Low-Overhead Magic State Distillation with Color Codes

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Based on **arXiv:2409.07707** with
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Outline

1. Introduction & Background
2. Scheme 1: Single-Level MSD via Faulty T-Measurements
3. Scheme 2: Cultivation-MSD
4. Performance Analysis
5. Conclusion & Future Work

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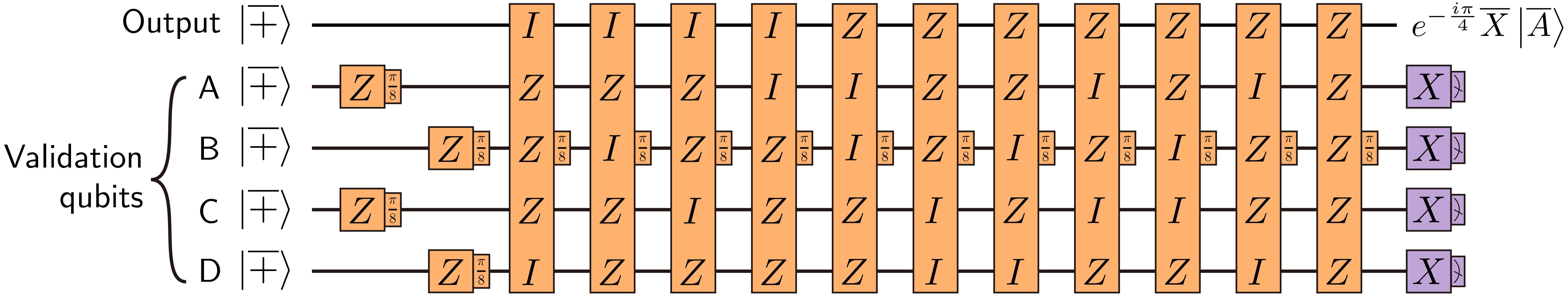
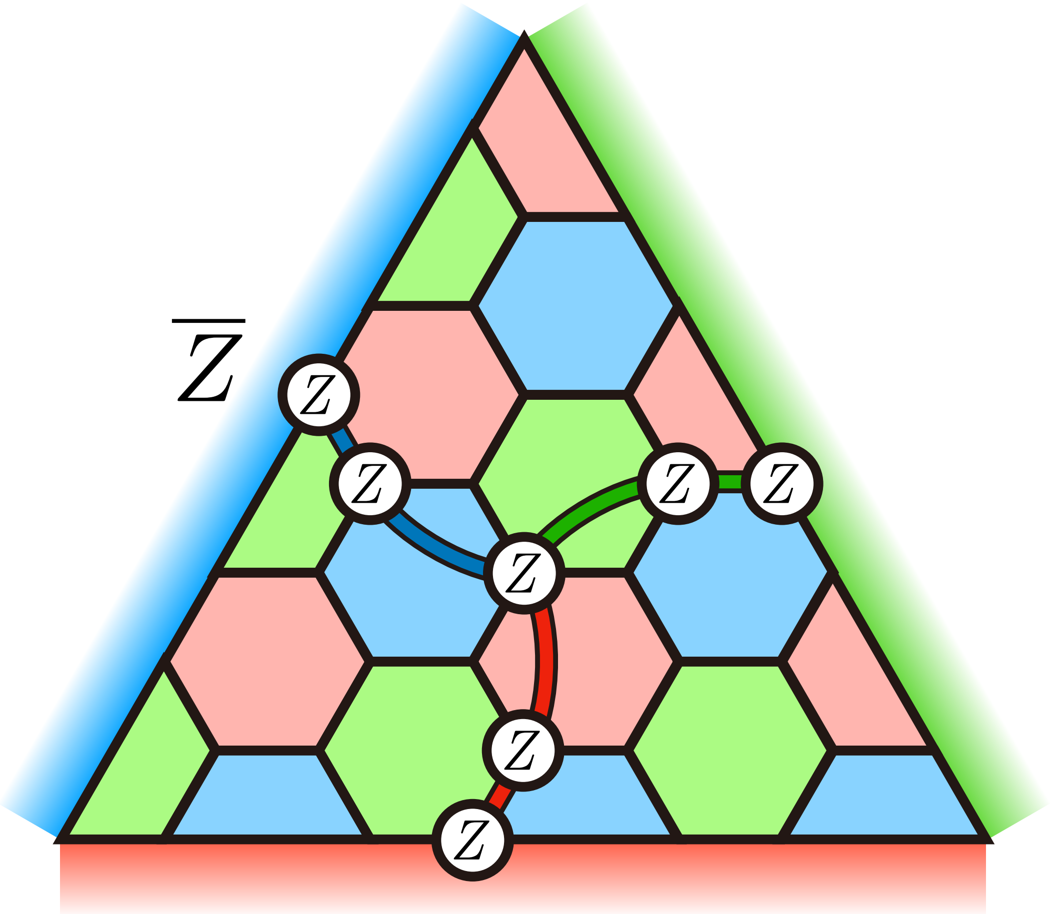
Motivation: The Need for Efficient Non-Clifford Gates

- Fault-tolerant quantum computing requires **universal gate sets**, including non-Clifford gates (e.g., T gate)
- **Implementing non-Clifford gates fault-tolerantly** is a major resource bottleneck.
- **Magic states**: Spacial resource states for implementing non-Clifford gates.
  ex) $|A\rangle := |0\rangle + e^{i\pi/4}|1\rangle$ + Clifford circuit → T gate
- **Magic State Distillation (MSD)** is a representative approach but is **resource-intensive**.
- Need for **code-specific end-to-end optimization** of MSD protocols.

