

Supplementary document of paper:
The optimal distance threshold of aerial FFR systems with multiple antennas

PROOF OF THEOREM 1

To begin, we investigate a crossover of the outage probabilities of full and partial FRs. That is, we find d_r^* and P_{out}^* satisfying the equalities given by $P_{\text{out},f}(d_r^*) = P_{\text{out},p}(d_r^*) = P_{\text{out}}^*$. From (6), (8), and (9), the equalities can be rewritten as

$$P_{\text{out}}^* = F_X(u_f(d_r^*)) = F_X(u_p(d_r^*)), \quad (22)$$

where

$$\begin{aligned} u_f(d_r) &= \frac{r_s N_t \lambda_s^\alpha}{p_s} (\xi d_r^\alpha + \mu) \cdot \left(2^{\frac{R}{r_s W}} - 1 \right) > 0, \\ u_p(d_r) &= \frac{r_s N_t \lambda_s^\alpha}{p_s} \left(\frac{\xi}{n} d_r^\alpha + \mu_n \right) \cdot \left(2^{\frac{nR}{r_s W}} - 1 \right) > 0. \end{aligned} \quad (23)$$

Since $F_X(x)$, given by (6), is a strictly increasing function of x , (22) can be calculated as

$$u_f(d_r^*) = u_p(d_r^*) = F_X^{-1}(P_{\text{out}}^*). \quad (24)$$

From (23) and the first equality in (24), we have

$$\xi (d_r^*)^\alpha + \mu = \left(\frac{\xi}{n} (d_r^*)^\alpha + \mu_n \right) \cdot \left(\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} \right). \quad (25)$$

From (25), it can be shown that the crossover in cell size, d_r^* , is given by (13). It is clear from $R > 0$ that the denominator of (13) is positive. Thus, we have $d_r^* > 0$ if and only if the numerator is positive, i.e., $\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} < \mu/\mu_n$. Since $2^{\frac{(n-k)R}{r_s W}}$ ($k = 1, \dots, n-1$) is a strictly increasing function of R , the inequality can be rewritten as $R < R_t$, where R_t is the data rate that satisfies (22). Namely, we obtain $d_r^* > 0$ (or, equivalently, a crossover exists) if and only if $R < R_t$. Substituting d_r^* , given by (13), into either (8) or (9), it can be shown that the crossover in outage probability, P_{out}^* , is expressed as (12). We have proven that a crossover, given by (12) and (13), exists if and only if $R < R_t$. In the following, we prove the inequalities given by (10) and (14).

i) Case of $R < R_t$: We prove (10) in this case. From (23), the derivatives of $u_f(d_r)$ and $u_p(d_r)$ are given by

$$\frac{\partial u_f(d_r)}{\partial d_r} = \frac{r_s N_t \lambda_s^\alpha \xi \alpha d_r^{\alpha-1}}{p_s} \cdot \left(2^{\frac{R}{r_s W}} - 1 \right) > 0, \quad \frac{\partial u_p(d_r)}{\partial d_r} = \frac{r_s N_t \lambda_s^\alpha \xi \alpha d_r^{\alpha-1}}{p_s n} \cdot \left(2^{\frac{nR}{r_s W}} - 1 \right) > 0, \quad (26)$$

where we have used the fact that λ_s and λ_i in (23) are constants, as stated below (4). From (26), $\partial u_p(d_r)/\partial d_r$ can be expressed as

$$\frac{\partial u_p(d_r)}{\partial d_r} = \frac{\partial u_f(d_r)}{\partial d_r} \left(\frac{1}{n} \sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} \right). \quad (27)$$

From the fact that $\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} > n$, Eqs. (26) and (27) lead to $\partial u_p(d_r)/\partial d_r > \partial u_f(d_r)/\partial d_r > 0$. Thus, from (24), we have

$$\begin{aligned} u_f(d_r^*) &> u_f(d_r) > u_p(d_r) \quad \text{for } 0 < d_r < d_r^*, \\ u_f(d_r^*) &< u_f(d_r) < u_p(d_r) \quad \text{for } d_r > d_r^* > 0. \end{aligned} \quad (28)$$

