

## Math Homework Week #2, Inner Product Space

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1. (1, 3.1)

$$(i) \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$$

$$= \frac{1}{4}(\|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 - \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle - \|y\|^2)$$

$$= \frac{1}{4}(4 \cdot \langle x, y \rangle)$$

$$= \langle x, y \rangle$$

$$(ii) \frac{1}{2}(\|x+y\|^2 + \|x-y\|^2)$$

$$= \frac{1}{2}(\|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2 + \|x\|^2 - \langle x, y \rangle - \langle y, x \rangle + \|y\|^2)$$

$$= \frac{1}{2}(2\|x\|^2 + 2\|y\|^2)$$

$$= \|x\|^2 + \|y\|^2$$

2. (2, 3.2)

$$= \frac{1}{4}(4[Re \langle x, y \rangle] + 4i[Im \langle x, y \rangle])$$

$$= \langle x, y \rangle$$

$$\text{Because, } i\|x+iy\|^2 - i\|x-iy\|^2 = -2(-\langle x, y \rangle + \overline{\langle x, y \rangle}) = 4i[Im \langle x, y \rangle]$$

3. (3, 3.3)

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx$$

$$(i) \cos\theta = \frac{\int_0^1 x \cdot x^5 dx}{\sqrt{\int_0^1 x^2 dx} \sqrt{\int_0^1 x^{10} dx}}$$

$$= \frac{[\frac{1}{7}x^7]_0^1}{\sqrt{[\frac{1}{3}x^3]_0^1} \sqrt{[\frac{1}{11}x^{11}]_0^1}} = \frac{\sqrt{33}}{7}$$

$$(ii) \cos\theta = \frac{\int_0^1 x^2 \cdot x^4 dx}{\sqrt{\int_0^1 x^4 dx} \sqrt{\int_0^1 x^8 dx}}$$

$$= \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}} \sqrt{\frac{1}{9}}} = \frac{\sqrt{45}}{7}$$

4. (4, 3.8)

(i)

$$(1) \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cdot \sin t dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} \sin 2t dt$$

$$= \frac{1}{2\pi} \left[ -\frac{1}{2} \cos 2t \right]_{-\pi}^{\pi}$$

$$= 0$$

$$(2) \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cdot \cos 2t \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t (2\cos^2 t - 1) \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (2\cos^3 t - \cos t) \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} (\cos 3t + \cos t) \, dt$$

$$= \frac{1}{\pi} \left[ \frac{1}{6} (\sin 3t) + \frac{1}{2} \sin t \right]_{-\pi}^{\pi}$$

$$= 0$$

$$(3) \frac{1}{\pi} \int_{-\pi}^{\pi} \cos t \cdot \sin 2t \, dt = \frac{1}{\pi} \int_{-\pi}^{\pi} 2\cos^2 t \cdot \sin t \, dt$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} 2(\sin t - \sin^3 t) \, dt$$

$$= \frac{1}{\pi} \left[ \frac{1}{2} (-\cos t) + \frac{1}{6} (-\cos 3t) \right]_{-\pi}^{\pi} = 0$$

$$(4) \int_{-\pi}^{\pi} \sin t \cdot \cos 2t \, dt = \int_{-\pi}^{\pi} \sin t (1 - 2\sin^2 t) \, dt$$

$$= \int_{-\pi}^{\pi} -\frac{1}{2} \sin t + \frac{3}{2} \sin 3t \, dt$$

$$= \left[ \frac{1}{2} \cos t - \frac{1}{2} \cos 3t \right]_{-\pi}^{\pi}$$

$$= 0$$

$$(5) \int_{-\pi}^{\pi} \sin t \cdot \sin 2t \, dt = \int_{-\pi}^{\pi} 2\sin^2 t \cos t \, dt = \int_{-\pi}^{\pi} 2(1 - \cos^2 t) \cos t \, dt$$

$$= \int_{-\pi}^{\pi} 2\cos t - 2 \cdot \frac{1}{4} (3\cos t + \cos 3t) \, dt = \int_{-\pi}^{\pi} \frac{1}{2} \cos t - \frac{1}{2} \cos 3t \, dt$$

$$= \left[ \frac{1}{2} \sin t - \frac{1}{6} \sin 3t \right]_{-\pi}^{\pi}$$

$$= 0$$

$$(6) \int_{-\pi}^{\pi} \cos 2t \cdot \sin 2t \, dt$$

$$\int_{-2\pi}^{2\pi} \cos t \cdot \sin t \left( \frac{1}{2} \right) \, dt = 0$$

Thus, S is orthogonal set.

