Math Homework Week #2, Inner Product Space

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1. (1, 3.1)
   (i) $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$

   $= \frac{1}{4}(\|x\|^2 + <x,y> + <y,x> + \|y\|^2 - \|x\|^2 + <x,y> + <y,x> - \|y\|^2)$

   $= \frac{1}{4}(4 \cdot <x,y>)$

   $= <x,y>$

   (ii) $\frac{1}{2}(\|x+y\|^2 + \|x-y\|^2)$

   $= \frac{1}{2}(\|x\|^2 + <x,y> + <y,x> + \|y\|^2 + \|x\|^2 - <x,y> - <y,x> + \|y\|^2)$

   $= \frac{1}{2}(2\|x\|^2 + 2\|y\|^2)$

   $= \|x\|^2 + \|y\|^2$

2. (2, 3.2)
   $= \frac{1}{4}(4[Re <x,y>] + 4i[Im <x,y>])$

   $= <x,y>$

   Because, $i\|x+iy\|^2 - i\|x-iy\|^2 = -2(-<x,y> + \overline{<x,y>}) = 4i[IM <x,y>]$

3. (3, 3.3)
   $<f,g> = \int_0^1 f(x)g(x)dx$

   (i) $cos\theta = \frac{\int_0^1 x \cdot x^5 dx}{\sqrt{\int_0^1 x^2 dx}\sqrt{\int_0^1 x^{10} dx}}$

   $= \frac{[\frac{1}{7}x^7]_0^1}{\sqrt{[\frac{1}{3}x^3]_0^1 [\frac{1}{11}x^{11}]_0^1}} = \frac{\sqrt{33}}{7}$

   (ii) $cos\theta = \frac{\int_0^1 x^2 \cdot x^4 dx}{\sqrt{\int_0^1 x^4 dx}\sqrt{\int_0^1 x^8 dx}}$

   $= \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}\frac{1}{9}}} = \frac{\sqrt{45}}{7}$

4. (4, 3.8)
   (i)
   (1) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot sint\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2} sin2t\ dt$

$= \frac{1}{2\pi}[-\frac{1}{2}cos2t]_{-\pi}^{pi}$
$= 0$

(2) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot cos2t\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} cost(2cos^2t - 1)\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi}(2cos^3t - cost)\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(cos3t + cost)\ dt$

$= \frac{1}{\pi}[\frac{1}{6}(sin3t) + \frac{1}{2}sint]_{-\pi}^{\pi}$

$= 0$

(3) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot sin2t\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} 2cos^2t \cdot sint\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} 2(sint - sin^3t)\ dt$

$= \frac{1}{\pi}[\frac{1}{2}(-cost) + \frac{1}{6}(-cos3t)]_{-\pi}^{\pi} = 0$

(4) $\int_{-\pi}^{\pi} sint \cdot cos2t\ dt = \int_{-\pi}^{\pi} sint(1 - 2sin^2t)\ dt$

$= \int_{-\pi}^{\pi} -\frac{1}{2}sint + \frac{3}{2}sin3t\ dt$

$= [\frac{1}{2}cost - \frac{1}{2}cost3t]_{-\pi}^{\pi}$

$= 0$

(5) $\int_{-\pi}^{\pi} sint \cdot sin2t\ dt = \int_{-\pi}^{\pi} 2sin^2tcost\ dt = \int_{-\pi}^{\pi} 2(1 - cos^2t)cost\ dt$

$= \int_{-\pi}^{\pi} 2cost - 2 \cdot \frac{1}{4}(3cost + cos3t)\ dt = \int_{-\pi}^{\pi} \frac{1}{2}cost - \frac{1}{2}cos3t\ dt$

$= [\frac{1}{2}sint - \frac{1}{6}sin3t]_{-\pi}^{\pi}$

$= 0$

(6) $\int_{-\pi}^{\pi} cos2t \cdot sin2t\ dt$

$\int_{-2\pi}^{2\pi} cost \cdot sint(\frac{1}{2})\ dt = 0$

Thus, S is orthogonal set.
For measure, $\frac{1}{\pi}\int_{-\pi}^{\pi} cos^2tdt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi} sin^2tdt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi} cos^22tdt = 1$
$\frac{1}{\pi}\int_{-\pi}^{\pi} sin^22tdt = 1$

Thus, it is orthonormal.

