Math, Problem Set #1 Answer, Probability Theory

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Due Monday, June 26 at 8:00am

1. **Exercises from chapter.** Do the following exercises in Chapter 3 of **?**: 3.6, 3.8, 3.11, 3.12 (watch this movie clip), 3.16, 3.33, 3.36.

(3.6) ($\Rightarrow$) For any $A \in \mathcal{F}$, so $A \subset \Omega$.
By the way, $\Omega = \bigcup_{i \in I} B_i$ and $B_i \cap B_j = \phi$ for all $i \neq j$
$\therefore A \subset \bigcup_{i \in I} B_i$
$\therefore A = \bigcup_{i \in I} A \cap B_i$
By the countable additivity, $P(\bigcup_{i \in I} A \cap B_i) = \sum_{i \in I} P(A \cap B_i)$

(3.8) ($\Rightarrow$) Claim: If A and B are independent, then A^c and B^c are also independent.
$P(A^c \cap B^c) = P[(A \cup B)^c] = 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 + P(A)P(B) - P(A) - P(B)$
$= P(B^c)P(A^c)$
We can extend this result to general n sets by mathematical induction.
$P(\bigcup_{k=1}^n E_k) + P(\bigcap_{k=1}^n E_k^c) = 1$
$P(\bigcup_{k=1}^n E_k) = 1 - P(\bigcap_{k=1}^n E_k^c)$
$= 1 - \Pi_{k=1}^n P(E_k^c)$ by independence.
$= 1 - \Pi_{k=1}^n 1 - P(E_k)$

(3.11) ($\Rightarrow$) $P(s = crime | \text{s tested } +) = \frac{P(s\ \ tested\ \ + | s=crime)P(s=crime)}{P(steted+)}$
$= (1)(\frac{1}{250})/(\frac{1}{3})$
$= \frac{3}{250}$

(3.12) ($\Rightarrow$) Assume that a player picks door 1, we can calculate the probability of changing the door. And also, we can extend this to general cases.
For general n doors,

X_i: the door that player chooses
E_i: the car behind the door X_i
M_i: the door that Monti Hall opens

$P(M) = P(M|E_1)P(E_1) + \ldots P(M|E_n)P(E_n)$
$= (\frac{1}{{}_{n-1}C_k})(\frac{1}{n}) + (\frac{1}{{}_{n-2}C_k})(\frac{1}{n})(n-k-1)$
$= \frac{1}{{}_{n-1}C_k}$

$P(E_1|M) = \frac{P(M \cap E_1)}{P(M)} = \frac{P(M|E_1)P(E_1)}{P(M)} = \frac{(\frac{1}{{}_{n-1}C_k})(\frac{1}{n})}{\frac{1}{{}_{n-1}C_k}} = \frac{1}{n}$

$P(E_2|M) = \frac{P(M \cap E_2)}{P(M)} = \frac{P(M|E_2)P(E_2)}{P(M)} = \frac{(\frac{1}{{}_{n-2}C_k})(\frac{1}{n})}{\frac{1}{{}_{n-1}C_k}} = \frac{n-1}{(n-k-1)(n)}$

If n=3, k=1, then, $\frac{2}{3}$
If n=10, k=8, then, $\frac{9}{10}$

Though a player picks other doors, the probability is same because of symmetry.

(3.16) $(\Rightarrow)$ $Var(X) = E[(X-\mu)^2]$
$= E(X^2 - 2X\mu + \mu^2)$
$= E(X^2) - 2E(X)\mu + E(\mu)^2$
$= E(X^2) - 2\mu^2 + \mu^2$
$= E(X^2) - \mu^2$

(3.33) $(\Rightarrow)$ Using Chebyshev's inequality,
$P(|\frac{X_1+X_2...+X_n}{n} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$
$P(|\frac{B}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}$
$\sigma^2 = p(1-p)$ and $\mu = p$

(3.36) $(\Rightarrow)$ $S_{6242} = \sum_{i=1}^{6242} X_i$
$E(X_i) = 0.801$
$Var(X_i) = (0.801)(0.199) = 0.159399$
$P(S_{6242} > 5500) = 1 - P(S_{6242} \leq 5500)$

CLT says that $P(\frac{S_{6242}-n\mu}{\sigma\sqrt{n}} \leq \gamma) \Rightarrow \Phi(\gamma)$
In this case, we can get γ
$\gamma = \frac{5500-(6242)(0.801)}{\sqrt{0.159399}\sqrt{6242}} \approx 15.8563$
$\Phi(15.8563) \approx 1$
$\therefore$ It means that $P(S_{6242} > 5500) \approx 0$

2. Construct examples of events A, B, and C, each of probability strictly between 0 and 1, such that

   (a) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(B \cap C) = P(B)P(C)$, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

   $(\Rightarrow)$ $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$

   $A = \{1, 2, 3, 4\}$
   $B = \{4, 5\}$

$C = \{1, 4, 6, 7\}$
$P(A) = \frac{1}{2}$
$P(B) = \frac{1}{4}$
$P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{8}$
$P(A \cap C) = \frac{1}{4}$
$P(B \cap C) = \frac{1}{8}$
$P(A \cap B \cap C) = \frac{1}{16}$

(b) $P(A \cap B) = P(A)P(B)$, $P(A \cap C) = P(A)P(C)$, $P(A \cap B \cap C) = P(A)P(B)P(C)$, but $P(B \cap C) \neq P(B)P(C)$. (Hint: You can let Ω be a set of eight equally likely points.)

