Math Homework Week #3, Spectral Theory

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1. (4.2)
   $p = ax^2 + bx + c$

   $D(p)(x) = 2ax + b$

   $$\therefore \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} b \\ 2a \\ 0 \end{bmatrix}$$

   $$adjoint = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^T$$

   $$det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} = -\lambda^3 = 0$$

   $\therefore \lambda = 0$
   So, algebraic multiplicities: 3
   geometric multiplicities: 3 - 2(rank) = 1

2. (4.4)
   $(i) A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^H = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$
   $\lambda - (a + d)\lambda + (ad - bc) = 0$
   $b = \bar{c}, c = \bar{b}, a = \bar{a}, d = \bar{d}$
   $\lambda^2 - (a + d)\lambda + (ad - b^2) = 0$
   $(a - d)^2 + 4c\bar{c} \geq 0$

   $(ii) A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} A^H = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$
   $b = -\bar{c}, c = -\bar{b}, a = -\bar{a}, d = -\bar{d}$
   $(a - d)^2 - 4b\bar{b} < 0$ ($\because a, d = 0$ or imaginary numbers.)

3. (4.6)
   Let diagonal entries of an upper-triangular matrix A be $d_1, d_2, d_3, \cdots, d_n$. Then, $(A - \lambda I)$ is also upper triangular matrix. Now, diagonal entries are $d_i - \lambda$. $|\pi_i(d_i - \lambda)| = 0$ and λ is d_i here. For lower triangular matrix, the proof is almost same.

4. (4.8)
   (i) In terms of linear combination, it is linear independent. For example, when

we calculate determinent of wronskian, it is not zero.

(ii) $D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} sinx \\ cosx \\ sin2x \\ cos2x \end{bmatrix}$

(iii) $span[(1,0,0,0),(0,1,0,0)]$ and $span[(0,0,1,0),(0,0,0,1)]$

5. (4. 13)
$P = \begin{bmatrix} 1 & 1 \\ 0.5 & -1 \end{bmatrix}$
$Then, P^{-1}AP = \frac{1}{detp} \begin{bmatrix} -15 & 0 \\ 0 & -6 \end{bmatrix}$
I calculated $P^{-1}AP$ by general $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then input numbers.

6. (4.15)
$A = P^{-1}BP$, then $A^k = P^{-1}B^kP$
If A is semisimple, then it is diagonalizable. By Thm 4.3.7 proof, $A = P^{-1}DPwhereD = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$
Then, $A^n = P^{-1}D^nP$, so $D^2 = diag(\lambda_1^n, \cdots, \lambda_n^n)$
Then, $(a_0 + a_1A + \cdots + a_nA^n)x = (a_0 + \lambda + \lambda^2 + \cdots + \lambda^n)x$
So, $f(A)x = f(\lambda)x$

7. (4.16)
(i) $\lim_{n\to\infty} A^n$ with respect to the 1-norm.
$P^{-1}AP = D$ where $P = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ from eigen vectors. So, we can easily calculate the answer $\begin{bmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{bmatrix}$

(ii) In any form of norms, it has to be same in a reason that we calculate before taking norm.

(iii) $f(x) = 3 + 5x + x^3$
$\therefore f(\lambda_1) = 9$ and $f(\lambda_2) = 5.064$

8. (4.18)
From $(Ax) = (\lambda x)$, $(Ax)^T = x^TA = (\lambda x)^T = x^T\lambda = \lambda x^T$

9. (4.20)
If A is Hermitian and orthonormally similar to B, then, B is also Hermitian.
$\Rightarrow B = U^HAU$ where U is orthonormal.
$B^H = U^HA^HU = U^HAU = B$ by A is Hermitian.

10. (4.24)
(i) $A = A^H$ $\rho(x) = \frac{<x,Ax>}{\|x\|_2} = \frac{x^H Ax}{\|x\|_2} = \frac{<x,\lambda x>}{\|x\|_2} = \frac{x^H \lambda x}{\|x\|_2} = \lambda = \frac{<A^H x,x>}{\|x\|_2} = \frac{\bar{\lambda} x^H x}{\|x\|_2} = \bar{\lambda}$
$A = -A^H$
$\rho(x) = \frac{<x,Ax>}{\|x\|_2} = \frac{<x,\lambda x>}{\|x\|_2} = \frac{\lambda x^H x}{\|x\|_2} = \lambda = \frac{<A^H x,x>}{\|x\|_2} = -\frac{<Ax,x>}{\|x\|_2} = -\frac{\bar{\lambda} x^H x}{\|x\|_2} = -\bar{\lambda}$

11. (4.25)
(i) $(x_1 x_1^H + \cdots + x_n x_n^H)(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = (a_1 x_1 + \cdots + a_n b_n)$
$\because x_j^H x_j = 1$ and by orthogonality.
By the way, $(a_1 x_1 + \cdots + a_n b_n)$ is arbitrary linear combination, so,
$(x_1 x_1^H + \cdots + x_n x_n^H)$ has to be I.

