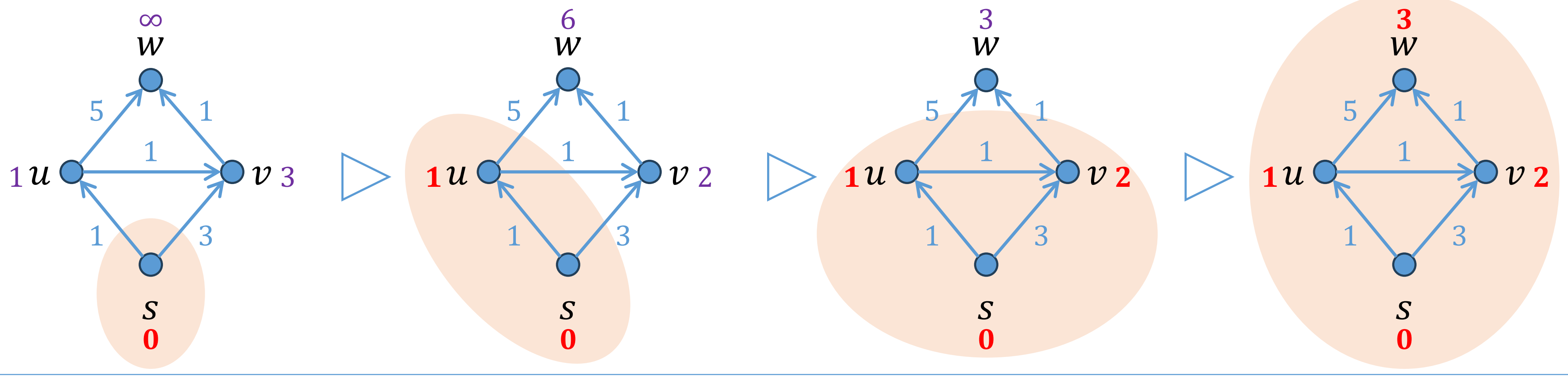
VisuAlgo (Dijkstra): https://visualgo.net/en/sssp?slide=7

Dijkstra's algorithm

```
**Dijkstra**$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}(d(u) + w(u,v))$
    • $d(v) = \min_{u \in R}(d(u) + w(u,v))$
    • Add $v$ to $R$
```



$\textbf{dist}(s,s) = 0$
$\textbf{dist}(s,u) = 1$
$\textbf{dist}(s,v) = 2$
$\textbf{dist}(s,w) = 3$

VisuAlgo (Dijkstra): https://visualgo.net/en/sssp?slide=7

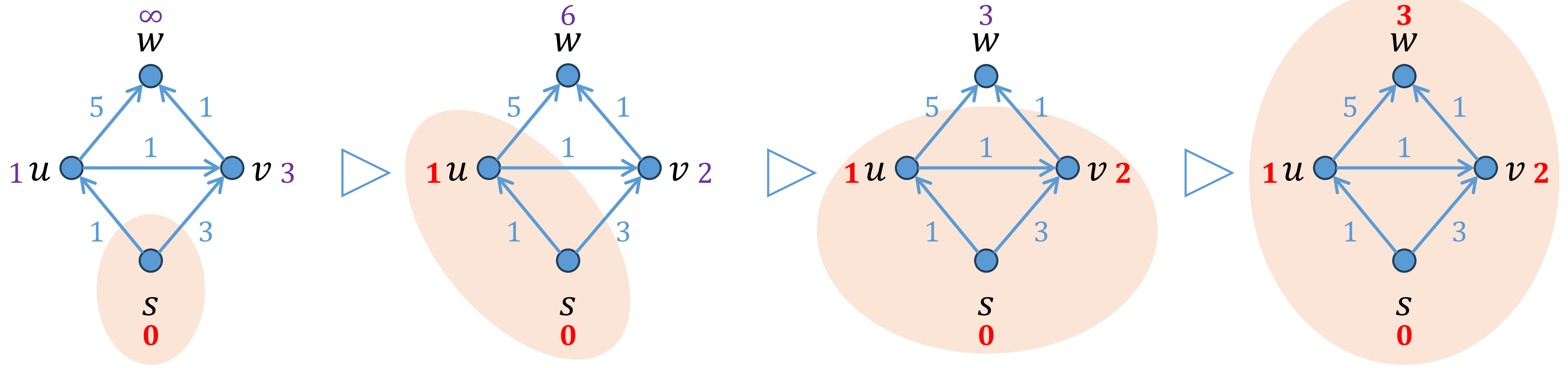
Loop invariant:

- At the start of an iteration:
  - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u\in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u\in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```



$\mathbf{dist}(s,s) = 0$
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VisuAlgo (Dijkstra): https://visualgo.net/en/sssp?slide=7

Loop invariant:

- At the start of an iteration:
    - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

$d(s) = 0$ is correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```

✓ **Initialization**

Maintenance

Termination

Loop invariant:

- At the start of an iteration:
    - $\forall u \in R,\ d(u) = \textbf{dist}(s,u)$

The computed distances are correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
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• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```

✓ Initialization

Maintenance

✓ **Termination**

At the end of the computation, $R = V$

Loop invariant:

- At the start of an iteration:
  - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```

Claim: For the selected vertex v:

- $\mathbf{dist}(s,v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$

✓ Initialization

Maintenance

✓ Termination

Proof of correctness

Loop invariant:
- At the start of an iteration:
  - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

```
Dijkstra(G = (V, E), s ∈ V)
• d(s) = 0
• R = {s}
• While R ≠ V
    • Select v ∈ V \ R to minimize min_{u∈R}(d(u) + w(u,v))
    • d(v) = min_{u∈R}(d(u) + w(u,v))
    • Add v to R
```

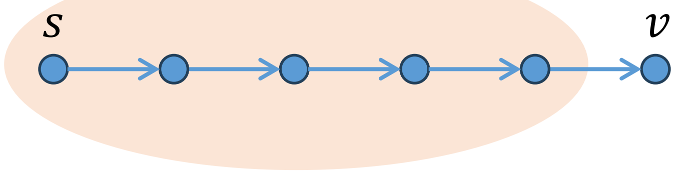
Claim: For the selected vertex v:
- $\mathbf{dist}(s,v) = \min_{u\in R}\big(d(u) + w(u,v)\big)$

✓ Initialization

Maintenance

✓ Termination

Proof: Consider a shortest path P from s to v.



All vertices in $P \setminus \{v\}$ must be in R.
- Otherwise, we should have selected the first vertex that is not in R.

Proof of correctness

Loop invariant:

- At the start of an iteration:
  - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

```
Dijkstra(G = (V, E), s ∈ V)
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u\in R}(d(u) + w(u,v))$
    • $d(v) = \min_{u\in R}(d(u) + w(u,v))$
    • Add $v$ to $R$
```

Claim: For the selected vertex v:

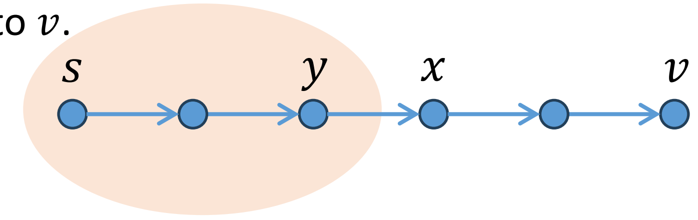
- $\mathbf{dist}(s,v) = \min_{u\in R}(d(u) + w(u,v))$

✓ Initialization

Maintenance

✓ Termination

Proof: Consider a shortest path P from s to v.



All vertices in $P \setminus \{v\}$ must be in R.

