Math Homework Week #5, Convex Analysis

OSM Lab, Eun-Seok Lee

1. (7.1)
   If S is a nonempty set,
   $\lambda_1 x_1 + \cdots + \lambda_k x_k \in conv(S), x_i \in S, k \in N$
   where $\lambda_i \geq 0$, and $\lambda_1 + \cdots + \lambda_k = 1$

   It means that $\lambda x + (1-\lambda)y \in con(S) \quad \forall \lambda$ s.t.$0 \leq \lambda \leq 1$
   It is the definition of convex set.

2. (7.2)
   1) hyperplane is convex.
   $P = \{x \in V | \langle a, x \rangle\}$ where $a \in V, a \neq 0$, and $b \in R$

   It means that, in the condition of $a_1 x_1 + \cdots + a_n x_n = b$,
   those $x_1, \cdots, x_n$ has to be memebers of P.
   By the definition of convex, $\lambda x + (1-\lambda)y \in P$
   $\because$ it is totally same expression to definition of hyperplane.

   2) half space is convex.
   $H = \{x \in V | \langle a, x \rangle \leq b\}$
   It means that, in the condition of $a_1 x_1 + \cdots + a_n x_n \leq b$,
   those $x_1, \cdots, x_n$ has to be memebers of P.
   By the definition of convex, $\lambda x + (1-\lambda)y \in P$
   $\because$ it is totally same expression to definition of hyperplane.

3. (7.4)
   (i) $\|x - p + p - y\|^2 = \|x - p\|^2 + \|p - y\|^2 + 2\|x - p\|\|p - y\| cos\theta$
   $= \|x - p\|^2 + \|p - y\|^2 + 2\langle x - p, p - y \rangle$

   (ii) $\|x - y\|^2 \geq \|x - p\|^2$ from (i) and assumption (7.14)
   $\therefore \|x - y\| > \|x - p\|$
   $\because y \neq p$
   $\therefore \|p - y\| > 0$

   (iii) $\|x - \lambda y - (1-\lambda)p\|^2 = \|x - p + \lambda(p - y)\|^2$
   $= \|x - p\|^2 + 2\|x - p\|\|\lambda(p - y)\| cos\theta + \lambda^2 \|y - p\|^2$
   $= \|x - p\|^2 + 2\lambda\langle x - p, p - y \rangle + \lambda^2 \|y - p\|^2$

(iv) $\frac{\|x-z\|^2-\|x-p\|^2}{\lambda} = 2\langle x-p, p-y\rangle + \lambda^2\|y-p\|^2$ from (iii)
By the definition of projection, LHS ≥ 0
Also, when $\lambda = 0$, $0 \leq \langle x-p, p-y\rangle$

4. (7.6)
$f(\lambda x + (1-\lambda)y \leq \lambda f(x) + (1-\lambda)f(y) = c$
RHS $= c$, Then, $x \in domain, y \in domain$, so, $(\lambda x + (1-\lambda)y) \in domain$ for satisfying convex function definition.

5. (7.7)
$\lambda_1\theta_1 f_1(x) + \lambda_1(1-\theta_1)f_1(y) + \cdots + \lambda_n\theta_n f_n(x) + \lambda_n(1-\theta_n)f_n(x) \geq \lambda_1 f_1(\theta_1 x + (1-\theta_1)y) + \cdots + \lambda_n f_n(\theta_n x + (1-\theta_n)y)$
$\therefore \lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$

6. (7.13)
If f is not constant, $\exists x$ and y s.t.$f(x) \neq f(y)$

1) Let x be the point of $max(f(x))$
If $y_1 < x$ and $f(y_1) \neq f(x)$, $f(\lambda x + (1-\lambda)y_1) \leq \lambda f(x) + (1-\lambda)f(y_1)$ and $f(y_1) \leq f(x)$
If $y_2 > x$ and $f(y_2) \neq f(x)$, $f(\lambda x + (1-\lambda)y_2) \leq \lambda f(x) + (1-\lambda)f(y_2)$ and $f(y_2) \leq f(x)$
Now, $\exists\lambda$ s.t. $f(x) = \lambda f(y_1) + (1-\lambda)f(y_2) > \lambda f(y_1) + (1-\lambda)f(y_2)$,
$(\because f(x) \neq f(y_1), f(y_2))$
then, it is contradiction to the convex definition.

2) If there is no maximum point of the function, it has to be increasing function as x goes to infinity or it has to be decreasing function as x goes to infinity. if function increases and decreases, it has to be concave by the proof of "1)". I will prove the case of the increasing function in a reason that the way of proving method is same.
(i) First, we can pick $x < y$ and $f(x) < f(y)$. If $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$, then, it should not be bounded because it is a line.
(ii) Second, we can pick $x < y$ and $f(x) < f(y)$. If $\lambda f(x) + (1-\lambda)f(y) < \lambda f(x) + (1-\lambda)f(y)$, then, $f(t) \geq f(x) + (t-a)\frac{f(y)-f(x)}{y-x}$ s.t. $t > y$ by convexity. Then, as t goes infinity, it diverges and this is contradiction.

I showed every possible cases. Therefore, Q.E.D..

7. (7.20)
By the assumption, $\lambda f(x) + (1-\lambda)f(y) = f(\lambda x + (1-\lambda)y)$
$\therefore f(0) + \lambda f(x) = f(\lambda x)$ when $y = 0$.
This is a general case and exactly affine form.

8. (7.21)
   $[\phi(f(x))]' = \phi'(f(x))f'(x) = 0$ by the chain rule.
   Then, $\phi(x)$ is strictly increasing, so $\phi'(f(x)) \neq 0$.
   Therefore, $f'(x) = 0$, and this is same FONC with ($min\ f(x)$).

References