Using $F_X(x)$ given by (6), the outage probabilities of (8) and (9) can be rewritten as

$$P_{\text{out},f}(d_r) = F_X(u_f(d_r)) \quad \text{and} \quad P_{\text{out},p}(d_r) = F_X(u_p(d_r)), \quad (29)$$

where $u_f(d_r)$ and $u_p(d_r)$ are given by (23). Since $F_X(x)$ is a strictly increasing function of x , Eqs. (22), (28), and (29) yield (10).

ii) Case of $R > R_t$: We prove (14) in this case. From (23), we have

$$\lim_{d_r \rightarrow 0} \frac{u_f(d_r)}{u_p(d_r)} = \frac{\mu}{\mu_n} \cdot \frac{1}{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}}} < 1, \quad (30)$$

where the inequality follows from the condition of $R > R_t$; that is, $\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} > \sum_{k=0}^{n-1} 2^{\frac{kR_t}{r_s W}} = \mu/\mu_n$ (refer to (22)). From (30) and $u_p(d_r) > 0$ given by (23), we have $\lim_{d_r \rightarrow 0} u_p(d_r) > \lim_{d_r \rightarrow 0} u_f(d_r)$. Further, from (23), we obtain $\lim_{d_r \rightarrow \infty} u_f(d_r)/u_p(d_r) = n/\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} < 1$. From this and $u_p(d_r) > 0$, we have $\lim_{d_r \rightarrow \infty} u_p(d_r) > \lim_{d_r \rightarrow \infty} u_f(d_r)$. From the fact that $F_X(x)$ is a strictly increasing function of x , and from (29), the inequalities of $\lim_{d_r \rightarrow 0} u_p(d_r) > \lim_{d_r \rightarrow 0} u_f(d_r)$ and $\lim_{d_r \rightarrow \infty} u_p(d_r) > \lim_{d_r \rightarrow \infty} u_f(d_r)$ respectively yield

$$\lim_{d_r \rightarrow 0} P_{\text{out},p}(d_r) > \lim_{d_r \rightarrow 0} P_{\text{out},f}(d_r) \quad \text{and} \quad \lim_{d_r \rightarrow \infty} P_{\text{out},p}(d_r) > \lim_{d_r \rightarrow \infty} P_{\text{out},f}(d_r). \quad (31)$$

Recall that the crossover of $P_{\text{out},f}(d_r)$ and $P_{\text{out},p}(d_r)$ does not exist in this case of $R > R_t$. Thus, (31) yields $P_{\text{out},p}(d_r) > P_{\text{out},f}(d_r)$ for $d_r > 0$. We have thus proven the second inequality in (14). We then prove the first inequality in (14). From (29), $\partial P_{\text{out},f}(d_r)/\partial d_r$ can be expressed as $\partial P_{\text{out},f}(d_r)/\partial d_r = \partial F_X(u_f(d_r))/\partial u_f(d_r) \cdot \partial u_f(d_r)/\partial d_r > 0$, where the inequality follows from

the fact that $F_X(x)$ is a strictly increasing function of x , and from (26). Thus, for $d_r > 0$, we have

$$\begin{aligned} P_{\text{out},f}(d_r) &> \lim_{d_r \rightarrow 0} P_{\text{out},f}(d_r) \\ &= 1 - Q_{N_t N_r} \left(\sqrt{2K N_t N_r}, \left\{ \frac{2(K+1)r_s N_t \lambda_s^\alpha}{p_s} \cdot \mu \left(2^{\frac{R}{r_s W}} - 1 \right) \right\}^{1/2} \right) \\ &> P_{\text{out}}^{\text{lb}}, \end{aligned} \quad (32)$$

where $P_{\text{out}}^{\text{lb}}$ is given by (15), and the second inequality follows from $R > R_t$. We have proven the first inequality in (14).

iii) Case of $R = R_t$: We prove (14). In this case, from (23), $u_f(d_r)/u_p(d_r)$ is expressed as

$$\frac{u_f(d_r)}{u_p(d_r)} = \frac{\xi d_r^\alpha + \mu}{\left(\frac{\xi d_r^\alpha}{n} + \mu_n \right) \cdot \left(\sum_{k=0}^{n-1} 2^{\frac{k R_t}{r_s W}} \right)}. \quad (33)$$

From (22), Eq. (33) can be rewritten as

$$\frac{u_f(d_r)}{u_p(d_r)} = \frac{\xi d_r^\alpha + \mu}{\frac{\xi d_r^\alpha}{n} \cdot \frac{\mu}{\mu_n} + \mu}. \quad (34)$$

