Projection of analytic surfaces in higher dimension

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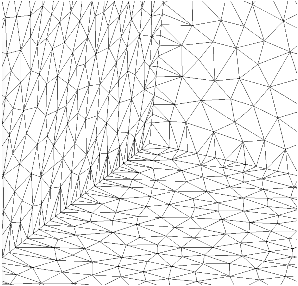
Description

- $F, G : \mathbb{R}^4 \to \mathbb{R}$ two real analytic functions
- $\mathcal{S} = \{(x, y, z, t) \in \mathbb{R}^4 \mid F(x, y, z, t) = G(x, y, z, t) = 0\}$ smooth surface
- $\mathfrak{p} : \mathbb{R}^4 \to \mathbb{R}^3$
  $(x, y, z, t) \mapsto (x, y, z)$ projection along the direction $(0, 0, 0, 1)$
- $\Omega = \mathfrak{p}(\mathcal{S})$ is a singular surface

Problem: Compute a triangulation isotopic to Ω.



Motivation

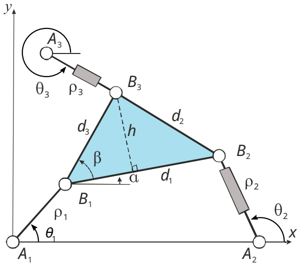
Previous works: Curve in 3D

Projection on $\mathbb{R}^3$ of surfaces in $\mathbb{R}^4$
- Generic singularities
- Contributions
- Algorithm

Motivation in Robotic

- $2RPR - RR$: parallel mechanism
- ρ_1 fixed
- Articular variables: ρ_2, ρ_3
- Pose variables: θ_1, α

E_W is a smooth variety of dimension 2 contained in a 4-dimensional space.

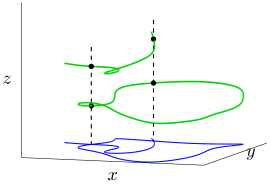


For a fix value of ρ_2 (for example) compute E_W

Projection on $\mathbb{R}^2$ of curves in $\mathbb{R}^3$

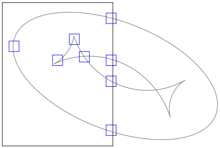
- Input: $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = G(x, y, z) = 0\}$ curve defined as the intersection of two surfaces
- $\mathfrak{p} : \mathbb{R}^3 \to \mathbb{R}^2$
- Output: piecewise linear curve isotopic to $\mathfrak{p}(\mathcal{C})$

[IMP,16] Imbach R, Moroz G and Pouget M: *Numeric and Certified Isolation of the Singularities of the Projection of a Smooth Space Curve*



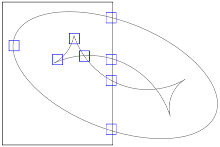
Algorithm for curve in 3D

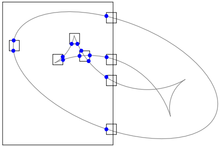
1. Isolate in boxes the special points
   - boundary points
   - x-critical points
   - singularities
2. Connect boxes



Algorithm for curve in 3D

1. Isolate in boxes the special points
   - boundary points
   - x-critical points
   - singularities
2. Connect boxes





Algorithm for curve in 3D

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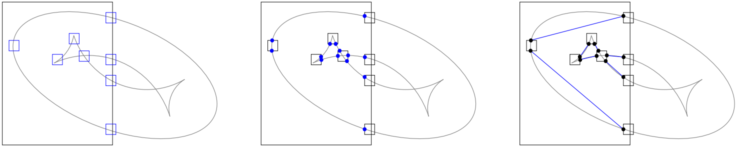
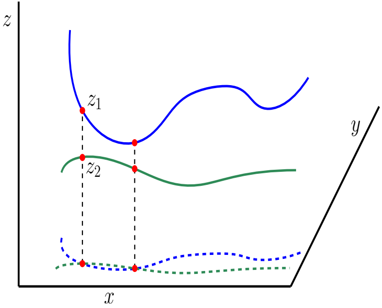


Characterizing singularities

- $\mathcal{B} = \{(x, y) \in \mathbb{R}^2 \mid \exists z \in \mathbb{R} \; s.t \; (x, y, z) \in \mathcal{C}\}$