(ii) compute $||t||$
$< t, t >= \frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt = \frac{2}{3}\pi^2$

(iii) $\sum_{i=1}^{m} < x_i, cos(3t) > x_i = 0$

(iv) $\sum_{i=1}^{m} < x_i, t > x_i = 2\pi(sint) - \pi(sin(2t))$

5. (5, 3.9) $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 cos\theta - x_2 sin\theta \\ x_1 sin\theta + x_2 cos\theta \end{bmatrix}$
$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 cos\theta - y_2 sin\theta \\ y_1 sin\theta + y_2 cos\theta \end{bmatrix}$

$(x_1 cos\theta - x_2 sin\theta)(y_1 cos\theta - y_2 sin\theta) + (x_1 sin\theta + x_2 cos\theta)(y_1 sin\theta + y_2 cos\theta) = x_1 y_1 + x_2 y_2$

6. (6, 3.10)
(i) $Q^H Q = \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix} \begin{bmatrix} q_1 & \dots & q_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$
(ii) $\|Qx\| = (Qx)^H Qx = x^H Q^H Qx = x^H x = ||x||$
(iii) $Q^{-1} Q Q^H = Q^{-1}$
$Q^H = Q^{-1}$
(iv) $QQ^H = I$ has a proof already above.
(v) $det(Q^H Q) = det(I)$
$det(Q^H) det(Q) = I$
$(detQ)^2 = 1$
$\therefore |det(Q)| = 1$

Converse is not true. Counter example is $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
(vi) $(Q_1 Q_2)(Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = QQ_1^H = I$
$(Q_1 Q_2)^H (Q_1 Q_2) = I$
$\therefore Q_1 Q_2$ is also orthonormal.

7. (7, 3.11)
Assume $(x_1 \cdots x_{n-1})$ are linearly independent.
Then, $x_n =$ span $(x_1, x_2 \cdots x_{n-1})$
If $(e_1, \cdots e_{n-1})$ are orthonormal basis, after doing G-S process, then,
$x_n =< x_n, e_1 > e_1 + < x_n, e_2 > e_2 + \cdots + < x_n, e_{n-1} > e_{n-1}$
Now, in the G-S process, for finding e_n

$e_n = \frac{x_n - p_n}{\|x_n - p_n\|}$, then , $x_n - p_n = 0$ So we will calculate $\frac{0}{\|0\|}$

8. (8, 3.16) From $A = QR$,
(i) If A is singular, some diagonal entries of R can be zero. In this case, we can have multiple Q.

   (ii) $A = Q_1R_1 = Q_2R_2$.
$K = R_1R_2^{-1} = Q_1^HQ_2$ and $(R_1^{-1})^HR_2^H = R_1R_2^{-1}$
diagonals of K are positive and 1. Also, it has to be lower triangular and upper triangular matrix.
$\therefore, R_1 = R_2, Q_1 = Q_2$

9. (9, 3.17)
$A^HAx = A^Hb$
$(\hat{Q}\hat{R})^H(\hat{Q}\hat{R})x = (\hat{Q}\hat{R})^Hb$
$\hat{R}^HRx = \hat{Q}\hat{R}^Hb$
$\hat{Q}^Hx = \hat{R}^Hb$

10. (10, 3.23)
$\|y\| = \|x + y - x\| \leq \|x\| + \|y - x\|$

    Thus, $\|y\| - \|x\| \leq \|y - x\|$

    $\|x\| = \|y + x - y\| \leq \|y\| + \|x - y\|$

    Thus, $\|x\| - \|y\| \leq \|x - y\|$

    Therefore, $|\|x\| - \|y\|| \leq \|x - y\|$

11. (11, 3.24)
(i)

    (1) $\int_a^b |0|dt = 0$ and $\int_a^b |f(t)|dt \geq 0$

    (2) $\int_a^b |af(t)|dt = a\int_a^b |f(t)|dt$

    (3) $\int_a^b |f(t)|dt + \int_a^b |g(t)|dt = \int_a^b |f(t)| + |g(t)|dt \geq \int_a^b |f(t) + g(t)|dt$

    (ii)

