Chapter 3.

Dynamic Programming

Foundations of Algorithms, 5th Ed.

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Contents

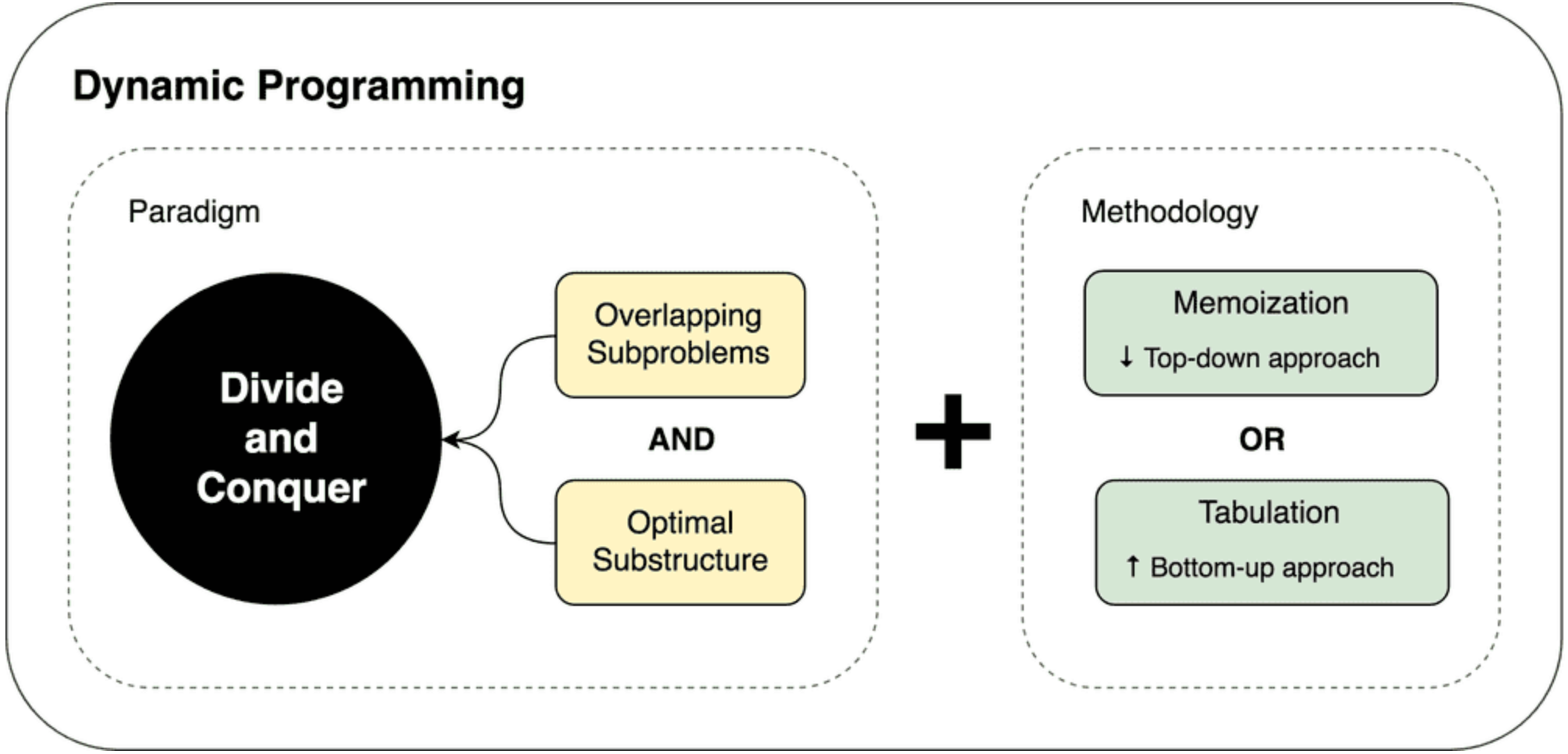
Dynamic Programming

- Dynamic Programming
  - is similar to divide-and-conquer in that
    - an instance of a problem is divided into smaller instances.
  - However, D.P. *solves small instances first*,
    - *store the results*, and later, whenever we need a result,
      - *look it up instead of recomputing* it.
  - The term "***dynamic programming***" comes from *control theory*,
    - "programming" means the use of an *array* (*table*)
      - in which a solution is constructed. (called *memoization*)
  - Dynamic Programming is a *bottom-up* approach,
    - whereas the divide-and-conquer is a *top-down* approach.

Dynamic Programming

- The steps in the design of Dynamic Programming:
  1. *Establish* a ***recursive property*** that gives
     - the solution to an instance of the problem. (*top-down* approach)
  2. *Solve* an instance of the problem *in a* ***bottom-up fashion***
     - by solving smaller instances first (using *memoization*).

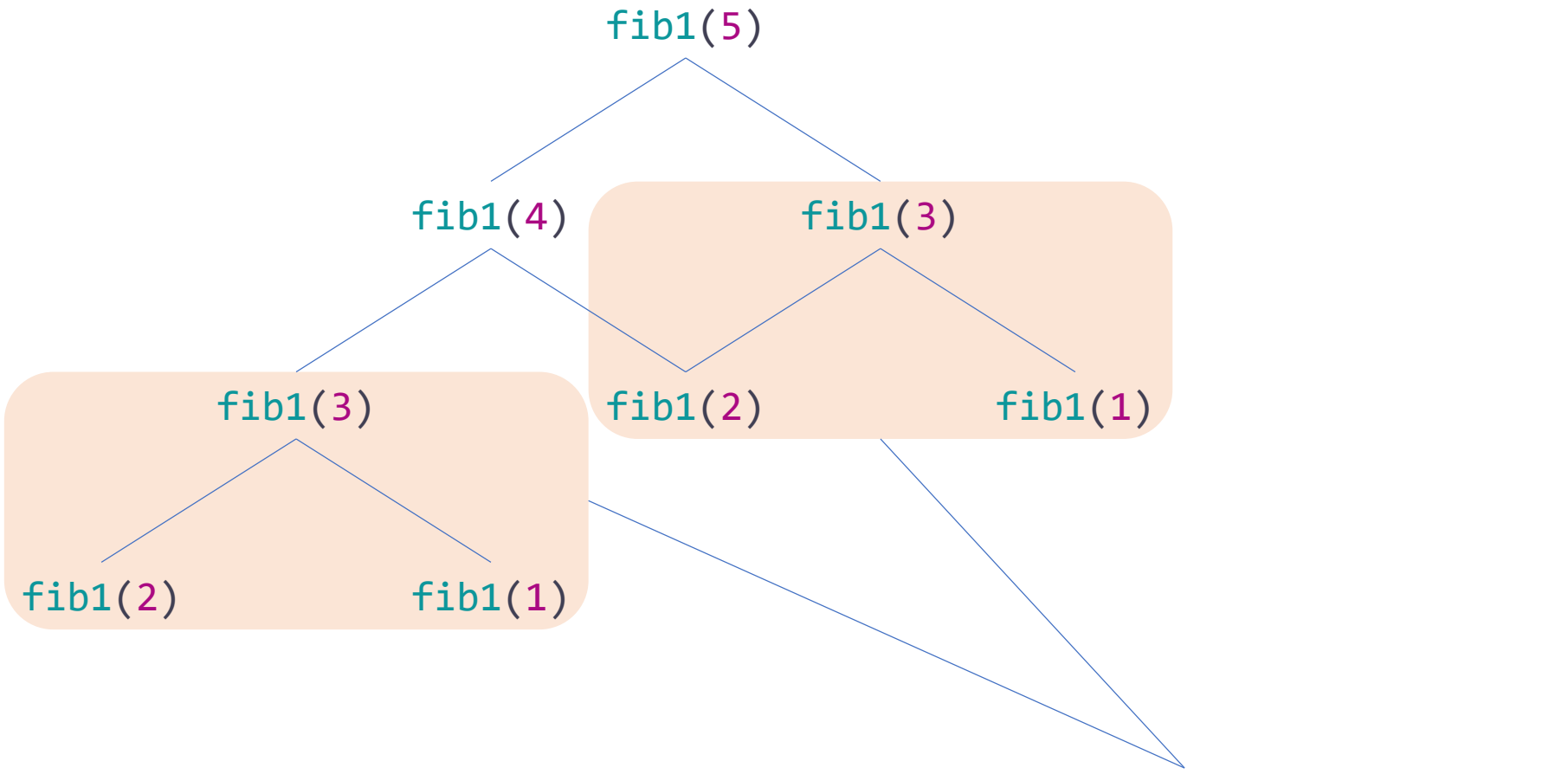
Dynamic Programming



https://trekhleb.dev/blog/2018/dynamic-programming-vs-divide-and-conquer/

Dynamic Programming

- Fibonacci Sequence Revisited:



Overlapping Subproblems

Dynamic Programming

- Using *Tabulation*

```
typedef unsigned long long longint;
vector<longint> F;

longint fib2(int n) {
    if (n >= F.size()) {
        int start = F.size();
        F.resize(n + 1);
        if (n < 2)
            F[n] = n;
        else {
            for (int i = start; i <= n; i++)
                F[i] = F[i - 1] + F[i - 2];
        }
    }
    return F[n];
}
```

Dynamic Programming

- Using *Memoization*

```
typedef unsigned long long longint;
map<int, longint> F;

longint fib3(int n) {
    if (F.find(n) != F.end())
        return F[n];
    else {
        if (n < 2)
            F[n] = n;
        else
            F[n] = fib3(n - 1) + fib3(n - 2);
        return F[n];
    }
}
```

3.1 The Binomial Coefficient

- The Binomial Coefficient Problem:
  - The definition of *binomial coefficient* is given by:
    - $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, for $0 \leq k \leq n$.
    - It is hard to compute directly because $n!$ is very large even for small n.
  - Using the *recursive property* of the binomial coefficient:
    - $$\binom{n}{k} = \begin{cases} \binom{n-1}{k-1} + \binom{n-1}{k} & 0 < k < n \\ 1 & k = 0 \text{ or } k = n \end{cases}$$
    - We can eliminate the need to compute $n!$ or $k!$.

