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COMPUTER GRAPHICS AND ANIMATION

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Unit-2: Two Dimensional and Three Dimensional Transformation

What is Transformation?

- Transformation is changing of Position, shape, size, or orientation of an object on display.

Two Types

1. Basic Transformation
2. Other Transformation

Basic Transformation

- Basic transformation includes three transformations
  - ✓ **Translation**,
  - ✓ **Rotation**, and
  - ✓ **Scaling**.
- These three transformations are known as basic transformation because with combination of these three transformations we can obtain any transformation.

Translation

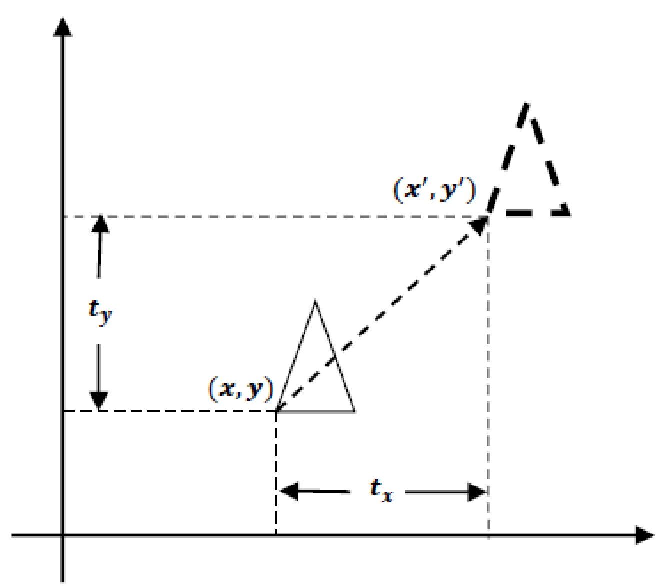

Figure: Translation of a vertex p (x, y) to P' (x', y') of triangle.

- It is a transformation that used to reposition the object along the straight line path from one coordinate location to another.
- It is rigid body transformation so we need to translate whole object.
- We translate two dimensional point by adding translation distance $\boldsymbol{t_x}$ and $\boldsymbol{t_y}$ to the original coordinate position $(\boldsymbol{x}, \boldsymbol{y})$ to move at new position $(\boldsymbol{x'}, \boldsymbol{y'})$ as:

$$\boldsymbol{x'} = \boldsymbol{x} + \boldsymbol{t_x}$$
$$\& \qquad \boldsymbol{y'} = \boldsymbol{y} + \boldsymbol{t_y}$$

- Translation distance pair $(\boldsymbol{t_x}, \boldsymbol{t_y})$ is called a **Translation Vector** or **Shift Vector**.
- We can represent it into single matrix equation in column vector as

$$\boldsymbol{P'} = \boldsymbol{P} + \boldsymbol{T}$$

vector is a quantity that has both magnitude (strength or size) and direction. It's like an arrow pointing in a specific way with a certain length.

<u>Matrix form</u>

$$\begin{bmatrix} \boldsymbol{x'} \\ \boldsymbol{y'} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} + \begin{bmatrix} \boldsymbol{t_x} \\ \boldsymbol{t_y} \end{bmatrix}$$

- We can also represent it in row vector form as:

$$\boldsymbol{P'} = \boldsymbol{P} + \boldsymbol{T}$$

$$[\boldsymbol{x'}\ \boldsymbol{y'}] = [\boldsymbol{x}\ \boldsymbol{y}] + [\boldsymbol{t_x}\ \boldsymbol{t_y}]$$

- Since column vector representation is standard mathematical notation and since many graphics package like **GKS (**Graphics Kernel System**)** and **PHIGS (**Programmer Hierarchical Graphics System,**)** uses column vector we will also follow column vector representation.

Example: - Translate the triangle [A (10, 10), B (15, 15), C (20, 10)] 2 unit in x direction and 1 unit in y-direction.

We know that

$$P' = P + T$$

$$P' = [P] + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

For point (10, 10)

$$A' = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$

For point (15, 15)

$$B' = \begin{bmatrix} 15 \\ 15 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$B' = \begin{bmatrix} 17 \\ 16 \end{bmatrix}$$

For point (10, 10)

$$C' = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$C' = \begin{bmatrix} 22 \\ 11 \end{bmatrix}$$

Hence, Final coordinates after translation are [A’ (12, 11), B’ (17, 16), C’ (22, 11)].

Rotation

- It is a transformation that used to reposition the object along the circular path in the XY - plane.
- I.e. move the object from one point to another point with an angle but the distance of that object / point from the origin should be same.

Key Notes:

- To generate a rotation we specify **a rotation angle → θ** and the position of the **Rotation Point** (**Pivot Point**) (x_r, y_r) about which the object is to be rotated.

- Positive value of rotation angle defines **counter clockwise rotation** (CCW) / **Anti-Clockwise** and negative value of rotation angle defines **clockwise** rotation.

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- Line perpendicular to rotating plane and passing through pivot point is called **axis of rotation**.

Rotation when pivot point is at coordinate origin (0, 0).

- Consider a point P(x, y) is original point and r is the constant distance from origin and ∅ is the original angular displacement from x-axis.
- Rotate P (x, y) with and angle $\boldsymbol{\theta}$ in anticlockwise direction and point obtained after rotation is P' (x', y')

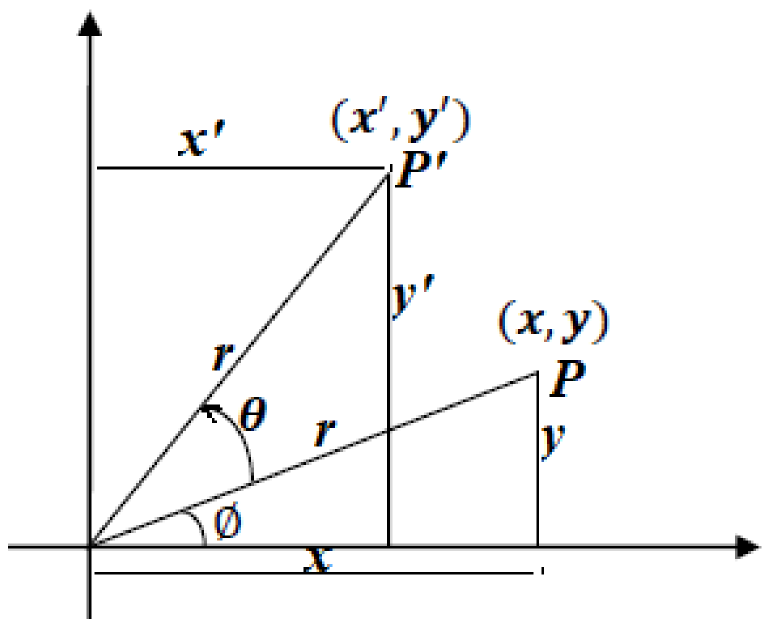

Figure: Rotation at origin (0,0)

From figure we can write.

Cos ∅ = b / h = x / r

$$\boldsymbol{x} = \boldsymbol{r} \cos \emptyset$$

Sin ∅ = p / h = y / r

$$\boldsymbol{y} = \boldsymbol{r} \sin \emptyset$$

Similarly,

Cos (θ + ∅) = x' / r

$$\boldsymbol{x'} = \boldsymbol{r}\ \mathbf{cos}(\boldsymbol{\theta} + \emptyset)$$
$$= \boldsymbol{r}\ \mathbf{cos}\ \emptyset\ \mathbf{cos}\ \boldsymbol{\theta} - \boldsymbol{r}\ \mathbf{sin}\ \emptyset\ \mathbf{sin}\ \boldsymbol{\theta}$$

Now replace $\boldsymbol{r}\ \mathbf{cos}\ \emptyset$ with $\boldsymbol{x}$ and $\boldsymbol{r}\ \mathbf{sin}\ \emptyset$ with $\boldsymbol{y}$ in above equation.

$$\boldsymbol{x'} = \boldsymbol{x}\ \mathbf{cos}\ \boldsymbol{\theta} - \boldsymbol{y}\ \mathbf{sin}\ \boldsymbol{\theta}$$

And

Sin (θ+∅) = y' / r

$$\boldsymbol{y'} = \boldsymbol{r}\ \mathbf{sin}(\emptyset + \boldsymbol{\theta})$$
$$= \boldsymbol{r}\ \mathbf{cos}\ \emptyset\ \mathbf{sin}\ \boldsymbol{\theta} + \boldsymbol{r}\ \mathbf{sin}\ \emptyset\ \mathbf{cos}\ \boldsymbol{\theta}$$

Now replace $\boldsymbol{r}\ \mathbf{cos}\ \emptyset$ with $\boldsymbol{x}$ and $\boldsymbol{r}\ \mathbf{sin}\ \emptyset$ with $\boldsymbol{y}$ in above equation.

$$\boldsymbol{y'} = \boldsymbol{x}\ \mathbf{sin}\ \boldsymbol{\theta} + \boldsymbol{y}\ \mathbf{cos}\ \boldsymbol{\theta}$$

Therefore,

$$\boxed{\begin{aligned}\boldsymbol{x'} &= \boldsymbol{x}\ \mathbf{cos}\ \boldsymbol{\theta} - \boldsymbol{y}\ \mathbf{sin}\ \boldsymbol{\theta}\\ \boldsymbol{y'} &= \boldsymbol{x}\ \mathbf{sin}\ \boldsymbol{\theta} + \boldsymbol{y}\ \mathbf{cos}\ \boldsymbol{\theta}\end{aligned}}$$

We can write it in the form of column vector matrix equation as;

$$\boldsymbol{P'} = \boldsymbol{R} \cdot \boldsymbol{P}$$

Matrix Form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

If the Rotation is Clockwise then,

$\boldsymbol{\theta}$ is negative

We know,

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Cos (-$\boldsymbol{\theta}$) = cos $\boldsymbol{\theta}$ and

Sin (-$\boldsymbol{\theta}$) = -sin $\boldsymbol{\theta}$

Example:

1. Apply Rotation on point (4, 3) and angle is 45 degree.
2. Locate the new position of the triangle [A (5, 4), B (8, 3), C (8, 8)] after its rotation by
   i. 90 degree clockwise about the origin.
   ii. 90 degree CCW at origin

Solution of i.

Here,

Given vertices of triangle are A (5, 4), B (8, 3) C (8, 8)

And, Rotation is clockwise hence we take $\theta = -90°$.

We know,

$P' = R \cdot P$

Let's write given vertices into matrix form P=

$$\begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$$

Then, P'=R.P

$$P' = \begin{bmatrix} \cos(-90) & -\sin(-90) \\ \sin(-90) & \cos(-90) \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \end{bmatrix}$$

$$P' = \begin{bmatrix} 4 & 3 & 8 \\ -5 & -8 & -8 \end{bmatrix}$$

Therefore, Final coordinates after rotation are [A' (4, -5), B' (3, -8), C' (8, -8)].

Scaling

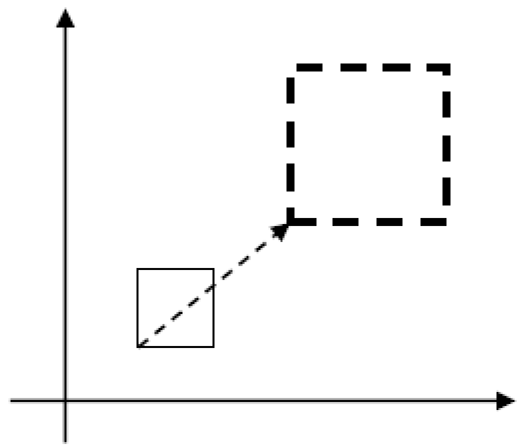

Figure: Scaling

- It is a transformation that used to alter the size of an object.
- This operation is carried out by multiplying coordinate value (x, y) with scaling factor (sx, sy) respectively.

So, equation for scaling is given by:

$$x' = x \cdot s_x$$
$$y' = y \cdot s_y$$

- These equation can be represented in column vector matrix equation as:

$$P' = S \cdot P$$

Matrix Representation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Any positive value can be assigned to (s_x, s_y).

