# Supplementary document of paper:
# The optimal distance threshold of MIMO aerial fractional frequency reuse networks

## Proof of Theorem 1

To begin, we investigate a crossover of the outage probabilities of full and partial FRs. That is, we find $d_{\mathrm{r}}^*$ and $P_{\mathrm{out}}^*$ satisfying the equalities given by $P_{\mathrm{out,\,f}}(d_{\mathrm{r}}^*) = P_{\mathrm{out,\,p}}(d_{\mathrm{r}}^*) = P_{\mathrm{out}}^*$. From (6), (8), and (9), the equalities can be rewritten as

$$P_{\mathrm{out}}^* = F_X(u_{\mathrm{f}}(d_{\mathrm{r}}^*)) = F_X(u_{\mathrm{p}}(d_{\mathrm{r}}^*)), \tag{22}$$

where

$$\begin{aligned} u_{\mathrm{f}}(d_{\mathrm{r}}) &= \frac{r_{\mathrm{s}} N_{\mathrm{t}} \lambda_{\mathrm{s}}^{\alpha}}{p_{\mathrm{s}}} \big(\xi \, d_{\mathrm{r}}^{\alpha} + \mu\big) \cdot \left(2^{\frac{R}{r_{\mathrm{s}} W}} - 1\right) > 0, \\ u_{\mathrm{p}}(d_{\mathrm{r}}) &= \frac{r_{\mathrm{s}} N_{\mathrm{t}} \lambda_{\mathrm{s}}^{\alpha}}{p_{\mathrm{s}}} \left(\frac{\xi}{n} \, d_{\mathrm{r}}^{\alpha} + \mu_n\right) \cdot \left(2^{\frac{nR}{r_{\mathrm{s}} W}} - 1\right) > 0. \end{aligned} \tag{23}$$

Since $F_X(x)$, given by (6), is a strictly increasing function of $x$, (22) can be calculated as

$$u_{\mathrm{f}}(d_{\mathrm{r}}^*) = u_{\mathrm{p}}(d_{\mathrm{r}}^*) = F_X^{-1}(P_{\mathrm{out}}^*). \tag{24}$$

From (23) and the first equality in (24), we have

$$\xi \, (d_{\mathrm{r}}^*)^{\alpha} + \mu = \left(\frac{\xi}{n} \, (d_{\mathrm{r}}^*)^{\alpha} + \mu_n\right) \cdot \left(\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}} W}}\right). \tag{25}$$

From (25), it can be shown that the crossover in cell size, $d_{\mathrm{r}}^*$, is given by (13). It is clear from $R > 0$ that the denominator of (13) is positive. Thus, we have $d_{\mathrm{r}}^* > 0$ if and only if the numerator is positive, i.e., $\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}} W}} < \mu/\mu_n$. Since $2^{\frac{(n-k)R}{r_{\mathrm{s}} W}}$ $(k = 1, \ldots, n-1)$ is a strictly increasing function of $R$, the inequality can be rewritten as $R < R_{\mathrm{t}}$, where $R_{\mathrm{t}}$ is the data rate that satisfies (22). Namely, we obtain $d_{\mathrm{r}}^* > 0$ (or, equivalently, a crossover exists) if and only if $R < R_{\mathrm{t}}$. Substituting $d_{\mathrm{r}}^*$, given by (13), into either (8) or (9), it can be shown that the crossover in outage probability, $P_{\mathrm{out}}^*$, is expressed as (12). We have proven that a crossover, given by (12) and (13), exists if and only if $R < R_{\mathrm{t}}$. In the following, we prove the inequalities given by (10) and (14).

i) Case of $R < R_{\mathrm{t}}$: We prove (10) in this case. From (23), the derivatives of $u_{\mathrm{f}}(d_{\mathrm{r}})$ and $u_{\mathrm{p}}(d_{\mathrm{r}})$ are given by

$$\frac{\partial u_{\mathrm{f}}(d_{\mathrm{r}})}{\partial d_{\mathrm{r}}} = \frac{r_{\mathrm{s}} N_{\mathrm{t}} \lambda_{\mathrm{s}}^{\alpha} \xi \alpha d_{\mathrm{r}}^{\alpha-1}}{p_{\mathrm{s}}} \cdot \left(2^{\frac{R}{r_{\mathrm{s}} W}} - 1\right) > 0, \;\; \frac{\partial u_{\mathrm{p}}(d_{\mathrm{r}})}{\partial d_{\mathrm{r}}} = \frac{r_{\mathrm{s}} N_{\mathrm{t}} \lambda_{\mathrm{s}}^{\alpha} \xi \alpha d_{\mathrm{r}}^{\alpha-1}}{p_{\mathrm{s}} n} \cdot \left(2^{\frac{nR}{r_{\mathrm{s}} W}} - 1\right) > 0, \tag{26}$$

