Deriving the Mean of for continuous amplitude Gaussian

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = 1$$

with respect to the two parameters μ and σ (RHS will then be zero).
The Gaussian pdf is defined as

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

where μ and σ are two parameters, with $\sigma > 0$. By definition of the mean we have

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

which using integral properties can be written as

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} (x+\mu) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \\ &= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \;+\; \int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \end{aligned}$$

For the first integral, call it I_1 we have using additivity

$$I_1 = \int_{-\infty}^{0} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx$$

Swapping the integration limits in the first we have

$$I_1 = -\int_{0}^{-\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx$$

and using again integral properties we have

$$\begin{aligned} I_1 &= \int_{0}^{\infty} (-x) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(-x)^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx \\ \Rightarrow I_1 &= -\int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx + \int_{0}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = 0 \end{aligned}$$

So we have that

$$E(X) = \int_{-\infty}^{\infty} \mu \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx$$

Multiply by $\sigma\sqrt{2}$ to obtain

$$E(X) = \int_{-\infty}^{\infty} \mu \frac{1}{\sqrt{\pi}} e^{-x^2} dx = \mu \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x^2} dx$$

Now

$$\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-x^2} dx = \lim_{t\to\infty} \frac{2}{\sqrt{\pi}} \int_{0}^{t} e^{-x^2} dx = \lim_{t\to\infty} \operatorname{erf}(t) = 1$$

where "erf" is the error function So we end up with

$$E(X) = \mu$$

hence the parameter μ is the mean of the distribution.

Now the Deriving the Variance

We have

$$\text{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

Applying the same tricks as before we have

$$\int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx$$

$$= \sigma\sqrt{2} \int_{-\infty}^{\infty} (\sigma\sqrt{2}x)^2 \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\sigma\sqrt{2}x)^2}{2\sigma^2}\right\} dx$$

$$= \sigma^2 \frac{4}{\sqrt{\pi}} \int_0^{\infty} x^2 e^{-x^2} dx$$

Define $t = x^2 \Rightarrow x = \sqrt{t}$ and $dt = 2xdx = 2\sqrt{t}dx \Rightarrow dx = (2\sqrt{t})^{-1}dt$

Substituting

$$[V(X) = \sigma^2 \frac{4}{\sqrt{\pi}} \int_0^{\infty} (\sqrt{t})^2 (2\sqrt{t})^{-1} e^{-t} dt = \sigma^2 \frac{4}{\sqrt{\pi}} \frac{1}{2} \int_0^{\infty} t^{\frac{3}{2}-1} e^{-t} dt$$

$$= \sigma^2 \frac{4}{\sqrt{\pi}} \frac{1}{2} \Gamma\left(\frac{3}{2}\right)$$

$$\Rightarrow V(X) = \sigma^2 \frac{4}{\sqrt{\pi}} \frac{1}{2} \frac{\sqrt{\pi}}{2} = \sigma^2$$

where $\Gamma()$ is the Gamma function. So the parameter σ is the square-root of the variance, i.e. the standard deviation.

Mean and Variance of a Uniform Distribution

Using the basic definition of expectation we may write:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx = \frac{1}{2(b-a)} \left[x^2\right]_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} \\ &= \frac{b+a}{2} \end{aligned}$$

Using the formula for the variance, we may write:

$$
\begin{aligned}
V(X) &= E(X^2) - [E(X)]^2 \\
&= \int_a^b x^2 \cdot \frac{1}{b-a} dx - \left(\frac{b+a}{2}\right)^2 = \frac{1}{3(b-a)} \left[x^3\right]_a^b - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$