- Values of (s_x, s_y) less than 1 reduce the size while values greater than 1 enlarge the size of object, and
- Object remains unchanged when values of both factor is 1.

- Same values of s_x and s_y will produce **Uniform Scaling**. And different values of s_x and s_y will produce **Differential Scaling**.
- Objects transformed with above equation are both scale and repositioned.
- Scaling factor with value less than 1 will move object closer to origin, while scaling factor with value greater than 1 will move object away from origin.

Example:

1. Scale an object about origin with 3 vertices (4,4), (3,2) and (5,2) along x axis and y axis by 2 unit .
2. Consider the square with left-bottom corner at (2, 2) and right-top corner at (6, 6). Apply the transformation which makes its size half.

   Solution:

   $P' = S \cdot P$

   $$P' = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix}$$

   $$P' = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \end{bmatrix}$$

   $$P' = \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

Hence, Final coordinate after scaling are [A’ (1, 1), B’ (3, 1), C’ (3, 3), D’ (1, 3)].

Matrix Representation and homogeneous coordinates

Why Homogeneous coordinate system?

- ✓ **Homogeneous coordinates** a coordinate system that algebraically treats all points in the projective plane (both Euclidean and ideal) equally.
- ✓ For example, the standard homogeneous coordinates [p_1,p_2,p_3] of a point P in the projective plane are of the form [x,y,1] if P is a point in the Euclidean plane z=1 whose Cartesian coordinates are (x,y,1), or are of the form [a,b,0] if P is the ideal point – the point at infinity – associated to all lines in the Euclidean plane z=1 with direction numbers a,b,0.
- ✓ Homogeneous coordinates are so called because they treat Euclidean and ideal points in the same way.
- ✓ Homogeneous coordinates are widely used in computer graphics because they enable affine and projective transformations to be described as matrix manipulations in a coherent way.

- Many graphics application involves sequence of geometric transformations.
- For example in design and picture construction application we perform Translation, Rotation, and scaling to fit the picture components into their proper positions.
- For efficient processing we will reformulate transformation sequences.
- We have matrix representation of basic transformation and we can express it in the general matrix form as:

$$\boldsymbol{P'} = \boldsymbol{M1} \cdot \boldsymbol{P} + \boldsymbol{M2}$$

Where,

$\boldsymbol{P}$ and $\boldsymbol{P'}$ are initial and final point position,
$\boldsymbol{M1}$ contains rotation and scaling terms and
$\boldsymbol{M2}$ contains translational terms associated with pivot point, fixed point and reposition.

- For efficient utilization we must calculate all sequence of transformation in one step and for that reason we reformulate above equation to eliminate the matrix addition associated with translation terms in matrix $\boldsymbol{M2}$**.**
- We can combine that thing by expanding 2X2 matrix representation into 3X3 matrices.
- It will allows us to convert all transformation into matrix multiplication but we need to represent vertex position $(\boldsymbol{x}, \boldsymbol{y})$ with homogeneous coordinate triple $(\boldsymbol{x_h}, \boldsymbol{y_h}, \boldsymbol{h})$ Where $\boldsymbol{x} = \boldsymbol{x_h} / \boldsymbol{h}$,

$$\boldsymbol{y} = \boldsymbol{y_h} / \boldsymbol{h}$$

Thus, we can also write triple as $(\boldsymbol{h} \cdot \boldsymbol{x}, \boldsymbol{h} \cdot \boldsymbol{y}, \boldsymbol{h})$.

- For two dimensional geometric transformation we can take value of $\boldsymbol{h}$ is any positive number so we can get infinite homogeneous representation for coordinate value $(\boldsymbol{x}, \boldsymbol{y})$.
- But convenient choice is set $\boldsymbol{h} = \boldsymbol{1}$ as it is multiplicative identity, than $(\boldsymbol{x}, \boldsymbol{y})$ is represented as $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{1})$.
- Expressing coordinates in homogeneous coordinates form allows us to represent all geometric transformation equations as matrix multiplication.

In short,

- ✓ To convert a 2×2 matrix to 3×3 matrix, we have to add an extra dummy coordinate. So, we can represent the point by 3 numbers instead of 2 numbers, which is called Homogenous Coordinate system.

- ✓ In this system, we can represent all the transformation equations in matrix multiplication.

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- ✓ Any Cartesian point P(X, Y) can be converted to homogenous coordinates by P’ (x_h, y_h, h). The ‘h’ is normally set to 1. If the value of ‘h’ is more the one value then all the co-ordinate values are scaled by this value.

Let’s see each representation with $\boldsymbol{h} = \mathbf{1}$

Translation

$$P' = T_{(t_x,t_y)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of translation matrix is obtain by putting $-\boldsymbol{t_x}$ & $-\boldsymbol{t_y}$ instead of $\boldsymbol{t_x}$ & $\boldsymbol{t_y}$.

Rotation:

$$P' = R_{(\theta)} \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of rotation matrix is obtained by replacing $\boldsymbol{\theta}$ by $-\boldsymbol{\theta}$.

Scaling

$$P' = S(s_x, s_y) \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

NOTE: - Inverse of scaling matrix is obtained by replacing s_x & s_y by 1 / s_x & 1 / s_y respectively

Composite Transformation:

- When more than one transformation are applied for performing a task then such transformation is called composite transformation.
- Forming product of transformation matrix is referred as concatenation or composition of matrices.
- Basic purpose of composing transformation is to gain efficiency by applying a single composed transformation to appoint, rather than applying series of transformation one after another.
- Composite transformation indicates combination of different basic transformation in sequence to obtain desire result or combination could be a sequence of two or more successive transformation (e.g. two successive translation, two successive rotation or two or more successive scaling)

Translations

If Two successive transformation vectors **(t_{x1}, t_{y1}) and (t_{x2}, t_{y2})** are applied to a point **P**, the final position or transformed location **P'** is calculated as:

$$P' = T(t_{x2}, t_{y2}) \cdot \{T(t_{x1}, t_{y1}) \cdot P\}$$

$$P' = \{T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})\} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = T(t_{x1} + t_{x2}, t_{y1} + t_{y2}) \cdot P\}$$

- Here P' and P are column vector of final and initial point coordinate respectively.

- This shows that **two successive translation are additive**.
- This concept can be extended for any number of successive translations

Example: Obtain the final coordinates after two translations on point P (2, 3) with translation vector (4, 3) and (−1, 2) respectively.

We know,

$$P' = T\ (t_{x1} + t_{x2}, t_{y1} + t_{y2}) \cdot P$$

$$P' = \begin{bmatrix} 1 & 0 & t_{x1} + t_{x2} \\ 0 & 1 & t_{y1} + t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} 1 & 0 & 4 + (-1) \\ 0 & 1 & 3 + 2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix}$$

Hence, Final Coordinates after translations are P' (5, 8).

Rotations

Two successive Rotations are performed as:

$$\boldsymbol{P' = R(\theta_2) \cdot \{R(\theta_1) \cdot P\}}$$
$$\boldsymbol{P' = \{R(\theta_2) \cdot R(\theta_1)\} \cdot P}$$

$$\boldsymbol{P' = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

$$\boldsymbol{P' = \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & -\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1 & 0 \\ \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1 & \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

$$\boldsymbol{P' = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & 0 \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

Where $\boldsymbol{P'}$ and $\boldsymbol{P}$ are column vector of final and initial point coordinate respectively.

- This shows that **two successive rotation are additive**.
- This concept can be extended for any number of successive rotations.

Example: Obtain the final coordinates after two rotations on point P (6, 9) with rotation angles are 30 and 60 degree respectively.

We have,

$P' = R\ (\theta_1 + \theta_2) \cdot P$

$$P' = \begin{bmatrix} cos(\theta_1+\theta_2) & -sin(\theta_1+\theta_2) & 0 \\ sin(\theta_1+\theta_2) & cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} cos(30+60) & -sin(30+60) & 0 \\ sin(30+60) & cos(30+60) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ 6 \\ 1 \end{bmatrix}$$

Hence, Final Coordinates after rotations are P’ (−9, 6).

Scaling

Two successive scaling are performed as:

$$\boldsymbol{P' = S(s_{x2},\, s_{y2}) \cdot \{S(s_{x1},\, s_{y1}) \cdot P\}}$$
$$\boldsymbol{P' = \{S(s_{x2},\, s_{y2}) \cdot S(s_{x1},\, s_{y1})\} \cdot P}$$

$$\boldsymbol{P' = \begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

$$\boldsymbol{P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P}$$

$\boldsymbol{P' = S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \cdot P}$

Where, $\boldsymbol{P'}$ and $\boldsymbol{P}$ are column vector of final and initial point coordinate respectively.

- This shows that **two successive scaling are multiplicative**.
- This concept can be extended for any number of successive scaling.

Example: Obtain the final coordinates after two scaling on line PQ [P (2, 2), Q (8, 8)] with scaling factors are (2, 2) and (3, 3) respectively.

$P' = S\ (s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2}) \cdot P$

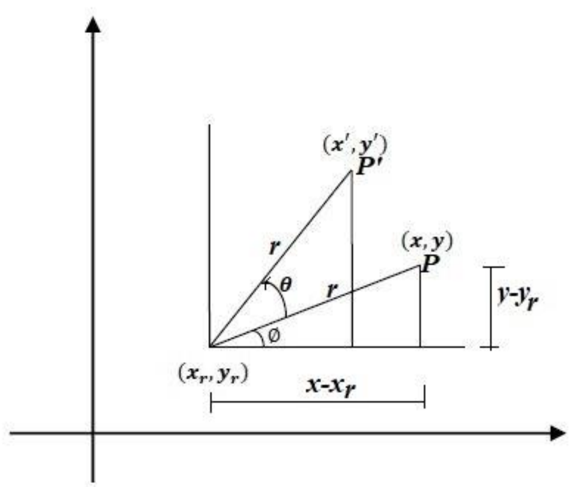
$$P' = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P = \begin{bmatrix} 2 \cdot 3 & 0 & 0 \\ 0 & 2 \cdot 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot P$$

$$P' = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 8 \\ 2 & 8 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 48 \\ 12 & 48 \\ 1 & 1 \end{bmatrix}$$

Hence, Final Coordinates after rotations are P’ (12, 12) and Q’ (48, 48).

Fixed Point Rotation

- Also called *Rotation of a point about an arbitrary pivot position*



In the context of rotation, an arbitrary pivot position refers to the ability to rotate an object around any point, not just its center or a pre-defined location.

Figure: Rotation about an arbitrary pivot point

❖ Transformation equation for rotation of a point about pivot point (x_r, y_r) is calculated in the similar way as rotation about an origin and they are:

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$
$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

❖ These equations are differing from rotation about origin and its matrix representation is also different.

For rotating object about arbitrary point (called pivot point) we need to apply following sequence of transformation.

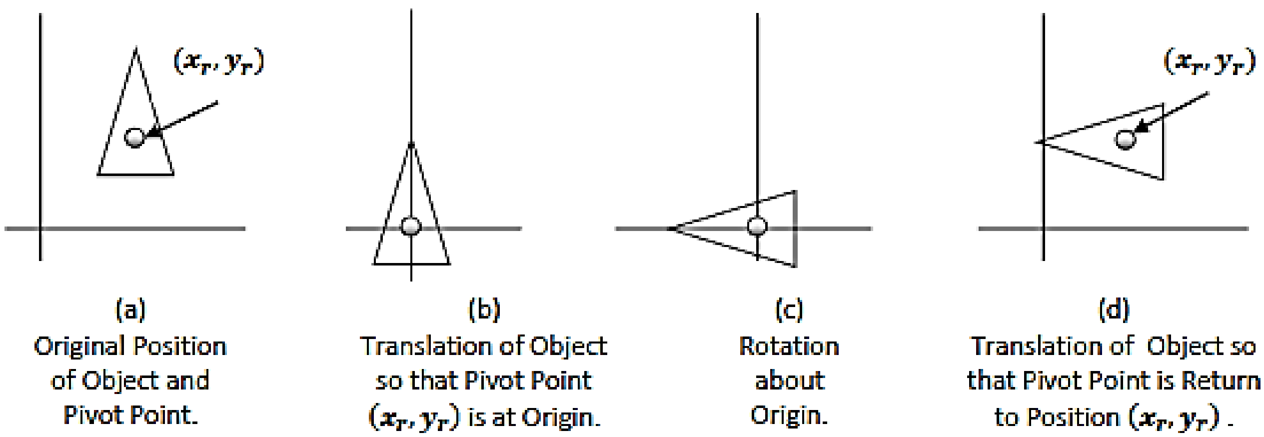


Figure: General Fixed point Rotation

Let P (x, y) is rotated to new position P' (x', y') by an angle θ about pivot point (x_r, y_r) then P' (x', y') can be calculated as:

DO

1. Translate the object so that the pivot-point coincides with the coordinate origin.
2. Rotate the object about the coordinate origin with specified angle.
3. Translate the object so that the pivot-point is returned to its original position (i.e. Inverse of step-1).