(ii) $AI = A(x_1 x_1^H + \cdots + x_n x_n^H)$
$\therefore A = Ax_1 x_1^H + \cdots + Ax_n x_n^H = \lambda x_1 x_1^H + \cdots + \lambda x_n x_n^H$

12. (4.27)
let $x_i = (0, \cdots, 1 + a * i, \cdots, 0)$.
Then, $x_i^H A x_I = ((1 + a * i)^H a_{ii} (1 + a * i) > 0$
$\therefore a_{ii} > 0$
($\because A$ is hermitian, so $A = A^H$)
Therefore, a_{ii} is real.

13. (4.28)
We already know that trace is one of inner product in Chapter 3. So, it has to satisfy Cauchy-Schwartz ineqaulity.
So, $tr(AB) \leq \sqrt{tr(A^2)tr(B^2)} = \sqrt{\sum(\lambda_i^A)^2 \sum(\lambda_i^B)^2} \leq \sqrt{(\sum \lambda_i^A)^2 (\sum \lambda_i^B)^2} = tr(A)tr(B)$

14. (4.31)
(i) $\|A\|_2 = sup\frac{\|Ax\|_2}{\|x\|_2} = sup\frac{\|U\Sigma V^H x\|_2}{\|x\|_2} = sup\frac{\|\Sigma V^H x\|_2}{\|x\|_2} = sup\frac{\|\Sigma V^H x\|_2}{\|V^H x\|_2} = sup\frac{(\sum |\sigma_i V^H x|^2)^{0.5}}{(\sum |V^H x|^2)^{0.5}}$
(ii) By the method of (i), now $\frac{1}{\sigma_n} = \|A^{-1}\|_2$ (It is the largest here.)
(iii) $\|A^H\|_2^2 = \sigma_1^2 = \|A^T\|_2^2 = \sigma_1^2 = \|A^H A\|_2 = \sigma_1^2 = (\|A\|_2)^2$
(iv) $\|UAV\|_2 = \|UU\Sigma V^H V\|_2 = \|\Sigma\|_2 = \|A\|_2$

15. (4.32)
(i) $\|UAV\|_F = \|AV\|_F = \|U\Sigma V^H V\|_F = \|\Sigma\|_F = \|V\Sigma\|_F = \|\Sigma^H V^H\|_F$
($\because$ trace property)
$= \|U\Sigma V^H\|_F = \|A\|_F$
(ii) $\|\Sigma\|_F = \sqrt{tr(\Sigma^H \Sigma)} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$

16. (4.33)
$\|A\|_2 = \|\Sigma\|_2 = sup|y^H \Sigma x| = \sigma_1$
I already showed $\|A\|_2 = \|\Sigma\|_2$ above.

17. (4.36)
$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda = 1$ or 3. From $A^H A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$,
$\sigma_1 \approx 3.6503$, $\sigma_2 \approx 0.8219$

18. (4.38)
(i) $AA^+A = U_1\Sigma_1V_1^HV_1\Sigma_1^{-1}U_1^HA = U_1U_1^HA = U_1U_1^HU_1\Sigma_1V_1^H = U_1\Sigma_1V_1^H = A$
(ii) $A^+AA^+ = V_1\Sigma^{-1}U_1^H = A^+$
(iii) $(AA^+)^H = U_1U_1^H = AA^+$ from (i) we can easily know it.
(iv) $(A^+A)^H = V_1V_1^H = V_1\Sigma^{-1}U_1^HU_1\Sigma_1V_1^H = A^+A$
(v) From (iii),
From orthogonal projection, $A(A^HA)^{-1}A^H$, we can know that $A(A^HA)^{-1}A^HA = AA^+A = A$
(vi) From (iv),
From orthogonal projection, $A^H(AA^H)^{-1}A$, we can know that $AA^H(AA^H)^{-1}A = AA^+A = A$

References