($\Rightarrow$) $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1, 2, 3, 4\}$
$B = \{3, 4, 5, 6\}$
$C = \{2, 3, 7, 8\}$

$P(A) = \frac{1}{2}$
$P(B) = \frac{1}{2}$
$P(C) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{4}$
$P(A \cap C) = \frac{1}{4}$
$P(B \cap C) = \frac{1}{8}$
$P(A \cap B \cap C) = \frac{1}{8}$

3. Prove that Benford's Law is, in fact, a well-defined discrete probability distribution.
($\Rightarrow$) (1) $0 \leq P(d) \leq 1$
(2) $P(\Omega) = P(1) + \ldots + P(9) = 1$
(3) Finite additivity statisfies

$\therefore$ It is a well-defined dicrete probability distribution.

4. A person tosses a fair coin until a tail appears for the first time. If the tail appears on the nth flip, the person wins 2^n dollars. Let the random variable X denote the player's winnings.

(a) (St. Petersburg paradox) Show that $E[X] = +\infty$.

($\Rightarrow$) $E(X) = \sum_{k=1}^{\infty} \frac{1}{2^k} 2^k = 1 + 1 + \ldots = \infty$

(b) Suppose the agent has log utility. Calculate $E[\ln X]$.

$(\Rightarrow)$ $E(lnX) = \sum_{k=1}^{\infty} \frac{1}{2^k} ln2^k$
$= ln2 \sum_{k=1}^{\infty} \frac{k}{2^k}$
$= 2ln2$

$\because$This is a typical arithmetico-geometric sequence.
$A_n = \sum_{k=1}^{n} (a + (k-1)d)r^k$
$A_n - rA_n = ar + \frac{dr^2(1-r^{n-1})}{(1-r)} - (a + (n-1)d)r^{n+1}$

as $n \to \infty$, then, $ar + \frac{dr^2}{(1-r)}$
In this case, $a = 1, r = \frac{1}{2}, d = 1,$
$\therefore \sum_{k=1}^{\infty} \frac{k}{2^k} = 2$

5. (Siegel's paradox) Suppose the exchange rate between USD and CHF is 1:1. Both a U.S. investor and a Swiss investor believe that a year from now the exchange rate will be either 1.25 : 1 or 1 : 1.25, with each scenario having a probability of 0.5. Both investors want to maximize their wealth in their respective home currency (a year from now) by investing in a risk-free asset; the risk-free interest rates in the U.S. and in Switzerland are the same. Where should the two investors invest?

$(\Rightarrow)$ They should invest the other country's currency because of the Jensen's inequalty.

$\frac{1}{2}(1.25 + \frac{1}{1.25}) = \frac{2.05}{2} = 1.025 > 1$

6. Consider a probability measure space with $\Omega = [0, 1]$.

(a) Construct a random variable X such that $E[X] < \infty$ but $E[X^2] = \infty$.

$(\Rightarrow)$ $f_X(x) = \begin{cases} \frac{2}{x^3}, & \text{if } x \in [1, \infty) \\ 0, & \text{if else} \end{cases}$
$\int_1^{\infty} x f_X(x)dx < \infty$
$\int_1^{\infty} x^2 f_X(x)dx = \infty$

(b) Construct random variables X and Y such that $P(X > Y) > \frac{1}{2}$ but $E[X] < E[Y]$.

$(\Rightarrow)$ $f_X(x) = \begin{cases} \frac{11}{20}, & \text{if } x \in [0, \frac{20}{11}] \\ 0, & \text{if else} \end{cases}$

$f_Y(y) = \begin{cases} e^{-y}, & \text{if } x \in [0, \infty) \\ 0, & \text{if else} \end{cases}$

$$P(X < Y) = \int \Pr[X < y]\, f_Y(y)\, \mathrm{d}y = \int F_X(y)\, f_Y(y)\, \mathrm{d}y$$

$\therefore P(X < Y) = \frac{11}{20}[1 - \frac{1}{e^{\frac{20}{11}}}] \approx 0.4607 < 0.5$
$\therefore P(X > Y) > 0.5$ but $E(X) < E(Y)$
$\because E(X) = \frac{10}{11} < 1$, but $E(Y) = \int_0^\infty ye^{-y}dy = 1$

(c) Construct random variables X, Y, and Z such that $P(X > Y)P(Y > Z)P(X > Z) > 0$ and $E(X) = E(Y) = E(Z) = 0$.