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Composite Matrix **(CM) = T. R $_{\theta}$. T^{-1}**

$$\mathbf{CM} = T_{(x_r, y_r)}\,(R_\theta)\;T_{(-x_r, -y_r)}$$

$$= \begin{bmatrix} 1 & 0 & x_r \\ 0 & 1 & y_r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_r \\ 0 & 1 & -y_r \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

P'= CM*P , Which is required expression

Where P' and P are column vector of final and initial point coordinate respectively and (x_r, y_r) are the coordinates of pivot-point.

Example: - Locate the new position of the triangle [A (5, 4), B (8, 3), C (8, 8)] after its rotation by 90^0 Clockwise about the centroid.

Solution:

Pivot point is centroid of the triangle so:

$$x_r = \frac{5+8+8}{3} = 7\,, \qquad y_r = \frac{4+3+8}{3} = 5$$

As rotation is clockwise we will take $\theta = -90°$.

$P' = R(x_r, y_r, \theta) \cdot P$

We have,

Composite Matrix **(CM) = T. R $_{\theta}$. T^{-1}**

$$= \mathbf{T}_{(x_r, y_r)}\,(\mathbf{R}_\theta)\;\boldsymbol{T}_{(-x_r, -y_r)}$$

$$= \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & -\sin(-90) & 0 \\ \sin(-90) & \cos(-90) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 12 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, P'= CM * P

$$= \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 12 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 10 \\ 7 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence, final coordinates after rotation are A' (6, 7), B' (5, 4), C' (10, 4)

<u>Or directly you can solve as:</u>

$$P' = R(x_r, y_r, \theta) \cdot P$$

$$P' = \begin{bmatrix} \cos\theta & -\sin\theta & x_r(1-\cos\theta) + y_r\sin\theta \\ \sin\theta & \cos\theta & y_r(1-\cos\theta) - x_r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

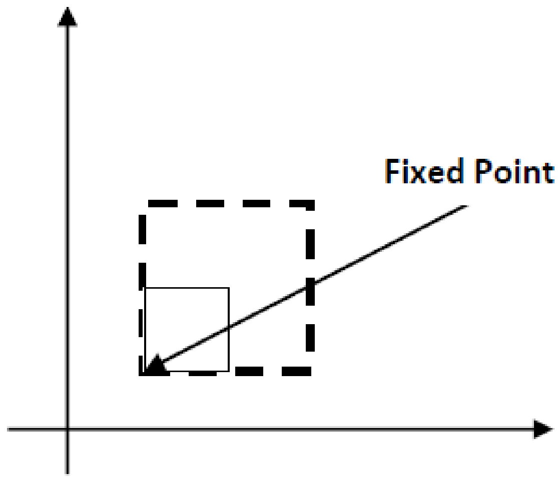
$$P' = \begin{bmatrix} \cos(-90) & -\sin(-90) & 7(1-\cos(-90)) + 5\sin(-90) \\ \sin(-90) & \cos(-90) & 5(1-\cos(-90)) - 7\sin(-90) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 8 & 8 \\ 4 & 3 & 8 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & 10 \\ 7 & 4 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

final coordinates after rotation are A' (6, 7), B' (5, 4), C' (10, 4)

Fixed point scaling:

We can control the position of object after scaling by keeping one position fixed called **Fix point** (x_f, y_f) that point will remain unchanged after the scaling transformation



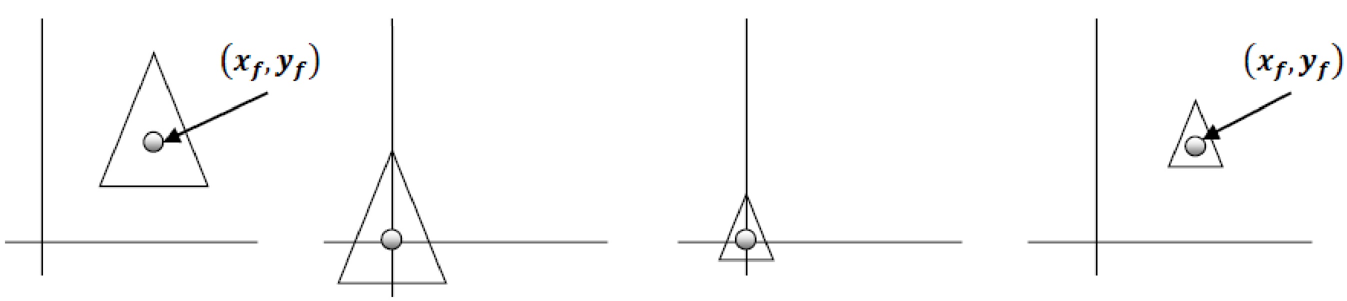
Equation for scaling with fixed point position as (xf, yf) is:

$x' = x_f + (x - x_f)sx$ $\qquad$ $y' = y_f + (y - y_f)sy$

$x' = x_f + xsx - x_f sx$ $\qquad$ $y' = y_f + ysy - y_f sy$

$x' = xsx + x_f(1 - sx)$ $\qquad$ $y' = ysy + y_f(1 - sy)$

For scaling object about arbitrary point (called pivot point) we need to apply following sequence of transformation



(a) Original Position of Object and Fixed Point

(b) Translate Object so that Fixed Point (x_f, y_f) is at Origin

(c) Scale Object with Respect to Origin

(d) Translate Object so that Fixed Point is Return to Position (x_f, y_f) .

Let P (x, y) is Scaled to new position P' (x', y') about pivot point (x_f, y_f) then P' (x', y') can be calculated as:

1. Translate the object so that the fixed-point coincides with the coordinate origin.
2. Scale the object with respect to the coordinate origin with specified scale factors.
3. Translate the object so that the fixed-point is returned to its original position (i.e. Inverse of step-1).

Composite Matrix (CM) = **T. S. T^{-1}**

$$= \boldsymbol{T}(\boldsymbol{x_f}, \boldsymbol{y_f}) \cdot \boldsymbol{S}(\boldsymbol{sx}, \boldsymbol{sy}) \cdot \boldsymbol{T}(-\boldsymbol{x_f}, -\boldsymbol{y_f})$$

$$\begin{bmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore,

P'= CM *P

Here $\boldsymbol{P'}$ and $\boldsymbol{P}$ are column vector of final and initial point coordinate respectively and $(\boldsymbol{x_f}, \boldsymbol{y_f})$ are the coordinates of fixed-point.

$X_f, y_f = (x_1+x_2 / 2 , y_1+y_2 / 2)$

Example: - Consider square with left-bottom corner at (2, 2) and right-top corner at (6, 6) apply the Transformation which makes its size half such that its center remains same.

As we want size half so value of scale factor are $s_x = 0.5$, $s_y = 0.5$ and Coordinates of square are [A (2, 2), B (6, 2), C (6, 6), D (2, 6)].

Fixed point is the center of square so:

$x_f, y_f = (x_1+x_2) / 2$ and $(y_1+y_2) / 2$ (calculate the mid-point of diagonal)

$= (4, 4)$

$P' = S(x_f\ , y_f\ , s_x, s_y\ ) \cdot P$

$$P' = \begin{bmatrix} s_x & 0 & x_f(1-s_x) \\ 0 & s_y & y_f(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & 0 & 4(1-0.5) \\ 0 & 0.5 & 4(1-0.5) \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & 0 & 2 \\ 0 & 0.5 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 6 & 2 \\ 2 & 2 & 6 & 6 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 3 & 5 & 5 & 3 \\ 3 & 3 & 5 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, Final coordinate after scaling are [A’ (3, 3), B’ (5, 3), C’ (5, 5), D’ (3, 5)]

OTHER TRANSFORMATION:

- ✓ Two types

  1. Shear
  2. Reflection

Shear

- ❖ A transformation that distorts the shape of an object is called **shear**.
- ❖ Two common shearing transformations are:
  - X-shear
  - Y-shear

X-Shear

- ❖ Changes are made to the x coordinate and preserves the y coordinates, which causes the vertical line to tilt right or left as shown in figure below:

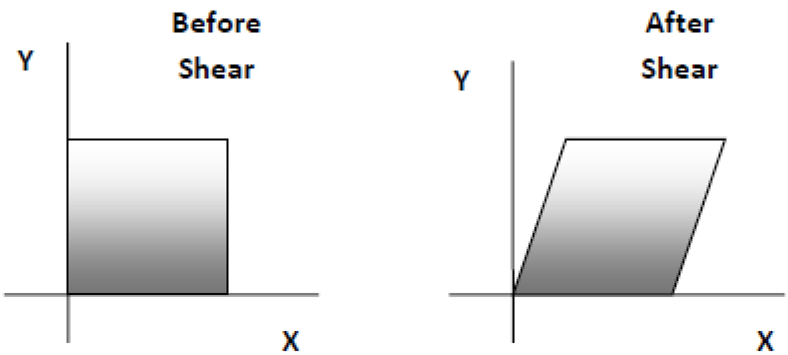
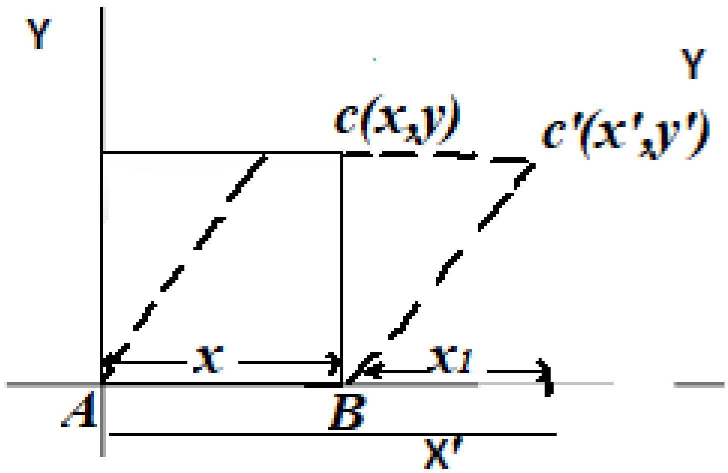


Figure: Share in X-direction



Shearing towards x direction / relative to x axis is given by

$Y' = y$

$X' = x + x_1$, **where x_1 α y** (it means coordinate position is shifted horizontally by an amount proportional to its distance y value from the axis)

And $x_1 = Sh_x .y$ Where, $\mathbf{Sh_x}$ is shearing constant and is any real number.

