Math Homework Week #3, Spectral Theory

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1. (4.2)
   $p = ax^2 + bx + c$

   $D(p)(x) = 2ax + b$

   $$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

   $$adjoint = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^T$$

   $$det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3 = 0$$

   $\therefore \lambda = 0$
   So, algebraic multiplicities: 3
   geometric multiplicities: 3 - 2(rank) = 1

2. (4.4)
   $(i) A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^H = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$
   $\lambda - (a+d)\lambda + (ad - bc) = 0$
   $b = \bar{c}, c = \bar{b}, a = \bar{a}, d = \bar{d}$
   $\lambda^2 - (a+d)\lambda + (ad - b^2) = 0$
   $(a-d)^2 + 4c\bar{c} \geq 0$

   $(ii) A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^H = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$
   $b = -\bar{c}, c = -\bar{b}, a = -\bar{a}, d = -\bar{d}$
   $(a-d)^2 - 4b\bar{b} < 0$ ($\because a, d = 0$ or imaginary numbers.)

3. (4.6)
   Let diagonal entries of an upper-triangular matrix A be $d_1, d_2, d_3, \cdots, d_n$. Then, $(A - \lambda I)$ is also upper triangular matrix. Now, diagonal entries are $d_i - \lambda$. $|\pi_i(d_i - \lambda)| = 0$ and λ is d_i here. For lower triangular matrix, the proof is almost same.

4. (4.8)
   (i) I think that it has to be $C^\infty([-\pi, \pi], R)$ such as Exercise 3.8. I think it might

be typo. If it is $C^\infty([-\pi,\pi],R)$, then I can prove as the following.

(1) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot sint\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}sin2t\ dt$

$= \frac{1}{2\pi}[-\frac{1}{2}cos2t]_{-\pi}^{pi}$
$= 0$

(2) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot cos2t\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} cost(2cos^2t - 1)\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi}(2cos^3t - cost)\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{2}(cos3t + cost)\ dt$

$= \frac{1}{\pi}[\frac{1}{6}(sin3t) + \frac{1}{2}sint]_{-\pi}^{\pi}$

$= 0$

(3) $\frac{1}{\pi}\int_{-\pi}^{\pi} cost \cdot sin2t\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} 2cos^2t \cdot sint\ dt$

$= \frac{1}{\pi}\int_{-\pi}^{\pi} 2(sint - sin^3t)\ dt$

$= \frac{1}{\pi}[\frac{1}{2}(-cost) + \frac{1}{6}(-cos3t)]_{-\pi}^{\pi} = 0$

(4) $\int_{-\pi}^{\pi} sint \cdot cos2t\ dt = \int_{-\pi}^{\pi} sint(1 - 2sin^2t)\ dt$

$= \int_{-\pi}^{\pi} -\frac{1}{2}sint + \frac{3}{2}sin3t\ dt$

$= [\frac{1}{2}cost - \frac{1}{2}cost3t]_{-\pi}^{\pi}$

$= 0$

(5) $\int_{-\pi}^{\pi} sint \cdot sin2t\ dt = \int_{-\pi}^{\pi} 2sin^2tcost\ dt = \int_{-\pi}^{\pi} 2(1 - cos^2t)cost\ dt$

$= \int_{-\pi}^{\pi} 2cost - 2 \cdot \frac{1}{4}(3cost + cos3t)\ dt = \int_{-\pi}^{\pi}\frac{1}{2}cost - \frac{1}{2}cos3t\ dt$

$= [\frac{1}{2}sint - \frac{1}{6}sin3t]_{-\pi}^{\pi}$

$= 0$

(6) $\int_{-\pi}^{\pi} cos2t \cdot sin2t\ dt$

$\int_{-2\pi}^{2\pi} cost \cdot sint(\frac{1}{2})\ dt = 0$

Thus, S is orthogonal set. By definition, it is linear independent.

(ii) $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} sinx \\ cosx \\ sin2x \\ cos2x \end{bmatrix}$

(iii) $span[(1,0,0,0),(0,1,0,0)]$ and $span[(0,0,1,0),(0,0,0,1)]$

5. (4. 13)
$P = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix}$
$Then, P^{-1}AP = \frac{1}{detp} \begin{bmatrix} -15 & 0 \\ 0 & -6 \end{bmatrix}$
I calculated $P^{-1}AP$ by general $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then input numbers.

6. (4.15)
$A = P^{-1}BP$, then $A^k = P^{-1}B^kP$
If A is semisimple, then it is diagonalizable. By Thm 4.3.7 proof, $A = P^{-1}DPwhereD = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$
Then, $A^n = P^{-1}D^nP$, so $D^2 = diag(\lambda_1^n, \cdots, \lambda_n^n)$
Then, $(a_0 + a_1A + \cdots + a_nA^n)x = (a_0 + \lambda + \lambda^2 + \cdots + \lambda^n)x$
So, $f(A)x = f(\lambda)x$

7. (4.16)
(i) $\lim_{n\to\infty} A^n$ with respect to the 1-norm.
$P^{-1}AP = D$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ from eigen vectors. So, we can easily calculate the answer $\begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$

(ii) In any form of norms, it has to be same in a reason that we calculate before taking norm.