3.1 The Binomial Coefficient

ALGORITHM 3.1: Binomial Coefficient Using Divide-and-Conquer

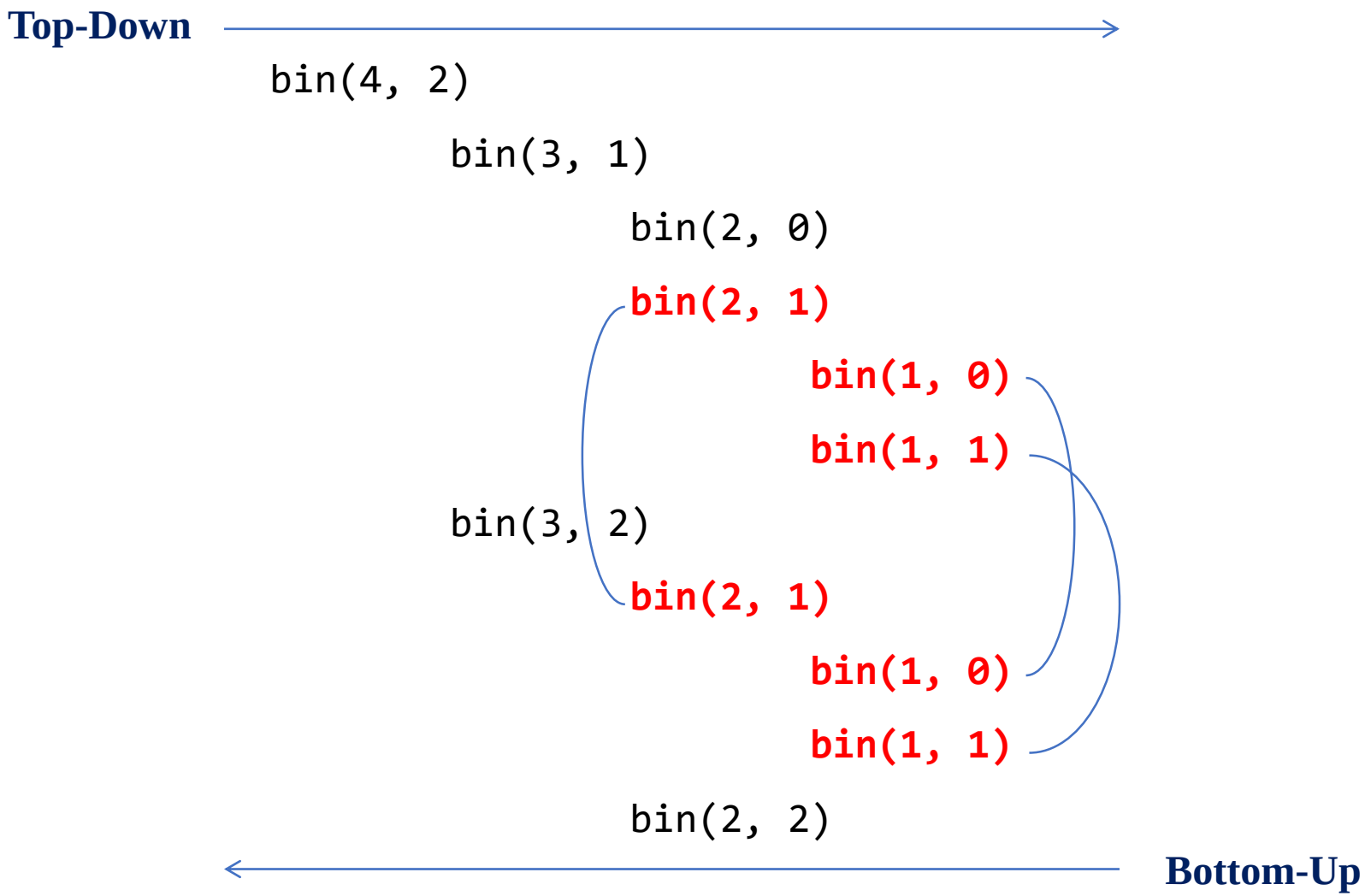
```
typedef unsigned long long LongInteger;

LongInteger bin(int n, int k)
{
    if (k == 0 || n == k)
        return 1;
    else
        return bin(n - 1, k) + bin(n - 1, k - 1);
}
```

3.1 The Binomial Coefficient

- The ***Inefficiency*** of Algorithm 3.1:
  - Algorithm 3.1 computes $2\binom{n}{k} - 1$ *terms* to determine $\binom{n}{k}$. (Exercise 3.1.2)
  - The problem is that
    - the *same instances* are *solved* in *each recursive call*.
    - (*overlapping subproblems*)
  - Recall that the *divide-and-conquer* approach is always *inefficient*
    - when an instance is divided into *two smaller instances*
      - that are *almost as large as* the original instance.

3.1 The Binomial Coefficient



3.1 The Binomial Coefficient

- Design a more efficient algorithm using Dyn. Prog.
  - *Establish* a *recursive property*:
    - Let B be an array that $B[i][j]$ contains $\binom{n}{k}$.
    - $B[i][j] = \begin{cases} B[i-1][j-1] + B[i-1][j] & 0 < j < i \\ 1 & j = 0 \; or \; j = i \end{cases}$
  - *Solve* an instance of the problem in a *bottom-up fashion.*
    - by *computing the rows* in B in sequence
      - *starting* with the *first* row.

3.1 The Binomial Coefficient

j, *k*

B[i][j]	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

i, *n*

$B[i-1][j-1]$ $B[i-1][j]$

$B[i][j]$

- You may recognize the array in this figure as *Pascal's triangle*.

3.1 The Binomial Coefficient

ALGORITHM 3.2: Binomial Coefficient Using Divide-and-Conquer

```
typedef unsigned long long LongInteger;

LongInteger bin2(int n, int k)
{
    vector<vector<LongInteger>> B(n + 1, vector<LongInteger>(n + 1));
    for (int i = 0; i <= n; i++)
        for (int j = 0; j <= min(i, k); j++)
            if (j == 0 || j == i)
                B[i][j] = 1;
            else
                B[i][j] = B[i - 1][j] + B[i - 1][j - 1];
    return B[n][k];
}
```

3.1 The Binomial Coefficient

- Time Complexity of Algorithm 3.2:
  - The parameters n and k are *not* the *size* of *input* to this algorithm.
    - Rather, they are the input, and the input size is
      - the *number of symbols* it takes to *encode* them.
  - For given n and k,
    - the number of passes for each value of i are:

i	0	1	2	3	...	k	$k+1$	...	n
Number of passes	1	2	3	4	...	$k+1$	$k+1$	...	$k+1$

3.1 The Binomial Coefficient

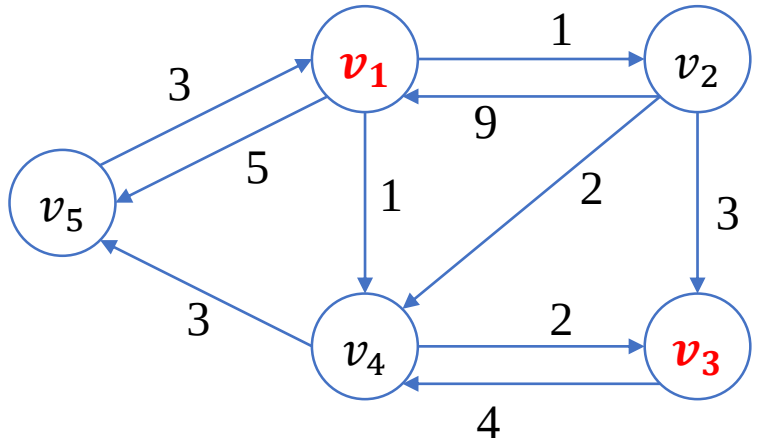
- Time Complexity of Algorithm 3.2:
  - The total number of passes is:
    - $1 + 2 + 3 + 4 + \cdots + k + (k + 1) + (k + 1) \ldots + (k + 1)$
    - $= \frac{k(k+1)}{2} + (n - k + 1)(k + 1)$
    - $= \frac{(2n-k+2)(k+1)}{2} \in \Theta(nk)$.
  - We can argue that we have developed a *much more efficient* algorithm,
    - by using *dynamic programming* instead of *divide-and-conquer*.

3.1 The Binomial Coefficient

- *Improving* the Performance of Algorithm 3.2
  - Use only a *one-dimensional array* indexed from 0 to k,
    - instead of creating the *entire two-dimensional* array B.
    - Once a row is computed,
      - we no longer need the values in the row that proceeds it.
    - By the way, how can we do it with *only one* array?
  - Can we *reduce* the size of one-dimensional array *in half*?
    - Take advantage of the fact that
      - $\binom{n}{k} = \binom{n}{n-k}$.

3.2 Floyd's Algorithm for Shortest Paths

- Finding the Shortest Paths in a Graph:
  - from *each vertex* to *all other* vertices (*All Pairs* Shortest Paths)



- Shortest path from v_1 to v_3?
  - $length[v_1, v_2, v_3] = 1 + 3 = 4$
  - $length[v_1, v_4, v_3] = 1 + 2 = 3$
  - $length[v_1, v_2, v_4, v_3] = 1 + 2 + 2 = 5$

3.2 Floyd's Algorithm for Shortest Paths

- The Shortest Paths as an ***Optimization Problem***:
  - There can be more than one *candidate solution*
    - to an instance of an optimization problem.
  - Each *candidate solution* has a *value* associated with it,
    - and a solution to the instance is any candidate solution
      - that has an *optimal value* (either *minimum* or *maximum*).
  - There can be more than one shortest paths from one vertex to another.
    - Our problem is to find *any one* of the shortest paths.

3.2 Floyd's Algorithm for Shortest Paths

- The **Brute-Force** Approach:
  - Determine, for each vertex,
    - the *lengths of all the paths* from that vertex to each other vertex,
      - and compute the *minimum* of these lengths.
  - The *total number of paths* from one vertex to another vertex is
    - $(n-2)(n-3)\cdots 1 = (n-2)!$ (*factorial time*)
    - This algorithm is *worse* than *exponential time*.
  - This is often the case with many optimization problems.
    - Our goal is to find a more efficient algorithm for this problem.

3.2 Floyd's Algorithm for Shortest Paths

- ***Floyd's Algorithm***:
  - A *cubic-time* algorithm for the problem of (*All Pairs*) *Shortest Paths*.
    - not *exponential*, but *polynomial*.
  - Using *dynamic programming*,
    - First, we develop an algorithm
      - that determines *only the lengths* of the shortest paths.
    - After, we modify it to *produce shortest paths* as well.

3.2 Floyd's Algorithm for Shortest Paths

- Data Structure for the Floyd's Algorithm
  - We represent a *weighted* (*directed*) graph containing n vertices
    - by an array (*adjacency matrix*) W where
      - $W[i][j] = \begin{cases} weight\ on\ edge & \text{if there is an edge from } v_i \text{ to } v_j \\ \infty & \text{if there is no edge from } v_i \text{ to } v_j \\ 0 & if\ i = j \end{cases}$
  - We also define an array D as a matrix that
    - contains the lengths of the shortest paths in the graph.