Therefore,

$$x' = x + shx \cdot y \,,$$
$$y' = y$$

Representation in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Shx & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Hence, **P’ = Sh_x * P**

Y-Shear

- Changes are made to the y coordinate and preserves the x coordinates, which causes the horizontal line to transform into which slopes up or down as shown in figure below:

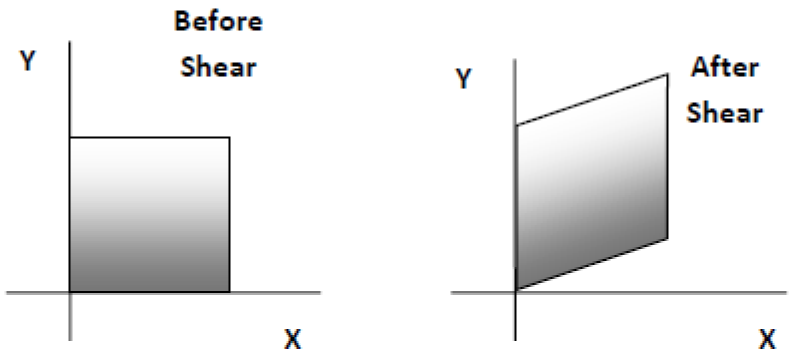


Figure: Y-share

Shearing towards y direction / relative to y axis is given by

X’ = x,
Y’ = y + Sh_y . x

Representation in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ Shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Hence, **P’ = Sh_y * P**

Shear in both direction:

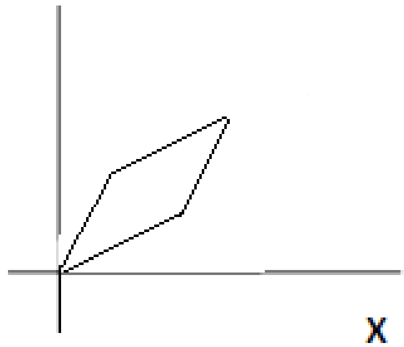


Figure: Shearing towards both (x and y) direction

- No co-ordinates preserved, both changed

$X' = x + Sh_x \cdot y$

$Y' = y + Sh_y \cdot x$

Representing in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Shx & 0 \\ Shy & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

X-direction Shearing relative to other reference line:

We can generate $x - direction$ shear relative to other reference line $y = y_{ref}$ with following equation:

$$\boldsymbol{x' = x + shx \cdot (y - yref)},$$

$$\boldsymbol{y' = y}$$

x' = x + x1 where x1 = Shx * y
y -> (y - yref)

Matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & Shx & -Shx\,.\,y\,ref \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

x' = x + Shx (y - yref)
x' = x + y*Shx - yref * shx

Y-direction Shearing relative to other reference line:

We can generate $y - direction$ shear relative to other reference line $x = xref$ with following equation:

$\boldsymbol{x' = x}$,

$\boldsymbol{y' = y + sh_y \cdot (x - x_{ref})}$

In matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ Shy & 1 & -Shy.Xref \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Numerical:

Problem-01: Shear the unit square in x direction with shear parameter ½ relative to line $y = -1$.

Solution:

Here $yref = -1$ and $shx = 0.5$

Coordinates of unit square are [A (0, 0), B (1, 0), C (1, 1), D (0, 1)]

$$P' = \begin{bmatrix} 1 & sh_x & -sh_x \cdot y_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0.5 & -0.5\cdot(-1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0.5 & 0.5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0.5 & 1.5 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, Final coordinate after shear are [A' (0.5, 0), B' (1.5, 0), C' (2, 1), D' (1, 1)]

Problem-02: Shear the unit square in y direction with shear parameter ½ relative to line $x = -1$.

Solution:

Here, $xref = -1$ and

$shy = 0.5$

Coordinates of unit square are [A (0, 0), B (1, 0), C (1, 1), D (0, 1)].

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ sh_y & 1 & -sh_y\cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5\cdot(-1) \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0.5 & 1 & 2 & 1.5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence, Final coordinate after share are [A' (0, 0.5), B' (1, 1), C' (1, 2), D' (0, 1.5)]

Reflection:

- Transformation that produces a mirror image of an object is reflection

  OR

  Providing a mirror image about an axis of an object is called reflection
- Reflection is a rotation operation with 180^0

  OR

  The mirror image for a two –dimensional reflection is generated relative to an ***axis of reflection*** by rotating the object 180o about the reflection axis
- In reflection size of object do not change.
- Basically, are of 3 types
  - About x-axis
  - About y-axis
  - About both (x, y) axis

X axis vane si tesko opposite kun xhaina, (y) so y negative. Yesari bhujyo vane 3d ma thakkai zun vanxha tyo chai negative hunxha

Reflection about *the x axis* (i.e. *y* = 0 line)

- In this Reflection
  - X- coordinate position same
  - Y- Coordinate position flip (change the sign)

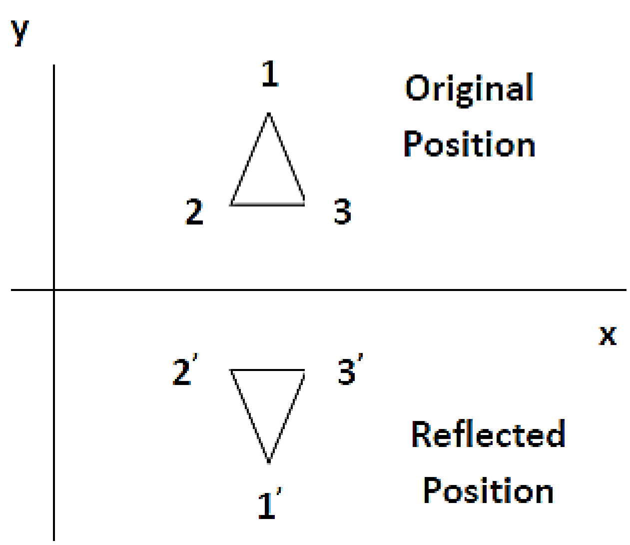


Figure: Reflection about x - axis.

i.e. $x' = x$ and $y' = -y$

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- Matrix Representation:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Hence, **P' = R_{fx} . P**

Reflection about the *the y axis* (i.e. line *x* = 0)

- In this Reflection
  - X- Coordinate position flip (change the sign)
  - Y- coordinate position same

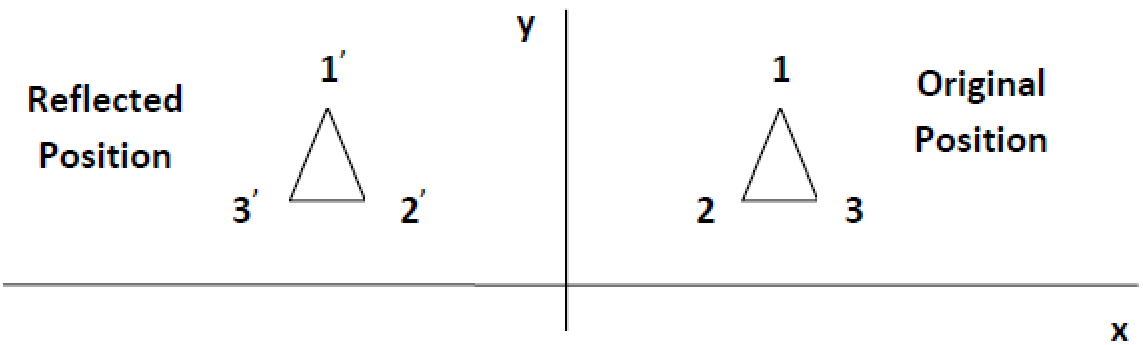

Figure: Reflection about y-axis

i.e. x' = -x and y'= y

In matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Hence, P' = R_{fy} . P

Reflection about the *Origin*.

- In this Reflection
  - X- Coordinate position flip (change the sign)
  - Y- coordinate position flip

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- i.e. both x and y coordinate flip

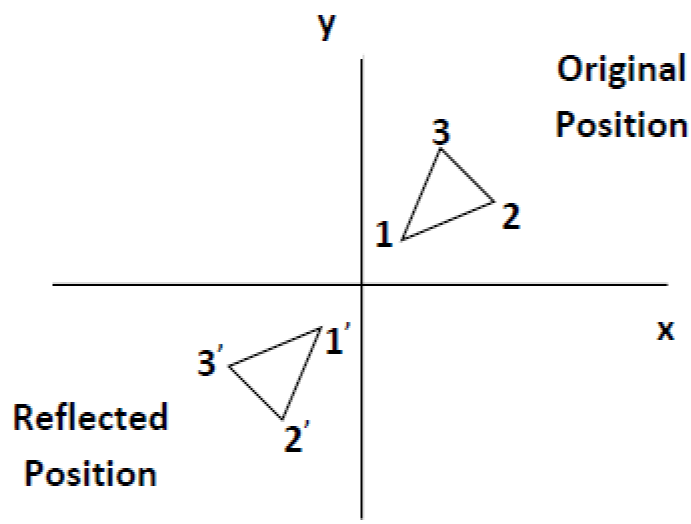


Figure: Reflection about Origin

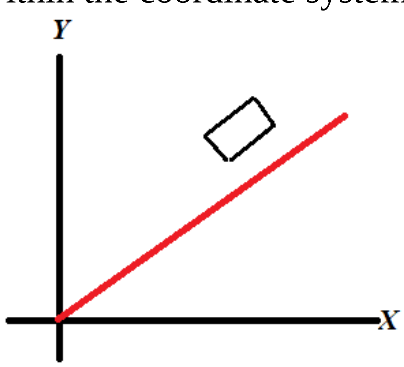
i.e x'= -x and y' = -y

Matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Reflection on any arbitrary axis:

- Any chosen axis within the coordinate system is arbitrary axis.
- Some arbitrary axis and reflection on them are as follows:



<u>Reflection about line y = x (i.e. θ = 45⁰)</u>

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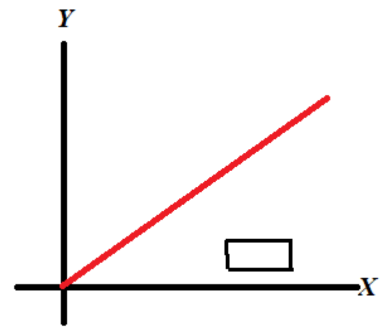
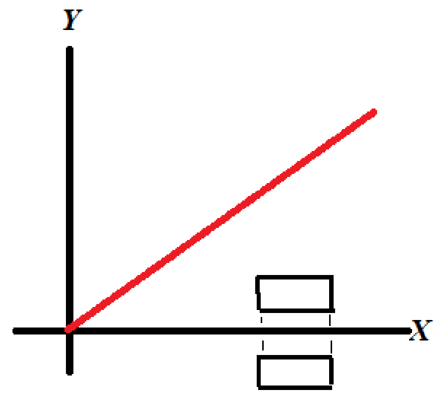


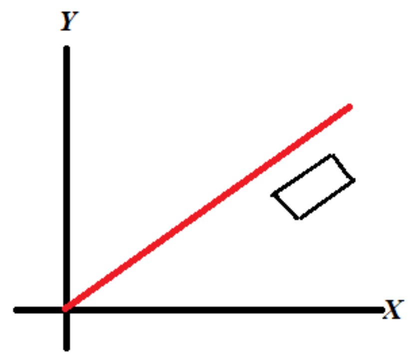
Do:

1. Rotate about origin in Clockwise direction by 45^0

   (This rotates the line y = x to x- axis)



2. Take Reflection against X-axis



3. Rotate in anticlockwise direction by same angle

<u>Therefore,</u>

CM = $R_{f(y = x)} = R_{(\theta)} * R_{fx} * R_{(-\theta)}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Trasnlate is used only for puting the pivot point at center, rotation here.