Further, we have

$$\frac{\mu}{\mu_n} = \sum_{k=0}^{n-1} 2^{\frac{k R_t}{r_s W}} > n. \quad (35)$$

From (34) and (35), we obtain $u_f(d_r)/u_p(d_r) < 1$. Combining this and $u_p(d_r) > 0$, as given by (23), we have $u_p(d_r) > u_f(d_r)$. Thus, from the fact that $F_X(x)$ is a strictly increasing function of x , and from (29), it follows that $P_{\text{out},p}(d_r) > P_{\text{out},f}(d_r)$; we have thus proven the second inequality in (14). For the case of $R = R_t$ considered here, the second inequality in (32) is replaced by the equality. Thus, we also obtain $P_{\text{out},f}(p_s) > P_{\text{out}}^{\text{lb}}$ for $R = R_t$, which is the first inequality in (14).

PROOF OF THEOREM 2

We rewrite P_{out}^* , given by (12), as $P_{\text{out}}^* = F_X(h(R))$, where $F_X(x)$ is given by (6), and $h(R)$ is defined as

$$h(R) = \frac{r_s N_t \lambda_s^\alpha}{p_s} \cdot \frac{\mu - n \mu_n}{\sum_{k=0}^{n-1} 2^{\frac{k R}{r_s W}} - n} \cdot \left(2^{\frac{n R}{r_s W}} - 1 \right). \quad (36)$$

The derivative of P_{out}^* with regard to R can be expressed as

$$\frac{\partial P_{\text{out}}^*}{\partial R} = \frac{\partial F_X(h(R))}{\partial h(R)} \cdot \frac{\partial h(R)}{\partial R}. \quad (37)$$

Since $F_X(x)$ is a strictly increasing function of x , we have $\partial P_{\text{out}}^*/\partial R > 0$ if $\partial h(R)/\partial R > 0$. In the following, we will prove that $\partial h(R)/\partial R > 0$. It can be shown from (36) that the derivative of $h(R)$ is expressed as

$$\frac{\partial h(R)}{\partial R} = \frac{N_t \lambda_s^\alpha \ln 2}{W} \cdot (\mu - n\mu_n) \cdot \frac{p(R)}{\left(2^{\frac{nR}{r_s W}} - n2^{\frac{R}{r_s W}} + n - 1\right)^2}, \quad (38)$$

where

$$p(R) = 2^{\frac{nR}{r_s W}} \left(2^{\frac{R}{r_s W}} - 1\right)^2 \cdot \left(\frac{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}}}{2^{\frac{(n-1)R}{2r_s W}}} + n\right) \cdot \left(\frac{\sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}}}{2^{\frac{(n-1)R}{2r_s W}}} - n\right). \quad (39)$$

Combining (22) and the condition that P_{out}^* exists (i.e., $R < R_t$) in Theorem 1, we obtain

$$\frac{\mu}{\mu_n} = \sum_{k=0}^{n-1} 2^{\frac{kR_t}{r_s W}} > \sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} > n. \quad (40)$$

From (40), we have $\mu > n\mu_n$. Thus, from (38), it follows that $\partial h(R)/\partial R > 0$ if $p(R) > 0$.

Below, we prove that $p(R) > 0$. Let $q(R) = \sum_{k=0}^{n-1} 2^{\frac{kR}{r_s W}} / 2^{\frac{(n-1)R}{2r_s W}} - n$ be the last factor of $p(R)$.

From (39), we have $p(R) > 0$ if $q(R) > 0$. The derivative of $q(R)$ can be expressed as

$$\frac{\partial q(R)}{\partial R} = \frac{\ln 2}{r_s W} \cdot \frac{\sum_{k=0}^{n-1} (k - \frac{n-1}{2}) 2^{\frac{kR}{r_s W}}}{2^{\frac{(n-1)R}{2r_s W}}}. \quad (41)$$

We denote the numerator of the second factor of $\partial q(R)/\partial R$ by $z(R) = \sum_{k=0}^{n-1} (k - \frac{n-1}{2}) 2^{\frac{kR}{r_s W}}$.