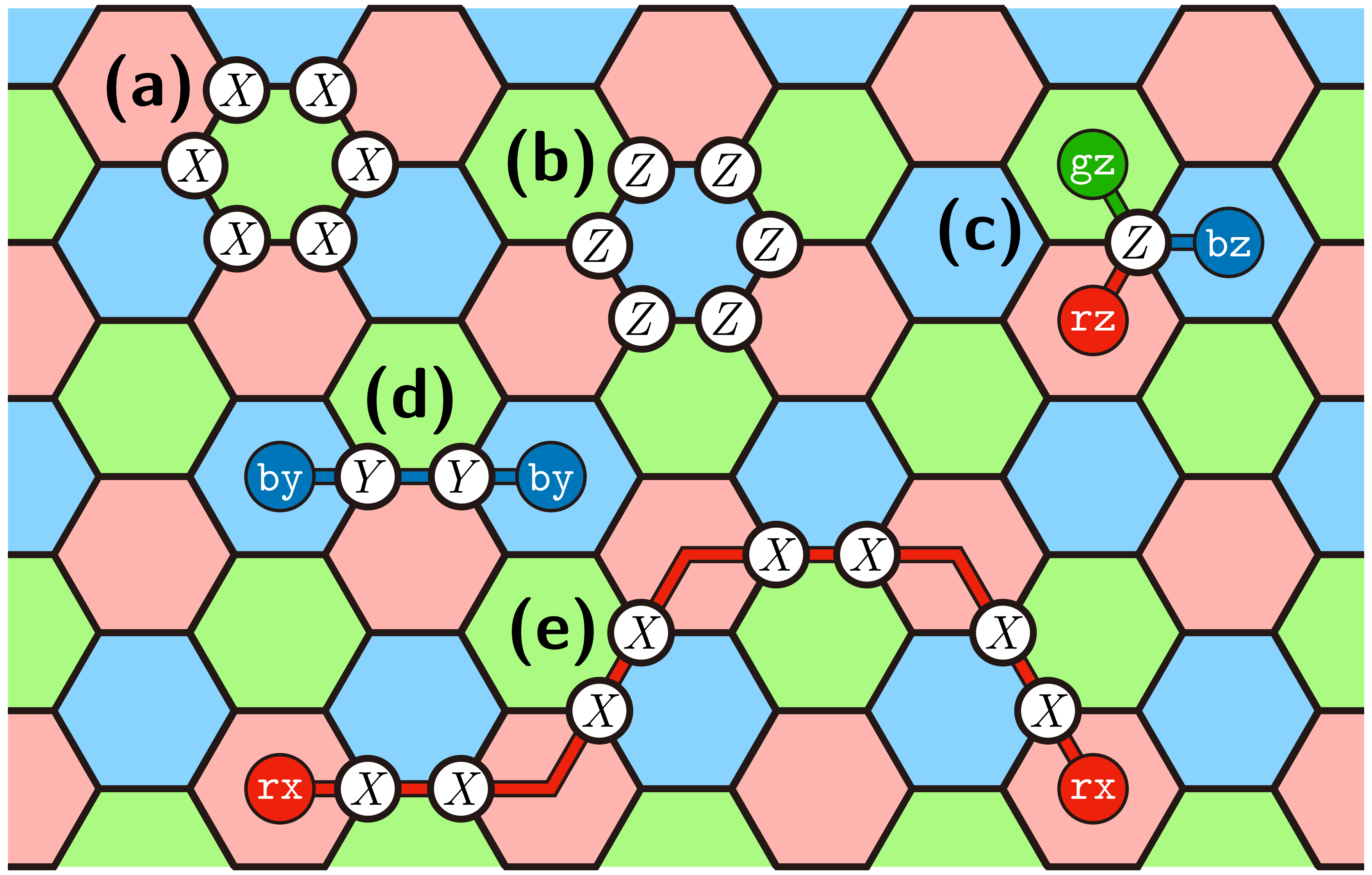


Why Color Codes? & Our Contribution



- **Color codes**: A promising topological QEC code family for FTQC.
  - High encoding rates (vs. surface codes).
  - Transversal Clifford gates.
  - Efficient lattice surgery Ref) Thomsen et al., PRR 2024
- Disadvantage: Low fault tolerance ($p_{\text{th}} \approx 0.5\,\%$ vs. $\sim 1\,\%$ in surface codes)
- This work: **Two optimized 15-to-1 MSD schemes** for 2D color codes using lattice surgery.
  - Scheme 1: **Faulty T-measurement** based ($O(p^3)$ infidelity).
  - Scheme 2: **Cultivation-MSD** (Ultra-low infidelity).
  - Key Result: **~100x resource reduction** vs. previous color code MSD

2D Color Codes: The Basics



- Color-code lattice Ref) Bombin & Martin-Delgado, PRL 2006
  - **3-valent**
  - **3-colorable** faces & edges
- Qubit on each vertex
- X-type and Z-type checks (stabilizers) on each face
- Errors are detected by check operators.

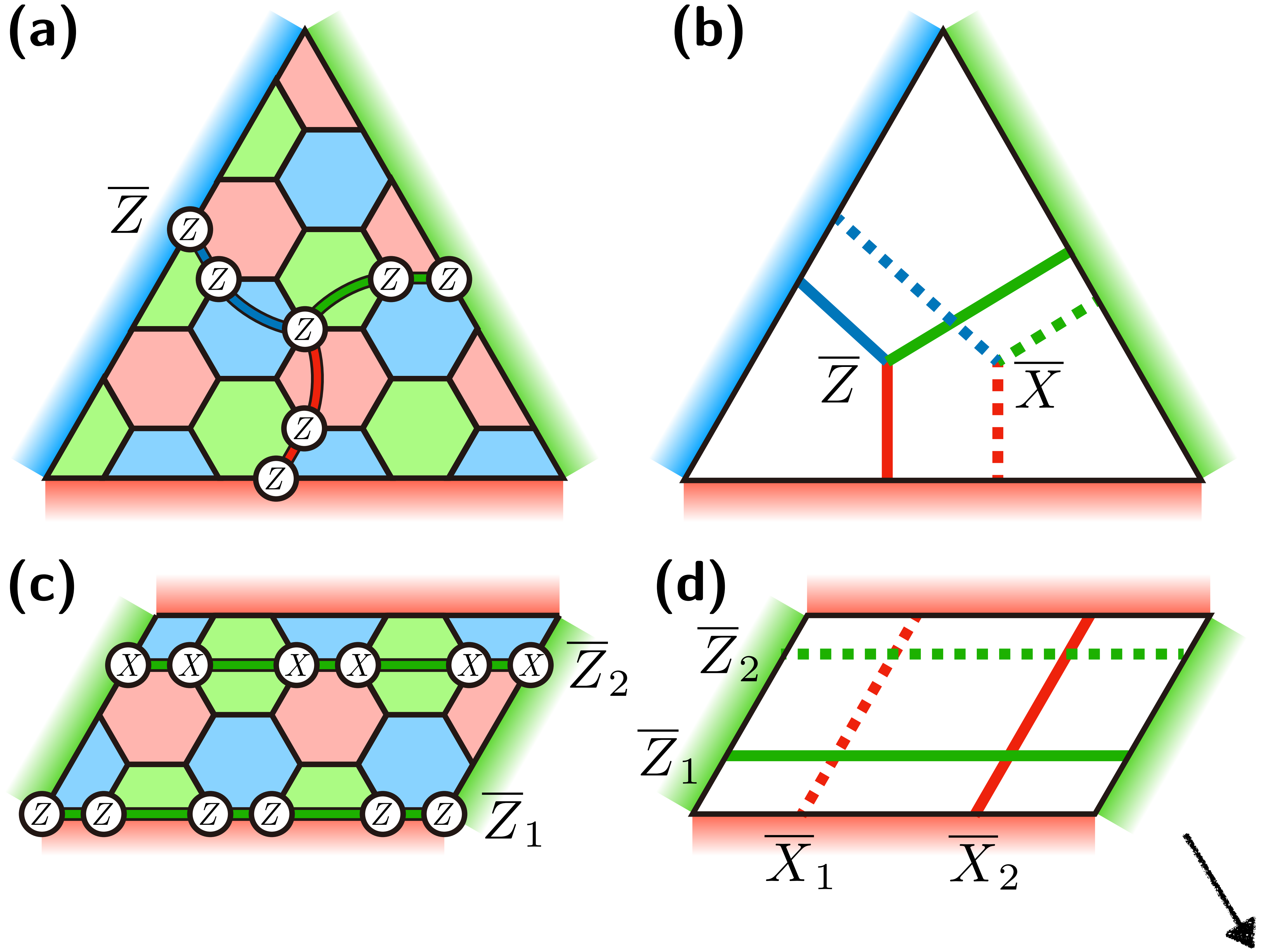
Logical Qubits

Triangular patch

(single logical qubit)

Rectangular patch

(two logical qubits)



Solid lines: Pauli-Z strings

Dotted lines: Pauli-X strings

Different code distances for logical X and Z errors.