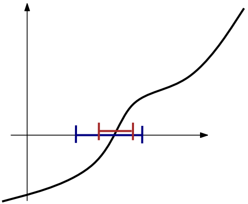


- $(x, y) \in \mathcal{B}$ is a node $\Leftrightarrow$ (x, y, z_1, z_2) satisfies:
  $F(x, y, z_1) = G(x, y, z_1) = F(x, y, z_2) = G(x, y, z_2) = 0$

Interval Newton operator

Newton operator
B an interval (resp. box) containing p
$N(x) = x - F'(p)^{-1}F(x)$

$$N(B) \subset B \Rightarrow F \text{ has a solution in } B$$

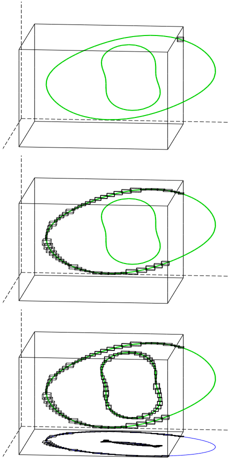


Regular system: A system is regular iff the Jacobian matrix has maximum rank.

Boxes are obtained by combined interval Newton and subdivision algorithm.

Certified path-tracker

- $F, G : \mathbb{R}^3 \to \mathbb{R}$ **B** a box of $\mathbb{R}^3$
- $\mathcal{C} = \{P \in \mathbf{B} \mid F(P) = G(P) = 0\}$ a smooth curve in $\mathbb{R}^3$



Certified path-tracker:
Input: $F, G : \mathbb{R}^3 \to \mathbb{R}$, **B**
Output: a sequence of boxes $\{\mathbf{B}_k\}_{\leq \ell}$ enclosing each component

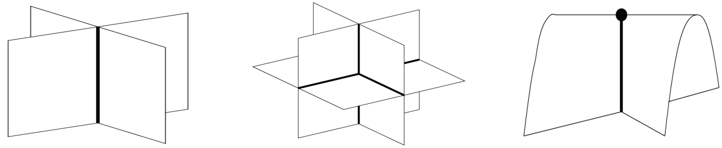
[MGGJ,13] Martin B, Goldsztejn A, Granvilliers L and Jermann C: *Certified parallelotope continuation for one-manifolds.* SIAM Journal on Numerical Analysis, 51(6) : 3373–3401, 2013.

Generic singularities

- Input: $\mathcal{S} = \{(x, y, z, t) \in \mathbb{R}^4 \mid F(x, y, z, t) = G(x, y, z, t) = 0\}$
- $\mathfrak{p} : (x, y, z, t) \mapsto (x, y, z)$ projection along the direction $(0, 0, 0, 1)$
- Output: Triangulation isotopic to $\Omega = \mathfrak{p}(\mathcal{S}) \subset \mathbb{R}^3$

- In mathematics, generic singularities are well known

[Gor,96] Goryunov V: *Local invariants of mappings of surfaces into three-space*.

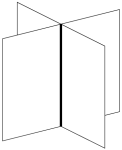


Contributions

1. Encode with systems all the singularities of the projected surface
2. The systems are regular under certain hypotheses

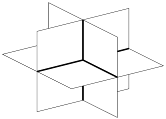
- *curve of double-points*

$$\begin{cases} F(x,y,z,t_i) = G(x,y,z,t_i) = 0 \\ i = 1,2 \end{cases}$$



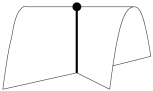
- *triple-point*

$$\begin{cases} F(x,y,z,t_i) = G(x,y,z,t_i) = 0 \\ i = 1,2,3 \end{cases}$$



- *cross-cap*

$$\begin{cases} F(x,y,z,t) & = & 0 \\ G(x,y,z,t) & = & 0 \\ \partial_t F(x,y,z,t) & = & 0 \\ \partial_t G(x,y,z,t) & = & 0 \end{cases}$$



Algorithm

1. Isolate the triple-points and cross-caps
   - enclose each triple-point and cross-cap in a special box
2. Draw the curve of double-points
   - Using the track paths in $3D$
3. Reconstruct the smooth part of Ω

[GoGr,08] Goldsztejn A and Granvilliers L: *A new Framework for sharp and Efficient Resolution of CNCSP with manifolds of Solutions*

THANK YOU!