Middle East Technical University Department of Computer Engineering

Discrete Computational Structures
Take Home Exam 1

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Question 1 (25 pts)

a) Given the sets A and B, prove that

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$$

using set membership notation and logical equivalences. Show each step clearly.

We must show both

$$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$$

and

$$(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$$

1) Suppose x $\in (A \cup B) - (A \cap B)$

2) By the definition of difference

$$x \in (A \vee B) \wedge x \notin (A \cap B)$$

3) By the defination of union and intersection

$$(x \in A \vee x \in B) \wedge \neg(x \in A \wedge x \in B)$$

4) By using De Morgan's Law for propositions

$$(x \in A \vee x \in B) \wedge (\neg(x \in A) \wedge \neg(x \in B))$$

5) By the definition of $\notin$

$$(x \in A \vee x \in B) \wedge (x \notin A \vee x \notin B)$$

6) Using Distributive Law

$$((x \in A \vee x \in B) \wedge x \notin A) \vee ((x \in A \vee x \in B) \wedge x \notin B)$$

7) Using Distributive Law x2

$$[(x \in A \wedge x \notin A) \vee (x \in B \wedge x \notin A)] \vee [(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin B)]$$

8) Using Complement Laws

$$[\varnothing \vee (x \in B \wedge x \notin A)] \vee [(x \in A \wedge x \notin B) \vee \varnothing]$$

9) Using Identity Law

$$(x \in B \wedge x \notin A) \vee (x \in A \wedge x \notin B)$$

10) By the definition of difference x2

$$(1)x \in (B - A) \vee (2)x \in (A - B)$$

11) Using Commutative Law

$$(2)x \in (A - B) \vee (1)x \in (B - A)$$

12) By the definition of union

$$(A - B) \cup (B - A)$$

Therefore $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

1) Suppose x $\in (A - B) \cup (B - A)$

2) By the definition of union

$$x \in (A - B) \vee x \in (B - A)$$

3) By the definition of difference x2

$$(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$$

4) Using Distribution Law

$$((x \in A \wedge x \notin B) \vee x \in B) \wedge ((x \in A \wedge x \notin B) \vee x \notin A)$$

5) Using Distribution Law x2

$$[(x \in A \vee x \in B) \wedge (x \notin B \vee x \in B)] \wedge [(x \in A \vee x \notin A) \wedge (x \notin B \vee x \notin A)]$$

6) Using Complement Law x2

$$[(x \in A \vee x \in B) \wedge U] \wedge [U \wedge (x \notin B \vee x \notin A)]$$

7) Using Identity Law x2

$$(x \in A \vee x \in B) \wedge (x \notin B \vee x \notin A)$$

8) By the definition of $\notin$

$$(x \in A \lor x \in B) \land (\neg(x \in B) \lor \neg(x \in A))$$

9) By the definition of De Morgan's Law for propositional logic

$$(x \in A \lor x \in B) \land \neg(x \in B \land x \in A)$$

10) By the definition of Union and Intersection

$$x \in (A \cup B) \land x \notin (B \cap A)$$

11) Using Commutative Law

$$x \in (A \cup B) \land x \notin (A \cap B)$$

12) By the definition of difference

$$(A \cup B) - (A \cap B)$$

Therefore $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

Hence, $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Question 2 (25 pts)

Prove that the set
$\{f \mid f: N \rightarrow \{0, 1\}$, f is a function$\}$ - $\{f \mid f: \{0, 1\} \rightarrow N$, f is a function$\}$ is uncountable.

Let say X to the set : $\{f \mid f: N \rightarrow \{0, 1\}$, f is a function$\}$

Let say Y to the set : $\{f \mid f: \{0, 1\} \rightarrow N$, f is a function$\}$

Any function f, where $N \rightarrow \{0, 1\}$ means the set of all binary strings.
I will show that this set is uncountable by the diagonalization argument.
The set X {includes a1, a2, a3 ...} where:
a1= a.11 a.12 a.13 ...
a2= a.21 a.22 a.23 ...
a3= a.31 a.32 a.33 ...
... goes on like this (any enumeration)
And there exists a binary string b = b1 b2 b3 ... such that:
b.i $\neq$ a.ii, For instance b.i = 0 if a.ii = 1, and b.i = 1 if a.ii = 0
b $\neq$ ai, i $\in N$ (a1, a2, a3 ...)