3.2 Floyd's Algorithm for Shortest Paths

W	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0

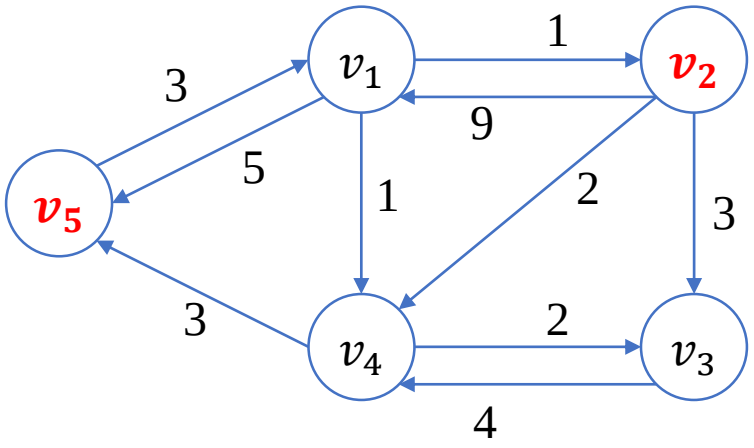
D	1	2	3	4	5
1	0	1	**3**	1	4
2	8	0	3	2	5
3	10	11	0	4	7
4	6	7	2	0	3
5	3	4	6	4	0

3.2 Floyd's Algorithm for Shortest Paths

- Solving the Problem:
  - If we can develop a way
    - to *calculate the values* in D *from those* in W,
      - we will have an algorithm for the Shortest Paths problem.
  - Let us create a *sequence* of $n + 1$ arrays $D^{(k)}$,
    - where $0 \leq k \leq n$ and where

$$D^{(k)}[i][j] = \text{length of a } \textit{shortest path} \text{ from } v_i \text{ to } v_j \text{ using } \textit{only vertices} \text{ in the set } \{v_1, v_2, \cdots, v_k\} \text{ as } \textit{intermediate vertices}.$$

3.2 Floyd's Algorithm for Shortest Paths



- $D^{(0)}[2][5] = length[v_2, v_5] = \infty$
- $D^{(1)}[2][5] = minimum(length[v_2, v_5], length[v_2, v_1, v_5])$
  $= minimum(\infty, 14) = 14.$
- $D^{(2)}[2][5] = D^{(1)}[2][5] = 14.$
- $D^{(3)}[2][5] = D^{(2)}[2][5] = 14.$
- $D^{(4)}[2][5] = minimum(length[v_2, v_1, v_5], length[v_2, v_4, v_5],$
  $length[v_2, v_1, v_4, v_5], length[v_2, v_3, v_4, v_5]$
  $= minimum(14{,}5, 13{,}10) = 5.$
- $D^{(5)}[2][5] = D^{(4)}[2][5] = 5.$

3.2 Floyd's Algorithm for Shortest Paths

- Solving the Problem:
  - Note that $D^{(n)}[i][j]$ is the length of a shortest path from v_i to v_j,
    - because it is the length of a shortest path from v_i to v_j
      - that is allowed to pass through *any of the other vertices*.
  - Therefore, we have established that
    - $D^{(0)} = W$ and $D^{(n)} = D$.
  - To determine D from W,
    - we need only find a way to obtain $D^{(n)}$ from $D^{(0)}$.

3.2 Floyd's Algorithm for Shortest Paths

- The steps for Dynamic Programming:
  - Establish a recursive property
    - with which we compute $D^{(k)}$ from $D^{(k-1)}$.
  - Solve an instance of the problem in a *bottom-up fashion*
    - by repeating the process for $k = 1$ to n.
    - This process creates the sequence:

$$D^0,\ D^1,\ D^2,\ \cdots\ ,\ D^n$$

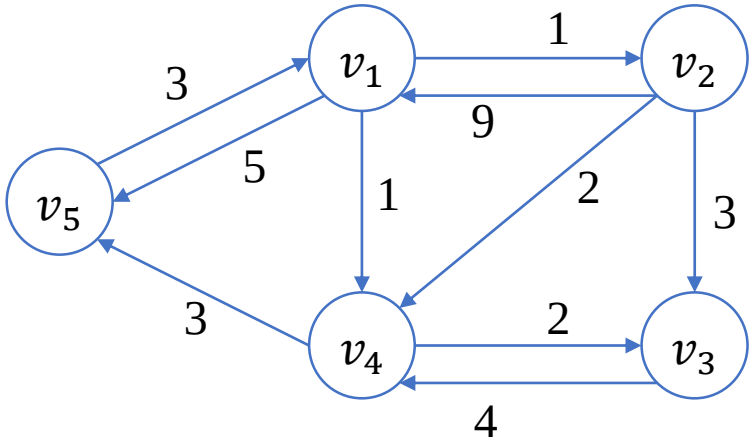
W $\qquad\qquad\qquad\qquad$ D

3.2 Floyd's Algorithm for Shortest Paths

- Two cases of Establishing Recursive Property:
  1. At least one shortest path from v_i to v_j does not use v_k,
     - using only vertices in $\{v_1, v_2, \cdots, v_k\}$ as intermediate vertices.
     - Then, $D^{(k)}[i][j] = D^{(k-1)}[i][j]$.
  2. All shortest paths from v_i to v_j do use v_k,
     - using only vertices in $\{v_1, v_2, \cdots, v_k\}$ as intermediate vertices.
     - Then, $D^{(k)}[i][j] = D^{(k-1)}[i][k] + D^{(k-1)}[k][j]$.

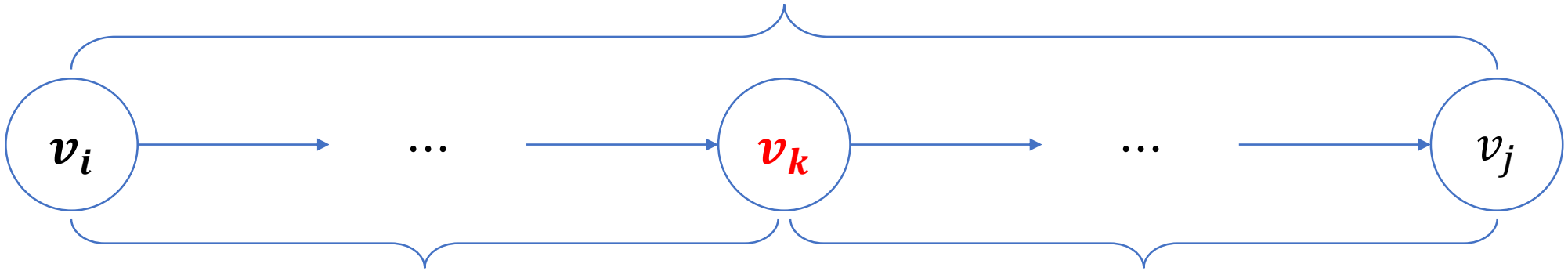
3.2 Floyd's Algorithm for Shortest Paths

- Examples of Two Cases:
  - **case 1**: $D^{(5)}[1][3] = D^{(4)}[1][3] = 3$,
    - because when we include vertex v_5,
      - the shortest path from v_1 to v_3 is still $[v_1, v_4, v_3]$.
  - **case 2**: $D^{(2)}[5][3] = D^{(1)}[5][2] + D^{(1)}[2][3] = 4 + 3 = 7$,
    - because v_k cannot be an intermediate vertex on the subpath from v_i to v_k,
      - that subpath uses only vertices in $\{v_1, v_2, \cdots, v_{k-1}\}$ as intermediates.



3.2 Floyd's Algorithm for Shortest Paths

A shortest path from v_i to v_j using only vertices in $\{v_1, v_2, \cdots, v_k\}$



A shortest path from v_i to v_k using only vertices in $\{v_1, v_2, \cdots, v_k\}$

A shortest path from v_k to v_j using only vertices in $\{v_1, v_2, \cdots, v_k\}$

3.2 Floyd's Algorithm for Shortest Paths

- Considering Two Cases:
  - Let $D^{(k)}[i][j]$ be the *minimum* of the values from either Case 1 or Case 2.
    - $D^{(k)}[i][j] = minimum(D^{(k-1)}[i][j], D^{(k-1)}[i][k] + D^{(k-1)}[k][j])$.
  - Now we have established a recursive property (Step 1).
    - To accomplish Step 2,
      - create the sequence of arrays: $D^0, D^1, D^2, \cdots, D^n$.

3.2 Floyd's Algorithm for Shortest Paths

$D^0 = W$

D^0	1	2	3	4	5
1	0	1	∞	1	5
2	9	0	3	2	∞
3	∞	∞	0	4	∞
4	∞	∞	2	0	3
5	3	∞	∞	∞	0

- $D^{(1)}[2][4] = minimum\left(D^{(0)}[2][4], D^{(0)}[2][1] + D^{(0)}[1][4]\right)$
  $= minimum(2,\ 9 + 1) = 2.$
- $D^{(1)}[5][2] = minimum\left(D^{(0)}[5][2], D^{(0)}[5][1] + D^{(0)}[1][2]\right)$
  $= minimum(\infty,\ 3 + 1) = 4.$
- $D^{(1)}[5][4] = minimum\left(D^{(0)}[5][4], D^{(0)}[5][1] + D^{(0)}[1][4]\right)$
  $= minimum(\infty,\ 3 + 1) = 4.$
- $D^{(2)}[5][4] = minimum\left(D^{(1)}[5][4], D^{(1)}[5][2] + D^{(1)}[2][4]\right)$
  $= minimum(4,\ 4 + 2) = 4.$

3.2 Floyd's Algorithm for Shortest Paths

ALGORITHM 3.3: Floyd's Algorithm for Shortest Paths

```
#define INF 0xffff

typedef vector<vector<int>> matrix_t;

void floyd(int n, matrix_t& W, matrix_t& D)
{
    for (int i = 1; i <= n; i++)
        for (int j = 1; j <= n; j++)
            D[i][j] = W[i][j];
    for (int k = 1; k <= n; k++)
        for (int i = 1; i <= n; i++)
            for (int j = 1; j <= n; j++)
                D[i][j] = min(D[i][j], D[i][k] + D[k][j]);
}
```

3.2 Floyd's Algorithm for Shortest Paths

- Why can we use only one array D?
  - because the *values* in the kth row and the kth column
    - are *not changed* during the kth iteration of the loop.
  - In the kth iteration, the algorithm assigns
    - $D[i][k] = minimum(D[i][k], D[i][k] + D[k][k])$, and
    - $D[k][j] = minimum(D[k][j], D[k][k] + D[k][j])$
  - During the kth iteration, $D[i][j]$ is computed
    - from only *its own value* and *values* in the kth row and the kth column.
    - These values have *maintained* their values from the $(k-1)$st iteration.

- Time Complexity of Floyd's algorithm: $T(n) = n^3 \in \Theta(n^3)$.