where we have used the fact that $\lambda_{\rm s}$ and $\lambda_i$ in (23) are constants, as stated below (4). From (26), $\partial u_{\rm p}(d_{\rm r})/\partial d_{\rm r}$ can be expressed as

$$\frac{\partial u_{\rm p}(d_{\rm r})}{\partial d_{\rm r}} = \frac{\partial u_{\rm f}(d_{\rm r})}{\partial d_{\rm r}} \left( \frac{1}{n} \sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\rm s}W}} \right). \tag{27}$$

From the fact that $\sum_{k=0}^{n-1} 2^{\frac{kR}{W}} > n$, Eqs. (26) and (27) lead to $\partial u_{\rm p}(d_{\rm r})/\partial d_{\rm r} > \partial u_{\rm f}(d_{\rm r})/\partial d_{\rm r} > 0$. Thus, from (24), we have

$$\begin{aligned} &u_{\rm f}(d_{\rm r}^*) > u_{\rm f}(d_{\rm r}) > u_{\rm p}(d_{\rm r}) \quad \text{for } \ 0 < d_{\rm r} < d_{\rm r}^*, \\ &u_{\rm f}(d_{\rm r}^*) < u_{\rm f}(d_{\rm r}) < u_{\rm p}(d_{\rm r}) \quad \text{for } \ d_{\rm r} > d_{\rm r}^* > 0. \end{aligned} \tag{28}$$

Using $F_X(x)$ given by (6), the outage probabilities of (8) and (9) can be rewritten as

$$P_{\rm out,\,f}(d_{\rm r}) = F_X\left(u_{\rm f}(d_{\rm r})\right) \quad \text{and} \quad P_{\rm out,\,p}(d_{\rm r}) = F_X\left(u_{\rm p}(d_{\rm r})\right), \tag{29}$$

where $u_{\rm f}(d_{\rm r})$ and $u_{\rm p}(d_{\rm r})$ are given by (23). Since $F_X(x)$ is a strictly increasing function of $x$, Eqs. (22), (28), and (29) yield (10).

ii) Case of $R > R_{\rm t}$: We prove (14) in this case. From (23), we have

$$\lim_{d_{\rm r}\to 0} \frac{u_{\rm f}(d_{\rm r})}{u_{\rm p}(d_{\rm r})} = \frac{\mu}{\mu_n} \cdot \frac{1}{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\rm s}W}}} < 1, \tag{30}$$

where the inequality follows from the condition of $R > R_{\rm t}$; that is, $\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\rm s}W}} > \sum_{k=0}^{n-1} 2^{\frac{kR_{\rm t}}{r_{\rm s}W}} = \mu/\mu_n$ (refer to (22)). From (30) and $u_{\rm p}(d_{\rm r}) > 0$ given by (23), we have $\lim_{d_{\rm r}\to 0} u_{\rm p}(d_{\rm r}) > \lim_{d_{\rm r}\to 0} u_{\rm f}(d_{\rm r})$. Further, from (23), we obtain $\lim_{d_{\rm r}\to\infty} u_{\rm f}(d_{\rm r})/u_{\rm p}(d_{\rm r}) = n/\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\rm s}W}} < 1$. From this and $u_{\rm p}(d_{\rm r}) > 0$, we have $\lim_{d_{\rm r}\to\infty} u_{\rm p}(d_{\rm r}) > \lim_{d_{\rm r}\to\infty} u_{\rm f}(d_{\rm r})$. From the fact that $F_X(x)$ is a strictly increasing function of $x$, and from (29), the inequalities of $\lim_{d_{\rm r}\to 0} u_{\rm p}(d_{\rm r}) > \lim_{d_{\rm r}\to 0} u_{\rm f}(d_{\rm r})$ and $\lim_{d_{\rm r}\to\infty} u_{\rm p}(d_{\rm r}) > \lim_{d_{\rm r}\to\infty} u_{\rm f}(d_{\rm r})$ respectively yield

$$\lim_{d_{\rm r}\to 0} P_{\rm out,\,p}(d_{\rm r}) > \lim_{d_{\rm r}\to 0} P_{\rm out,\,f}(d_{\rm r}) \quad \text{and} \quad \lim_{d_{\rm r}\to\infty} P_{\rm out,\,p}(d_{\rm r}) > \lim_{d_{\rm r}\to\infty} P_{\rm out,\,f}(d_{\rm r}). \tag{31}$$