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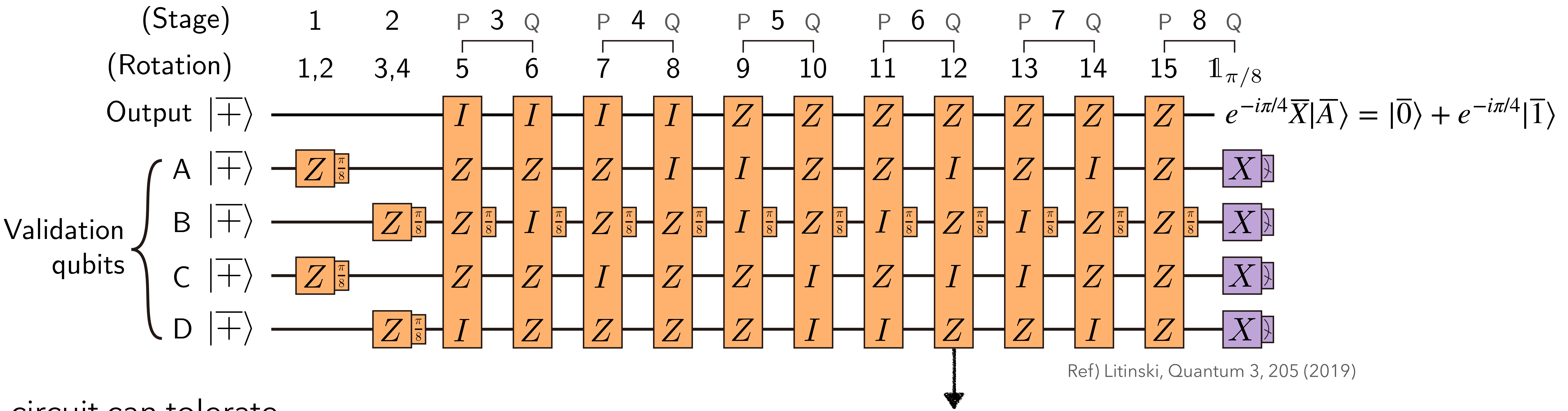
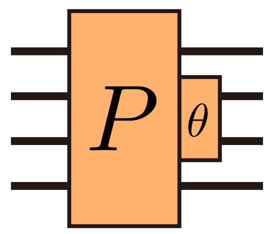
15-to-1 MSD circuit



Ref) Litinski, Quantum 3, 205 (2019)

Faulty rotation gate $\overline{P}_{\pi/8}$  $= \overline{P}_\theta = e^{-i\theta\overline{P}}$

The circuit can tolerate

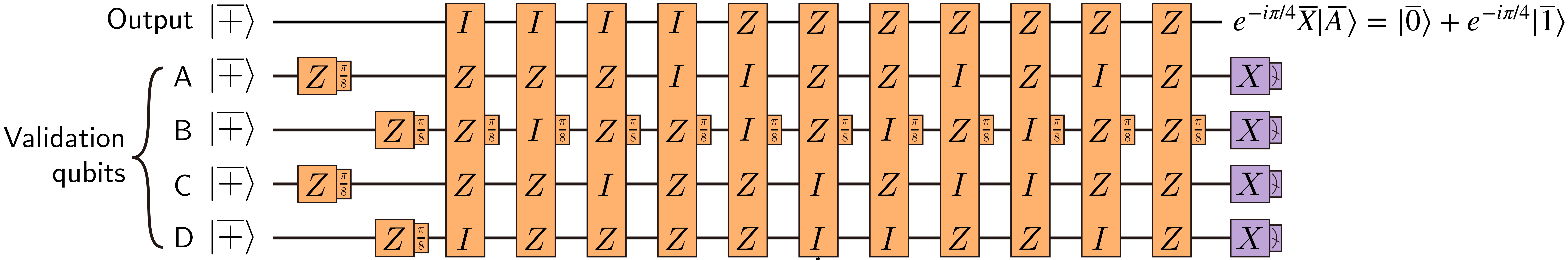
- at most two rotation errors,
- any $\overline{Z}$ errors on validation qubits,
- **any single-location $\overline{X}$ error on a validation qubit** (our finding!)

→ Allows using smaller distances d_X, d_Z for validation qubits

Single-Level MSD Scheme

Configuration of rotation gates

(Stage)	1	2	P 3 Q		P 4 Q		P 5 Q		P 6 Q		P 7 Q		P 8 Q	
(Rotation)	1,2	3,4	5	6	7	8	9	10	11	12	13	14	15	$\mathbb{1}_{\pi/8}$



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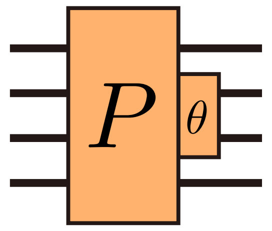
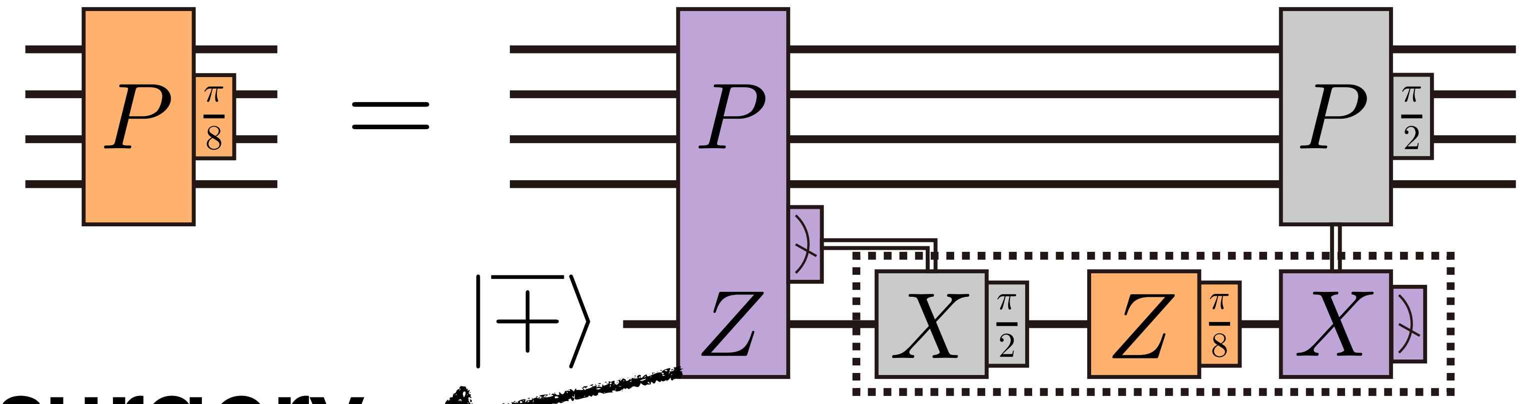