For measure,  $\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 t \, dt = 1$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 t \, dt = 1$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2 2t \, dt = 1$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2 2t \, dt = 1$$

Thus, it is orthonormal.

$$(ii) \text{ compute } ||t|| \\ \langle t, t \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} t^2 dt = \frac{2}{3} \pi^2$$

$$(iii) \sum_{i=1}^m \langle x_i, \cos(3t) \rangle x_i = 0$$

$$(iv) \sum_{i=1}^m \langle x_i, t \rangle x_i = 2\pi(\sin t) - \pi(\sin(2t))$$

$$5. (5, 3.9) \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_1 \sin\theta + x_2 \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \cos\theta - y_2 \sin\theta \\ y_1 \sin\theta + y_2 \cos\theta \end{bmatrix}$$

$$(x_1 \cos\theta - x_2 \sin\theta)(y_1 \cos\theta - y_2 \sin\theta) + (x_1 \sin\theta + x_2 \cos\theta)(y_1 \sin\theta + y_2 \cos\theta) = x_1 y_1 + x_2 y_2$$

6. (6, 3.10)

$$(i) Q^H Q = \begin{bmatrix} q_1^H \\ \vdots \\ q_n^H \end{bmatrix} \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

$$(ii) \|Qx\| = (Qx)^H Qx = x^H Q^H Qx = x^H x = \|x\|$$

$$(iii) Q^{-1} Q Q^H = Q^{-1}$$

$$Q^H = Q^{-1}$$

$$(iv) Q Q^H = I \text{ has a proof already above.}$$

$$(v) \det(Q^H Q) = \det(I)$$

$$\det(Q^H) \det(Q) = I$$

$$(\det Q)^2 = 1$$

$$\therefore |\det(Q)| = 1$$

Converse is not true. Counter example is  $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$$(vi) (Q_1 Q_2)(Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = Q Q_1^H = I$$

$$(Q_1 Q_2)^H (Q_1 Q_2) = I$$

$$\therefore Q_1 Q_2 \text{ is also orthonormal.}$$

7. (7, 3.11)

Assume  $(x_1 \cdots x_{n-1})$  are linearly independent.

Then,  $x_n = \text{span}(x_1, x_2 \cdots x_{n-1})$

If  $(e_1, \cdots e_{n-1})$  are orthonormal basis, after doing G-S process, then,

$$x_n = \langle x_n, e_1 \rangle e_1 + \langle x_n, e_2 \rangle e_2 + \cdots + \langle x_n, e_{n-1} \rangle e_{n-1}$$

Now, in the G-S process, for finding  $e_n$

$e_n = \frac{x_n - p_n}{\|x_n - p_n\|}$ , then ,  $x_n - p_n = 0$  So we will calculate  $\frac{0}{\|0\|}$

8. (8, 3.16) From  $A = QR$ ,

(i) If  $A$  is singular, some diagonal entries of  $R$  can be zero. In this case, we can have multiple  $Q$ .

(ii)  $A = Q_1 R_1 = Q_2 R_2$ .

$K = R_1 R_2^{-1} = Q_1^H Q_2$  and  $(R_1^{-1})^H R_2^H = R_1 R_2^{-1}$

diagonals of  $K$  are positive and 1. Also, it has to be lower triangular and upper triangular matrix.