3.2 Floyd's Algorithm for Shortest Paths

- Producing the *Shortest Paths*:
  - Define an array P, where

$$P[i][j] = \begin{cases} \text{highest index of } \textit{an itermediate vertex} \text{ on the shortest path} \\ \text{from } v_i \text{ to } v_j, \text{ if at least one intermediate vertex exists.} \\ 0, \text{ if no intermediate vertex exists.} \end{cases}$$

P	1	2	3	4	5
1	0	0	4	0	4
2	5	0	0	0	4
3	5	5	0	0	4
4	5	5	0	0	0
5	0	1	4	1	0

ALGORITHM 3.4: Floyd's Algorithm for Shortest Paths 2

```
void floyd2(int n, matrix_t& W, matrix_t& D, matrix_t& P)
{
    for (int i = 1; i <= n; i++)
        for (int j = 1; j <= n; j++) {
            D[i][j] = W[i][j];
            P[i][j] = 0;
        }
    for (int k = 1; k <= n; k++)
        for (int i = 1; i <= n; i++)
            for (int j = 1; j <= n; j++)
                if (D[i][j] > D[i][k] + D[k][j]) {
                    D[i][j] = D[i][k] + D[k][j];
                    P[i][j] = k;
                }
}
```

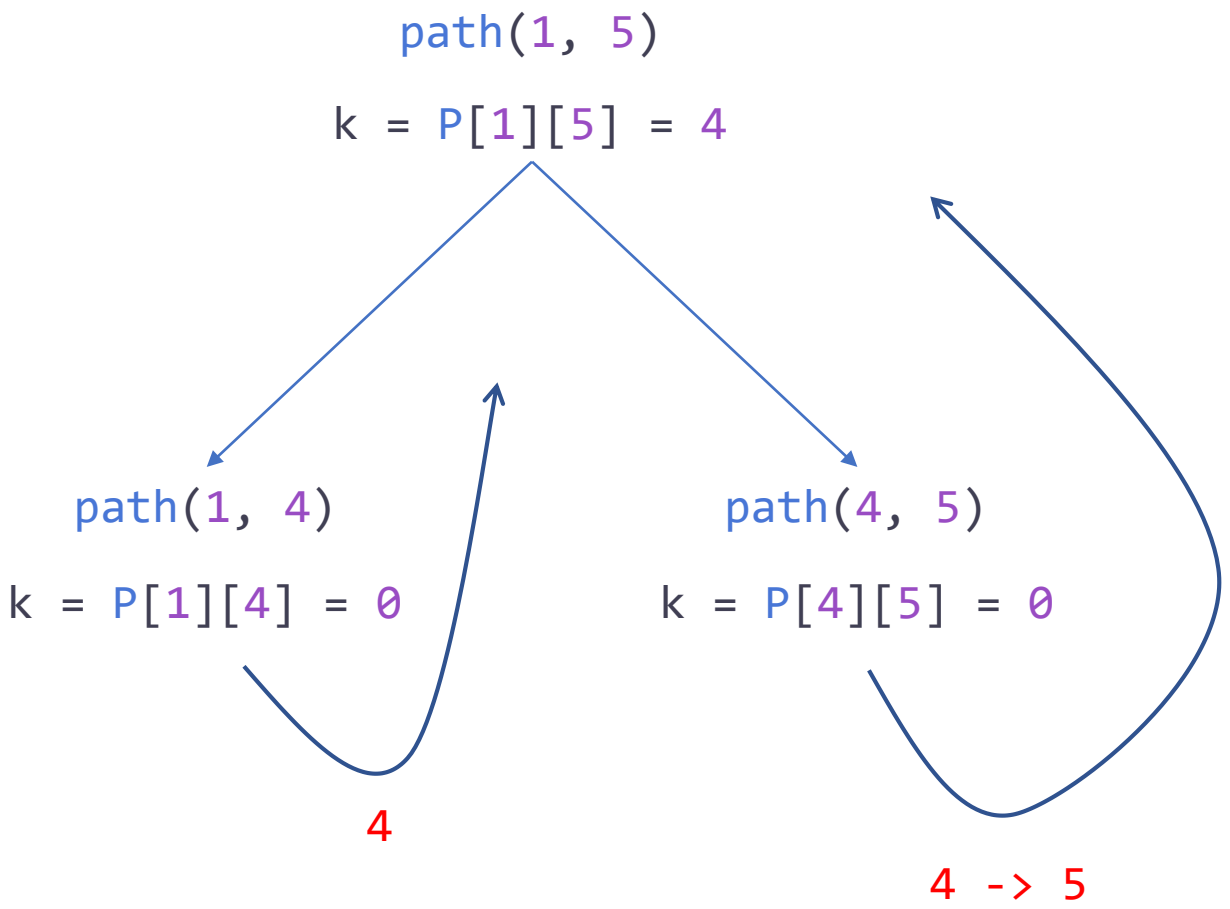
3.2 Floyd's Algorithm for Shortest Paths

ALGORITHM 3.5: Print Shortest Path

```
void path(matrix_t& P, int u, int v, vector<int>& p)
{
    int k = P[u][v];
    if (k != 0) {
        path(P, u, k, p);
        p.push_back(k);
        path(P, k, v, p);
    }
}
```

3.2 Floyd's Algorithm for Shortest Paths

P	1	2	3	4	5
1	0	0	4	0	4
2	5	0	0	0	4
3	5	5	0	0	4
4	5	5	0	0	0
5	0	1	4	1	0



3.3 Dynamic Programming and Optimization Problems

- The design of a D.P. algorithm for an *optimization problem*:
  1. *Establish* a recursive property
     - that gives the optimal solution to an instance of the problem.
  2. *Compute* the *value* of an *optimal solution* in a bottom-up fashion.
  3. *Construct* an *optimal solution* in a bottom-up fashion.

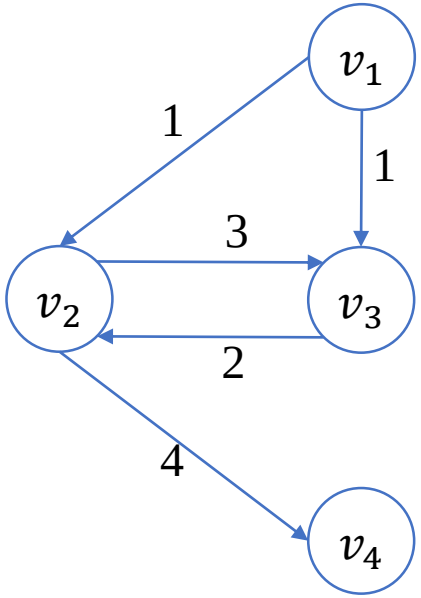
  - Note that *Steps 2* and *3* are ordinarily accomplished
    - at *about the same point* in the algorithm

3.3 Dynamic Programming and Optimization Problems

- ***The Principle of Optimality***:
  - An *optimization problem* can be solved using *dynamic programming*,
    - if and only if the *principle of optimality applies* in the problem.
  - The principle of optimality is said to *apply* in a problem
    - if an *optimal solution* to an instance of a problem
      - *always contains optimal solutions to all subinstances*.

3.3 Dynamic Programming and Optimization Problems

- Example: the *Longest Paths* problem
  - Finding the *longest **simple** paths* from each vertex to all other vertices.
  - We restrict the path to be *simple*,
    - because *a cycle* can always create an arbitrarily long path
      - by repeatedly passing through the cycle.
  - Then, we can show that the principle of optimality *does not apply*.



- The optimal (longest) simple path from v_1 to v_4 is $[v_1, v_3, v_2, v_4]$.
- However, the subpath $[v_1, v_3]$ is not an optimal path from v_1 to v_4
  - $length\ [v_1, v_3] = 1$ and $length[v_1, v_2, v_3] = 4$.

3.4 Chained Matrix Multiplication

- The *Multiplication* of *Chained Matrices*:
  - Suppose we want to multiply a 2×3 matrix and a 3×4 matrix:
    - $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 29 & 35 & 41 & 38 \\ 74 & 89 & 104 & 83 \end{bmatrix}$
  - The resultant matrix is a 2×4 matrix.
  - If we use the standard method of multiplying matrices
    - it takes *three* elementary *multiplications*.
    - For 29 in the product: 3 multiplications $(1 \times 7 + 2 \times 2 + 3 \times 6)$.
  - Because there $2 \times 4 = 8$ entries in the product,
    - the *total number* of elementary multiplication is $2 \times 3 \times 4 = 24$.

3.4 Chained Matrix Multiplication

- The number of *elementary multiplications*:
  - In general, to multiply an $i \times j$ matrix with a $j \times k$ matrix,
    - the number of elementary multiplications is $i \times j \times k$.
  - Consider the multiplication of the following four matrices:
    - $A \times B \times C \times D$
    - $(20 \times 2) \quad (2 \times 30) \quad (30 \times 12) \quad (12 \times 8)$
  - Matrix multiplication is an *associative operation*,
    - meaning the *order* in which we multiply *does not matter*.
    - for example, $A(B(CD)) = (AB)(CD)$.

3.4 Chained Matrix Multiplication

- The number of elementary multiplications:
  - There are *five different orders* with different number multiplications.
    - $A(B(CD))$: 3,680
    - $(AB)(CD)$: 8,880
    - $A((BC)D)$: 1,232 (*optimal order*)
    - $((AB)C)D$: 10,320
    - $(A(BC))D$: 3,120
  - *Our goal* is to develop an algorithm that
    - determines the *optimal order* for multiplying n chained matrices.

3.4 Chained Matrix Multiplication

- The *Brute-Force* Approach:
  - Consider *all possible orders* and *take* the *minimum*.
  - We will show that this algorithm is *at least exponential-time*.
    - For a given n matrices: $A_1, A_2, \cdots, A_n$,
    - let t_n be the number of different orders.
    - Then, $t_n \geq t_{n-1} + t_{n-1} = 2t_{n-1}$, and $t_2 = 1$.
    - Therefore, $t_n \geq 2^{n-2}$. (*at least exponential*)

3.4 Chained Matrix Multiplication

- Does the *principle of optimality* apply?
  - Show that the *optimal order* for multiplying n matrices includes
    - the *optimal order* for multiplying any *subset* of the n matrices.
  - Then, we can use dynamic programming to construct a solution.