$$= \overline{P}_\theta = e^{-i\theta\overline{P}}$$

Five logical qubits

Layout

Faulty rotation gate $\overline{P}_{\pi/8}$

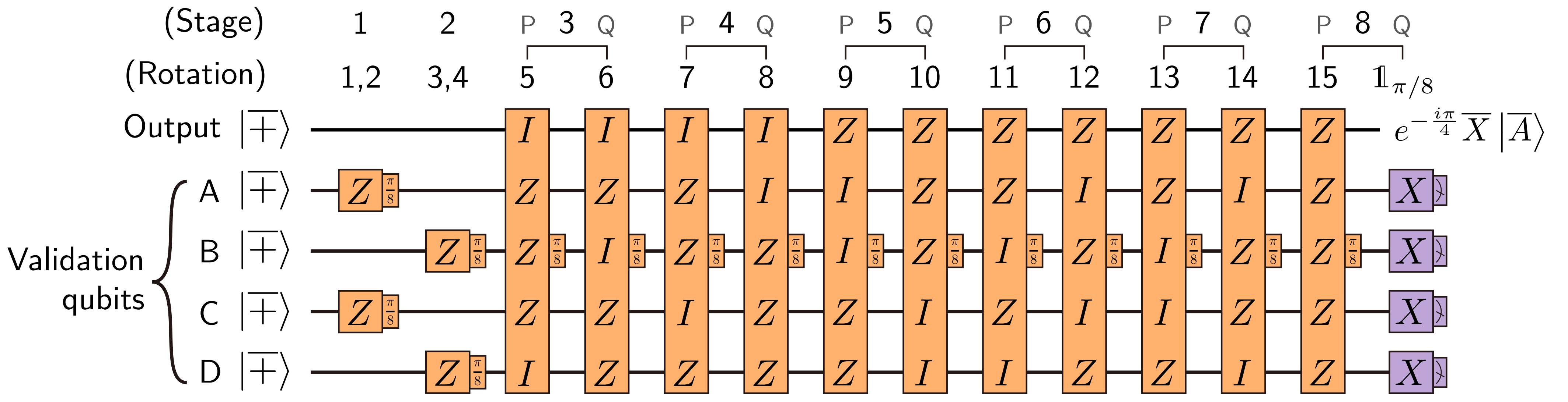


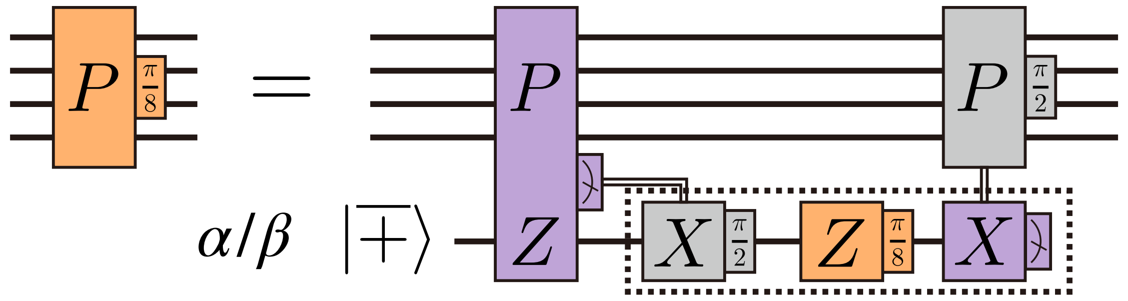
Faulty T-measurement

Lattice surgery

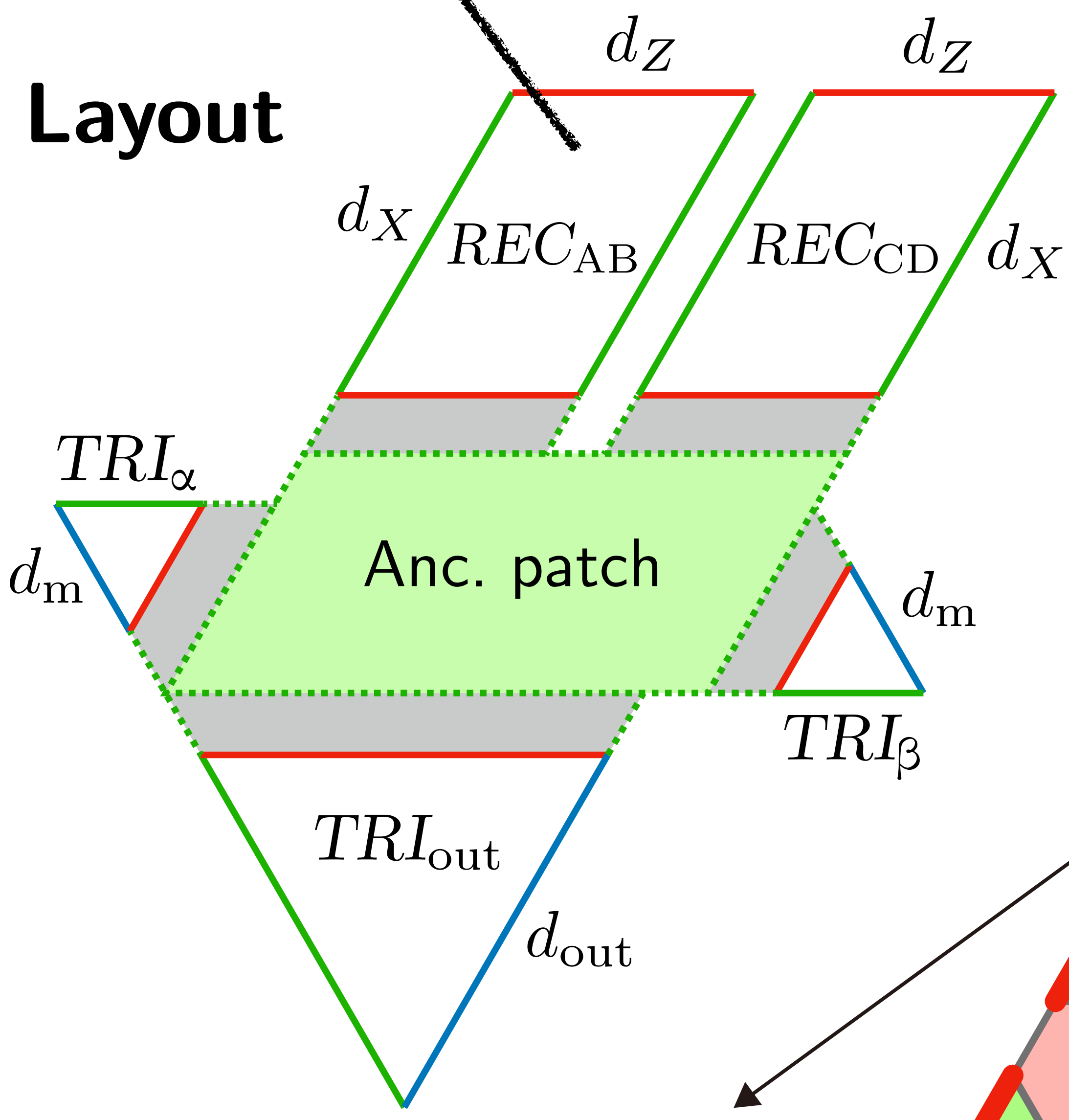
Single-Level MSD Scheme

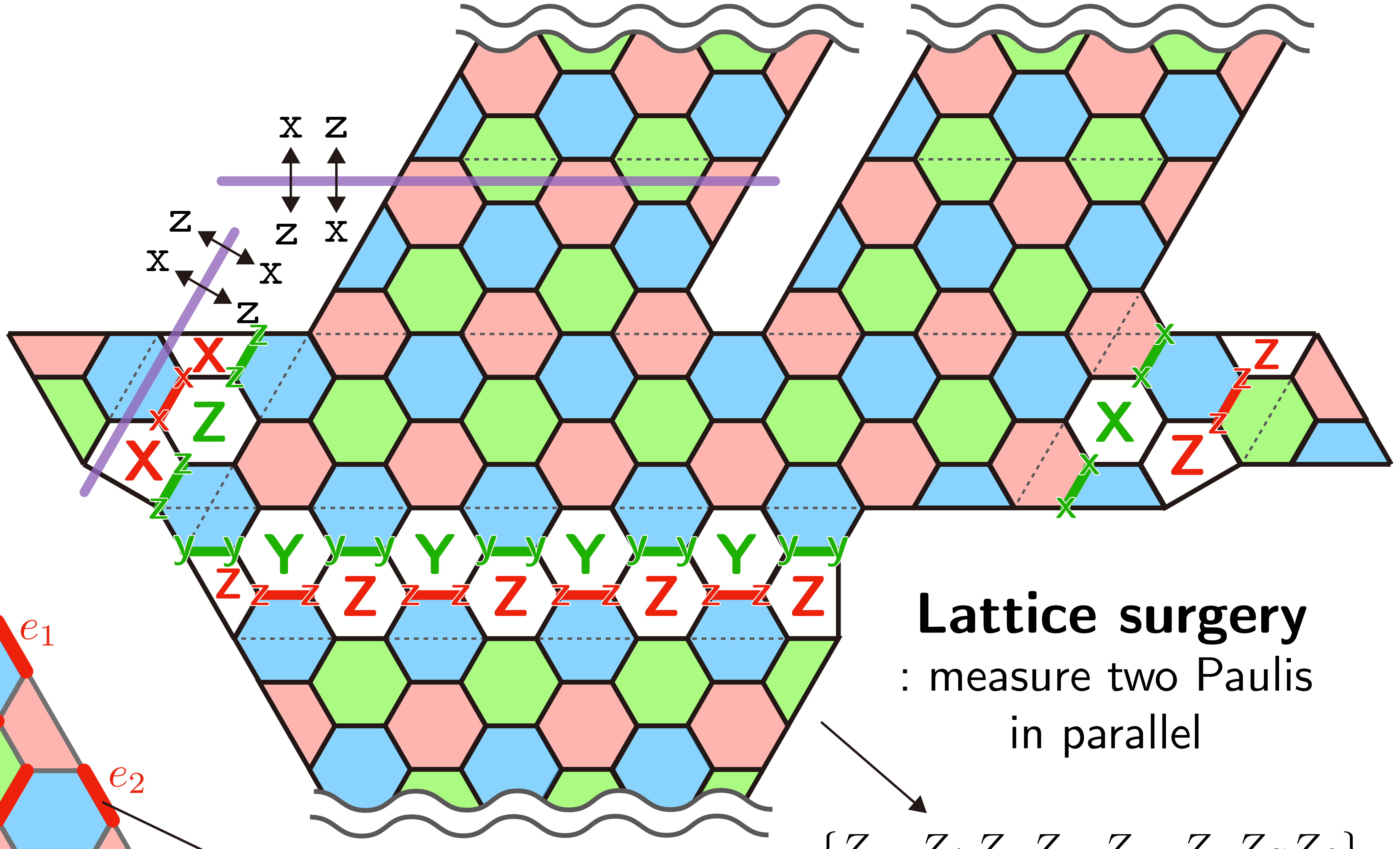
1) Independently adjustable d_X and d_Z
2) Anc. patch adjacent to only Z-boundaries



