Q1.

(a) ① it's obvious that for any vertice v in I, $v \in V$.

② for all $v_i, v_j \in I$, $(i<j)$ edge $(v_i, v_j) \notin E$. because considering the defination of I, $(\forall j<i)[(v_j, v_i) \notin E]$

So set I meets the two requirements. Hence I is an independent set.

(b) The condition of v in I is v must be ahead of all its neighbours.

v's neighbour number is $d(v)$, if v in the set I, the size of I will increase by 1.

Hence, we just sum up all the vertices with the possibility of the vertice in I.

$$E[|I|] = E[\sum_{v\in V} 1 \cdot P(v)] = \sum_{v \in V} E[1 \cdot P(v)] = \sum_{v\in V} E[\frac{1}{d(v)+1}] = \sum_{v \in V} \frac{1}{d(v)+1}$$

($P(v)$ is the possibility of v in set I. $\frac{1}{d(v)+1}$)

(c) in part (B). we have $E[|I|] = \sum_{v\in V} \frac{1}{d(v)+1}$

assume there doesn't exist an independent set of size at least $\sum_{v\in V} \frac{1}{d(v)+1}$

So we can get for any set I, $|I| < \sum_{v \in V} \frac{1}{d(v)+1}$

Then caculate the expectation of $|I|$

$$\Rightarrow E[|I|] = \sum_{I} |I| \cdot P(I) < \sum_{v\in V} \frac{1}{d(v)+1} \sum_{I} P(I) = \sum_{v\in V} \frac{1}{d(v)+1}$$

it contridicts the original conclusion.

So. there exists an independent set of size at least $\sum_{v\in V} \frac{1}{d(v)+1}$

2.

$$E = \sum_{i=0}^{+\infty} i \cdot 2^{-i-1} = \sum_{i=1}^{+\infty} i \cdot 2^{-i-1} = \frac{1}{2} \sum_{i=1}^{+\infty} i \cdot 2^{-i}$$

$$\Rightarrow 2E = \sum_{i=1}^{+\infty} i \cdot 2^{-i} = 1\cdot 2^{-1} + 2 \cdot 2^{-2} + 3 \cdot 2^{-3} + \dots + i \cdot 2^{-i} \ (i \to +\infty) \quad ①$$

$$E = \sum_{i=1}^{+\infty} i \cdot 2^{-i-1} = 1 \cdot 2^{-2} + 2 \cdot 2^{-3} + \dots + (i-1) \cdot 2^{-i} + i \cdot 2^{-i-1} \ (i \to +\infty) \quad ②$$

① - ② $\Rightarrow$

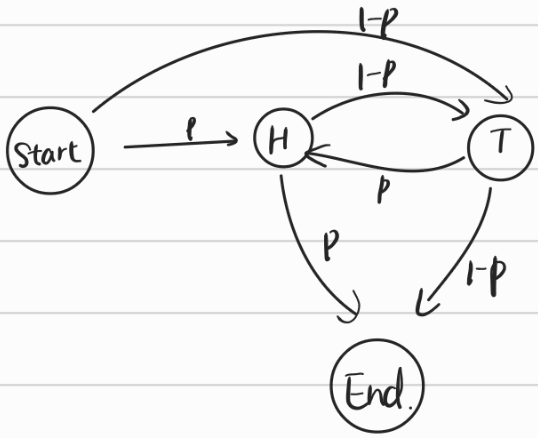
$$E = \sum_{i=1}^{+\infty} 2^{-i} - i \cdot 2^{-i-1}$$

$$\lim_{i \to +\infty} E = \sum_{i=1}^{+\infty} 2^{-i} - i \cdot 2^{-i-1} = 1$$

3. p heads

$1-p$ tails

This is a question about Markov chain.



P_H, the possibility of HH appearing ahead of TT. Starting from H.
P_T, the possibility of HH appearing ahead of TT. starting from T.

$$\begin{cases} P_H = p + (1-p) P_T \\ P_T = p \cdot P_H + 0 \cdot (1-p) \end{cases} \Rightarrow \begin{cases} P_H = p + (1-p) P_T \\ P_T = p \cdot P_H \end{cases}$$

$$\Rightarrow P_H = p + (1-p) \cdot p \cdot P_H$$

$$\Rightarrow P_H (1 - p + p^2) = p \Rightarrow P_H = \frac{p}{1-p+p^2}$$

$$P_T = \frac{p^2}{1-p+p^2}.$$

P_0, starting from S_0

$$P_0 = p \cdot P_H + (1-p) P_T$$

$$\Rightarrow P_0 = \frac{p^2 + p^2 - p^3}{1-p+p^2} = \frac{2p^2 - p^3}{1-p+p^2}.$$