$\therefore, R_1 = R_2, Q_1 = Q_2$

9. (9, 3.17)

$$A^H A x = A^H b$$

$$(\hat{Q} \hat{R})^H (\hat{Q} \hat{R}) x = (\hat{Q} \hat{R})^H b$$

$$\hat{R}^H R x = \hat{Q} \hat{R}^H b$$

$$\hat{Q}^H x = \hat{R}^H b$$

10. (10, 3.23)

$$\|y\| = \|x + y - x\| \leq \|x\| + \|y - x\|$$

$$\text{Thus, } \|y\| - \|x\| \leq \|y - x\|$$

$$\|x\| = \|y + x - y\| \leq \|y\| + \|x - y\|$$

$$\text{Thus, } \|x\| - \|y\| \leq \|x - y\|$$

$$\text{Therefore, } \left| \|x\| - \|y\| \right| \leq \|x - y\|$$

11. (11, 3.24)

(i)

$$(1) \int_a^b |0| dt = 0 \text{ and } \int_a^b |f(t)| dt \geq 0$$

$$(2) \int_a^b |af(t)| dt = a \int_a^b |f(t)| dt$$

$$(3) \int_a^b |f(t)| dt + \int_a^b |g(t)| dt = \int_a^b |f(t)| + |g(t)| dt \geq \int_a^b |f(t) + g(t)| dt$$

(ii)

$$(1) (\int_a^b |0|^2 dt)^{0.5} = 0 \text{ and } (\int_a^b |f(t)|^2 dt)^{0.5} \geq 0$$

$$(2) (\int_a^b |af(t)|^2 dt)^{0.5} = a (\int_a^b |f(t)|^2 dt)^{0.5}$$

$$(3) (\int_a^b |f(t)|^2 dt)^{0.5} + (\int_a^b |g(t)|^2 dt)^{0.5} \geq (\int_a^b |f(t) + g(t)|^2 dt)^{0.5}$$

(iii)

$$(1) \sup|0| = 0 \text{ and } \sup|f(x)| \geq 0$$

$$(2) \sup|af(x)| = a \cdot \sup|f(x)|$$

$$(3) \sup|f(x)| + \sup|f(y)| \geq \sup|f(x) + f(y)|$$

$$\because |f(x)| + |f(y)| \geq |f(x) + f(y)|$$

12. (12, 3.26)

For equivalence relation,

$$1) a \sim a$$

$$2) a \sim b \text{ iff } b \sim a$$

$$3) \text{ if } a \sim b \text{ and } b \sim c, \text{ then } a \sim c$$

$$1) \text{ There is } 0 < m < M \text{ s.t. } m\|X\|_a \leq \|X\|_a \leq M\|X\|_a$$

$$2) \text{ There is } 0 < m < M \text{ s.t. } m\|X\|_a \leq \|X\|_b \leq M\|X\|_a$$

$$\text{There is } 0 < \frac{1}{M} < \frac{1}{m} \text{ s.t. } \frac{1}{M}\|X\|_b \leq \|X\|_a \leq \frac{1}{m}\|X\|_b$$

$$3) \text{ There is } 0 < m < M \text{ s.t. } m\|X\|_a \leq \|X\|_b \leq M\|X\|_a$$

$$\text{There is } 0 < m^* < M^* \text{ s.t. } m^*\|X\|_b \leq \|X\|_c \leq M^*\|X\|_b$$

$$\text{Then, there is } 0 < \frac{m}{m^*} < MM^* \text{ s.t. } \frac{m}{m^*}\|X\|_a \leq \|X\|_c \leq MM^*\|X\|_a$$

$$(i) 1) \|X\|_2 \leq \|X\|_1$$

$$(|x_1|^2 + \dots + |x_n|^2) < (|x_1| + \dots + |x_n|)^2 \because \text{interaction terms}$$

$$2) \|X\|_1 \leq \sqrt{n}\|X\|_2$$

$$\|X\|_1 = \sum_{i=1}^n x_i 1 \leq (\sum_{i=1}^n (x_i)^2)^{0.5} (\sum_{i=1}^n 1)^{0.5} \leq \sqrt{n}\|X\|_2 \text{ by C-S inequality.}$$

$$(ii) \|X\|_\infty = \sup(|x_1|, \dots, |x_n|)$$

$$1) \|X\|_\infty \leq \|X\|_1 \text{ This is trivial.}$$

$$2) \|X\|_2 \leq \sqrt{n} \cdot \sup(|x_1|, \dots, |x_n|)$$

$$(|x_1|^2 + \dots + |x_n|^2)^{0.5} \leq |x_1| + \dots + |x_n|$$

$$|x_1|^2 + \dots + |x_n|^2 \leq (|x_1| + \dots + |x_n|)^2 \leq n[\sup(|x_1|, \dots, |x_n|)]^2$$

