CS3230 Semester 2 2024/2025

Assignment 05
Randomized Algorithms

Due: Sunday, 9th Mar 2025, 11:59 pm SGT.

Instructions:

- Canvas Assignment Submission page: `Assignments/Assignment 5`.
- Please upload PDFs containing your solutions (hand-written & scanned, or typed) by the due date.
- Name the file **Assignment5_SID.pdf**, where SID should be replaced by your student ID.
- You may discuss the problems with your classmates at a high level only. You should write up your solutions on your own (any copying from your co-students or usage of Internet or AI tools is not allowed). Please note the names of your collaborators or any other sources in your submission; failure to do so would be considered plagiarism.
- Question listed as "graded for correctness" (worth 6 points) require complete answers. Other questions (worth 1 point each) will be graded only based on reasonable attempts. However, you should still do these questions, as they are practice questions, which would be useful for exams as well as for your knowledge.

1. (6 points; graded for correctness) Consider an undirected simple graph $G = (V, E)$. A set $V' \subseteq V$ is called an *independent set* of G if for all $u, v \in V'$, edge (u, v)[1] is not in E. Let $d(u)$ denote the degree of u, that is, $|\{v : (u, v) \in E\}|$.

   This question shows that some independent set of G is of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$.

   Consider a uniform random order of the vertices in V: $v_1, v_2, \ldots, v_n$. Let I be the set of vertices v all of whose neighbours occur after v in this order. That is, $I = \{v_i : (\forall j < i)[(v_i, v_j) \notin E]\}$.

   (a) (2 points) Show that I is an independent set.

   (b) (3 points) Show that the expected size of I is $\sum_{v \in V} \frac{1}{d(v)+1}$.

   (c) (1 point) Conclude that you can always choose an ordering of the vertices in V for which I is of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$, and thus there exists an independent set of size at least $\sum_{v \in V} \frac{1}{d(v)+1}$.

   Note: If you cannot show earlier parts, but can do later parts assuming earlier parts, you will be given credit for the later parts.

2. (1 point) Suppose that for each integer $i \geq 0$, the probability of getting an input of length i is 2^{-i-1}. What is the expected length of the input?

3. (1 point) A certain coin has probability p of coming up heads. Suppose you keep tossing the coin until two consecutive heads or two consecutive tails come up. What is the probability that you see two consecutive heads before seeing two consecutive tails?

[1] For undirected graphs, sometimes the set notation $\{u, v\}$ is used for an edge instead of (u, v).