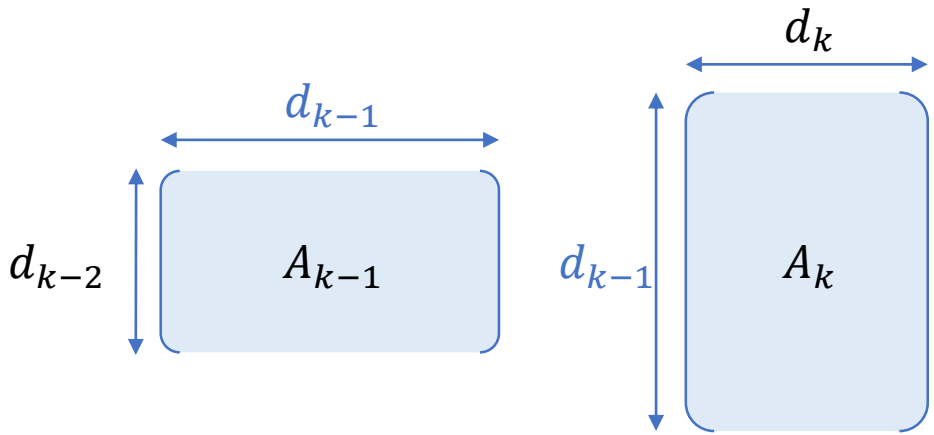
$$A_1\left(\left(\left(A_2A_3\right)A_4\right)A_5\right)A_6\right)$$

- If this order is *optimal* for multiplying $A_1A_2A_3A_4A_5A_6$,
- then the *optimal order* for multiplying $A_2A_3A_4$ is this.

$$\left(A_2A_3\right)A_4\right)$$

3.4 Chained Matrix Multiplication

- Multiplying A_{k-1} times A_k:
  - The number of columns in A_{k-1}
    - must equal to the number of rows in A_k.
  - If we let d_0 be the number of rows in A_1
    - and d_k be the number of columns in A_k for $1 \leq k \leq n$,
    - then, the dimension of A_k is $d_{k-1} \times d_k$.



3.4 Chained Matrix Multiplication

- Creating a sequence of arrays:
  - For $1 \leq i \leq j \leq n$, let

$$M[i][j] = \begin{cases} \textit{minimum number} \text{ of } \textit{multiplications} \text{ needed to multiply} \\ A_i \text{ through } A_j, \text{ if } i < j. \\ 0, \text{ if } i = j. \end{cases}$$

A_1	$\times$	A_2	$\times$	A_3	$\times$	A_4	$\times$	A_5	$\times$	A_6
(5×2)		(2×3)		(3×4)		(4×6)		(6×7)		(7×8)
d_0 d_1		d_2		d_3		d_4		d_5		d_6

  - To multiply A_4, A_5, and A_6, we have two orders:
    - $(A_4A_5)A_6$: $d_3d_4d_5 + d_3d_5d_6 = 392$
    - $A_4(A_5A_6)$: $d_4d_5d_6 + d_3d_4d_6 = 528$
  - Therefore, $M[4][6] = minimum(392, 528) = 392$.

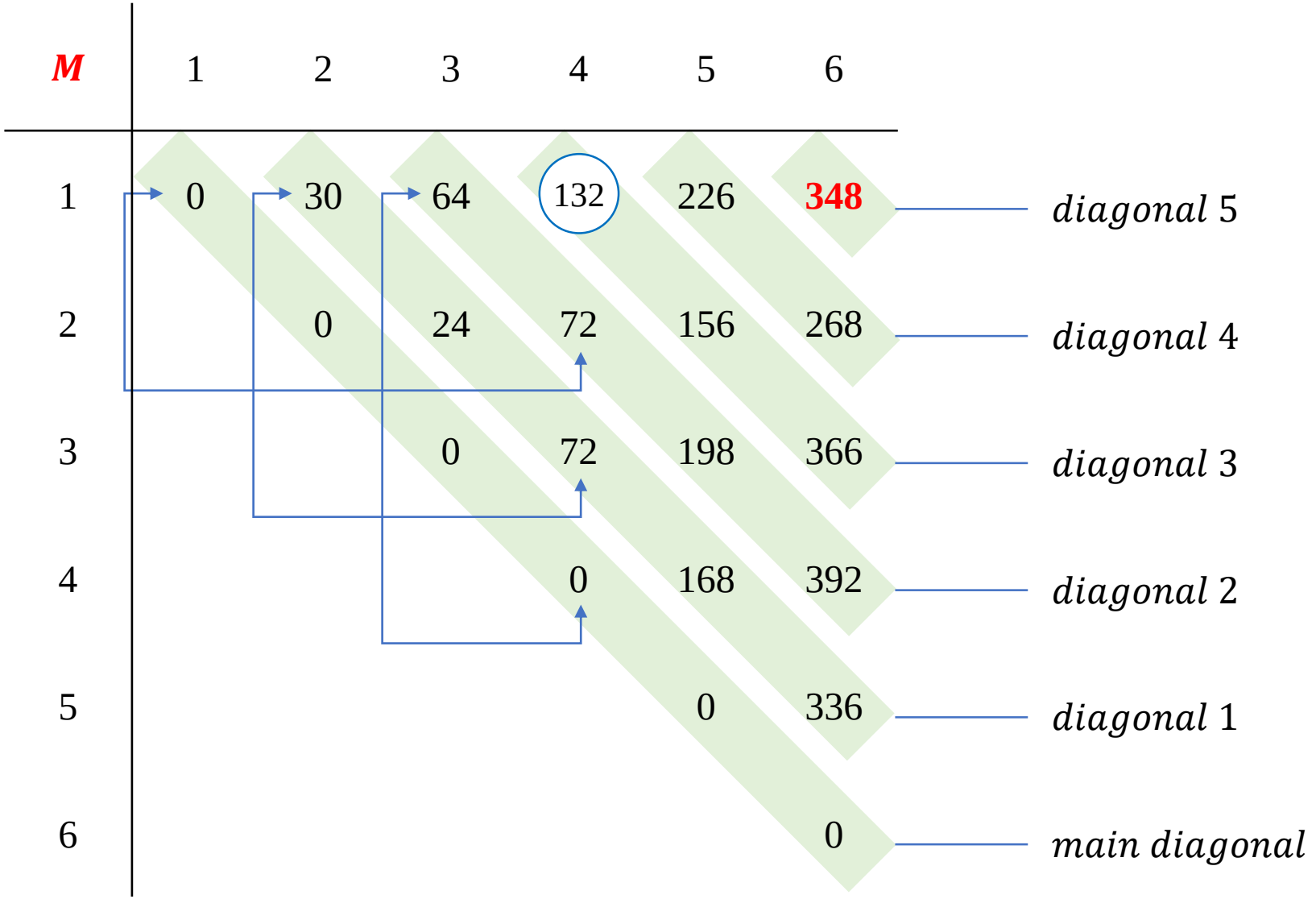
3.4 Chained Matrix Multiplication

- Establish the recursive property:
  - The *optimal order* for multiplying *six* matrices
    - must have *one of these factorizations*:
      1. $(A_1)(A_2A_3A_4A_5A_6)$
      2. $(A_1A_2)(A_3A_4A_5A_6)$
      3. $(A_1A_2A_3)(A_4A_5A_6)$
      4. $(A_1A_2A_3A_4)(A_5A_6)$
      5. $(A_1A_2A_3A_4A_5)(A_6)$
  - Of these *factorizations*,
    - the one that *yields* the *minimum* number of multiplications
      - must be the optimal one.

3.4 Chained Matrix Multiplication

- Establish the *recursive property*:
  - The number of multiplications for the kth factorization
    - is the *minimum number* needed to *obtain each factor*
      - ***plus*** the *number* needed to *multiply two factors*.
    - that is, $M[1][k] + M[k+1][6] + d_0 d_k d_6$.
  - We have established that
    - $M[1][6] = \underset{1 \le k \le 5}{\text{minimum}}(M[1][k] + M[k+1][6] + d_0 d_k d_6)$.
  - We can *generalize* this result into:
    - $M[i][j] = \underset{i \le k \le j-1}{\text{minimum}}\left(M[i][k] + M[k+1][j] + d_{i-1} d_k d_j\right)$, if $i < j$.
    - $M[i][i] = 0$.

3.4 Chained Matrix Multiplication



3.4 Chained Matrix Multiplication

$M[i][i] = 0$ for $1 \leq i \leq 6$

$$M[1][2] = \underset{1 \leq k \leq 1}{\text{minimum}}(M[1][k] + M[k+1][2] + d_0 d_k d_2)$$
$$= M[1][1] + M[2][2] + d_0 d_1 d_2 = 0 + 0 + 5 \times 2 \times 3 = 30.$$

$$M[1][3] = \underset{1 \leq k \leq 2}{\text{minimum}}(M[1][k] + M[k+1][3] + d_0 d_k d_3)$$
$$= \text{minimum}(M[1][1] + M[2][3] + d_0 d_1 d_3, M[1][2] + M[3][3] + d_0 d_2 d_3)$$
$$= \text{minimum}(0 + 24 + 5 \times 2 \times 4,\ 30 + 0 + 5 \times 3 \times 4) = 64.$$

$$M[1][4] = \underset{1 \leq k \leq 3}{\text{minimum}}(M[1][k] + M[k+1][4] + d_0 d_k d_4)$$
$$= \text{minimum}(M[1][1] + M[2][4] + d_0 d_1 d_4, M[1][2] + M[3][4] + d_0 d_2 d_4, M[1][3] + M[4][4] + d_0 d_3 d_4)$$
$$= \text{minimum}(0 + 72 + 5 \times 2 \times 6,\ 30 + 72 + 5 \times 3 \times 6,\ 64 + 0 + 5 \times 4 \times 6) = 132.$$

$M[1][6] = 348.$

3.4 Chained Matrix Multiplication

ALGORITHM 3.6: Minimum Multiplications

```
int minmult(int n, int d[], int P[][], int M[][]) {
    int i, j, k, diagonal;

    for (i = 1; i <= n; i++)
        M[i][i] = 0;
    for (diagonal = 1; diagonal <= n - 1; diagonal++)
        for (i = 1; i <= n - diagonal; i++) {
            j = i + diagonal;
            M[i][j] = minimum_{i≤k≤j−1}(M[i][k] + M[k + 1][j] + d_{i−1} d_k d_j);
            P[i][j] = a value of k that gave the minimum;
        }
    return M[1][n];
}
```

3.4 Chained Matrix Multiplication

```
void minmult(int n, vector<int>& d, matrix_t& M, matrix_t& P)
{
    for (int i = 1; i <= n; i++)
        M[i][i] = 0;
    for (int diagonal = 1; diagonal <= n - 1; diagonal++)
        for (int i = 1; i <= n - diagonal; i++) {
            int j = i + diagonal, k;
            M[i][j] = minimum(i, j, k, d, M);
            P[i][j] = k;
        }
}
```

3.4 Chained Matrix Multiplication

```
int minimum(int i, int j, int& mink, vector<int>& d, matrix_t& M)
{
    int minValue = INF, value;
    for (int k = i; k <= j - 1; k++) {
        value = M[i][k] + M[k + 1][j] + d[i - 1] * d[k] * d[j];
        if (minValue > value) {
            minValue = value;
            mink = k;
        }
    }
    return minValue;
}
```

3.4 Chained Matrix Multiplication

- Time Complexity of Algorithm 3.6 (Every-Case)
  - Basic Operation: the *instructions* executed for each value of k.
  - Input Size: n, the number of matrices to be multiplied.
  - Since we have a nested loop,
    - the number of passes through the k loop is
      - $j - 1 - i + 1 = i + diagonal - 1 - i + 1 = diagonal.$
    - the total number of times that the basic operation is done equals

$$\sum_{diagonal=1}^{n-1} [(n - diagonal) \times diagonal] = \frac{n(n-1)(n+1)}{6} \in \Theta(n^3)$$

3.4 Chained Matrix Multiplication

- Obtaining the *Optimal Order* from the array P.