- Otherwise, we should have selected the $\underbrace{\text{first vertex}}_{x}$ that is not in R.

$$\min_{u\in R}(\underbrace{d(u)}_{\mathbf{dist}(s,u)} + w(u,v)) \geq \mathbf{dist}(s,v) > \underbrace{d(y)}_{\mathbf{dist}(s,y)} + w(y,x) \geq \min_{u\in R}(d(u) + w(u,x))$$

Loop invariant:
- At the start of an iteration:
  - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}(d(u) + w(u,v))$
    • $d(v) = \min_{u \in R}(d(u) + w(u,v))$
    • Add $v$ to $R$
```

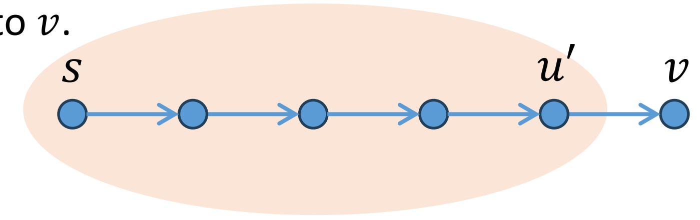
Claim: For the selected vertex v:
✓ $\mathbf{dist}(s,v) = \min_{u \in R}(d(u) + w(u,v))$

✓ Initialization

Maintenance

✓ Termination

Proof: Consider a shortest path P from s to v.



All vertices in $P \setminus \{v\}$ must be in R.

$$\min_{u \in R}\Big(\underbrace{d(u)}_{\mathbf{dist}(s,u)} + w(u,v)\Big) \geq \mathbf{dist}(s,v) = \underbrace{d(u')}_{\mathbf{dist}(s,u')} + w(u',v) \geq \min_{u \in R}(d(u) + w(u,v))$$

Loop invariant:

- At the start of an iteration:
    - $\forall u \in R,\ d(u) = \mathbf{dist}(s,u)$

The computed distances are correct.

Proof of correctness

```
Dijkstra$(G = (V,E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```

Claim: For the selected vertex v:

✓ $\mathbf{dist}(s,v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$

✓ Initialization

✓ **Maintenance**

✓ Termination

Question 2

Who is the **Master of Algorithms** pictured below?

- Stephen Cook
- Edsger Dijkstra
- Robert Tarjan
- Avi Wigderson

Answer

Edsger Dijkstra

- 1972 Turing Award.
- Advocacy of structured programming:
    - A programming paradigm that makes use of structured control flow as opposed to unstructured jumps to different sections.
    - “Go To statement considered harmful”

“Computer Science is no more about computers than astronomy is about telescopes.”
— Edsger Dijkstra

Source:
- https://en.wikipedia.org/wiki/Edsger_W._Dijkstra
- https://amturing.acm.org/award_winners/dijkstra_1053701.cfm

Answer

What is the shortest way to travel from Rotterdam to Groningen, in general: from given city to given city. It is the algorithm for the shortest path, which I designed in about twenty minutes. One morning I was shopping in Amsterdam with my young fiancée, and tired, we sat down on the café terrace to drink a cup of coffee and I was just thinking about whether I could do this, and I then designed the algorithm for the shortest path. As I said, it was a twenty-minute invention. In fact, it was published in '59, three years later. The publication is still readable, it is, in fact, quite nice. One of the reasons that it is so nice was that I designed it without pencil and paper. I learned later that one of the advantages of designing without pencil and paper is that you are almost forced to avoid all avoidable complexities. Eventually, that algorithm became to my great amazement, one of the cornerstones of my fame.

— Edsger Dijkstra, in an interview with Philip L. Frana, Communications of the ACM, 2001

Source:
- https://en.wikipedia.org/wiki/Dijkstra%27s_algorithm

Efficiency

```
**Dijkstra**$(G = (V, E), s \in V)$
• $d(s) = 0$
• $R = \{s\}$
• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R} \big(d(u) + w(u, v)\big)$
    • $d(v) = \min_{u \in R} \big(d(u) + w(u, v)\big)$
    • Add $v$ to $R$
```

Efficiency

```
Dijkstra($G = (V, E), s \in V$)
• $d(s) = 0$
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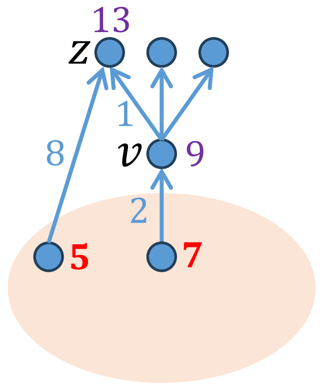
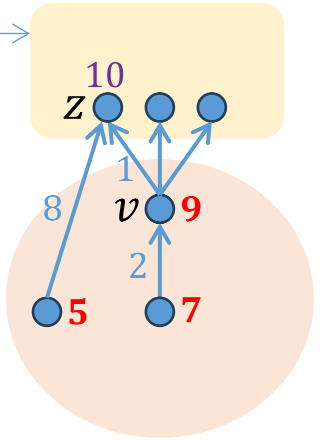
• While $R \neq V$
    • Select $v \in V \setminus R$ to minimize $\min_{u \in R}\big(d(u) + w(u,v)\big)$
    • $d(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$
    • Add $v$ to $R$
```