$(\Rightarrow) X \sim U[-1,1], Y \sim U[-2,2], Z \sim U[-3,3]$

7. Let the random variables X and Z be independent with $X \sim N(0,1)$ and $P(Z = 1) = P(Z = -1) = \frac{1}{2}$. Define $Y = XZ$ as the product of X and Z. Prove or disprove each of the following statements.

   (a) $Y \sim N(0,1)$.
   $(\Rightarrow)$ True
   $P(Y \le y) = \frac{1}{2}P(X \le y) + \frac{1}{2}P(-X \le y) = P(X \le y)$

   (b) $P(|X| = |Y|) = 1$.
   $(\Rightarrow)$ True
   $Y = XZ$, $\therefore Y = X$ or $-X$.
   Then, $|Y| = |X| \therefore P(|X| = |Y|) = 1$

   (c) X and Y are not independent.
   $(\Rightarrow)$ True
   $f_XY(x,y) \neq f_X(x)f_Y(y)$ when $x = 0, y = 1$

   (d) $Cov[X,Y] = 0$.
   $(\Rightarrow)$ True
   $Cov[X,Y] = E(XY) - E(X)E(Y) = E(X^2Z) - E(X)E(XZ) = 0$

   (e) If X and Y are normally distributed random variables with $Cov[X,Y] = 0$, then X and Y must be independent.
   $(\Rightarrow)$
   By (a),(b),(c),(d), it is not true. (c) is one of counter example of this statement.

8. Let the random variables X_i, $i = 1, 2, \ldots, n$, be i.i.d. having the uniform distribution on $[0,1]$, denoted $X_i \sim U[0,1]$. Consider the random variables $m = \min\{X_1, X_2, \ldots, X_n\}$ and $M = \max\{X_1, X_2, \ldots, X_n\}$. For both random variables m and M, derive their respective cumulative distribution (cdf), probability density function (pdf), and expected value.

   $(\Rightarrow)$

$$F_{X_i}(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \end{cases}$$

1) m

$$F_m(x) = \begin{cases} 1 - [1-x]^n, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \end{cases}$$

$$f_m(x) = \begin{cases} n(1-x)^{n-1}, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \\ 0, & \text{if } x > 1 \end{cases}$$

$$E(m) = \int_0^1 xn(1-x)^{n-1}dx = \frac{1}{n+1}$$

1) M

$$F_M(x) = \begin{cases} x^n, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \\ 1, & \text{if } x > 1 \end{cases}$$

$$f_M(x) = \begin{cases} n[x]^{n-1}, & \text{if } x \in [0,1] \\ 0, & \text{if } x < 0 \\ 0, & \text{if } x > 1 \end{cases}$$

$$E(M) = \int_0^1 xn(x)^{n-1}dx = \frac{n}{n+1}$$

9. You want to simulate a dynamic economy (e.g., an OLG model) with two possible states in each period, a "good" state and a "bad" state. In each period, the probability of both shocks is $\frac{1}{2}$. Across periods the shocks are independent. Answer the following questions using the Central Limit Theorem and the Chebyshev Inequality.

   (a) What is the probability that the number of good states over 1000 periods differs from 500 by at most 2%?

   ($\Rightarrow$)

   $S_{1000} = \sum_{i=1}^{1000} X_i$ $\qquad$ $X_i = \begin{cases} 1, & \text{if good} \\ 0, & \text{if bad} \end{cases}$

   $\mu = 0.5$ $\sigma^2 = 0.25$

   BY CLT,
   $P(S_{1000} \leq 500(1+0.02)) - P(S_{1000} \leq 500(1-0.02)) = \Phi(\frac{2}{\sqrt{10}}) - \Phi(\frac{-2}{\sqrt{10}})$
   $\approx 0.7357 - 0.2643 = 0.4714$

(b) Over how many periods do you need to simulate the economy to have a probability of at least 0.99 that the proportion of good states differs from $\frac{1}{2}$ by less than 1%?

($\Rightarrow$)
$P(|\frac{B}{n} - p| \geq \epsilon) \leq \frac{p(1-p)}{n\epsilon^2}$
$\sigma^2 = p(1-p) = \frac{1}{4}$ and $\mu = p = \frac{1}{2}$
$\epsilon = (0.005)$
$\frac{p(1-p)}{n\epsilon^2} = 0.01$
$\therefore$ We can easily get "n" from here.

$\therefore n = 1,000,000$

10. If $E[X] < 0$ and $\theta \neq 0$ is such that $E[e^{\theta X}] = 1$, prove that $\theta > 0$.

($\Rightarrow$)
$\log E(e^{\theta x}) = 0$
$E(\log e^{\theta x}) = E(\theta x)$

By Jensen's inequality,
$E(\theta x) \leq \log E(e^{\theta x})$
By the way, right side is 0.
$\therefore E(\theta x) \leq 0$
$\therefore \theta E(x) \leq 0$
Finally, $\theta > 0 \because E[X] < 0$

References