P	1	2	3	4	5	6
1		1	1	1	1	1
2			2	3	4	5
3				3	4	5
4					4	5
5						5
6						

- P[1][6] = 1: P[1][1] & P[2][6] •$(A_1A_2A_3A_4A_5A_6)$
- P[2][6] = 6: P[2][5] & P[6][6] •$(A_1)(A_2A_3A_4A_5A_6)$
- P[2][5] = 4: P[2][4] & P[5][5] •$(A_1)((A_2A_3A_4A_5)A_6)$
- P[2][4] = 3: P[2][3] & P[4][4] •$(A_1)\left(((A_2A_3A_4)A_5)A_6\right)$
- P[2][3] = 2: P[2][2] & P[3][3] •$(A_1)\left(((A_2A_3)A_4)A_5)A_6\right)$

3.4 Chained Matrix Multiplication

ALGORITHM 3.7: Print Optimal Order

```
void order(int i, int j, matrix_t& P, string& s)
{
    if (i == j)
        s += string("A") + to_string(i);
    else {
        int k = P[i][j];
        s += string("(");
        order(i, k, P, s);
        order(k + 1, j, P, s);
        s += string(")");
    }
}
```

3.5 Optimal Binary Search Trees

- A ***binary search tree*** (**BST**) is
  - a *binary tree* of items (*keys*) that come from an *ordered* set, such that

1. Each node contains *one unique key*.
2. The keys in the *left subtree* of a given node
   - are *less than* or *equal to* the key in that node.
3. The keys in the *right subtree* of a given node
   - are *greater than* or *equal to* the key in that node.

3.5 Optimal Binary Search Trees

- ***Optimal*** Binary Search Tree
  - Our goal is to organize the keys in a BST so that
    - the *average time* it takes to locate a key is *minimized*.
  - Optimal BST is a tree that is organized in this fashion.
  - This problem is *not hard*
    - if all keys have the *same probability* of being the *search key*.
  - We are concerned with the case
    - where the keys *do **not** have the same probability*.
  - We will discuss the case in which
    - it is known that the *search key is **in** the tree*.

3.5 Optimal Binary Search Trees

ALGORITHM 3.8: Search Binary Tree

```
void search(node_ptr tree, int keyin, node_ptr& p)
{
    bool found;

    p = tree;
    found = false;
    while (!found) {
        if (p->key == keyin)
            found = true;
        else if (keyin < p->key)
            p = p->left;
        else //  keyin > p->key
            p = p->right;
    }
}
```

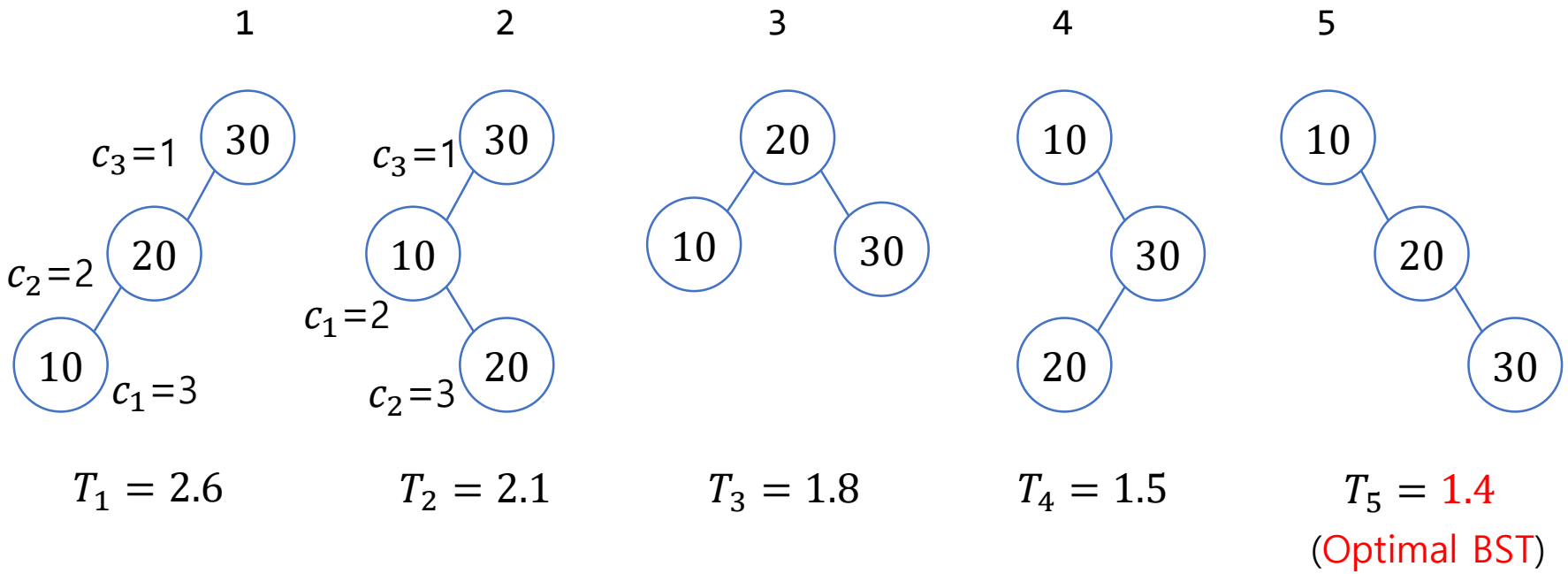
```
typedef struct node *node_ptr;
typedef struct node {
    int key;
    node_ptr left;
    node_ptr right;
} node_t;
```

3.5 Optimal Binary Search Trees

- Evaluating the search time of a BST
  - The *search time* is the *number of comparisons*
    - done by procedure *search* to locate a key.
  - Our goal is to determine a binary search tree
    - for which the *average search time* is *minimal*.
  - Let $K_1, K_2, \cdots, K_n$ be the n keys in order,
    - and let p_i be the *probability* that K_i is the *search key*.
  - The ***average search time*** is $T_{avg} = \sum_{i=1}^{n} c_i \, p_i$
    - where c_i be the *number of comparisons* needed to find K_i in a given tree.

3.5 Optimal Binary Search Trees

- An example:
  - $n = 3$, K=[10, 20, 30], $p = [0.7, 0.2, 0.1]$



$T_1 = 2.6$ $T_2 = 2.1$ $T_3 = 1.8$ $T_4 = 1.5$ $T_5 = 1.4$

(Optimal BST)

3.5 Optimal Binary Search Trees

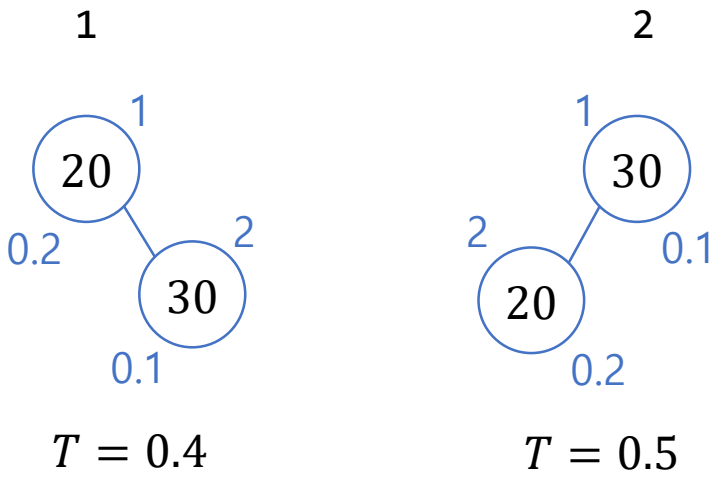
- Finding an Optimal BST
  - by considering all binary search trees.
  - The number of *different* BSTs with a depth of $n-1$ is 2^{n-1}.
    - If a BST's depth is $n-1$,
      - a node can be either to the left or to the right of its parent.
    - It means there are *two possibilities at each of those levels*.
  - The time complexity of this brute-force approach is *exponential*.

3.5 Optimal Binary Search Trees

- To apply the Dynamic Programming
  - Suppose that keys K_i through K_j are arranged in a tree
    - that minimizes $\sum_{m=i}^{j} c_m p_m$.
  - We will call such a tree ***optimal*** for those keys
    - and denote the *optimal value* by $A[i][j]$.
  - Because it takes one comparison to locate a key containing one key,
    - $A[i][i] = p_i$.

3.5 Optimal Binary Search Trees

- An example:
  - $n = 3$, K=[10, 20, 30], $p = [0.7, 0.2, 0.1]$, then determine $A[2][3]$.