Observation: No need to compute $x(v) = \min_{u \in R}\big(d(u) + w(u,v)\big)$ from scratch for every iteration.

Adding v to R can only affect the x-value of these vertices.

- $\underset{10}{x(z)} \leftarrow \min\{\underset{13}{x(z)}, \underset{9}{d(v)} + \underset{1}{w(v,z)}\}$





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```
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Observation: No need to compute $x(v) = \min_{u \in R}\left(d(u) + w(u,v)\right)$ from scratch for every iteration.

```
**Dijkstra**$(G = (V,E), s \in V)$
• $d(s) = 0$
• $x(v) = \infty$ for all $v \in V \setminus \{s\}$
• $R = \{s\}$
• For all $z \in V$ such that $(s,z) \in E$
    • $x(z) = \min\{x(z), d(s) + w(s,z)\}$
• **While** $R \neq V$
    • Select $v \in V \setminus R$ to minimize $x(v)$
    • $d(v) = x(v)$
    • Add $v$ to $R$
    • For all $z \in V \setminus R$ such that $(v,z) \in E$
        • $x(z) = \min\{x(z), d(v) + w(v,z)\}$
```

Efficiency

```
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• $d(s) = 0$
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        • $x(z) = \min\{x(z), d(v) + w(v, z)\}$
```

$\triangleright$

```
Dijkstra$(G = (V, E), s \in V)$
• $R = \emptyset$
• $x(v) = \infty$ for all $v \in V \setminus \{s\}$
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        • $x(z) = \min\{x(z), d(v) + w(v, z)\}$
```

Efficiency

- The algorithm can be implemented using a priority queue.

```
Dijkstra($G = (V, E), s \in V$)
• $R = \emptyset$
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• $x(s) = 0$
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    • For all $z \in V \setminus R$ such that $(v, z) \in E$
        • $x(z) = \min\{x(z), d(v) + w(v, z)\}$
```

$|V|$ **insert**

$|V|$ **extract-min**

At most $|E|$ **decrease-key**

Efficiency

- The algorithm can be implemented using a priority queue.

	Insert	Extract-min	Decrease-key	Time complexity of Dijkstra's algorithm
Binary heap	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O\big((\lvert E\rvert + \lvert V\rvert)\log\lvert V\rvert\big)$
Fibonacci heap	$O(1)$	$O(\log n)$ amortized	$O(1)$ amortized	$O(\lvert E\rvert + \lvert V\rvert\log\lvert V\rvert)$

https://en.wikipedia.org/wiki/Priority_queue

Amortized = average over n operations.