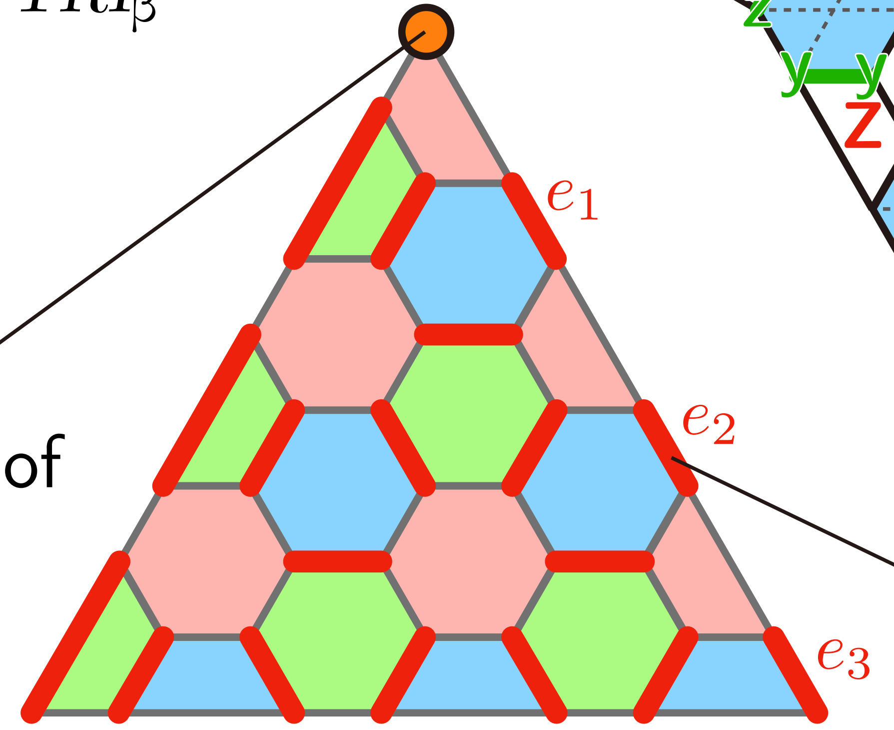


Layout





Measure in the basis of
$\{Z_{-\lambda\lambda_Z\pi/8}|\pm\rangle\}$



Faulty T-measurement in α and β

Bell measurement
$\{X \otimes X, Z \otimes Z\}$

Lattice surgery
: measure two Paulis
in parallel

$\{Z_{\text{out}} Z_{\text{A}} Z_{\text{D}} Z_\alpha, Z_{\text{out}} Z_{\text{B}} Z_{\text{C}} Z_\beta\}$

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Limitations of Single-Level Scheme & Integrating with Cultivation

- Single-level scheme infidelity bounded:
  $\sim 35p^3_{\mathrm{faulty.T.meas}} = O(p^3)$
- Need lower infidelities for complex algorithms.
- **Magic State Cultivation**
  : Distillation-free magic state preparation via transversal Cliffords + post-selection.
  Ref) Chamberland & Noh, npjQI (2021); Itogawa et al., arXiv:2403.03991 (2024);, Gidney et al., arXiv:2409.17595 (2024).
  - Highly **resource-efficient**.
  - **Scalability limited** by exponential retry cost with distance.
  - Cannot reach very small infidelities ($\lessapprox 10^{-9}$ at $p = 10^{-3}$)