Recall that the crossover of $P_{\rm out,\,f}(d_{\rm r})$ and $P_{\rm out,\,p}(d_{\rm r})$ does not exist in this case of $R > R_{\rm t}$. Thus, (31) yields $P_{\rm out,\,p}(d_{\rm r}) > P_{\rm out,\,f}(d_{\rm r})$ for $d_{\rm r} > 0$. We have thus proven the second inequality in (14). We then prove the first inequality in (14). From (29), $\partial P_{\rm out,\,f}(d_{\rm r})/\partial d_{\rm r}$ can be expressed as $\partial P_{\rm out,\,f}(d_{\rm r})/\partial d_{\rm r} = \partial F_X(u_{\rm f}(d_{\rm r}))/\partial u_{\rm f}(d_{\rm r}) \cdot \partial u_{\rm f}(d_{\rm r})/\partial d_{\rm r} > 0$, where the inequality follows from the fact that $F_X(x)$ is a strictly increasing function of $x$, and from (26). Thus, for $d_\mathrm{r} > 0$, we have

$$
\begin{aligned}
P_{\mathrm{out,\,f}}(d_\mathrm{r}) &> \lim_{d_\mathrm{r}\to 0} P_{\mathrm{out,\,f}}(d_\mathrm{r}) \\
&= 1 - Q_{N_\mathrm{t}N_\mathrm{r}}\left(\sqrt{2KN_\mathrm{t}N_\mathrm{r}}\,,\ \left\{\frac{2(K+1)r_\mathrm{s}N_\mathrm{t}\lambda_\mathrm{s}^\alpha}{p_\mathrm{s}}\cdot\mu\left(2^{\frac{R}{r_\mathrm{s}W}}-1\right)\right\}^{1/2}\right) \\
&> P_\mathrm{out}^\mathrm{lb}\,, \qquad (32)
\end{aligned}
$$

where $P_\mathrm{out}^\mathrm{lb}$ is given by (15), and the second inequality follows from $R > R_\mathrm{t}$. We have proven the first inequality in (14).

iii) Case of $R = R_\mathrm{t}$: We prove (14). In this case, from (23), $u_\mathrm{f}(d_\mathrm{r})/u_\mathrm{p}(d_\mathrm{r})$ is expressed as

$$
\frac{u_\mathrm{f}(d_\mathrm{r})}{u_\mathrm{p}(d_\mathrm{r})} = \frac{\xi d_\mathrm{r}^\alpha + \mu}{\left(\frac{\xi d_\mathrm{r}^\alpha}{n} + \mu_n\right)\cdot\left(\sum_{k=0}^{n-1} 2^{\frac{kR_\mathrm{t}}{r_\mathrm{s}W}}\right)}\,. \qquad (33)
$$

From (22), Eq. (33) can be rewritten as

$$
\frac{u_\mathrm{f}(d_\mathrm{r})}{u_\mathrm{p}(d_\mathrm{r})} = \frac{\xi d_\mathrm{r}^\alpha + \mu}{\frac{\xi d_\mathrm{r}^\alpha}{n}\cdot\frac{\mu}{\mu_n} + \mu}\,. \qquad (34)
$$

Further, we have

$$
\frac{\mu}{\mu_n} = \sum_{k=0}^{n-1} 2^{\frac{kR_\mathrm{t}}{r_\mathrm{s}W}} > n. \qquad (35)
$$

From (34) and (35), we obtain $u_\mathrm{f}(d_\mathrm{r})/u_\mathrm{p}(d_\mathrm{r}) < 1$. Combining this and $u_\mathrm{p}(d_\mathrm{r}) > 0$, as given by (23), we have $u_\mathrm{p}(d_\mathrm{r}) > u_\mathrm{f}(d_\mathrm{r})$. Thus, from the fact that $F_X(x)$ is a strictly increasing function of $x$, and from (29), it follows that $P_{\mathrm{out,\,p}}(d_\mathrm{r}) > P_{\mathrm{out,\,f}}(d_\mathrm{r})$; we have thus proven the second inequality in (14). For the case of $R = R_\mathrm{t}$ considered here, the second inequality in (32) is replaced by the equality. Thus, we also obtain $P_{\mathrm{out,\,f}}(p_\mathrm{s}) > P_\mathrm{out}^\mathrm{lb}$ for $R = R_\mathrm{t}$, which is the first inequality in (14).