13. (13, 3.28)

(i)

$$1) \frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_1$$

$$\frac{1}{\sqrt{n}}\|A\|_2 = \sup \frac{\|AX\|_2}{\sqrt{n}\|X\|_2} \leq \sup \frac{\|AX\|_1}{\|X\|_1}$$

$$\sqrt{n}\|X\|_2 \geq \|X\|_1 \text{ by 3.26 (i)}$$

$$\|AX\|_2 \leq \|AX\|_1 \text{ by 3.26 (i)}$$

$$\begin{aligned}
2) \quad & \|A\|_1 \leq \sqrt{n} \|A\|_2 \\
& \sup \frac{\|AX\|_1}{\|X\|_1} \leq \sqrt{n} \sup \frac{\|AX\|_2}{\|X\|_2} \\
& \|X\|_1 \geq \|X\|_2 \text{ by 3.26 (i)} \\
& \sqrt{n} \|AX\|_2 \geq \|AX\|_1 \text{ by 3.26 (i)}
\end{aligned}$$

$$\begin{aligned}
(ii) \quad & \frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \\
& \sup \frac{\|AX\|_\infty}{\sqrt{n} \|X\|_\infty} \leq \sup \frac{\|AX\|_2}{\|X\|_2} \\
& \|A\|_2 \leq \sqrt{n} \|A\|_\infty \\
& \sup \frac{\|AX\|_2}{\|X\|_2} \leq \sup \frac{\sqrt{n} \|AX\|_\infty}{\|X\|_\infty} \\
& \text{Now, same logic above applies.}
\end{aligned}$$

14. (14, 3.29)

$$\begin{aligned}
& \rightarrow \text{By the property } \|Qx\| = \|x\|, \|Q\| = \frac{\|Qx\|}{\|x\|} = \frac{\|x\|}{\|x\|} = 1. \\
& \rightarrow \|R_x\| = \sup \frac{\|R_x A\|_2}{\|A\|_2} \\
& = \sup \frac{\|Ax\|_2}{\|A\|_2} \\
& \sup \frac{\|Ax\|_2}{\sup \frac{\|Ax\|_2}{\|x\|_2}} \leq \|x\|_2 \\
& \text{Now, combining hint given, } \|R_x\|_2 \geq \|x\|_2
\end{aligned}$$

15. (15, 3.30)

$$\begin{aligned}
1) \quad & \|SAS^{-1}\| \geq \text{and } A = 0 \leftrightarrow \|SAS^{-1}\| = 0 \\
2) \quad & \|S(tA)S^{-1}\| = |t| \cdot \|SAS^{-1}\| \\
3) \quad & \|S(A+B)S^{-1}\| \leq \|SAS^{-1}\| + \|SBS^{-1}\| \\
4) \quad & \|SABS^{-1}\| = \|SAS^{-1}SBS^{-1}\| \leq \|SAS^{-1}\| \cdot \|SBS^{-1}\|
\end{aligned}$$