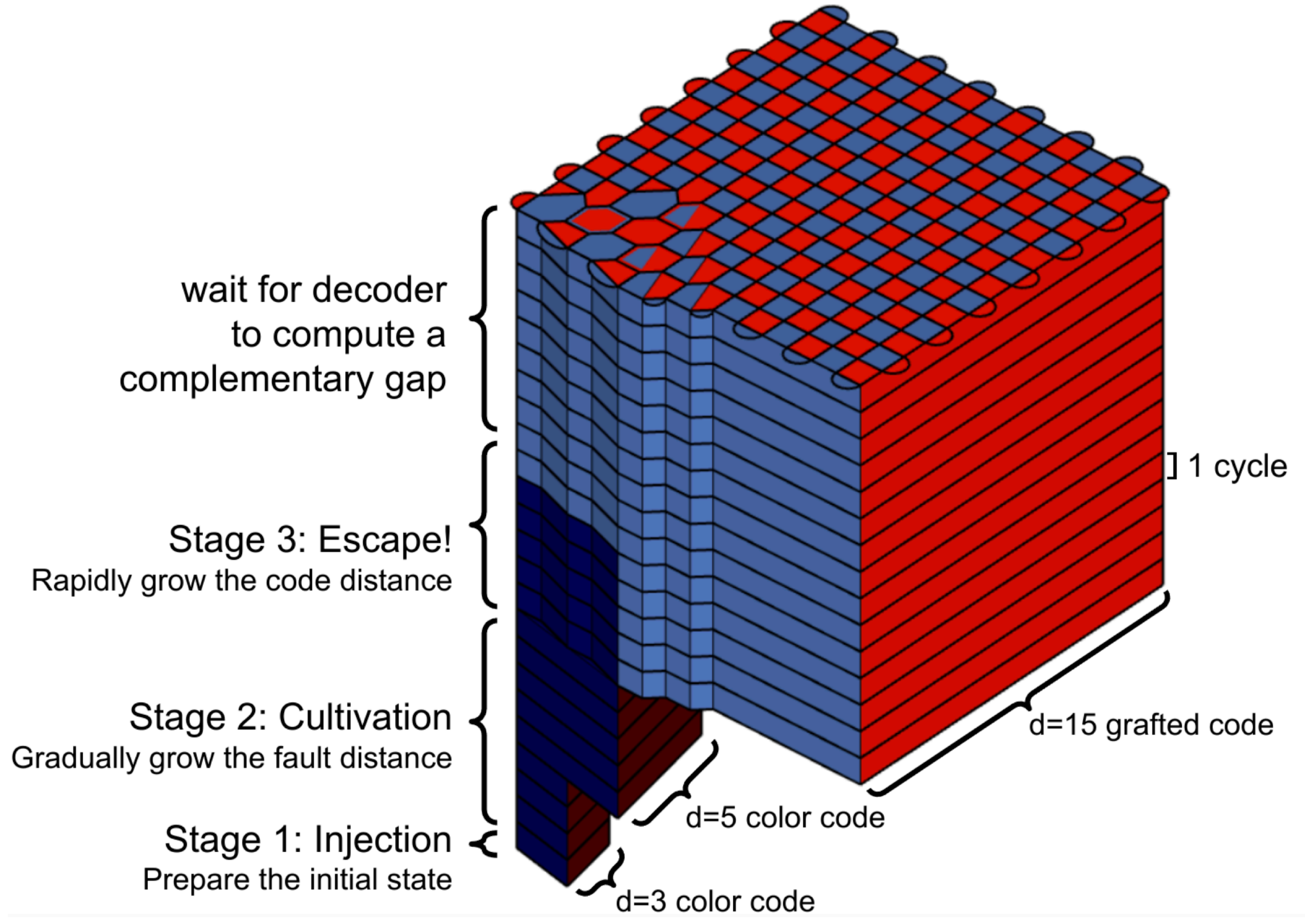
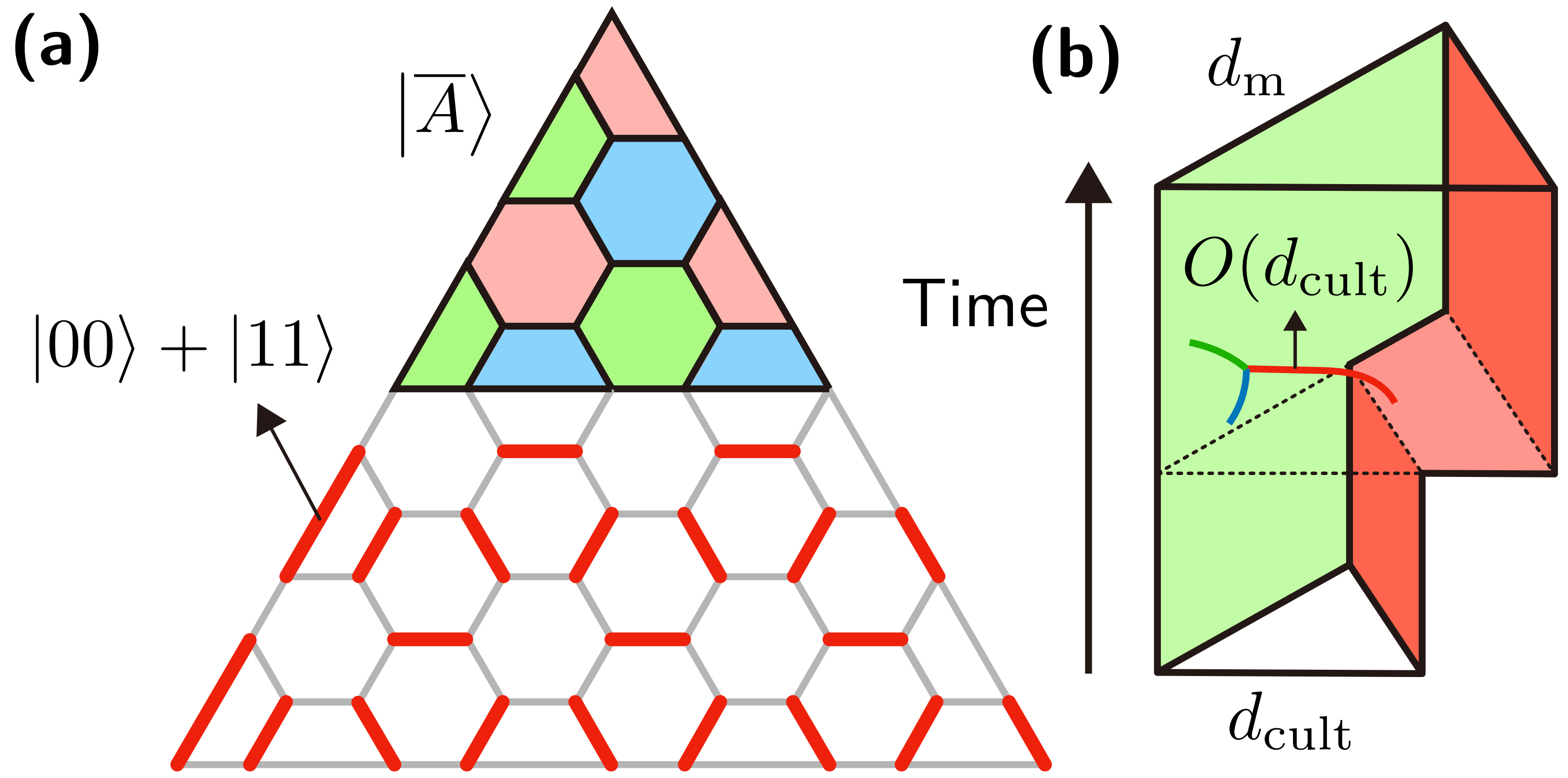
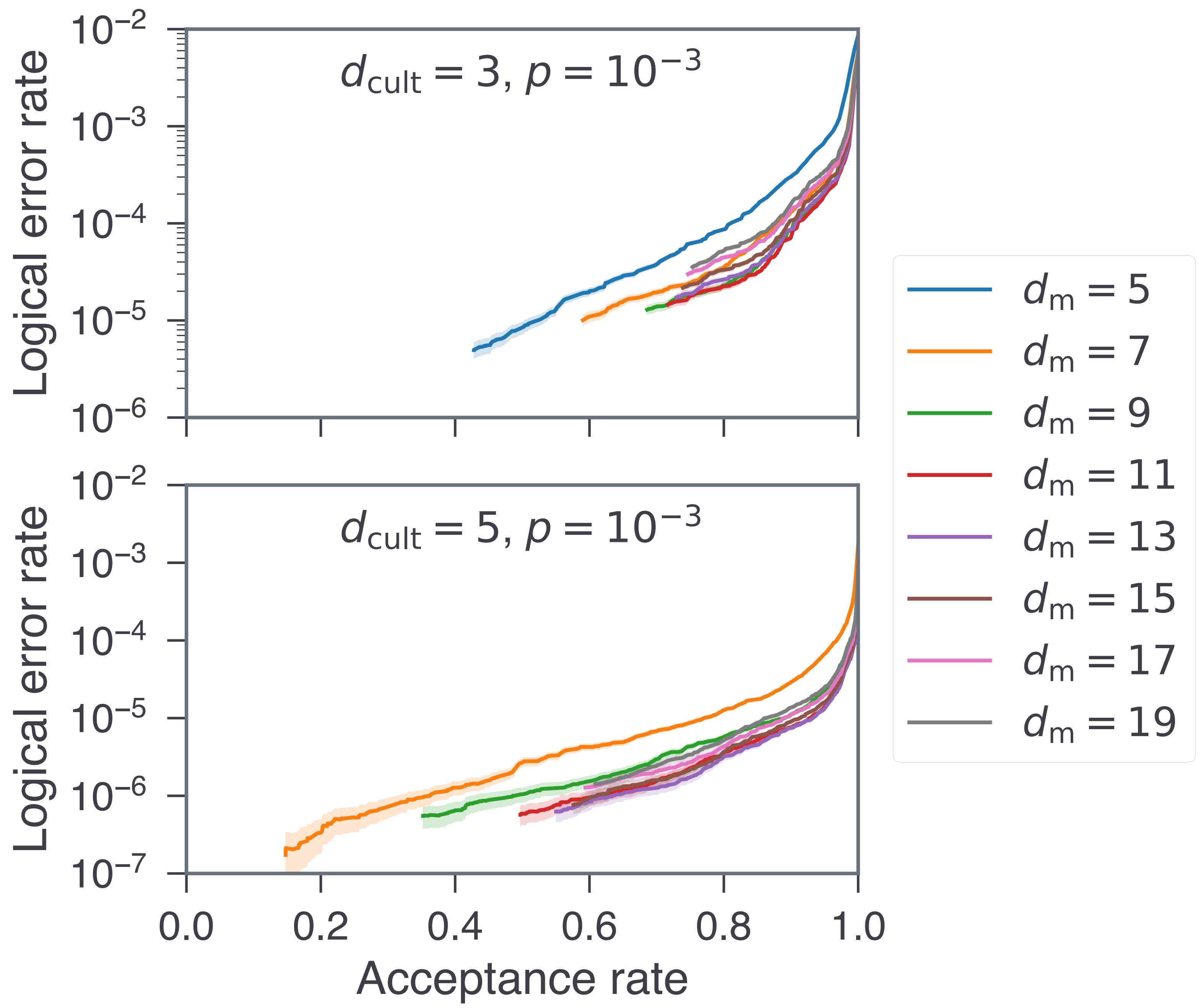