$|V|$ **insert**

$|V|$ **extract-min**

At most $|E|$ **decrease-key**

```
Dijkstra($G = (V, E), s \in V$)
• $R = \emptyset$
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```

Recursive algorithms

Fib(n)
- If $n \leq 1$, return n.
- Else, return $\mathbf{Fib}(n-1) + \mathbf{Fib}(n-2)$.

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of a recursive algorithm:
  - **Base case:** without any recursive calls
    - Show that the algorithm is correct for the base case.
  - **Inductive step:** with recursive calls
    - Assume that the algorithm is correct for any input of size smaller than n.
    - Show that the algorithm is correct for any input of size n.

Recursive algorithms

Fib(n)
- If $n \leq 1$, return n.
- Else, return $\mathbf{Fib}(n-1) + \mathbf{Fib}(n-2)$.

- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

If $n \leq 1$, the algorithm is correct:
- $\mathbf{Fib}(0) = 0$
- $\mathbf{Fib}(1) = 1$

- Analysis of a recursive algorithm:
  - **Base case:**
    - Show that the algorithm is correct for the base case.
  - **Inductive step:**
    - Assume that the algorithm is correct for any input of size smaller than n.
    - Show that the algorithm is correct for any input of size n.

Recursive algorithms

$\mathbf{Fib}(n)$
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- **Goal:** For a given algorithm $\mathcal{A}$, prove that it is correct.

- Analysis of a <u>recursive algorithm</u>:
  - **Base case:**
    - Show that the algorithm is correct for the base case.
  - **Inductive step:**
    - Assume that the algorithm is correct for any input of size smaller than n.
    - Show that the algorithm is correct for any input of size n.

If $n \leq 1$, the algorithm is correct:
- $\mathbf{Fib}(0) = 0$
- $\mathbf{Fib}(1) = 1$

If $n > 1$, the algorithm is correct:
- $\mathbf{Fib}(n) = \mathbf{Fib}(n-1) + \mathbf{Fib}(n-2)$

Searching in a sorted array

- **Input:**
  - A sorted array A.
  - Two indices:
    - lb (lower bound)
    - ub (upper bound)
  - A number x.

- **Goal:**
  - Decide if $x \in A[\mathrm{lb}..\mathrm{ub}]$

If $(\mathrm{lb} > \mathrm{ub})$, the answer is **NO**.

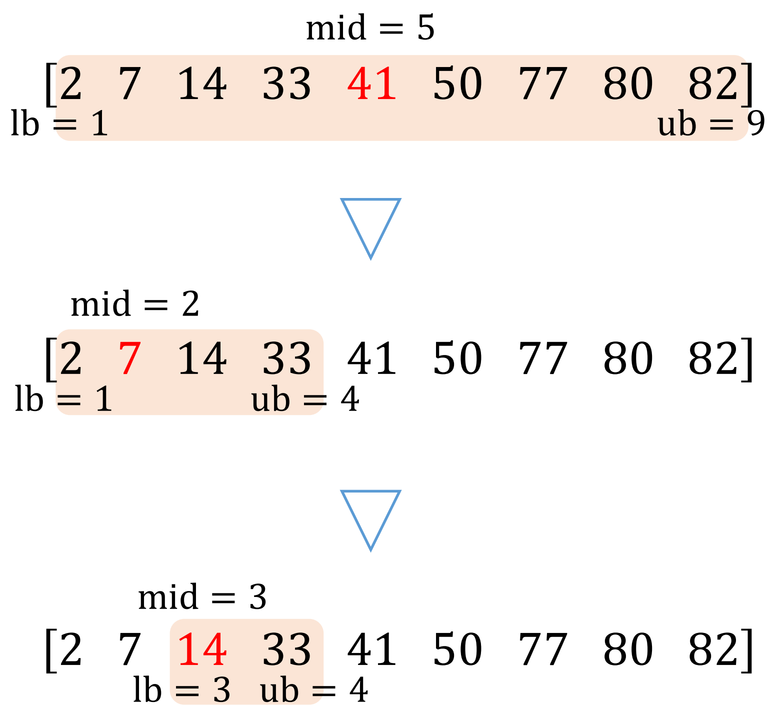
Searching in a sorted array

- **Input:**
  - A sorted array A.
  - Two indices:
    - lb (lower bound)
    - ub (upper bound)
  - A number x.
- **Goal:**
  - Decide if $x \in A[\mathrm{lb}..\mathrm{ub}]$

```
**BinarySearch**$(A, \mathrm{lb}, \mathrm{ub}, x)$
• **If** $(\mathrm{lb} > \mathrm{ub})$, return **NO**.
• **Else**
    • $\mathrm{mid} \leftarrow \left\lfloor \frac{\mathrm{lb}+\mathrm{ub}}{2} \right\rfloor$.
    • **If** $(x = A[\mathrm{mid}])$, return **YES**.
    • **If** $(x > A[\mathrm{mid}])$, **BinarySearch**$(A, \mathrm{mid}+1, \mathrm{ub}, x)$.
    • **If** $(x < A[\mathrm{mid}])$, **BinarySearch**$(A, \mathrm{lb}, \mathrm{mid}-1, x)$.
```

Searching in a sorted array

$x = 14$



YES

