A Few Useful Facts

Properties of functions

- Exponentials
- Logarithms
- Summations
- Limits

Exponentials

$$a^{-1} = 1/a$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

$$e^x \geq 1 + x$$

Notation: Logarithms

- **Binary log:** $\log n = \log_2 n$
- **Natural log:** $\ln n = \log_e n$
- **Exponentiation:** $\log^k n = (\log n)^k$
- **Composition:** $\log \log n = \log(\log n)$

$$
\begin{aligned}
a &= b^{\log_b a} \\
\log_c(ab) &= \log_c a + \log_c b \\
\log_b a^n &= n \log_b a \\
\log_b a &= \frac{\log_c a}{\log_c b} \\
\log_b(1/a) &= -\log_b a \\
\log_b a &= \frac{1}{\log_a b} \\
a^{\log_b c} &= c^{\log_b a}
\end{aligned}
$$

Base of logarithms and exponentiation in asymptotics

- $\log n \in \Theta(\ln n) = \Theta(\log_{10} n)$
- However, $4^n \notin \Theta(2^n)$

Sterling’s Approximation

- $n! = \sqrt{2\,\pi n}\left(\frac{n}{e}\right)^{n}\left(1 + \mathrm{O}\left(\frac{1}{n}\right)\right)$
- $\log(n!) \in \Theta(n \log n)$

Summations and Geometric Series

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

For $0 < x < 1,$ $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\sum_{k=0}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \in \Theta(n^2)$$

Limit

- Assume f(n), g(n)>0.

- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = 0 \rightarrow f(n) = o\big(g(n)\big)$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) < \infty \rightarrow f(n) = O\big(g(n)\big)$
- $0 < \lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) < \infty \rightarrow f(n) = \Theta\big(g(n)\big)$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) > 0 \rightarrow f(n) = \Omega\big(g(n)\big)$
- $\lim_{n\to\infty}\left(\frac{f(n)}{g(n)}\right) = \infty \rightarrow f(n) = \omega\big(g(n)\big)$

L’Hospital’s Rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Example

$$\lim_{n\to\infty} \frac{n \ln n}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n}$$

$$= 0$$

Thus, $n \ln n \in \mathrm{o}(\mathrm{n}^2)$