Therefore there does not exist an enumeration counting each element in X

Hence the set X is UNCOUNTABLE.

Any function f, where $\{f \mid f: \{0, 1\} \rightarrow N$, is determined by its values at 0 and 1, f(0) and f(1). f(0) can be any Natural number so does f(1). Therefore f is actually cartesian product of two sets with cardinality of N. Lets enumarate these sets as {a1,a2,a3...} and {b1,b2,b3...}. I arrange these elements in an infinite matrix and use "zigzag" method to traverse this matrix. For instance $\{(a1, b1),(a2, b1),(a1, b2),(a3, b1),(a2, b2),(a1, b3)...\}$. Hence f is countable.

Therefore the set Y is COUNTABLE.

By the previous two boxes, X is UNCOUNTABLE, and Y is COUNTABLE.

Assume X - Y is countable.
The union of countably many countable sets is countable; thus (X-Y)∪Y is countable. (1)
Since X $\subset (X - Y)\cup$Y , then X must be countable too. $\perp$But it can not be, becuase we know that X is uncountable.

Therefore X - Y is UNCOUNTABLE

(Proof of (1):
Since (X-Y) is countable, we can enumerate (X-Y)={a1,a2,a3,...}.

Since Y is countable we can enumerate Y={b1,b2,...}
And now we can enumerate (X-Y)∪Y as {a1,b1,a2,b2,...} and thus (X-Y)∪Y is countable.)

In the previous box, it is shown that X-Y which is {f | f : N → {0, 1}, f is a function} - {f | f : {0, 1} → N, f is a function} is UNCOUNTABLE

Question 3) 25pts

Prove that $f(n) = 4^n + 5n^2 logn$ is not $O(2^n)$

Definition: f(x) is O(g(x)) if there exists such constants c and k such that f(x) $f(x) \leq c \cdot g(x)$ where $x \geq k$

Assume $f(n) = 4^n + 5n^2 logn$ is $O(2^n)$, and $n > 1$
$\frac{f(n)}{g(n)} \leq c$, where $\frac{f(n)}{g(n)} = \frac{2^{2n}+5n^2 logn}{2^n} = 2^n + \frac{5n^2 logn}{2^n}$

Since $x > 1$, $\quad 2^n + \frac{5n^2 logn}{2^n} > 2^n + 0 = 2^n$

$n > \log_2(c)$ implies $2^n > c$ and $f(n) > c \cdot 2^n$

$\perp$ Contradiction with the definition of the Big O Notation

Therefore when $n > 1, n > k$, and $n > \log_2(c)$ f(n)$>$ $c \cdot 2^n$ which contradicts with the definition of the Big O Notation.

$f(n) = 4^n + 5n^2 logn$ is NOT $O(2^n)$

Question 4) 25pts

$$x > 2, n > 2, (2x-1)^n - x^2 \equiv -x - 1 (mod(x-1))$$

Determine the value of x

1	$2x - 1 = 1 + (2 \cdot (x-1))$	
2	$2x - 1 \equiv 1(mod(x-1))$	(1)
3	$x \equiv 1(mod(x-1))$	(2)
4	$(2x-1)^n - x^2 \equiv 1^n - 1^2$	$(2,3)$
5	$0 \equiv -x - 1(mod(x-1))$	$(Premise, 4)$
6	$x + 1 \equiv 0(mod(x-1))$	(5)
7	$2 \equiv 0(mod(x-1))$	$(3,6)$
8	$2 \equiv x - 1(mod(x-1))$	(7)
9	$x = 3$	$(x > 2, 8)$