```
**BinarySearch**$(A, \mathrm{lb}, \mathrm{ub}, x)$
• **If** $(\mathrm{lb} > \mathrm{ub})$, return **NO**.
• **Else**
    • $\mathrm{mid} \leftarrow \left\lfloor \frac{\mathrm{lb}+\mathrm{ub}}{2} \right\rfloor$.
    • **If** $(x = A[\mathrm{mid}])$, return **YES**.
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```

Proof of correctness

- **Induction** on the array size:
  - $n = \mathrm{ub} - \mathrm{lb} + 1$

```
BinarySearch($A, \mathrm{lb}, \mathrm{ub}, x$)
• If ($\mathrm{lb} > \mathrm{ub}$), return NO.
• Else
    • $\mathrm{mid} \leftarrow \left\lfloor \frac{\mathrm{lb}+\mathrm{ub}}{2} \right\rfloor$.
    • If ($x = A[\mathrm{mid}]$), return YES.
    • If ($x > A[\mathrm{mid}]$), BinarySearch($A, \mathrm{mid}+1, \mathrm{ub}, x$).
    • If ($x < A[\mathrm{mid}]$), BinarySearch($A, \mathrm{lb}, \mathrm{mid}-1, x$).
```

Proof of correctness

- **Induction** on the array size:
  - $n = \mathrm{ub} - \mathrm{lb} + 1$

- **Base case:**
  - If $n < 1$, the algorithm correctly returns **NO**.

```
BinarySearch$(A, \mathrm{lb}, \mathrm{ub}, x)$
• If $(\mathrm{lb} > \mathrm{ub})$, return NO.
• Else
    • $\mathrm{mid} \leftarrow \left\lfloor \frac{\mathrm{lb}+\mathrm{ub}}{2} \right\rfloor$.
    • If $(x = A[\mathrm{mid}])$, return YES.
    • If $(x > A[\mathrm{mid}])$, BinarySearch$(A, \mathrm{mid}+1, \mathrm{ub}, x)$.
    • If $(x < A[\mathrm{mid}])$, BinarySearch$(A, \mathrm{lb}, \mathrm{mid}-1, x)$.
```

Proof of correctness

- **Induction** on the array size:
  - $n = \mathrm{ub} - \mathrm{lb} + 1$

- **Inductive step:**
  - Assume the algorithm works correctly for any input of size smaller than n.

```
BinarySearch(A, lb, ub, x)
• If (lb > ub), return NO.
• Else
    • mid ← ⌊(lb+ub)/2⌋.
    • If (x = A[mid]), return YES.
    • If (x > A[mid]), BinarySearch(A, mid + 1, ub, x).
    • If (x < A[mid]), BinarySearch(A, lb, mid − 1, x).
```

If $(x = A[\mathrm{mid}])$, the answer must be **YES**.

If $(x > A[\mathrm{mid}])$, then:

- $(x \in A[\mathrm{lb..ub}])$ if and only if $(x \in A[\mathrm{mid} + 1..\mathrm{ub}])$. A is sorted
- Therefore, the answer must be $\mathbf{BinarySearch}(A, \mathrm{mid} + 1, \mathrm{ub}, x)$. Induction hypothesis

The case of $(x < A[\mathrm{mid}])$ is similar.

Efficiency

- Input size:
  - $n = \mathrm{ub} - \mathrm{lb} + 1$
- Subproblem input size:
  - $\leq \lfloor n/2 \rfloor$
- Depth of recursion:
  - $O(\log n)$
- Time complexity:
  - $T(n) \in O(\log n)$

```
**BinarySearch**$(A, \mathrm{lb}, \mathrm{ub}, x)$
• **If** $(\mathrm{lb} > \mathrm{ub})$, return **NO**.
• **Else**
    • $\mathrm{mid} \leftarrow \left\lfloor \frac{\mathrm{lb}+\mathrm{ub}}{2} \right\rfloor$.
    • **If** $(x = A[\mathrm{mid}])$, return **YES**.
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    • **If** $(x < A[\mathrm{mid}])$, **BinarySearch**$(A, \mathrm{lb}, \mathrm{mid}-1, x)$.
```

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