## PROOF OF THEOREM 2

We rewrite $P_\mathrm{out}^*$, given by (12), as $P_\mathrm{out}^* = F_X(h(R))$, where $F_X(x)$ is given by (6), and $h(R)$ is defined as

$$
h(R) = \frac{r_\mathrm{s}N_\mathrm{t}\lambda_\mathrm{s}^\alpha}{p_\mathrm{s}}\cdot\frac{\mu - n\mu_n}{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_\mathrm{s}W}} - n}\cdot\left(2^{\frac{nR}{r_\mathrm{s}W}} - 1\right)\,. \qquad (36)
$$

The derivative of $P^*_{\mathrm{out}}$ with regard to $R$ can be expressed as

$$\frac{\partial P^*_{\mathrm{out}}}{\partial R} = \frac{\partial F_X(h(R))}{\partial h(R)} \cdot \frac{\partial h(R)}{\partial R}\,. \tag{37}$$

Since $F_X(x)$ is a strictly increasing function of $x$, we have $\partial P^*_{\mathrm{out}}/\partial R > 0$ if $\partial h(R)/\partial R > 0$. In the following, we will prove that $\partial h(R)/\partial R > 0$. It can be shown from (36) that the derivative of $h(R)$ is expressed as

$$\frac{\partial h(R)}{\partial R} = \frac{N_{\mathrm{t}}\lambda_{\mathrm{s}}^{\alpha}\ln 2}{W} \cdot \left(\mu - n\mu_n\right) \cdot \frac{p(R)}{\left(2^{\frac{nR}{r_{\mathrm{s}}W}} - n2^{\frac{R}{r_{\mathrm{s}}W}} + n - 1\right)^2}\,, \tag{38}$$

where

$$p(R) = 2^{\frac{nR}{r_{\mathrm{s}}W}}\left(2^{\frac{R}{r_{\mathrm{s}}W}} - 1\right)^2 \cdot \left(\frac{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}}W}}}{2^{\frac{(n-1)R}{2r_{\mathrm{s}}W}}} + n\right) \cdot \left(\frac{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}}W}}}{2^{\frac{(n-1)R}{2r_{\mathrm{s}}W}}} - n\right). \tag{39}$$

Combining (22) and the condition that $P^*_{\mathrm{out}}$ exists (i.e., $R < R_{\mathrm{t}}$) in Theorem 1, we obtain

$$\frac{\mu}{\mu_n} = \sum_{k=0}^{n-1} 2^{\frac{kR_{\mathrm{t}}}{r_{\mathrm{s}}W}} > \sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}}W}} > n. \tag{40}$$

From (40), we have $\mu > n\mu_n$. Thus, from (38), it follows that $\partial h(R)/\partial R > 0$ if $p(R) > 0$. Below, we prove that $p(R) > 0$. Let $q(R) = \sum_{k=0}^{n-1} 2^{\frac{kR}{r_{\mathrm{s}}W}}/2^{\frac{(n-1)R}{2r_{\mathrm{s}}W}} - n$ be the last factor of $p(R)$. From (39), we have $p(R) > 0$ if $q(R) > 0$. The derivative of $q(R)$ can be expressed as

$$\frac{\partial q(R)}{\partial R} = \frac{\ln 2}{r_{\mathrm{s}}W} \cdot \frac{\sum_{k=0}^{n-1}(k - \frac{n-1}{2})2^{\frac{kR}{r_{\mathrm{s}}W}}}{2^{\frac{(n-1)R}{2r_{\mathrm{s}}W}}}\,. \tag{41}$$

We denote the numerator of the second factor of $\partial q(R)/\partial R$ by $z(R) = \sum_{k=0}^{n-1}\left(k - \frac{n-1}{2}\right)2^{\frac{kR}{r_{\mathrm{s}}W}}$. In the following, we show that $z(R) > 0$.

i) The case where $n\ (>1)$ is even: In this case, $z(R)$ can be rewritten as

$$z(R) = \sum_{k=0}^{\frac{n}{2}-1}\left(k - \frac{n-1}{2}\right)2^{\frac{kR}{r_{\mathrm{s}}W}} + \sum_{k=\frac{n}{2}}^{n-1}\left(k - \frac{n-1}{2}\right)2^{\frac{kR}{r_{\mathrm{s}}W}}. \tag{42}$$