Figure from [Gidney et al., arXiv:2409.17595 (2024)]

The Growing Operation Bottleneck

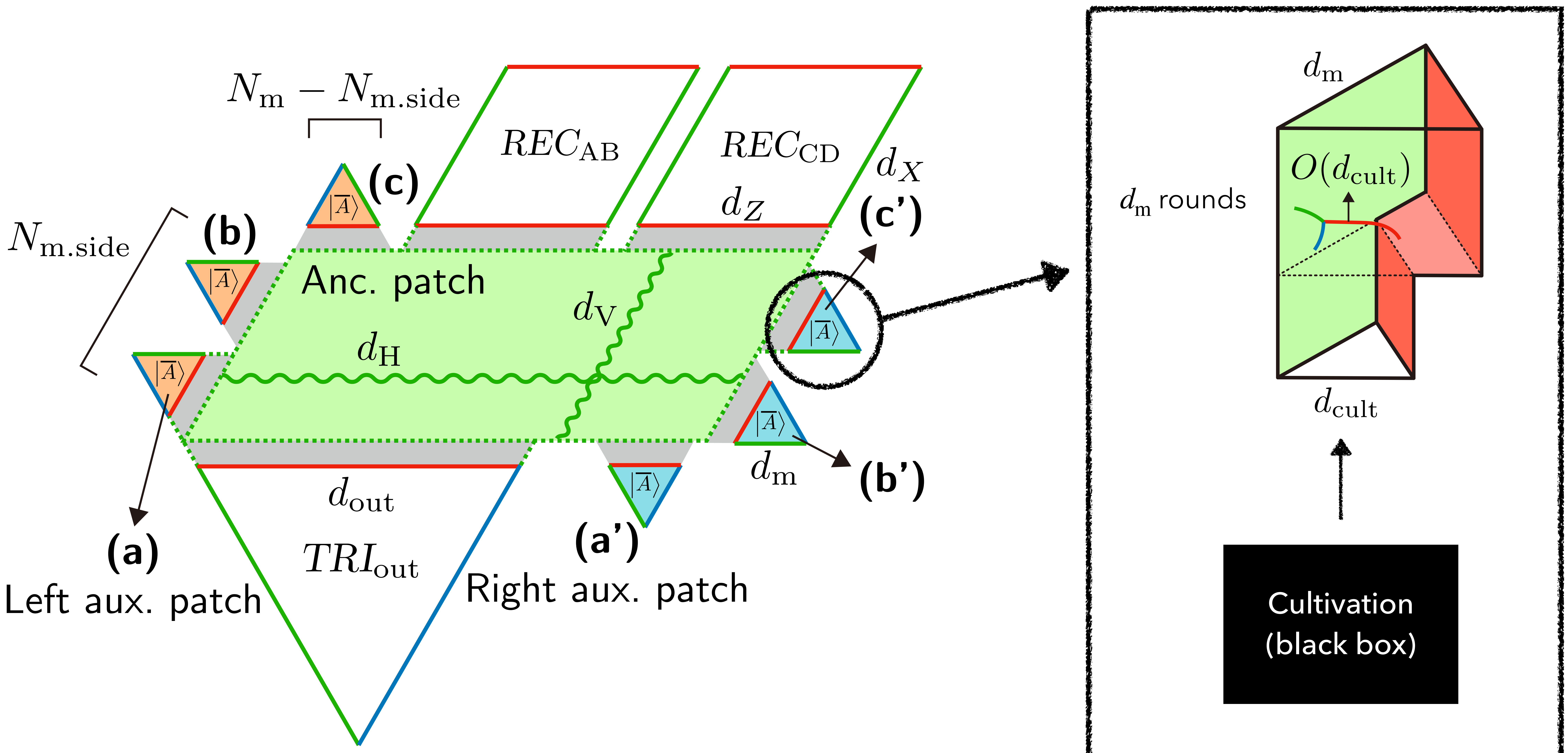
- Cultivated states are on small distance-d_{cult} patches (not sufficient to store high-quality magic states!)
- Need to **grow** the patch to a larger distance d_{m} for storage/use.
- Problem: Fault tolerance of growing depends on d_{cult}, not d_{m}! Can damage the cultivated states.
- Solution: Use **post-selection** based on the **logical gap** (confidence measure).
  - Logical gap: Log-likelihood ratio difference between a correction and a complementary correction.
  - Can be calculated by the **concatenated MWPM decoder**.





Ref) Lee et al., Quantum (2025)

Cultivation-MSD Scheme Overview



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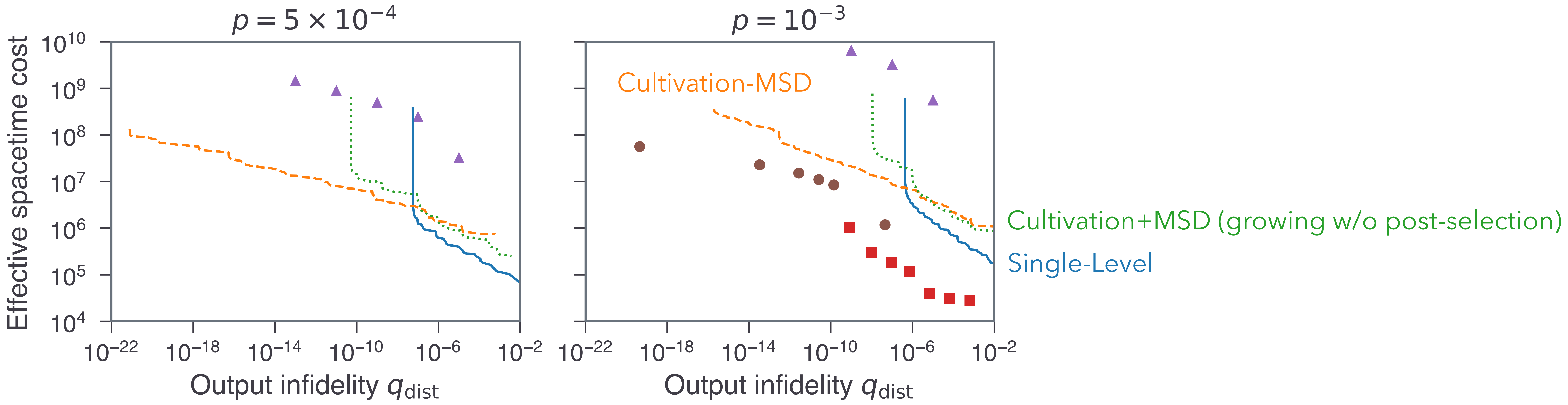
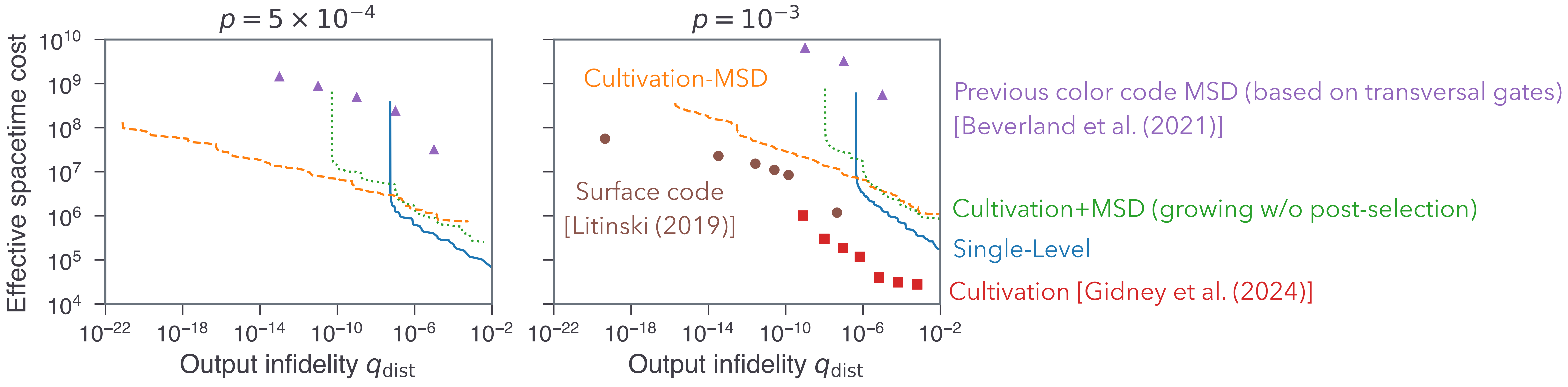
Performance Analysis