    (1) $(\int_a^b |0|^2dt)^{0.5} = 0$ and $(\int_a^b |f(t)|^2dt)^{0.5} \geq 0$

    (2) $(\int_a^b |af(t)|^2dt)^{0.5} = a(\int_a^b |f(t)|^2dt)^{0.5}$

    (3) $(\int_a^b |f(t)|^2dt)^{0.5} + (\int_a^b |g(t)|^2dt)^{0.5} \geq (\int_a^b |f(t) + g(t)|^2dt)^{0.5}$

(iii)

(1) $sup|0| = 0$ and $sup|f(x)| \geq 0$

(2) $sup|af(x)| = a \cdot sup|f(x)|$

(3) $sup|f(x)| + sup|f(y)| \geq sup|f(x) + f(y)|$

$\because |f(x)| + |f(y)| \geq |f(x) + f(y)|$

12. (12, 3.26)
For equivalence relation,
1) $a \sim a$
2) $a \sim b$ iff $b \sim a$
3) if $a \sim b$ and $b \sim c$, then $a \sim c$

1) There is $0 < m < M$ s.t $m||X||a \leq ||X||a \leq M||X||a$
2) There is $0 < m < M$ s.t $m||X||a \leq ||X||b \leq M||X||a$
There is $0 < \frac{1}{M} < \frac{1}{m}$ s.t $\frac{1}{M}||X||b \leq ||X||a \leq \frac{1}{m}||X||b$
3) There is There is $0 < m < M$ s.t $m||X||a \leq ||X||b \leq M||X||a$
There is $0 < m^* < M^*$ s.t $m^*||X||b \leq ||X||c \leq M^*||X||b$
Then, there is $0 < \frac{m}{m^*} < MM^*$ s.t $\frac{m}{m^*}||X||a \leq ||X||c \leq MM^*||X||a$

(i) 1) $\|X\|_2 \leq \|X\|_1$
$(|x_1|^2 + \cdots + |x_n|^2) < (|x_1| + \cdots + |x_n|)^2 \because$ interaction terms
2) $\|X\|_1 \leq \sqrt{n}\|X\|_2$
$\|X\|_1 = \sum_{i=1}^n x_i 1 \leq (\sum_{i=1}^n (x_i)^2)^{0.5} (\sum_{i=1}^n 1)^{0.5} \leq \sqrt{n}\|X\|_2$ by C-S ineqaulity.

(ii)$\|X\|_\infty = sup(|x_1|, ..., |x_n|)$
1) $\|X\|_\infty \leq \|X\|_1$ This is trivial.
2) $\|X\|_2 \leq \sqrt{n} \cdot sup(|x_1|, ..., |x_n|)$
$(|x_1|^2 + \cdots + |x_n|^2)^{0.5} \leq |x_1| + \cdots + |x_n|$
$|x_1|^2 + \cdots + |x_n|^2 \leq (|x_1| + \cdots + |x_n|)^2 \leq n[sup(|x_1|, \cdots, |x_n|)]^2$

13. (13, 3.28)
(i)
1) $\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1$
$\frac{1}{\sqrt{n}}\|A\|_2 = sup\frac{\|AX\|_2}{\sqrt{n}\|X\|_2} \leq sup\frac{\|AX\|_1}{\|X\|_1}$
$\sqrt{n}\|X\|_2 \geq \|X\|_1$ by 3.26 (i)
$\|AX\|_2 \leq \|AX\|_1$ by 3.26 (i)

2) $\|A\|_1 \leq \sqrt{n}\|A\|_2$
$sup\frac{\|AX\|_1}{\|X\|_1} \leq \sqrt{n}sup\frac{\|AX\|_2}{\|X\|_2}$
$\|X\|_1 \geq \|X\|_2$ by 3.26 (i)
$\sqrt{n}\|AX\|_2 \geq \|AX\|_1$ by 3.26 (i)

(ii) $\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2$
$sup\frac{\|AX\|_\infty}{\sqrt{n}\|X\|_\infty} \leq sup\frac{\|AX\|_2}{\|X\|_2}$
$\|A\|_2 \leq \sqrt{n}\|A\|_\infty$
$sup\frac{\|AX\|_2}{\|X\|_2} \leq sup\frac{\sqrt{n}\|AX\|_\infty}{\|X\|_\infty}$
Now, same logic above applies.