16. (16, 3.37)

$$\begin{aligned}
& p = ax^2 + bx + c \\
& p(1) = 2a + b = L(p) \\
& L(p) = \langle q, p \rangle = \int_0^1 (qp) dx = 2a + b \\
& \text{After setting } q = dx^2 + ex + f, \text{ Then, after integration, we got the identity} \\
& \text{equation like the next one.} \\
& \left(\frac{1}{5}d + \frac{1}{4}e + \frac{1}{3}f\right)a + \left(\frac{1}{4}d + \frac{1}{3}e + \frac{1}{2}f\right)b + \left(\frac{1}{3}d + \frac{1}{2}e + f\right)c = 2a + b
\end{aligned}$$

$$\therefore d = 180, e = -168, f = 24$$

17. (17, 3.38)

$$\begin{aligned}
& p = ax^2 + bx + c \\
& D(p)(x) = 2ax + b \\
& \therefore \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a \\ b \\ 0 \end{bmatrix}
\end{aligned}$$

$$adjoint = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T$$

18. (18, 3.39)

$$(i) \langle (S+T)^*x, y \rangle = \langle x, (S+T)y \rangle$$

$$= \langle x, Sy \rangle + \langle y, Ty \rangle = \langle S^*x, y \rangle + \langle T^*x, y \rangle = \langle (S^* + T^*)x, y \rangle$$

$$(ii) \langle (S^*)^*x, y \rangle = \langle x, S^*y \rangle = \langle Sx, y \rangle$$

$$(iii) \langle (ST)^*x, y \rangle = \langle x, STy \rangle = \langle S^*x, Ty \rangle = \langle T^*S^*x, y \rangle$$

$$(iv) (T^*)^{-1}T^* = 1 \text{ and } ((T^*)^{-1}T^*)^* = 1^* = 1$$

$$\text{Thus, } T^{**}((T^*)^{-1})^* = 1$$

$$T((T^*)^{-1})^* = 1$$

$$((T^*)^{-1})^* = T^{-1}$$

$$(T^*)^{-1} = (T^{-1})^*$$

19. (19, 3.40)

$$\langle A, B \rangle = \text{tr}(A^T B)$$

$$(i) A^* = A^H$$

$$\langle AB, C \rangle = \langle B, A^*C \rangle$$

$$\text{tr}((AB)^H C) = \text{tr}(B^H A^*C)$$

$$\text{tr}(B^H A^H C) = \text{tr}(B^H A^*C)$$

$$\therefore A^H = A^*$$

$$(ii) \langle A_2, A_3 A_1 \rangle = \langle A_2 A_1^*, A_3 \rangle$$

$$\text{tr}(A_2^H A_3 A_1) = \text{tr}((A_2 A_1^*)^H A_3) ??$$

$$\rightarrow \text{tr}(A_1^H A_2 A_3^H) = \text{tr}((A_2 A_1^*)^H A_3)$$

$$(iii) T_A(X) = AX - XA$$

$$\langle X, T_{A^*}X \rangle = \text{tr}[X^*(A^*X - XA^*)]$$

$$= \text{tr}[X^*XA - AX X^*]$$

$$= \text{tr}[X^*(XA - AX)]$$

$$= \text{tr}[(XA - AX)^*X]$$

$$= \langle T_A(X), X \rangle$$

$$= \langle X, (T_A)^*(X) \rangle$$

$$\therefore (T_A)^* = T_{A^*}$$

20. (20, 3.44)

$$\text{If } Ax = b \text{ satisfies, then, } 0 = 0^T \hat{x} = (y^T A)x = y^T Ax = y^T b = \langle y, b \rangle \neq 0$$

21. (21, 3.45)

First, I want to establish general cases of symmetric and skew-symmetric matrix

like below.

$$X = A + A^T \text{ and } Y = B - B^T$$

$$\text{tr}[(A + A^T)^T(B - B^T)] = \text{tr}[AB + A^T B - AB^T - A^T B^T] = 0 \text{ by property of trace.}$$

Then, it easily proves by this.

22. (22, 3.46)

$$(i) Ax \in R(A)$$

$$\text{Also } x \in N(A^H A).$$

$$\text{It means that } A^H Ax = 0$$

$$\therefore A^H(Ax) = 0$$

$$Ax \in N(A^H)$$

$$(ii) A^H Ax = 0 \rightarrow (A^H)^{-1} A^H Ax = 0 \rightarrow Ax = 0$$

$$(iii) N(A^H A) = N(A), \text{ so its dimension has to be same.}$$

$$(iv) Ax = 0.$$

If A has linear independent column, then  $x = 0$ .

so,  $A^H Ax = 0$  and  $\text{null}(A^H A)$  is 0.