In the following, we show that $z(R) > 0$.

i) The case where $n (> 1)$ is even: In this case, $z(R)$ can be rewritten as

$$z(R) = \sum_{k=0}^{\frac{n}{2}-1} \left(k - \frac{n-1}{2}\right) 2^{\frac{kR}{r_s W}} + \sum_{k=\frac{n}{2}}^{n-1} \left(k - \frac{n-1}{2}\right) 2^{\frac{kR}{r_s W}}. \quad (42)$$

Letting $j = -k-1+n/2$ and $j = k-n/2$ in the first and second summations of (42), respectively, we have $z(R) = \sum_{j=0}^{\frac{n}{2}-1} (j + \frac{1}{2}) 2^{\frac{(n/2-1-j)R}{r_s W}} (2^{\frac{(2j+1)R}{r_s W}} - 1)$, which shows that $z(R) > 0$.

ii) The case where $n (> 1)$ is odd: In this case, $z(R)$ can be expressed as

$$z(R) = \sum_{k=0}^{\frac{n-1}{2}-1} \left(k - \frac{n-1}{2}\right) 2^{\frac{kR}{r_s W}} + \sum_{k=\frac{n-1}{2}+1}^{n-1} \left(k - \frac{n-1}{2}\right) 2^{\frac{kR}{r_s W}}. \quad (43)$$

If we let $j = (n - 3)/2 - k$ and $j = k - (n + 1)/2$ in the first and second summations of (43), respectively, we obtain $z(R) = \sum_{j=0}^{\frac{n-3}{2}} (j + 1) \cdot 2^{\frac{((n-3)/2-j)R}{r_s W}} \left(2^{\frac{2(j+1)R}{r_s W}} - 1 \right)$, which indicates that $z(R) > 0$.

We have thus proven that $z(R) > 0$, which yields $\partial q(R)/\partial R > 0$ in (41). Further, from the definition of $q(R)$, given below (40), we have $q(0) = 0$. Hence, $q(R) > 0$ holds for $R > 0$. Thus, we have $p(R) > 0$ in (39), which leads to $\partial h(R)/\partial R > 0$ as stated below (40). Finally, we have $\partial P_{\text{out}}^*/\partial R > 0$ in (37), as indicated below (37).

PROOF OF THEOREM 3

Recall that P_{out}^* , given by (12), exists if and only if $R < R_t$, as indicated by Theorem 1. Also recall that $\partial P_{\text{out}}^*/\partial R > 0$ in Theorem 2. Thus, the largest crossover satisfies

$$\begin{aligned} \max_{R < R_t} P_{\text{out}}^* &< \lim_{R \rightarrow R_t} P_{\text{out}}^* = 1 - Q_{N_t N_r} \left(\sqrt{2K N_t N_r}, \left\{ \frac{2(K+1)r_s N_t \lambda_s^\alpha}{p_s} \right. \right. \\ &\quad \left. \left. \times \frac{\mu - n\mu_n}{\sum_{k=0}^{n-1} 2^{\frac{kR_t}{r_s W}} - n} \cdot \left(\sum_{k=0}^{n-1} 2^{\frac{kR_t}{r_s W}} \right) \cdot \left(2^{\frac{R_t}{r_s W}} - 1 \right) \right\}^{1/2} \right). \end{aligned} \quad (44)$$

From (22), Eq. (44) can be rewritten as

$$\begin{aligned} \lim_{R \rightarrow R_t} P_{\text{out}}^* &= 1 - Q_{N_t N_r} \left(\sqrt{2K N_t N_r}, \left\{ \frac{2(K+1)r_s N_t \lambda_s^\alpha}{p_s} \right. \right. \\ &\quad \left. \left. \times \frac{\mu - n\mu_n}{(\mu/\mu_n) - n} \cdot \frac{\mu}{\mu_n} \cdot \left(2^{\frac{R_t}{r_s W}} - 1 \right) \right\}^{1/2} \right), \end{aligned} \quad (45)$$

which is identical to $P_{\text{out}}^{\text{lb}}$ given by (15). We have thus proven the first inequality in (17). The second inequality in (17) readily follows from (14) in Theorem 1. The equality given below (17) (i.e., $P_{\text{out}}^{\text{lb}} = \lim_{d_r \rightarrow 0} P_{\text{out},f}(d_r)$ at $R = R_t$) can easily be shown from (8).