14. (14, 3.29)
$\rightarrow$ By the property $||Qx|| = ||x||$, $||Q|| = \frac{||Qx||}{||x||} = \frac{||x||}{|x||} = 1$.
$\rightarrow ||R_x|| = sup\frac{\|RxA\|_2}{\|A\|_2}$
$= sup\frac{\|Ax\|_2}{\|A\|_2}$
$sup\frac{\|Ax\|_2}{sup\frac{\|Ax\|_2}{\|x\|_2}} \leq \|x\|_2$
Now, combining hint given, $\|R_x\|_2 \geq \|x\|_2$

15. (15, 3.30)
1) $||SAS^{-1}|| \geq$ and $A = 0 \leftrightarrow ||SAS^{-1}|| = 0$
2) $||S(tA)S^{-1}|| = |t| \cdot ||SAS^{-1}||$
3) $||S(A+B)S^{-1}|| \leq ||SAS^{-1}|| + ||SBS^{-1}||$
4) $||SABS^{-1}|| = ||SAS^{-1}SBS^{-1}|| \leq ||SAS^{-1}|| \cdot ||SBS^{-1}||$

16. (16, 3.37)
$p = ax^2 + bx + c$
$p(1) = 2a + b = L(p)$
$L(p) =< q, p >= \int_0^1 (qp)dx = 2a + b$
After setting $q = dx^2 + ex + f$, Then, after integration, we got the identity eqation like the next one.
$(\frac{1}{5}d + \frac{1}{4}e + \frac{1}{3}f)a + (\frac{1}{4}d + \frac{1}{3}e + \frac{1}{2}f)b + (\frac{1}{3}d + \frac{1}{2}e + f)c = 2a + b$

$\therefore d = 180, e = -168, f = 24$

17. (17, 3.38)
$p = ax^2 + bx + c$
$D(p)(x) = 2ax + b$
$$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

$$adjoint = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^T$$

18. (18, 3.39)
    (i) $< (S+T)^*x, y > = < x, (S+T)y >$

    $= < x, Sy > + < y, Ty > = < S^*, x > + < T^*x, y > = < (S^* + T^*)x, y >$

    (ii) $< (S^*)^*x, y > = < x, S^*y > = < Sx, y >$

    (iii) $< (ST)^*x, y > = < x, STy > = < S^*x, Ty > = < T^*S^*x, y >$

    (iv) $(T^*)^{-1}T^* = 1$ and $((T^*)^{-1}T^*)^* = 1^* = 1$
    Thus, $T^{**}((T^*)^{-1})^* = 1$
    $T((T^*)^{-1})^* = 1$
    $((T^*)^{-1})^* = T^{-1}$
    $(T^*)^{-1} = (T^{-1})^*$

19. (19, 3.40)
    $< A, B > = tr(A^T B)$
    (i) $A^* = A^H$
    $< AB, C > = < B, A^*C >$
    $tr((AB)^H C) = tr(B^H A^* C)$
    $tr(B^H A^H C) = tr(B^H A^* C)$
    $\therefore A^H = A^*$

    (ii) $< A_2, A_3A_1 > = < A_2A_1^*, A_3 >$
    $tr(A_2^H A_3 A_1) = tr((A_2A_1^*)^H A_3)$??
    $\rightarrow tr(A_1^H A_2 A_3^H) = tr((A_2A_1^*)^H A_3)$
    (iii) $T_A(X) = AX - XA$
    $< X, T_{A^*}X > = tr[X^*(A^*X - XA^*)]$
    $= tr[X^*XA - AXX^*]$
    $= tr[X^*(XA - AX)]$
    $= tr[(XA - AX)^*X]$
    $= < T_A(X), X >$
    $= < X, (T_A)^*(X) >$
    $\therefore (T_A)^* = T_{A^*}$

20. (20, 3.44)
    If $Ax = b$ satifies, then, $0 = 0^T\hat{x} = (y^TA)x = y^TAx = y^Tb = < y, b > \neq 0$

21. (21, 3.45)
    First, I want to establish general cases of symmetric and skew-symmetric matrix

like below.
$X = A + A^T$ and $Y = B - B^T$
$tr[(A + A^T)^T(B - B^T)] = tr[AB + A^TB - AB^T - A^TB^T] = 0$ by property of trace.
Then, it easily proves by this.