$A^H A$  is nonsingular.

23. (23, 3.47)

$$P = A(A^H A)^{(-1)} A^H$$

$$(i) P^2 = A(A^H A)^{(-1)} A^H A(A^H A)^{(-1)} A^H = A(A^H A)^{(-1)} A^H = P$$

$$(ii) P^H = (A(A^H A)^{(-1)} A^H)^H = A(A^H A)^{(-1)} A^H$$

$$(iii) \text{rank}(P) = n$$

$$\text{rank}(P) \leq \text{rank}(A) \text{ and } PA = A \text{ so, } \text{rank}(A) \leq \text{rank}(P)$$

$$\therefore \text{rank}(P) = \text{rank}(A)$$

24. (24, 3.48)

$$(i) P(A + B) = \frac{(A+B) + (A+B)^T}{2} = P(A) + P(B)$$

$$P(tA) = \frac{tA + (tA)^T}{2} = tP(A)$$

$$(ii) \frac{\frac{A+(A)^T}{2} + \frac{A+(A)^T}{2}}{2} = P(A)$$

$$(iii) P^* = P$$

$$\langle P(A), B \rangle = \langle A, P^*(B) \rangle$$

$$\langle P(A), B \rangle = \langle \frac{A+(A)^T}{2}, B \rangle = \text{tr}[\frac{A+(A)^T}{2} B] = \text{tr}[\frac{A^T}{2} B + \frac{A}{2} B] = \text{first}$$

$$\langle A, P^*(B) \rangle = \langle A, \frac{B+(B)^T}{2} \rangle = \text{tr}[A \frac{B+(B)^T}{2}] = \text{tr}[\frac{AB}{2} + \frac{AB^T}{2}] = \text{second}$$

first and second are same.

$$(iv) N(P) = \text{skew}_n(R)?$$

Let the matrix X be being playing as operator P.

$$XA = 0 \rightarrow \frac{A+(A)^T}{2} = 0 \rightarrow A = -A^T$$

$$(v) R(P) = \text{Sym}_n(R)$$

$$XA = B \rightarrow \frac{A+(A)^T}{2} = B$$

$$B = B^T$$

$$(vi) \|A - \frac{A+(A)^T}{2}\| = \|\frac{A-(A)^T}{2}\| = \sqrt{\langle \frac{A-(A)^T}{2}, \frac{A-(A)^T}{2} \rangle}$$



$$= \sqrt{\text{tr}[\frac{A-(A)^T}{2} \frac{A-(A)^T}{2}]} = \sqrt{\text{tr}[\frac{1}{4}(A^T A - AA - A^T A^T + AA^T)]} = \sqrt{\frac{\text{tr}(A^T A) - \text{tr}(A^2)}{2}}$$

25. (25, 3.50)

I used the familiar notations for me first.

$$rx_2 + sy_2 = 1$$

$$y^2 = Y \text{ and } x^2 = X$$

$$sY + rX = 1$$

$$sY = -rX + 1$$

$$Y = -\frac{r}{s}X + \frac{1}{s}$$

$$= \beta_1 X + \beta_0$$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \frac{\hat{1}}{s} \\ -\frac{\hat{r}}{s} \end{bmatrix} = (\hat{X}^T \hat{X})^{-1} (\hat{X})^T y$$

where  $\hat{x} = (1, X) = (1, x^2)$

Thus,  $\hat{s} = \frac{1}{\hat{\beta}_0}$ ,  $\hat{r} = -\hat{s}\hat{\beta}_1 = -\frac{\hat{\beta}_1}{\hat{\beta}_0}$

$\therefore A, \mathbf{x}, \mathbf{b}$  are as in the following. ( $X = A, y = \mathbf{b}$ ) here.

$$A = \begin{bmatrix} 1 & x_1^2 \\ \vdots & \vdots \\ 1 & x_n^2 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \frac{1}{s} \\ r \\ -\frac{r}{s} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} y_1^2 \\ \vdots \\ y_n^2 \end{bmatrix}$$

## References