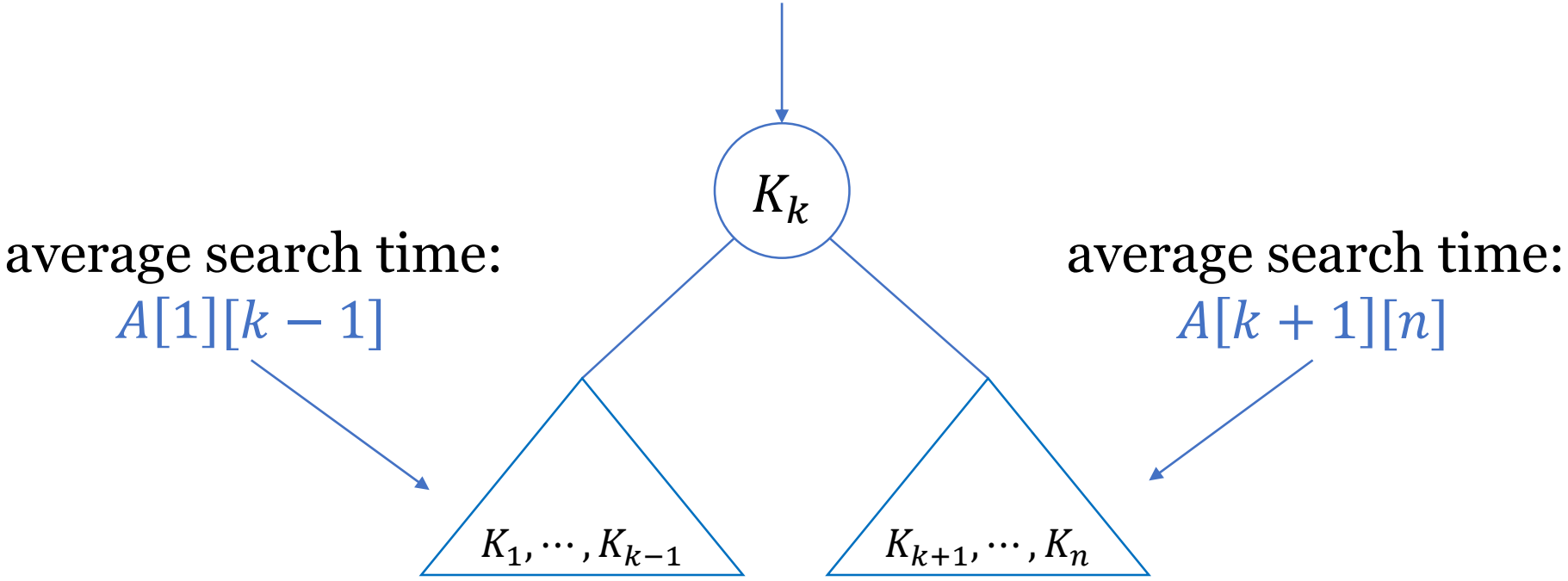
$T = 0.4$ $T = 0.5$

- $A[2][3] = 0.4$
  1. $1 \times p_2 + 2 \times p_3 = 0.4$ *(optimal)*
  2. $2 \times p_2 + 1 \times p_3 = 0.5$

- Note that the *optimal tree* is
  - the *right subtree* of the optimal tree obtained from the previous tree.
- The *principle of optimality* applies:
  - Any subtree of an optimal tree must be *optimal* for the key in that subtree.

3.5 Optimal Binary Search Trees

For each key, there is *one additional comparison* at the root.



3.5 Optimal Binary Search Trees

- Establish the recursive property:
  - The average search time for *tree k* is
    - $A[1][k-1]+A[k+1][n]+\sum_{m=1}^{n} p_m$.
  - The average search time for the *optimal tree* is given by
    - $A[1][n] = \underset{1 \le k \le n}{\text{minimum}}(A[1][k-1]+A[k+1][n])+\sum_{m=1}^{n} p_m$,
      - where $A[1][0]$ and $A[n+1][n]$ are defined to be 0.
  - In general,
    - $A[i][j] = \underset{i \le k \le j}{minimum}(A[i][k-1] + A[k+1][j]) + \sum_{m=i}^{j} p_m$, for $i < j$.
    - $A[i][j] = p_i$,
    - $A[i][i-1] = A[j+1][j] = 0$.

3.5 Optimal Binary Search Trees

- Determining an *Optimal* BST:
  - We proceed by computing *in sequence* the values on *each diagonal*.
    - because $A[i][j]$ is computed
      - from entries in the ith row but to the left of $A[i][j]$,
      - and from entries in the jth column but beneath $A[i][j]$.
  - Let an array R contain
    - the indices of the keys chosen for the root at each step.
    - $R[i][j]$: the index of the key in the root of an optimal tree
      - containing the ith through the jth keys.

3.5 Optimal Binary Search Trees

ALGORITHM 3.9: Optimal Binary Search Tree

```
void optsearchtree(int n, float p[], float &minavg, int R[][MAX]) {
    int i, j, k, diagonal;
    float A[MAX][MAX];
    for (i = 1; i <= n; i++) {
        A[i][i] = p[i];  A[i][i - 1] = 0;
        R[i][i] = i;     R[i][i - 1] = 0;
    }
    A[n + 1][n] = 0;
    R[n + 1][n] = 0;
    for (diagonal = 1; diagonal <= n - 1; diagonal++)
        for (i = 1; i <= n - diagonal; i++) {
            j = i + diagonal;
            A[i][j] = minimum_{i≤k≤j}(A[i][k − 1] + A[k + 1][j]) + Σ_{m=i}^{j} p_m;
            R[i][j] = a value of k that gave the minimum;
        }
    minavg = A[1][n];
}
```

3.5 Optimal Binary Search Trees

```
void optsearchtree(int n, vector<int>& p, matrix_t& A, matrix_t& R)
{
    for (int i = 1; i <= n; i++) {
        A[i][i - 1] = 0; A[i][i] = p[i];
        R[i][i - 1] = 0; R[i][i] = i;
    }
    A[n + 1][n] = R[n + 1][n] = 0;

    for (int diagonal = 1; diagonal <= n - 1; diagonal++)
        for (int i = 1; i <= n - diagonal; i++) {
            int j = i + diagonal;
            A[i][j] = minimum(i, j, k, p, A);
            R[i][j] = k;
        }
}
```

ALGORITHM 3.10: Build Optimal Binary Search Tree

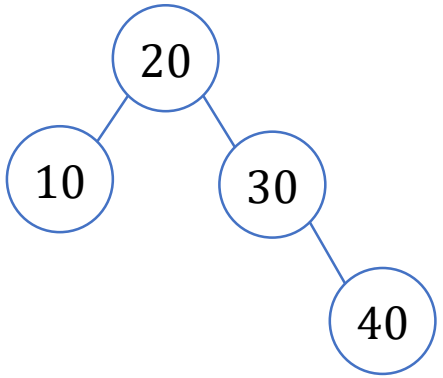
```
node_ptr tree(int i, int j, vector<int>& keys, matrix_t& R)
{
    int k = R[i][j];
    if (k == 0)
        return NULL;
    else {
        node_ptr node = create_node(keys[k]);
        node->left = tree(i, k - 1, keys, R);
        node->right = tree(k + 1, j, keys, R);
        return node;
    }
}
```

3.5 Optimal Binary Search Trees

- An example:
  - $n = 4, K = [10, 20, 30, 40], p = [3, 3, 1, 1]$.

A	1	2	3	4
1	3	9	11	14
2		3	5	8
3			1	3
4				1

R	1	2	3	4
1	1	1	2	2
2		2	2	2
3			3	3
4				4




```
preorder: 20 10 30 40
inorder:  10 20 30 40
```

3.5 Optimal Binary Search Trees

- Time Complexity of Algorithm 3.9:
  - Basic Operation: the instructions executed for *each value of* k.
    - They include a comparison to test for the minimum.
  - Input Size: n, the *number of keys*.
  - The control of this algorithm is almost *identical* to Algorithm 3.6.
    - The only difference is that,
      - the basic operation is done $diagonal + 1$ times.
  - Therefore,
    - $T(n) = \frac{n(n-1)(n+4)}{6} \in \Theta(n^3)$

3.6 The Traveling Salesperson Problem

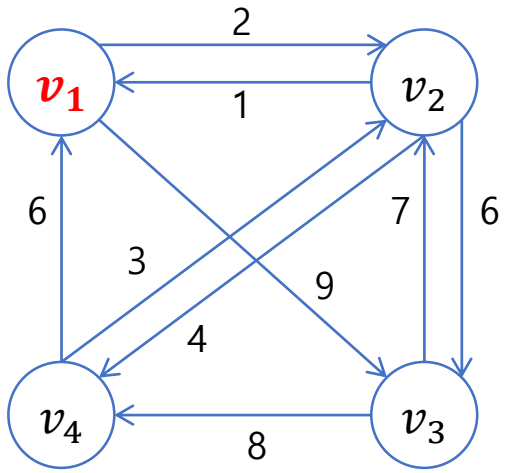
- The *Traveling Salesperson Problem* (**TSP**)
  - Suppose a *salesperson* is planning a *sales trip* that includes *20 cities*.
    - *Each city* is *connected* to some of the other cities by a road.
  - To *minimize* travel time,
    - we want to determine a *shortest route* that
      - *starts* at the salesperson's *home city*,
      - *visits* each of the cities *once*,
      - and *ends up* at the *home* city.

3.6 The Traveling Salesperson Problem

- Generalization of the TSP:
  - In a *weighted, directed* graph,
    - find an *optimal tour* when at least one tour exists.
  - A ***tour*** in a directed graph is *a path* from *a vertex* to *itself*
    - that *passes through* each of the other vertices *exactly once*.
    - also called a *Hamiltonian circuit*.
  - An ***optimal tour*** in a *weighted*, directed graph is
    - such a path of *minimum length*.

3.6 The Traveling Salesperson Problem

- An example:
  - We will consider v_1 to be the *starting vertex*.
  - There are *three tours* and the lengths are:
    - $length[v_1, v_2, v_3, v_4, v_1] = 22$
    - $length[v_1, v_3, v_2, v_4, v_1] = 26$
    - $length[v_1, v_3, v_4, v_2, v_1] = 21$ (optimal tour)



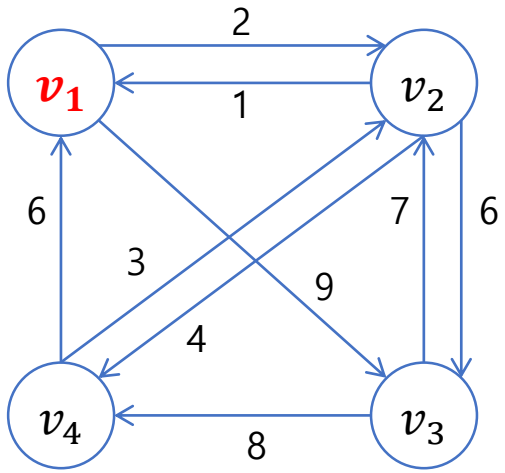
  - In general,
    - there can be an edge from *every vertex* to *every other vertex*.
    - the total number of tours is $(n-1)(n-2)\cdots 1 = (n-1)!$
    - *Factorial time complexity*: worse than *exponential*!

3.6 The Traveling Salesperson Problem

- The *principle of optimality* applies:
  - Notice that if v_k is the *first vertex* after v_1 on an *optimal tour*,
    - the *subpath* of the tour from v_k to v_1
    - must be a *shortest path* from v_k to v_1
    - that passes through each of the other vertices *exactly once*.
  - Therefore, we can design an algorithm for the TSP
    - using the dynamic programming.

3.6 The Traveling Salesperson Problem

- Formalization of the TSP:
  - Let W be an *adjacency matrix* for a given graph $G = (V, E)$.
  - Let V be a set of all the vertices.
  - Let A be a subset of V.
  - Let $D[v_i][A]$ be the length of a shortest path from v_i to v_1
    - *passing through* each vertex in A *exactly once*.

