Digital Image Processing (EE535)

Assignment #1

Problem 1. Prove the following: ($*$: Convolution)

$$h(m,n)*[u_1(m,n)*u_2(m,n)]=[h(m,n)*u_1(m,n)]*u_2(m,n)$$

Problem 2.

a. Determine the convolution of $x(m,n)$ of Example 2.1 (p.15 in Jain's book) with each of the following arrays, where the boxed element denotes the (0, 0) location.

i.

$$\begin{matrix} 0 & -1 & 1 \\ -1 & \boxed{4} & -1 \\ 0 & -1 & 0 \end{matrix}$$

ii.

$$\begin{matrix} 1 & \boxed{2} & 3 \end{matrix}$$

b. Write convolution equation (refer to example 2.6.) using the doubly Toeplitz block matrix for kernel a.i.

c. Show that in general the convolution of two arrays of sizes $(M_1 \times N_1)$ and $(M_2 \times N_2)$ yields an array of size $(M_1+M_2-1)\times(N_1+N_2-1)$.

Problem 3.

Prove the properties of the Fourier transform of two-dimensional sequences listed below.

Property	Function	Fourier transform
	$f(x,y)$	$F(\xi_1,\xi_2)$
Multiplication	$g(x,y)=h(x,y)f(x,y)$	$G(\xi_1,\xi_2)=H(\xi_1,\xi_2)*F(\xi_1,\xi_2)$
Spatial correlation	$c(x,y)=h(x,y)\otimes f(x,y)$	$C(\xi_1,\xi_2)=H(-\xi_1,-\xi_2)*F(\xi_1,\xi_2)$

Problem 4.

Prove the properties of the Fourier transform of two-dimensional sequences listed below.

Property	Sequence	Transform
	$x(m,n),y(m,n),h(m,n),\cdots$	$X(\omega_1,\omega_2),Y(\omega_1,\omega_2),H(\omega_1,\omega_2),\cdots$
Inner product	$I=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}x(m,n)y^*(m,n)$	$I=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}X(\omega_1,\omega_2)Y^*(\omega_1,\omega_2)d\omega_1 d\omega_2$

Problem 5. Show each of the following.

a. A circulant matrix is Toeplitz, but the converse is not true.

b. The product of two Toeplitz matrices need not be Toeplitz.

Problem 6. $S(\omega_1,\omega_2)$: SDF (spectral density function)

a. Prove $S(\omega_1,\omega_2)=S^*(\omega_1,\omega_2)$

b. Prove $S(\omega_1,\omega_2)\geq 0,\ \forall\omega_1,\omega_2$

c. Prove $S_u(\omega_1,\omega_2)=|H(\omega_1,\omega_2)|^2 S_\varepsilon(\omega_1,\omega_2)$

d. Show that $S(\omega_1,\omega_2)=\dfrac{\sigma^2(1-\rho_1^2)(1-\rho_2^2)}{(1+\rho_1^2-2\rho_1\cos\omega_1)(1+\rho_2^2-2\rho_2\cos\omega_2)}$ is the SDF of the random fields whose covariance function is the separable function given by
$r(m,n)=\sigma^2\rho_1^{|m|}\rho_2^{|n|},\quad |\rho_1|<1,\quad |\rho_2|<1$

Problem 7.

Let $g(x,y)$ denote a corrupted image formed by the addition of noise $\eta(x,y)$ to a noiseless image $f(x,y)$; that is

$$g(x,y)=f(x,y)+\eta(x,y),$$

where the assumption is that at every pair of coordinates (x,y) the noise is uncorrelated and has zero average value. If an image $\bar{g}(x,y)$ is formed by averaging *K* different noisy images,

$$\bar{g}(x,y)=\frac{1}{K}\sum_{i=1}^{K}g_i(x,y),\quad (g_i(x,y)\text{: a set of noisy images}),$$

then it follows that

$$E\{\bar{g}(x,y)\}=f(x,y)\ \cdots(1)\qquad\text{and}\qquad \sigma^2_{\bar{g}(x,y)}=\frac{1}{K}\sigma^2_{\eta(x,y)}\ \cdots(2)$$

where $E\{\bar{g}(x,y)\}$ is the expected value of $\bar{g}(x,y)$, and $\sigma^2_{\bar{g}(x,y)}$ and $\sigma^2_{\eta(x,y)}$ are the variances of $\bar{g}(x,y)$ and $\eta(x,y)$, respectively. Prove the validity of the equations (1) and (2). (Hint. The expected value of a sum is the sum of the expected values.)

Problem 8.

Prove that Gaussian random variables which are uncorrelated are also independent.