Hence, **P' = CM * P**

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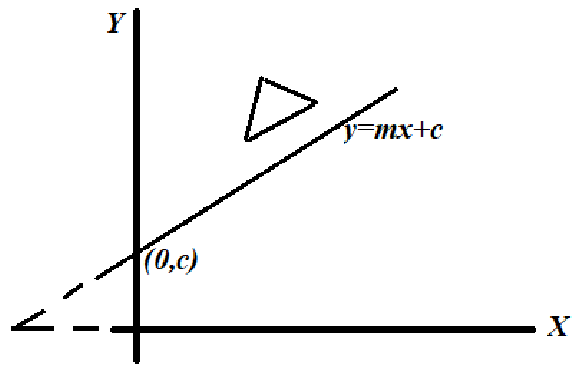
Reflection about y = mx + b

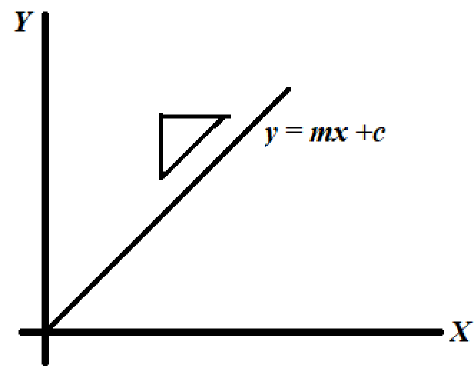
Here, initial position of object at

$$y = mx + c$$

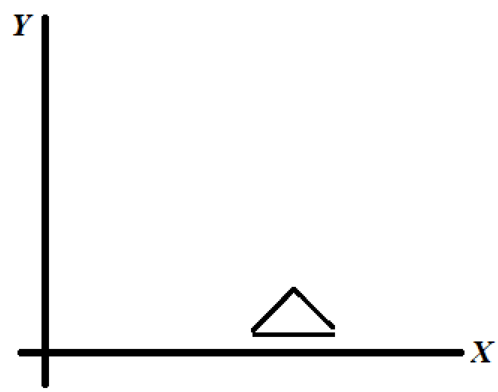


Do:

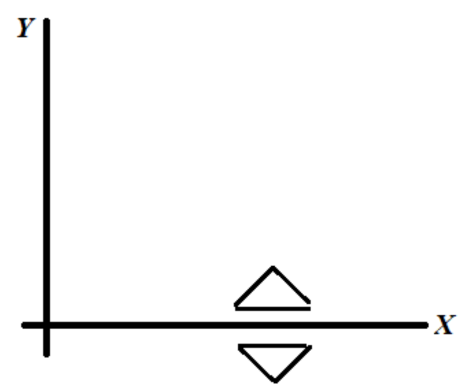
1. Translate the line and object so that the line passes through origin.



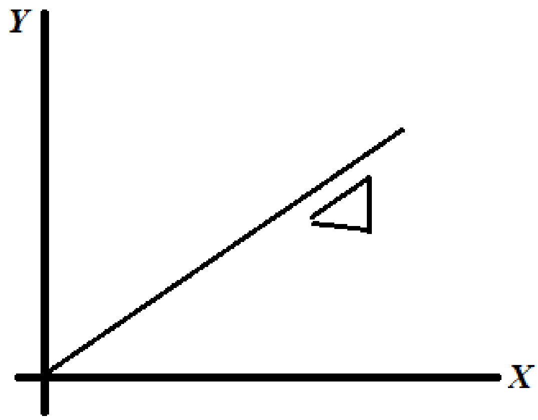
2. Rotate the line and object about origin until the line coincides with one of the coordinate axis



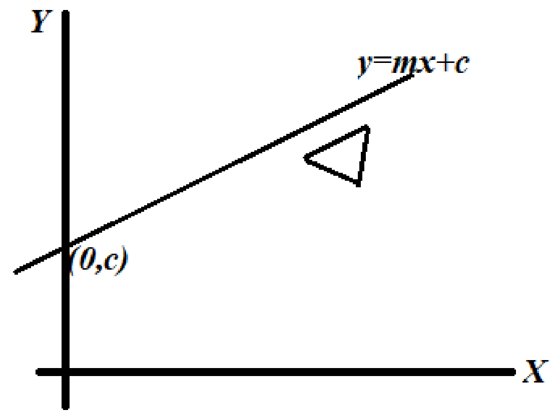
3. Reflect the object through about that axis



4. Apply the Inverse Rotation



5. Translate back to original Position



Therefore,

Composite Transformation Matrix for the Reflection is:

$CM = T_{(0,c)} \,.\, R_{\theta} .\, R_{x} \,.\, R_{-\theta} \,.\, T_{(0,-c)}$

Here, line y=mx+c

Slope (m) = tan θ

We have,

$\cos^2 \theta = 1 /$ $\frac{1}{\sqrt{m^2+1}}$ $(\tan^2 \theta + 1) = 1/ (m^2 + 1)$

$\cos \theta =$

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Similarly,

$\text{Sin}^2\,\theta + \cos^2\theta = 1$ $\quad 1 - \frac{1}{m^2+1} = \frac{m^2+1-1}{m^2+1}$

$\text{Sin}^2\,\theta = 1 - \cos^2\theta =$

$$\frac{m}{\sqrt{m^2+1}}$$

Sin θ =

Hence, CM = $\mathbf{T_{(0,c)} \cdot R_{\theta} \cdot R_x \cdot R_{-\theta} \cdot T_{(0,-c)}}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & -c\sin\theta \\ -\sin\theta & \cos\theta & -c\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & -c\sin\theta \\ \sin\theta & -\cos\theta & c\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{(m^2+1)}} & \frac{-m}{\sqrt{(m^2+1)}} & 0 \\ \frac{-m}{\sqrt{(m^2+1)}} & \frac{1}{\sqrt{(m^2+1)}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{(m^2+1)}} & \frac{m}{\sqrt{(m^2+1)}} & \frac{-cm}{\sqrt{(m^2+1)}} \\ \frac{m}{\sqrt{(m^2+1)}} & \frac{-1}{\sqrt{(m^2+1)}} & \frac{c}{\sqrt{(m^2+1)}} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & \frac{-2cm}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & \frac{c-cm^2}{m^2+1} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-m^2}{m^2+1} & \frac{2m}{m^2+1} & \frac{-2mc}{m^2+1} \\ \frac{2m}{m^2+1} & \frac{m^2-1}{m^2+1} & \frac{2c}{m^2+1} \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, **P' = CM * P**

Numerical Session:

3-Dimensional Transformation

- In computer Graphics, 3-D (three dimensions or three-dimensional) describes an image that provides the perception of depth.
- 3D Graphics 3D computer graphics or three-dimensional computer graphics are graphics that use a three-dimensional representation of geometric data that is stored in the computer for the purposes of performing calculations and rendering 2D images.
- 2D is "flat", using the horizontal and vertical (X and Y) dimensions, the image has only two dimensions.
- 3D adds the depth (Z) dimension. This third dimension allows for rotation and visualization from multiple perspectives. It is essentially the difference between a photo and a sculpture.

What are the issue in 3D that makes it more complex than 2D?

When we model and display a three-dimensional scene, there are many more considerations we must take into account besides just including coordinate values as 2D, some of them are:

- Relatively more co-ordinate points are necessary in comparison with 2D.
- Object boundaries can be constructed with various combinations of plane and curved surfaces.
- Consideration of projection (dimension change with distance) and transparency.
- Many considerations on visible surface detection and remove the hidden surfaces

3D Geometric Transformation

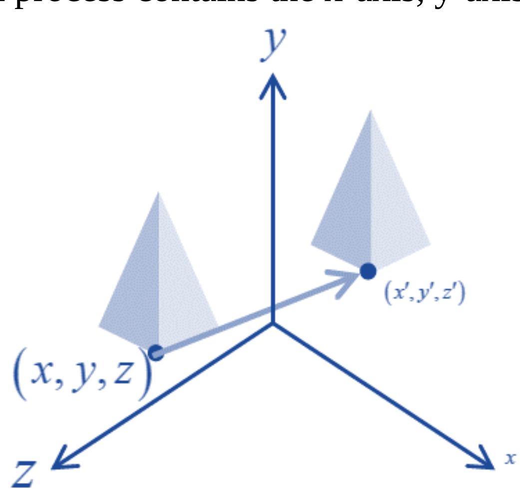
- Methods for geometric transformations and object modeling in three dimensions are extended from two-dimensional methods by including considerations for the z coordinate.
- Translation of an object in 3D is obtained by specifying a three dimensional translation vector, which determines how much the object is to be moved in each of the three coordinate directions.

Homogeneous Representation in 3D

- 2D transformations can be represented by 3 x 3 matrices using homogeneous coordinates.
- Similarly, 3D transformations can be represented by 4 x 4 matrices, by using homogeneous coordinate representations of points in 2 spaces as well.
- Thus, instead of representing a point as (x, y, z), we represent it as (x, y, z, H),
  - Where two these quadruples represent the same point it one is a non-zero multiple of the other the quadruple (0, 0, 0, 0) is not allowed.
- A standard representation of a point (x, y, z, H) with H not zero is given by (x/H, y/H, z/H, 1).
- Transforming the point to (x, y, z, 1) form is called homogenizing.

Translation

- A translation moves all points in an object along the same straight line path to new positions.
- 3DTranslation process contains the x-axis, y-axis, and z-axis.



Let's translate a point from P (x, y, z) to Q (x' y', z') and Translation distance towards x, y and z axis are t_x, t_y, t_z then the path is represented by a vector, called the translation or shift vector. We can write the components as:

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$
$$1=1$$

We can also represent the 3D Translation in matrix form

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Example: A Point has coordinates P (1, 2, 3) in x, y, z-direction. Apply the translation with a distance of 2 towards x-axis, 3 towards y-axis, and 4 towards the z-axis. Find the new coordinates of the point?

Solution:

We have,
Point P = (x_0, y_0, z_0) = (1, 2, 3)
Shift Vector = (T_x, T_y, T_z)
Let us assume the new coordinates of P = (x_1, y_1, z_1)
Now we are going to add translation vector and given coordinates, then

$X_1 = x_0 + T_x = (1 + 2) = 3$
$Y_1 = y_0 + T_y = (2 + 3) = 5$
$Z_1 = z_0 + T_z = (3 + 4) = 7$
Thus, the new coordinates are = (3, 5, 7)

Scaling

- Scaling changes the size of an object and involves the scale factors.

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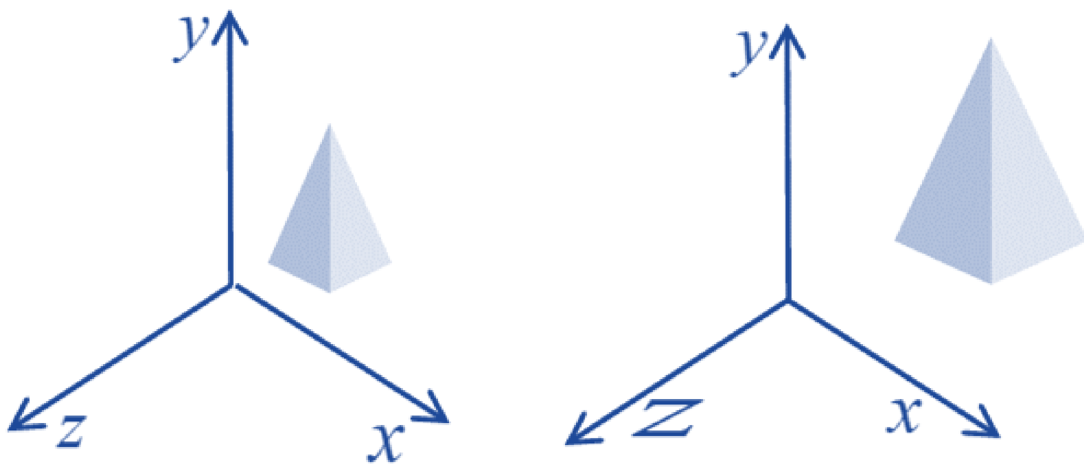
- The 2D and 3D scaling are similar, but the key difference is that the 3D plane also includes the z-axis along with the x and y-axis.

- The scaling factor towards x, y and z axis is denoted by **'S_x' 'S$_y$' and 'S_z' respectively.**



The object before scaling Object after scaling

The increment and decrement of an object is depends on two conditions. They are

i. **If scaling factor (S_x, S_y, S_z) > 1, then the size of the object increased.**
ii. **If scaling factor (S_x, S_y, S_z) < 1, then the size of the object decreased.**

Let us assume,
The initial coordinates of object = P (x, y, z)
Scaling factor for x-axis = S_x
Scaling factor for y-axis = S_y
Scaling factor for z-axis = S_z
The coordinates after Scaling = Q (x', y', z')

- **We can represent the 3D Scaling in the form of equation-**

$$X_1 = x.\ S_x$$
$$Y_1 = y.\ S_y$$
$$Z_1 = z.\ S_z$$

❖ Matrix representation of 3D Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{pmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Therefore, P' = S.P

Example: A 3D object that have coordinates points P(1, 4, 4), Q(4, 4, 6), R(4, 1, 2), T(1, 1, 1) and the scaling parameters are 3 along with x-axis, 4 along with y-axis and 4 along with z-axis. Apply scaling to find the new coordinates of the object?