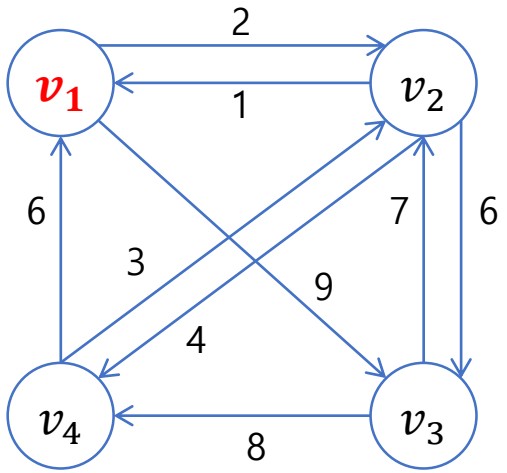


- $V = \{v_1, v_2, v_3, v_4\}$
- If $A = \{v_3\}$,
  - $D[v_2][A] = lengh[v_2, v_3, v_1] = \infty$.
- If $A = \{v_3, v_4\}$,
  - $D[v_2][A] = \text{minimum}(lengh[v_2, v_3, v_4, v_1], lengh[v_2, v_4, v_3, v_1])$
    $= \text{minimum}(20, \infty) = 20$.

3.6 The Traveling Salesperson Problem

- Establish the *recursive property*:
  - Because $V - \{v_1, v_j\}$ contains
    - all the vertices except v_1 and v_j and the principle of optimality applies,
  - The length of an optimal tour is
    - $\underset{2 \le j \le n}{\text{minimum}}\left(W[1][j] + D[v_j][V - \{v_1, v_j\}]\right)$.
  - In general, for $i \neq 1$ and $v_i \notin A$,
    - $D[v_i][A] = \underset{j:v_j \in A}{\text{minimum}}\left(W[i][j] + D[v_j][A - \{v_j\}]\right)$, if $A \neq \phi$
    - $D[v_i][\phi] = W[i][1]$.

3.6 The Traveling Salesperson Problem



W	1	2	3	4
1	0	2	9	∞
2	1	0	6	4
3	∞	7	0	8
4	6	3	∞	0

$A = \phi$:
- $D[v_2][\phi] = W[2][1] = 1$
- $D[v_3][\phi] = W[3][1] = \infty$
- $D[v_4][\phi] = W[4][1] = 6$

$A = \{v_2\}$:
- $D[v_3][A] = \underset{j:v_j \in A}{\text{minimum}}\left(W[3][j] + D[v_j][A - \{v_j\}]\right)$
  $= W[3][2] + D[v_2][\phi] = 7 + 1 = 8$

$A = \{v_2, v_3\}$:
- $D[v_4][A] = \underset{j:v_j \in A}{\text{minimum}}\left(W[4][j] + D[v_j][A - \{v_j\}]\right)$
  $= \text{minimum}(W[4][2] + D[v_2][\{v_3\}], W[4][3] + D[v_3][\{v_2\}])$
  $= \text{minimum}(3 + \infty, \infty + 8) = \infty$

$A = \{v_2, v_3, v_4\}$:
- $D[v_1][A] = \underset{j:v_j \in A}{\text{minimum}}\left(W[1][j] + D[v_j][A - \{v_j\}]\right)$
  $= \text{minimum}(W[1][2] + D[v_2][\{v_3, v_4\}],$
  $W[1][3] + D[v_3][\{v_2, v_4\}],$
  $W[1][4] + D[v_4][\{v_2, v_3\}])$
  $= \text{minimum}(2 + 20, 9 + 12, \infty + \infty) = 21$

3.6 The Traveling Salesperson Problem

ALGORITHM 3.11: The Dynamic Programming Algorithm for the TSP

```
void travel(int n, int W[][MAX], int P[][MAX], int &minlength) {
    int i, j, k, D[MAX][MAX_POW];

    for (i = 2; i <= n; i++)
        D[i][ϕ] = W[i][1];
    for (k = 1; k <= n - 2; k++)
        for (all subsets A ⊆ V − {v1} containing k vertices)
            for (i such that i ≠ 1 and v_i is not in A) {
                D[i][A] = minimum_{j:v_j∈A}(W[i][j] + D[j][A − {v_j}]);
                P[i][A] = value of j that gave the minimum;
            }
    D[1][V − {v1}] = minimum_{2≤j≤n}(W[1][j] + D[j][V − {v1, v_j}]);
    P[1][V − {v1}] = value of j that gave the minimum;
    minlength = D[1][V − {v1}];
}
```

3.6 The Traveling Salesperson Problem

```
void travel(int n, matrix_t& W, matrix_t& D, matrix_t& P, int &minlength) {
    int i, j, k, A;

    int subset_size = pow(2, n - 1);
    for (i = 2; i <= n; i++)
        D[i][0] = W[i][1];
    for (k = 1; k <= n - 2; k++)
        for (A = 0; A < subset_size; A++) {
            if (count(A) != k) continue;
            for (i = 2; i <= n; i++) {
                if (isIn(i, A)) continue;
                D[i][A] = minimum(n, i, j, A, W, D);
                P[i][A] = j;
            }
        }
    A = subset_size - 1; // A = V - {v1}
    D[1][A] = minimum(n, 1, j, A, W, D);
    P[1][A] = j;
    minlength = D[1][A];
}
```

3.6 The Traveling Salesperson Problem

- Handling Subsets using Bitwise Operations:
  - All subsets $A \subseteq V - \{v_1\}$: 0, 1, 2, 3, 4, 5, 6, 7

$A = \phi$: 000 = 0

$A = \{v_2\}$: 001 = 1

$A = \{v_3\}$: 010 = 2

$A = \{v_4\}$: 100 = 4

$A = \{v_2, v_3\}$: 011 = 3

$A = \{v_2, v_4\}$: 101 = 5

$A = \{v_3, v_4\}$: 110 = 6

$A = \{v_2, v_3, v_4\}$: 111 = 7

2^{n-1}

v_4 v_3 v_2

↓ ↓ ↓

1 0 1

↓ ↓

$A = \{ v_4 \quad v_2 \} = 5$

3.6 The Traveling Salesperson Problem

- Handling Subsets using Bitwise Operations:
  - All subsets A containing k vertices

```
int subset_size = pow(2, n - 1);

for (k = 1; k <= n - 2; k++)
    for (A = 0; A < subset_size; A++) {
        if (count(A) != k) continue;
        ……
    }
```

```
int count(int A) {
    int cnt = 0;
    for (; A != 0; A >>= 1)
        if (A & 1) cnt++;
    return cnt;
}
```

3.6 The Traveling Salesperson Problem

- Handling Subsets using Bitwise Operations:
  - for i such that $i \neq 1$ and v_i is not in A

```
for (i = 2; i <= n; i++) {
    if (isIn(i, A)) continue;
    ……
}
```

```
bool isIn(int i, int A) {
    return (A & (1 << (i - 2))) != 0;
}
```

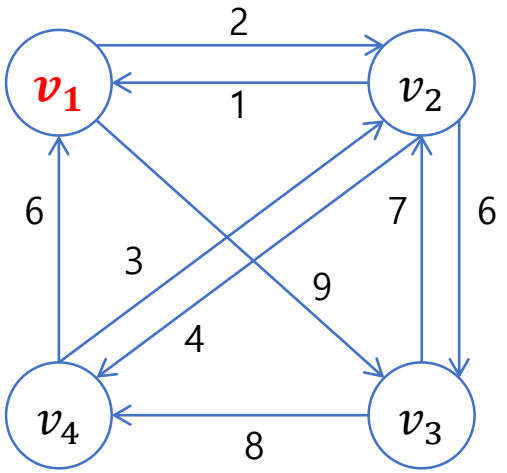
3.6 The Traveling Salesperson Problem

- Handling Subsets using Bitwise Operations:
  - Minimum & Set Difference: $A - \{v_j\}$

```
int minimum(int n, int i, int &minJ, int A, matrix_t& W, matrix_t& D) {
    int minV = INF, value;
    for (int j = 2; j <= n; j++) {
        if (!isIn(j, A)) continue;
        int value = W[i][j] + D[j][diff(A, j)];
        if (minV > value) {
            minV = value;
            minJ = j;
        }
    }
    return minV;
}
```

```
int diff(int A, int j) {
    return (A & ~(1 << (j - 2)));
}
```

3.6 The Traveling Salesperson Problem



W	1	2	3	4
1	0	2	9	∞
2	1	0	6	4
3	∞	7	0	8
4	6	3	∞	0

D	0	1	2	3	4	5	6	7
1								**21**
2	1		∞		10		20	
3	∞	8			14	12		
4	6	4	∞	∞				

P	0	1	2	3	4	5	6	7
1								3
2	0		2		4		3	
3	0	2			4	4		
4	0	2	2	4				

3.6 The Traveling Salesperson Problem

- Retrieving an *optimal tour* from the array P:
  - $P[i][A]$: the index of the first vertex after v_i
    - on a shortest path from v_i to v_1
    - that passes through all vertices in A exactly once.

P	0	1	2	3	4	5	6	7
1								3
2			2		4		3	
3		2			4	4		
4		2	2	4				

$P[1][7], A = \{v_2, v_3, v_4\}$:

$P[3][5], A = \{v_2, v_4\}$:

$P[4][1], A = \{v_2\}$:

$P[2][1], A = \{\phi\}$:

Optimal tour: v_1, v_3, v_4, v_2, v_1

3.6 The Traveling Salesperson Problem

```
void tour(int v, int A, matrix_t& P) {
    int k =  P[v][A];
    if (A == 0)
        cout << "1";
    else {
        cout << k << " -> ";
        tour(k, diff(A, k), P);
    }
}
```

```
cout << "optimal tour: 1 -> ";
tour(1, pow(2, n - 1) - 1, P);
```

$V - \{v_1\}$

3.6 The Traveling Salesperson Problem

- Time Complexity of Algorithm 3.11:
  - Basic Operation: the instructions executed *for each value of* v_j.
  - Input Size: n, the *number of vertices* in the graph.
  - The total number of times the basic operation is done is given by

$$T(n) = \sum_{k=1}^{n-2} (n-1-k)k\binom{n-1}{k}$$

$$= (n-1)\sum_{k=1}^{n-2} k\binom{n-2}{k}$$

$$= (n-1)(n-2)2^{n-3} \in \Theta(n^2 2^n)$$

3.6 The Traveling Salesperson Problem

- What we have gained:
  - Our algorithm is still $\Theta(n^2 2^n)$ which is *exponential*.
  - When there are 20 cities and it takes 1 μs to process 1 city,
    - Brute-force approach: $(n-1)! = 19!\ \mu s = 3857\ years$
    - Dyn. Prog.: $(n-1)(n-2)2^{n-3} = 19 \times 18 \times 2^{17}\ \mu s = 45\ seconds$
  - Can we solve the TSP in *polynomial* time?
    - *Nobody solved* it in *polynomial* time.
    - *Nobody proved* that it is *impossible* to solve it in polynomial time.

Any Questions?