(iii) $f(x) = 3 + 5x + x^3$
$\therefore f(\lambda_1) = 9$ and $f(\lambda_2) = 5.064$

8. (4.18)
From $(Ax) = (\lambda x)$, $(Ax)^T = x^TA = (\lambda x)^T = x^T\lambda = \lambda x^T$

9. (4.20)
If A is Hermitian and orthonormally similar to B, then, B is also Hermitian.
$\Rightarrow B = U^HAU$ where U is orthonormal.
$B^H = U^HA^HU = U^HAU = B$ by A is Hermitian.

10. (4.24)
(i) $A = A^H$ $\rho(x) = \frac{<x,Ax>}{\|x\|_2} = \frac{x^HAx}{\|x\|_2} = \frac{<x,\lambda x>}{\|x\|_2} = \frac{x^H\lambda x}{\|x\|_2} = \lambda = \frac{<A^Hx,x>}{\|x\|_2} = \frac{\bar{\lambda}x^Hx}{\|x\|_2} = \bar{\lambda}$

$A = -A^H$
$\rho(x) = \frac{<x,Ax>}{\|x\|_2} = \frac{<x,\lambda x>}{\|x\|_2} = \frac{\lambda x^H x}{\|x\|_2} = \lambda = \frac{<A^H x,x>}{\|x\|_2} = -\frac{<Ax,x>}{\|x\|_2} = -\frac{\bar{\lambda} x^H x}{\|x\|_2} = -\bar{\lambda}$

11. (4.25)
(i) $(x_1 x_1^H + \cdots + x_n x_n^H)(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = (a_1 x_1 + \cdots + a_n b_n)$
$\because x_j^H x_j = 1$ and by orthogonality.
By the way, $(a_1 x_1 + \cdots + a_n b_n)$ is arbitrary linear combination, so, $(x_1 x_1^H + \cdots + x_n x_n^H)$ has to be I.

(ii) $AI = A(x_1 x_1^H + \cdots + x_n x_n^H)$
$\therefore A = Ax_1 x_1^H + \cdots + Ax_n x_n^H = \lambda x_1 x_1^H + \cdots + \lambda x_n x_n^H$

12. (4.27)
let $x_i = (0, \cdots, 1 + a * i, \cdots, 0)$.
Then, $x_i^H A x_I = ((1 + a * i)^H a_{ii} (1 + a * i) > 0$
$\therefore a_{ii} > 0$
($\because A$ is hermitian, so $A = A^H$)
Therefore, a_{ii} is real.

13. (4.28)
We already know that trace is one of inner product in Chapter 3. So, it has to satisfy Cauchy-Schwartz ineqaulity.
So, $tr(AB) \leq \sqrt{tr(A^2)tr(B^2)} = \sqrt{\sum(\lambda_i^A)^2 \sum(\lambda_i^B)^2} \leq \sqrt{(\sum \lambda_i^A)^2 (\sum \lambda_i^B)^2} = tr(A)tr(B)$

14. (4.31)
(i) $\|A\|_2 = sup \frac{\|Ax\|_2}{\|x\|_2} = sup \frac{\|U \Sigma V^H x\|_2}{\|x\|_2} = sup \frac{\|\Sigma V^H x\|_2}{\|x\|_2} = sup \frac{\|\Sigma V^H x\|_2}{\|V^H x\|_2} = sup \frac{(\sum |\sigma_i V^H x|^2)^{0.5}}{(\sum |V^H x|^2)^{0.5}}$
(ii) By the method of (i), now $\frac{1}{\sigma_n} = \|A^{-1}\|_2$ (It is the largest here.) (iii) $\|A^H\|_2^2 = \sigma_1^2 = \|A^T\|_2 = \sigma_1^2 = \|A^H A\|_2 = \sigma_1^2 = (\|A\|_2)^2$
(iv) $\|UAV\|_2 = \|UU\Sigma V^H V\|_2 = \|\Sigma\|_2 = \|A\|_2$

15. (4.32)
(i) $\|UAV\|_F = \|AV\|_F = \|U \Sigma V^H V\|_F = \|\Sigma\|_F = \|V\Sigma\|_F = \|\Sigma^H V^H\|_F$
($\because$ trace property)
$= \|U \Sigma V^H\|_F = \|A\|_F$
(ii) $\|\Sigma\|_F = \sqrt{tr(\Sigma^H \Sigma)} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$

16. (4.33)
$\|A\|_2 = \|\Sigma\|_2 = sup|y^H \Sigma x| = \sigma_1$
I already showed $\|A\|_2 = \|\Sigma\|_2$ above.

17. (4.36)
$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$
$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1$ or 3. From $A^H A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$,
$\sigma_1 \approx 3.6503$, $\sigma_2 \approx 0.8219$

18. (4.38)
    (i) $AA^+A = U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^HA = U_1U_1^HA = U_1U_1^HU_1\Sigma_1V_1^H = U_1\Sigma_1V_1^H = A$
    (ii) $A^+AA^+ = V_1\Sigma^{-1}U_1^H = A^+$
    (iii) $(AA^+)^H = U_1U_1^H = AA^+$ from (i) we can easily know it.
    (iv) $(A^+A)^H = V_1V_1^H = V_1\Sigma^{-1}U_1^HU_1\Sigma_1V_1^H = A^+A$
    (v) From (iii), it has to be real.
    From orthogonal projection, $A(A^HA)^{-1}A^H$, we can know that $A(A^HA)^{-1}A^HA = AA^+A = A$
    (vi) From (iv), it has to be real.
    From orthogonal projection, $A^H(AA^H)^{-1}A$, we can know that $AA^H(AA^H)^{-1}A = AA^+A = A$

References