Solution: We have,
The initial coordinates of object = P (1, 4, 4), Q (4, 4, 6), R (4, 1, 2), S (1, 1, 1)
Scaling factor along with x-axis (S_x) = 3
Scaling factor along with y-axis (S_y) = 4
Scaling factor along with z-axis (S_z) = 4
Let the new coordinates after scaling = **(x', y', z')**

For coordinate P:

$X' = x.\ S_x = 1 \times 3 = 3$
$Y' = y.S_y = 4 \times 4 = 16$
$Z' = z.\ S_z = 4 \times 4 = 16$

The new coordinates = (3, 16, 16)

For coordinate Q:

$X' = x.S_x = 4 \times 3 = 12$
$Y' = y.S_y = 4 \times 4 = 16$
$Z' = z.S_z = 6 \times 4 = 24$

The new coordinates = (12, 16, 24)

For coordinate R:

$X' = x.S_x = 4 \times 3 = 12$
$Y' = y.S_y = 1 \times 4 = 4$

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$Z' = z.S_z = 2 \times 4 = 8$

The new coordinates = (12, 4, 8)

For coordinate S:

$X' = x.S_x = 1 \times 3 = 3$
$Y' = y.S_y = 1 \times 4 = 4$
$Z' = z.S_z = 1 \times 4 = 4$

The new coordinates = (3, 4, 4)

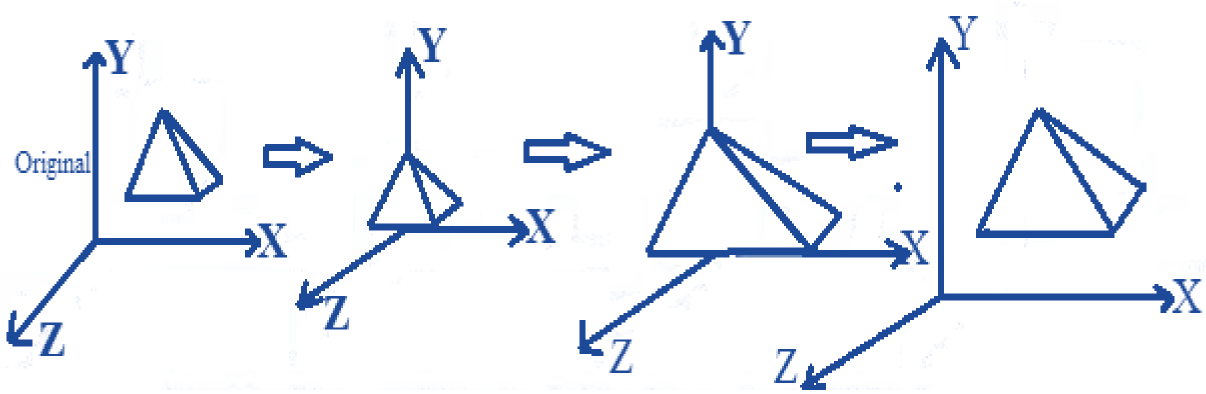
Thus, the new coordinates after scaling = P (3, 16, 16), Q (12, 16, 24), R (12, 4, 8), S (3, 4, 4).

Fixed Point Scaling in 3D

Let fixed point (x_f, y_f, z_f).

Do:

1. Translate fixed point to the origin
2. Scale object relative to the coordinate origin
3. Translate fixed point back to its original position



$CM = \mathbf{T}_{(x, y, z)}.\ \mathbf{S}_{(x, y, z)}.\ \mathbf{T}_{(-x, -y, -z)}$

$$= \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

P' = CM * P

Shearing

- Shearing transformations are used to modify object shapes.
- The basic difference between 2D and 3D Shearing is that the 3D plane also includes the z-axis.
- Let the point P (x, y,z) is obtained P' (x', y', z') after applying shear with shearing factor for x, y and z axis are **'Sh_x,' 'Sh_y,' and 'Sh_z.' respectively.**

Basically, Shearing in 3D Geometry are categorized into 3 different types

a) X- axis Shearing

- In this transformation alters Y and Z coordinate values by an amount that is proportional to the X value while leaving the X value unchanged i.e.

x'= x
y'= y + Shy * x
z'= z + shz * x

In 2d,
x' = x + Shx * y
y' = y

3D matrix representation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ shy & 1 & 0 & 0 \\ shz & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

b) Y -axis shearing

This transformation alters Y and Z coordinate values by an amount that is proportional to the Y value while leaving the Y value unchanged i.e.

y'= y
x'= x + Shx * y
z'= z +Shz * y

3D Matrix Representation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & shx & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & shz & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

a) Z -axis Shearing

This transformation alters x and y coordinate values by amount that is proportional to the z value while leaving z co-ordinate unchanged.

x' = x + Shx * z
y' = y + Shy * z
z' = z

3D Matrix Representation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & shx & 0 \\ 0 & 1 & shy & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Share Parameters Shx and Shy can be assigned any real values

Example.

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Given a 3D triangle with points (0, 0, 0), (1, 1, 2) and (1, 1, 3). Apply shear parameter 2 on X axis, 2 on Y axis and 3 on Z axis and find out the new coordinates of the object.

Solution

Given that,

Old corner coordinates of the triangle = A (0, 0, 0), B(1, 1, 2), C(1, 1, 3)
Shearing parameter towards X direction (Sh_x) = 2
Shearing parameter towards Y direction (Sh_y) = 2
Shearing parameter towards Y direction (Sh_z) = 3

<u>Shearing in X Axis</u>

<u>For Coordinates A (0, 0, 0)</u>

Let the new coordinates of corner A after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X = 0$
$Y' = Y + Sh_y . X = 0 + 2 \text{ x } 0 = 0$
$Z' = Z + Sh_z . X = 0 + 3 \text{ x } 0 = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

<u>For Coordinates B(1, 1, 2)</u>

Let the new coordinates of corner B after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X = 1$
$Y' = Y + Sh_y \text{ x } X = 1 + 2 \text{ x } 1 = 3$
$Z' = Z + Sh_z \text{ x } X = 2 + 3 \text{ x } 1 = 5$

Thus, New coordinates of corner B after shearing = (1, 3, 5).

<u>For Coordinates C (1, 1, 3)</u>

Let the new coordinates of corner C after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X = 1$
$Y' = Y + Sh_y \text{ x } X = 1 + 2 \text{ x } 1 = 3$

$Z' = Z + Sh_z \times X = 3 + 3 \times 1 = 6$

Thus, New coordinates of corner C after shearing = (1, 3, 6).

Thus, New coordinates of the triangle after shearing in X axis = A (0, 0, 0), B(1, 3, 5), C(1, 3, 6).

Shearing in Y Axis

For Coordinates A (0, 0, 0)

Let the new coordinates of corner A after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X + Sh_x \times Y = 0 + 2 \times 0 = 0$

$Y' = Y = 0$

$Z' = Z + Sh_z \times Y = 0 + 3 \times 0 = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

For Coordinates B (1, 1, 2)

Let the new coordinates of corner B after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X + Sh_x \times Y = 1 + 2 \times 1 = 3$

$Y' = Y = 1$

$Z' = Z + Sh_z \times Y = 2 + 3 \times 1 = 5$

Thus, New coordinates of corner B after shearing = (3, 1, 5).

For Coordinates C (1, 1, 3)

Let the new coordinates of corner C after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X + Sh_x \times Y = 1 + 2 \times 1 = 3$

$Y' = Y = 1$

$Z' = Z + Sh_z \times Y = 3 + 3 \times 1 = 6$

Thus, New coordinates of corner C after shearing = (3, 1, and 6).

Thus, New coordinates of the triangle after shearing in Y axis = A (0, 0, 0), B (3, 1, 5), C (3, 1, 6).

Shearing in Z Axis

For Coordinates A(0, 0, 0)

Let the new coordinates of corner A after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X + Sh_x \times Z = 0 + 2 \times 0 = 0$
$Y' = Y + Sh_y \times Z = 0 + 2 \times 0 = 0$
$Z' = Z = 0$

Thus, New coordinates of corner A after shearing = (0, 0, 0).

For Coordinates B(1, 1, 2)

Let the new coordinates of corner B after shearing = (X', Y', Z').

Applying the shearing equations, we have,

$X' = X + Sh_x \times Z = 1 + 2 \times 2 = 5$
$Y' = Y + Sh_y \times Z = 1 + 2 \times 2 = 5$
$Z' = Z = 2$

Thus, New coordinates of corner B after shearing = (5, 5, 2).

For Coordinates C(1, 1, 3)

Let the new coordinates of corner C after shearing = (X', Y', Z').

Applying the shearing equations, we have-

$X' = X + Sh_x \times Z = 1 + 2 \times 3 = 7$
$Y' = Y + Sh_y \times Z = 1 + 2 \times 3 = 7$
$Z' = Z = 3$

Thus, New coordinates of corner C after shearing = (7, 7, 3).

Thus, New coordinates of the triangle after shearing in Z axis = A (0, 0, 0), B (5, 5, 2), C (7, 7, 3).

Reflection

- A Three-dimensional can be performed relative to a selected reflection axes or with respect to selected reflected plane.
- Three-dimensional reflection matrices are set up similar to those for two dimensional.
- Reflection relative to given axis are equivalent to 180^0 rotation about that axis.
- The reflection planes are either XY, XZ or YZ (i.e. 3 types)
- The reflected object is always formed on the other side of mirror.

Consider a point object O has to be reflected in a 3D plane.

Let, Initial coordinates of the object O = (X, Y, Z) and new coordinates of the reflected object O after reflection = (X', Y', Z')

Reflection Relative to XY Plane (i.e. Reflection along Z axis)

- X and y value same, z value flip

OR, this transformation changes the sign of the z coordinates, leaving the x and y coordinate values unchanged i.e.

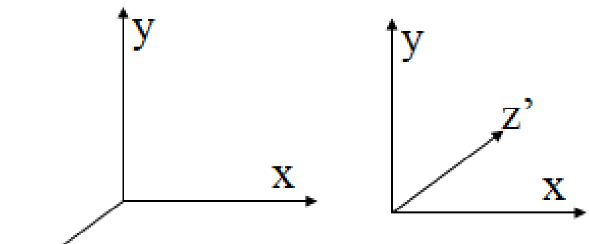


Fig. 3D reflection relative to the XY- Plane

X' = X
Y' = Y
Z' = -Z

In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection Relative to YZ Plane (i.e. Reflection along X axis)

Y and Z value unchanged, X value flip

OR, this transformation changes the sign of the x coordinates, leaving the y and z coordinate values unchanged.

This reflection is achieved by using the following reflection equations

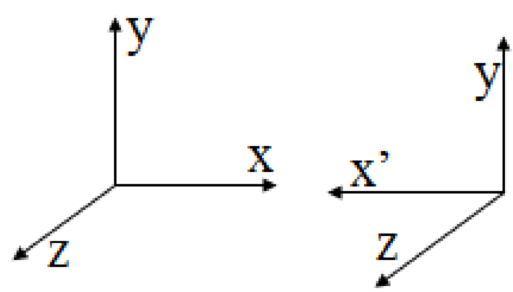


Fig. 3D reflection relative to the YZ- Plane

X' = -X
Y' = Y
Z' = Z

In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Reflection Relative to XZ Plane (i.e. Reflection along Y axis)

- X and Z values are unchanged while Y value flip.

OR, this transformation changes the sign of the y coordinates, leaving the x and z coordinate values unchanged.

This reflection is achieved by using the following reflection equations

X’ = X
Y’ = -Y
Z’ = Z

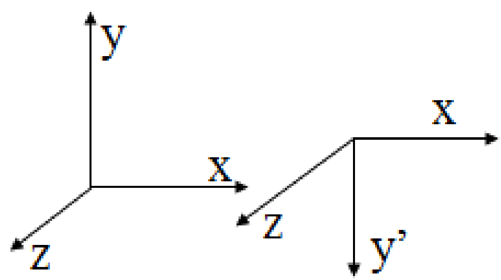


Fig. 3D reflection relative to the ZX- Plane

In Matrix form, the above reflection equations may be represented as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Example.