Letting $j = -k-1+n/2$ and $j = k-n/2$ in the first and second summations of (42), respectively, we have $z(R) = \sum_{j=0}^{\frac{n}{2}-1}\left(j + \frac{1}{2}\right)2^{\frac{(n/2-1-j)R}{r_{\mathrm{s}}W}}\left(2^{\frac{(2j+1)R}{r_{\mathrm{s}}W}} - 1\right)$, which shows that $z(R) > 0$.

ii) The case where $n\ (>1)$ is odd: In this case, $z(R)$ can be expressed as

$$z(R) = \sum_{k=0}^{\frac{n-1}{2}-1}\left(k - \frac{n-1}{2}\right)2^{\frac{kR}{r_{\mathrm{s}}W}} + \sum_{k=\frac{n-1}{2}+1}^{n-1}\left(k - \frac{n-1}{2}\right)2^{\frac{kR}{r_{\mathrm{s}}W}}. \tag{43}$$

If we let $j = (n-3)/2 - k$ and $j = k - (n+1)/2$ in the first and second summations of (43), respectively, we obtain $z(R) = \sum_{j=0}^{\frac{n-3}{2}} (j+1) \cdot 2^{\frac{((n-3)/2-j)R}{r_{\mathrm{s}}W}} \big(2^{\frac{2(j+1)R}{r_{\mathrm{s}}W}} - 1\big)$, which indicates that $z(R) > 0$.

We have thus proven that $z(R) > 0$, which yields $\partial q(R)/\partial R > 0$ in (41). Further, from the definition of $q(R)$, given below (40), we have $q(0) = 0$. Hence, $q(R) > 0$ holds for $R > 0$. Thus, we have $p(R) > 0$ in (39), which leads to $\partial h(R)/\partial R > 0$ as stated below (40). Finally, we have $\partial P_{\mathrm{out}}^{*}/\partial R > 0$ in (37), as indicated below (37).

## Proof of Theorem 3

Recall that $P_{\mathrm{out}}^{*}$, given by (12), exists if and only if $R < R_{\mathrm{t}}$, as indicated by Theorem 1. Also recall that $\partial P_{\mathrm{out}}^{*}/\partial R > 0$ in Theorem 2. Thus, the largest crossover satisfies

$$\max_{R<R_{\mathrm{t}}} P_{\mathrm{out}}^{*} < \lim_{R\to R_{\mathrm{t}}} P_{\mathrm{out}}^{*} = 1 - Q_{N_{\mathrm{t}}N_{\mathrm{r}}}\left(\sqrt{2KN_{\mathrm{t}}N_{\mathrm{r}}}\,, \left\{\frac{2(K+1)r_{\mathrm{s}}N_{\mathrm{t}}\lambda_{\mathrm{s}}^{\alpha}}{p_{\mathrm{s}}} \times \frac{\mu - n\mu_n}{\sum_{k=0}^{n-1} 2^{\frac{kR_{\mathrm{t}}}{r_{\mathrm{s}}W}} - n} \cdot \left(\sum_{k=0}^{n-1} 2^{\frac{kR_{\mathrm{t}}}{r_{\mathrm{s}}W}}\right) \cdot \left(2^{\frac{R_{\mathrm{t}}}{r_{\mathrm{s}}W}} - 1\right)\right\}^{1/2}\right). \tag{44}$$

From (22), Eq. (44) can be rewritten as

$$\lim_{R\to R_{\mathrm{t}}} P_{\mathrm{out}}^{*} = 1 - Q_{N_{\mathrm{t}}N_{\mathrm{r}}}\left(\sqrt{2KN_{\mathrm{t}}N_{\mathrm{r}}}\,, \left\{\frac{2(K+1)r_{\mathrm{s}}N_{\mathrm{t}}\lambda_{\mathrm{s}}^{\alpha}}{p_{\mathrm{s}}} \times \frac{\mu - n\mu_n}{(\mu/\mu_n) - n} \cdot \frac{\mu}{\mu_n} \cdot \left(2^{\frac{R_{\mathrm{t}}}{r_{\mathrm{s}}W}} - 1\right)\right\}^{1/2}\right), \tag{45}$$

which is identical to $P_{\mathrm{out}}^{\mathrm{lb}}$ given by (15). We have thus proven the first inequality in (17). The second inequality in (17) readily follows from (14) in Theorem 1. The equality given below (17) (i.e., $P_{\mathrm{out}}^{\mathrm{lb}} = \lim_{d_{\mathrm{r}}\to 0} P_{\mathrm{out,f}}(d_{\mathrm{r}})$ at $R = R_{\mathrm{t}}$) can easily be shown from (8).