22. (22, 3.46)
(i) $Ax \in R(A)$
Also $x \in N(A^HA)$.
It means that $A^HAx = 0$
$\therefore A^H(Ax) = 0$
$Ax \in N(A^H)$
(ii) $A^HAx = 0 \to (A^H)^{-1}A^HAx = 0 \to Ax = 0$
(iii) $N(A^HA) = N(A)$, so its dimenstion has to be same.
(iv) $Ax = 0$.
If A has linear independent column, then $x = 0$.
so, $A^HAx = 0$ and $null(A^HA)$ is 0.
A^HA is nonsingular.

23. (23, 3.47)
$P = A(A^HA)^{(-1)}A^H$
(i) $P^2 = A(A^HA)^{(-1)}A^HA(A^HA)^{(-1)}A^H = A(A^HA)^{(-1)}A^H = P$
(ii) $P^H = (A(A^HA)^{(-1)}A^H)^H = A(A^HA)^{(-1)}A^H$
(iii) $rank(P) = n$
$rank(P) \leq rank(A)$ and $PA = A$ so, $rank(A) \leq rank(P)$
$\therefore rank(P) = rank(A)$

24. (24, 3.48)
(i) $P(A + B) = \frac{(A+B)+(A+B)^T}{2} = P(A) + P(B)$
$P(tA) = \frac{tA+(tA)^T}{2} = tP(A)$
(ii) $\frac{\frac{A+(A)^T}{2} + \frac{A+(A)^T}{2}}{2} = P(A)$
(iii) $P^* = P$
$< P(A), B >=< A, P^*(B) >$
$< P(A), B >=< \frac{A+(A)^T}{2}, B >= tr[\frac{A+(A)^T}{2}B] = tr[\frac{A^T}{2}B + \frac{A}{2}B] - -first$
$< A, P^*(B) >=< A, \frac{B+(B)^T}{2} >= tr[A\frac{B+(B)^T}{2}] = tr[\frac{AB}{2} + \frac{AB^T}{2}] - -second$
first and second are same.
(iv) $N(P) = skew_n(R)$?
Let the matrix X be being playing as operator P.
$XA = 0 \to \frac{A+(A)^T}{2} = 0 \to A = -A^T$
(v) $R(P) = Sym_n(R)$
$XA = B \to \frac{A+(A)^T}{2} = B$
$B = B^T$
(vi) $\|A - \frac{A+(A)^T}{2}\| = \|\frac{A-(A)^T}{2}\| = \sqrt{< \frac{A-(A)^T}{2}, \frac{A-(A)^T}{2} >}$

$= \sqrt{tr[\frac{A-(A)^T}{2}\frac{A-(A)^T}{2}]} = \sqrt{tr[\frac{1}{4}(A^TA - AA - A^TA^T + AA^T)]} = \sqrt{\frac{tr(A^TA)-tr(A^2)}{2}}$

25. (25, 3.50)
I used the familiar notations for me first.
$rx_2 + sy_2 = 1$

$y^2 = Y \; and \; x^2 = X$

$sY + rX = 1$

$sY = -rX + 1$

$Y = -\frac{r}{s}X + \frac{1}{s}$

$= \beta_1 X + \beta_0$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \hat{\frac{1}{s}} \\ -\hat{\frac{r}{s}} \end{bmatrix} = (\hat{X}^T\hat{X})^{-1}(\hat{X})^T y$$

where $\hat{x} = (1, X) = (1, x^2)$

Thus, $\hat{s} = \frac{1}{\beta_0}, \; \hat{r} = -\hat{s}\hat{\beta}_1 = -\frac{\hat{\beta}_1}{\hat{\beta}_0}$

$\therefore A, \mathbf{x}, \mathbf{b}$ are as in the following. $(X = A, y = \mathbf{b})$ here.
$A = \begin{bmatrix} 1 & x_1^2 \\ \vdots & \vdots \\ 1 & x_n^2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \frac{1}{s} \\ -\frac{r}{s} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} y_1^2 \\ \vdots \\ y_n^2 \end{bmatrix}$

References