Given a 3D triangle with coordinate points A (3, 4, 1), B (6, 4, 2), C (5, 6, 3). Apply the reflection on the XY plane and find out the new coordinates of the object.

Solution

Given that,

Old corner coordinates of the triangle = A (3, 4, 1), B (6, 4, 2), C (5, 6, 3)
Reflection has to be taken on the XY plane

For Coordinates A (3, 4, 1)

Let the new coordinates of corner A after reflection = (X', Y', Z').

Applying the reflection equations, we have-

X' = X = 3
Y' = Y = 4
Z' = -Z= -1

Thus, New coordinates of corner A after reflection = (3, 4, -1).

For Coordinates B (6, 4, 2)

Let the new coordinates of corner B after reflection = (X', Y', Z')

Applying the reflection equations, we have-

X' = X = 6
Y' = Y = 4
Z' = -Z= -2

Thus, New coordinates of corner B after reflection = (6, 4, -2).

For Coordinates C (5, 6, 3)

Let the new coordinates of corner C after reflection = (X', Y', Z').

Applying the reflection equations, we have-

X' = X = 5
Y' = Y = 6
Z' = -Z= -3

Thus, New coordinates of corner C after reflection = (5, 6, -3).

Thus, New coordinates of the triangle after reflection = A (3, 4, -1), B (6, 4, -2), C(5, 6, -3).

Example-2

Given a 3D triangle with coordinate points A (3, 4, 1), B (6, 4, 2), C (5, 6, 3). Apply the reflection on the XZ plane and find out the new coordinates of the object.

Solution

Given that,

Old corner coordinates of the triangle = A (3, 4, 1), B (6, 4, 2), C(5, 6, 3)
Reflection has to be taken on the XZ plane

For Coordinates A (3, 4, 1)

Let the new coordinates of corner A after reflection = (X', Y', Z').

Applying the reflection equations, we have-

X' = X = 3
Y' = -Y = -4
Z' = Z= 1

Thus, New coordinates of corner A after reflection = (3, -4, 1).

For Coordinates B (6, 4, 2)

Let the new coordinates of corner B after reflection = (X', Y', Z').

Applying the reflection equations, we have-

X' = X = 6
Y' = -Y = -4
Z' = Z= 2

Thus, New coordinates of corner B after reflection = (6, -4, 2).

For Coordinates C (5, 6, 3)

Let the new coordinates of corner C after reflection = (X', Y', Z').

Applying the reflection equations, we have-

X' = X = 5
Y' = -Y = -6
Z' = Z= 3

Thus, New coordinates of corner C after reflection = (5, -6, 3).

Thus, New coordinates of the triangle after reflection = A (3, -4, 1), B (6, -4, 2), C (5, -6, 3).

Reflection about any axis parallel to one of the coordinate axes

Do:

1. Translate object so that reflection axis coincides with the parallel coordinate axis.
2. Perform specified reflection about that axis
3. Translate object back to its original Position

$$\mathbf{CM = T^{-1}. R. T}$$

$$\mathbf{P' = CM. P}$$

Reflection about any arbitrary plane in 3D Space

- Reflection about any arbitrary plane in 3D is similar to the reflection about any arbitrary line
- But the difference is that, we have to characterize the rotation by any normal vector 'N' in that plane.

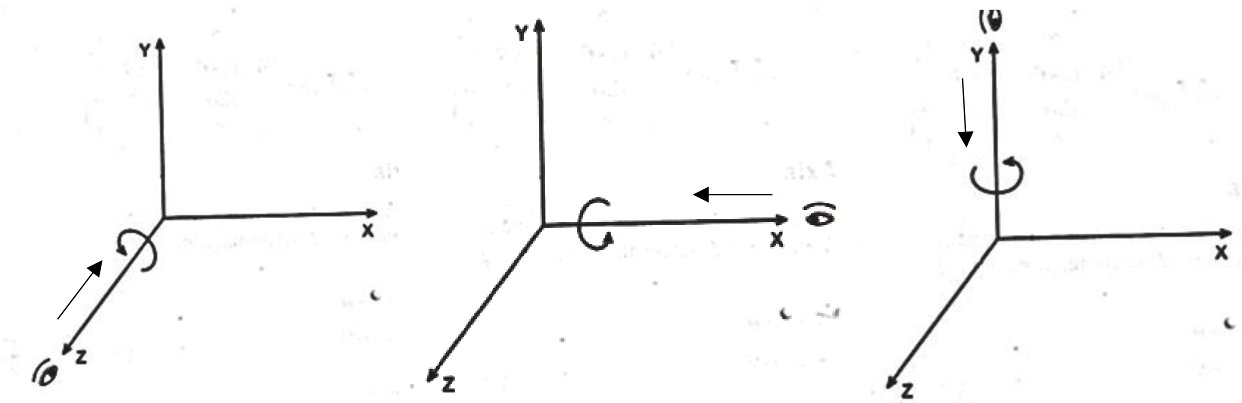
Step 1: Translate the reflection plane to the origin of the coordinate system
Step 2: Perform appropriate rotations to make the normal vector of the reflection plane at the origin until it coincides with the z-axis.
Step 3: After that reflect the object through the z= 0 coordinate plane.
Step 4: Perform the inverse of the rotation transformation
Step 5: Perform the inverse of the translation

Rotation in 3D

- In CG, 3D rotation is a process of rotating an object to an angle in a three dimensional plane.
- Rotation is moving of an object about an angle.
- Movement can be anticlockwise or clockwise.
- 3D rotation is complex as compared to the 2D rotation. For 2D we describe the angle of rotation, but for a 3D angle of rotation and axis of rotation are required. The axis can be either x or y or z.

- ❖ To determine a rotation transformation for an object in 3D space, following information is required:
  - ✓ Angle of rotation.
  - ✓ Pivot point
  - ✓ Direction of rotation
  - ✓ Axis of rotation

90 deg or 45 deg
middle point
clock wise / anti clockwise
x , y, z



Figures: Coordinate axis Rotations

Let's consider a origin as the center of rotation and a point P(x, y, z) is rotated through an angle about any one of the axes to get the transformed point P'(x', y', z'), then the equation for each rotation can be obtained as follows.

<u>3D Z-axis Rotation</u>

- ❖ This rotation is achieved by using the following rotation equations
- ❖ Two dimension rotation equations can be easily convert into 3D z- axis rotation equation.
- ❖ Rotation about z axis we leave z coordinate unchanged.

$$X' = x \cos \theta - y \sin \theta$$
$$Y' = x \sin \theta + y \cos \theta$$
$$z' = z$$

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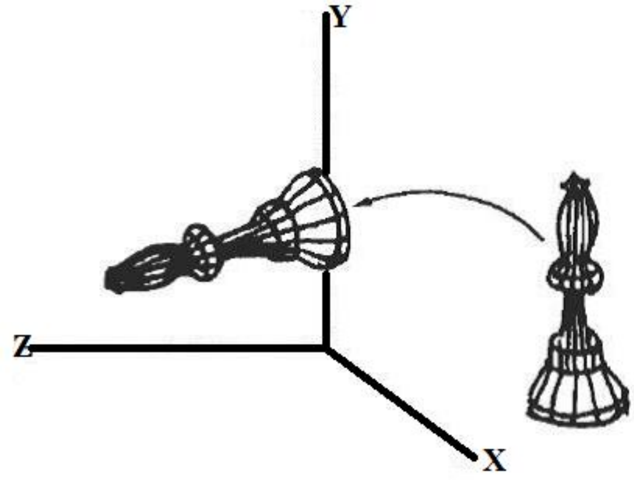
Z' = z

In matrix representation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Therefore, **P' = R$_{z(\theta)}$. P**

3D X-axis Rotation

This rotation is achieved by using the following rotation equations:

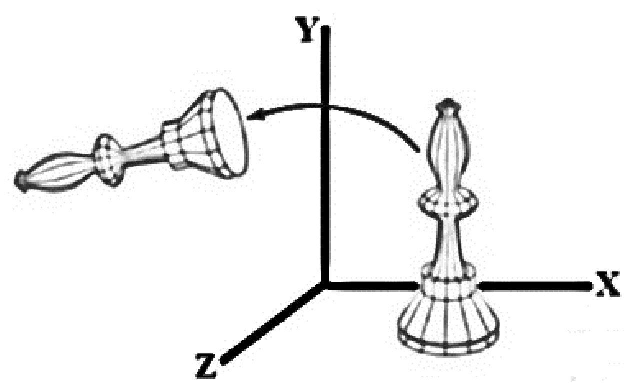
X' = x

Y' = y cos θ – z sin θ

Z' = y sin θ + z cos θ

In matrix Representation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



Therefore, **P' = R$_{x(\theta)}$. P**

3D Y-axis Rotation

Equations for this rotation are as:

X' = z sin θ + x cos θ
Y' = y
z' = zcos0 - xsin0

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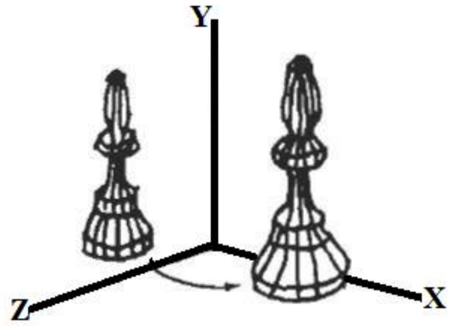
Z' = z cos θ – x sin θ

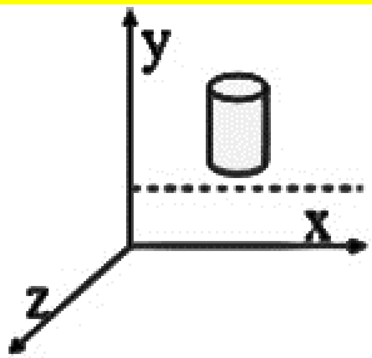
Matrix representation of these equation,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



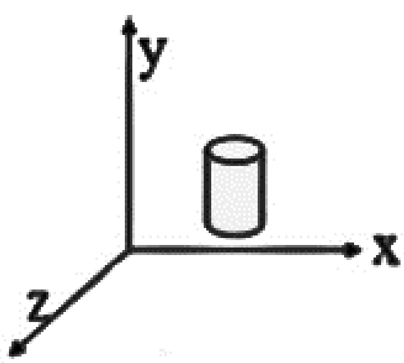
Therefore, **P' = $\mathbf{R}_{y(\theta)}$. P**

Rotation about an axis parallel to one of the coordinate axes

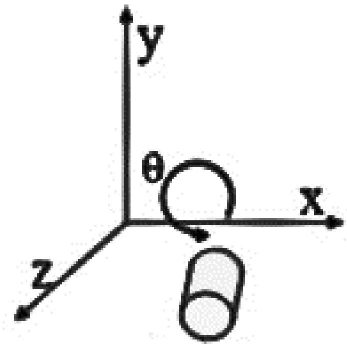


<u>DO:</u>

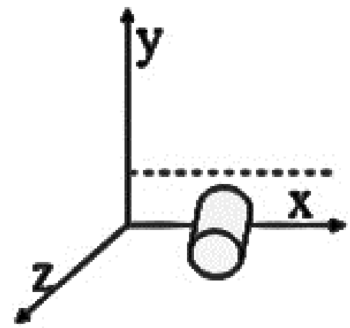
1. Translate object so that rotation axis coincides with the parallel coordinate axis.



2. Perform specified rotation about that axis



3. Translate object back to its original Position

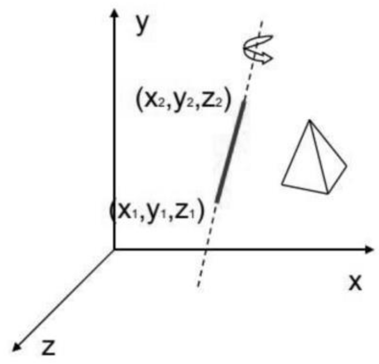


Therefore, CM= T R T^{-1}

P'= CM.P

Rotation about an arbitrary axis

- ❖ A rotation matrix for any axis that does not coincide with a coordinate axis can be set up as a composite Transformation involving combinations of translation and the coordinate-axes rotations.



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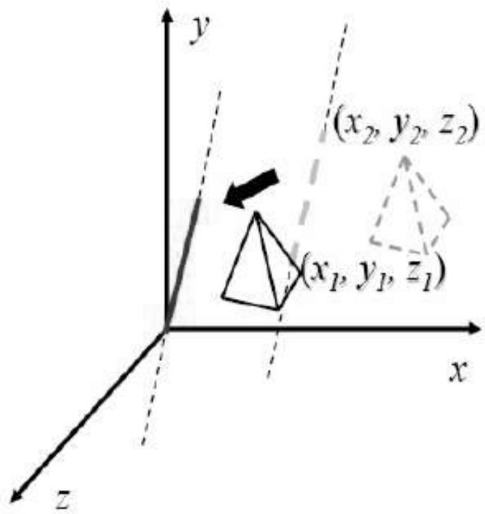
DO:

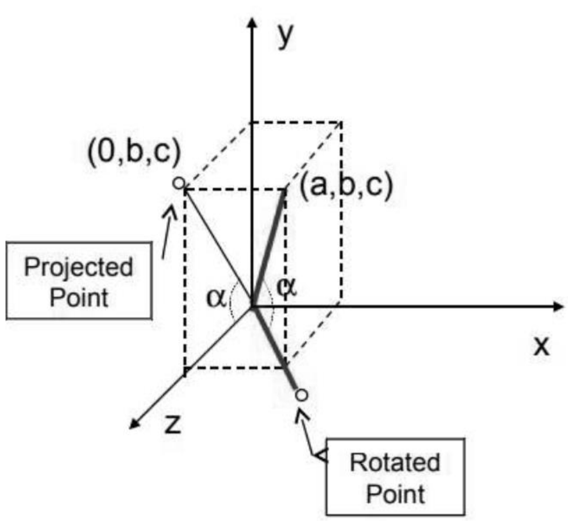
Step-1: Translate the arbitrary axis so that it passes through origin.
Step-2: Align the arbitrary axis on any major co-ordinate axis (z-axis)
Step-3: Rotate the object about yz-plane or z-axis
Step-4: Perform inverse rotation about y-axis & then x-axis
Step-5: Perform inverse translation

Step-1: Translate



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-2: Rotate about X-axis by an angle α



$$\sin\alpha = \frac{b}{\sqrt{b^2+c^2}} = \frac{b}{d}$$

$$\cos\alpha = \frac{c}{\sqrt{b^2+c^2}} = \frac{c}{d}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c/d & -b/d & 0 \\ 0 & b/d & c/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-3: Rotate about Y-axis by an angle β

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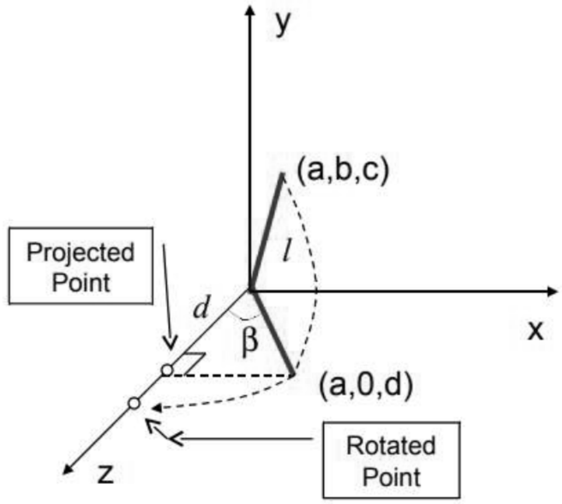


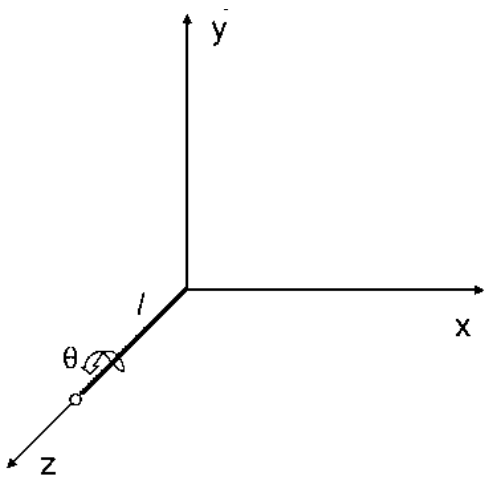
$$\sin\beta = \frac{a}{l}, \qquad \cos\beta = \frac{d}{l}$$

$$l^2 = a^2 + b^2 + c^2 = a^2 + d^2$$

$$d = \sqrt{b^2 + c^2}$$

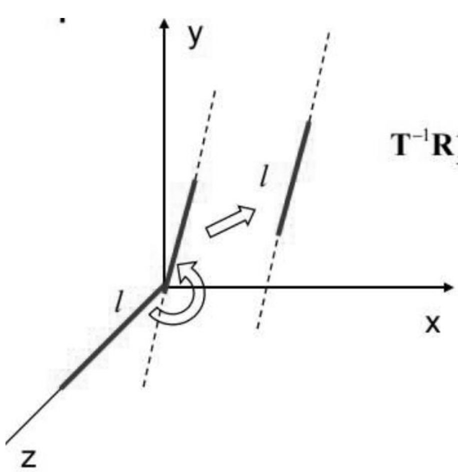
$$\mathbf{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & -\sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d/l & 0 & -a/l & 0 \\ 0 & 1 & 0 & 0 \\ a/l & 0 & d/l & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-4: Rotate about Z-axis by an angle θ



$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-5 Apply the inverse transformation



$$\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta) = \begin{bmatrix} 1 & 0 & 0 & -x_1 \\ 0 & 1 & 0 & -y_1 \\ 0 & 0 & 1 & -z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

Composite matrix (CM) = $\mathbf{T\,R_x\,R_y\,R_z\,R_y^{-1}\,R_x^{-1}\,T^{-1}}$

Numerical: Find the new co-ordinates of a unit cube 90 degree rotated about an axis defined by its end points A(2,1,0) and B(3,3,1).

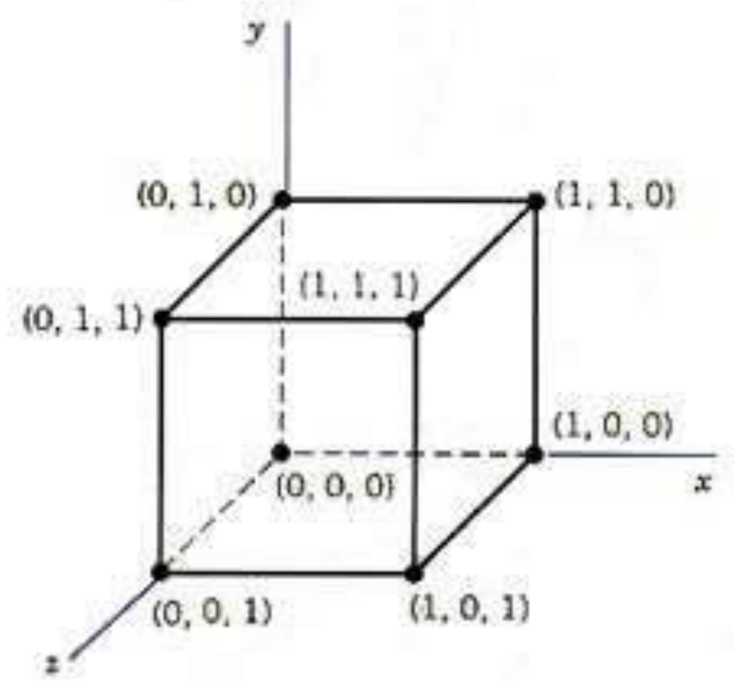
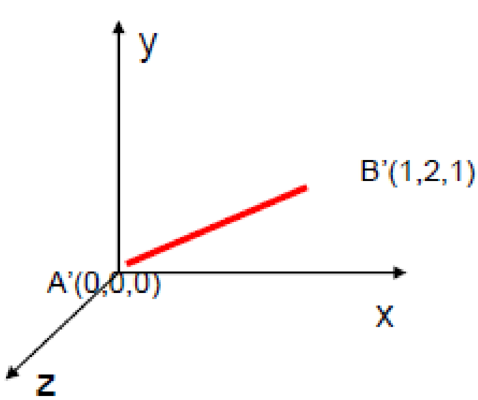


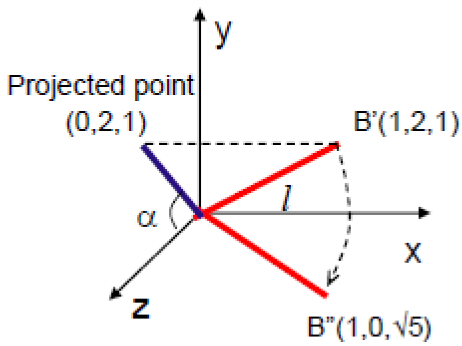
Figure: A Unit Cube

Step -1 Translate Point A to the Origin



$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-2: Rotate axix A'B' about the X axis by and angle α, until it lies on the XZ plane.



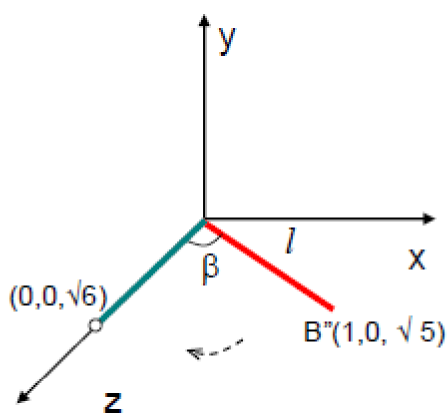
$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\cos\alpha = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$l = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\mathbf{R}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-3: Rotate axis A'B'' about the Y axis by and angle β until it coincides with the Z axis.



$$\sin\beta = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

$$\cos\beta = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

$$\mathbf{R}_y(\beta) = \begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step-4: Rotate the Cube 90 degree about Z axis.

$$\mathbf{R}_z(90°)=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finally, the Composite Matrix for the Rotation about the arbitrary axis AB becomes,

$$\mathbf{R}(\theta)=\mathbf{T}^{-1}\mathbf{R}_x^{-1}(\alpha)\mathbf{R}_y^{-1}(\beta)\mathbf{R}_z(90°)\mathbf{R}_y(\beta)\mathbf{R}_x(\alpha)\mathbf{T}$$

$$\mathbf{R}(\theta)=\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 \\ 0 & -\frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & \frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{30}}{6} & 0 & -\frac{\sqrt{6}}{6} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{30}}{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{5}}{5} & -\frac{2\sqrt{5}}{5} & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

Multiplying R (θ) by the point matrix of the original cube we get P’

$$[\mathbf{P}'] = \mathbf{R}(\theta) \cdot [\mathbf{P}]$$

$$[\mathbf{P}'] = \begin{bmatrix} 0.166 & -0.075 & 0.983 & 1.742 \\ 0.742 & 0.667 & 0.075 & -1.151 \\ -0.650 & 0.741 & 0.167 & 0.560 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.650 & 1.667 & 1.834 & 2.816 & 2.725 & 1.742 & 1.909 & 2.891 \\ -0.558 & -0.484 & 0.258 & 0.184 & -1.225 & -1.151 & -0.409 & -0.483 \\ 1.467 & 1.301 & 0.650 & 0.817 & 0.726 & 0.560 & -0.091 & 0.076 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Rotation about any arbitrary plane in 3D Space

- The rotation about any arbitrary plane perform same operation as the rotation about any arbitrary line, the only difference is that we have to characterize the rotation by any normal vector 'N' in that plane.

DO:

Step 1: Translate the rotation plane to the origin of the coordinate system
Step 2: Perform appropriate rotations to make the normal vector of the rotation plane at the origin until it coincides with the z-axis.
Step 3: After that rotate the object through the z= 0 coordinate plane.
Step 4: Perform the inverse of the rotation transformation
Step 5: Perform the inverse of the translation

End of Unit-2