- **Single-Level MSD**: infidelity $\geq 35(7p/3)^3$
- **Cultivation-MSD**: Achieves much lower infidelities (2×10^{-16} at $p = 0.1\,\%$)
- Growing w/ post-selection is **essential** for Cultivation-MSD performance

Comparison with Other Schemes



- vs. **Prior Color Code MSD** (based on transversal gates): $\sim 100\times$ lower spacetime cost.
- vs. **Cultivation** Alone: Cultivation is cheaper for moderate infidelity ($\gtrapprox 10^{-9}$), but Cultivation-MSD reaches much lower infidelities ($\sim 10^{-16}$).
- vs. **Surface Code MSD**: Our schemes have <1 order of magnitude higher cost.
  - Reason: Underperforming color code decoders. (Threshold: $\sim 0.2-0.6\,\%$ vs $\sim 1\,\%$).

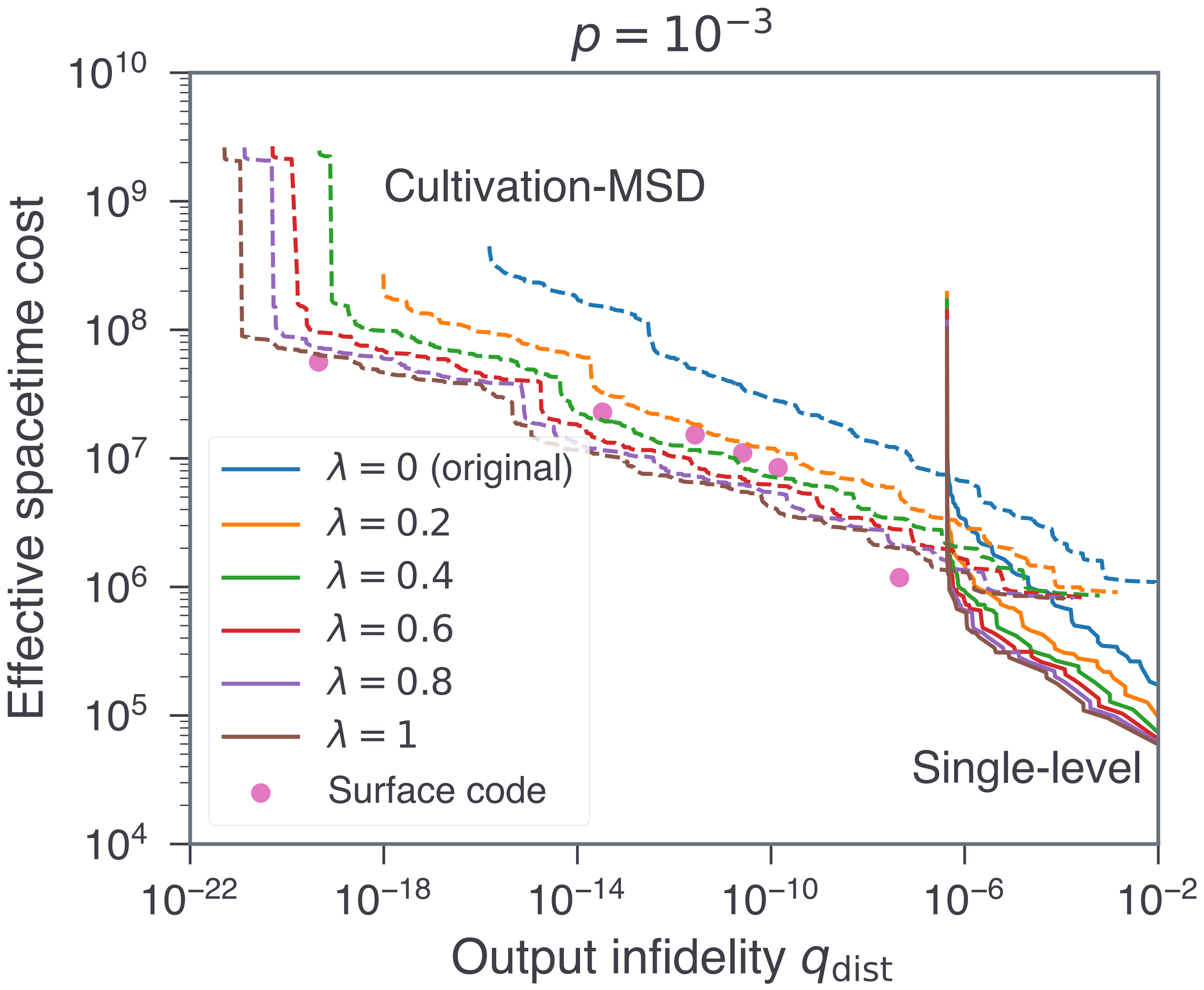
Required Improvement in Color Code Decoders

- Ansatz for logical error rates (when p is sufficiently small)

$$p_{\mathrm{L}} = \alpha \left(\frac{p}{p_{\mathrm{th}}} \right)^{\beta d + \eta}$$

- Scaling threshold p_{th}
  : Can be different from the "cross" threshold

- Assume improvement in p_{th}
  $p_{\mathrm{th}} \rightarrow (1 - \lambda) p_{\mathrm{th}} + \lambda (1\%)$



$\boxed{\lambda = 0.4}$

tri. patch:	$0.24\,\% \rightarrow 0.54\,\%$
rec. patch:	$0.37\,\% \rightarrow 0.62\,\%$
stability experiment:	$0.62\,\% \rightarrow 0.77\,\%$

(Concatenated MWPM decoder)

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Take-Home Messages

- We need **end-to-end MSD schemes optimized for individual QEC codes**, not just their high-level structures.
- For color codes, we can exploit their advantages:
  - Resource-efficient lattice surgery (enabling simultaneous measurement of commuting Pauli operators).
  - Various types of logical patches (triangular / rectangular).
  - Transversal Clifford gate (which enables cultivation).
- Two optimized schemes:
  - **Scheme 1 (Single-Level MSD via Faulty T-measurement)**: $O(p^3)$ infidelity
  - **Scheme 2 (Cultivation-MSD)**: Ultra-low infidelity (taking advantage of post-selection for growing)
- **100x cost reduction** vs. prior color code MSD
- Cultivation is really promising, but we eventually need MSD to further improve the quality of magic states.
- **Color code decoder improvements** are key to compete with surface codes.

Thank you

arXiv:2409.07707

Stephen Bartlett (USYD)

Felix Thomsen (USYD)

Nicholas Fazio (USYD)

Benjamin Brown (IBM)

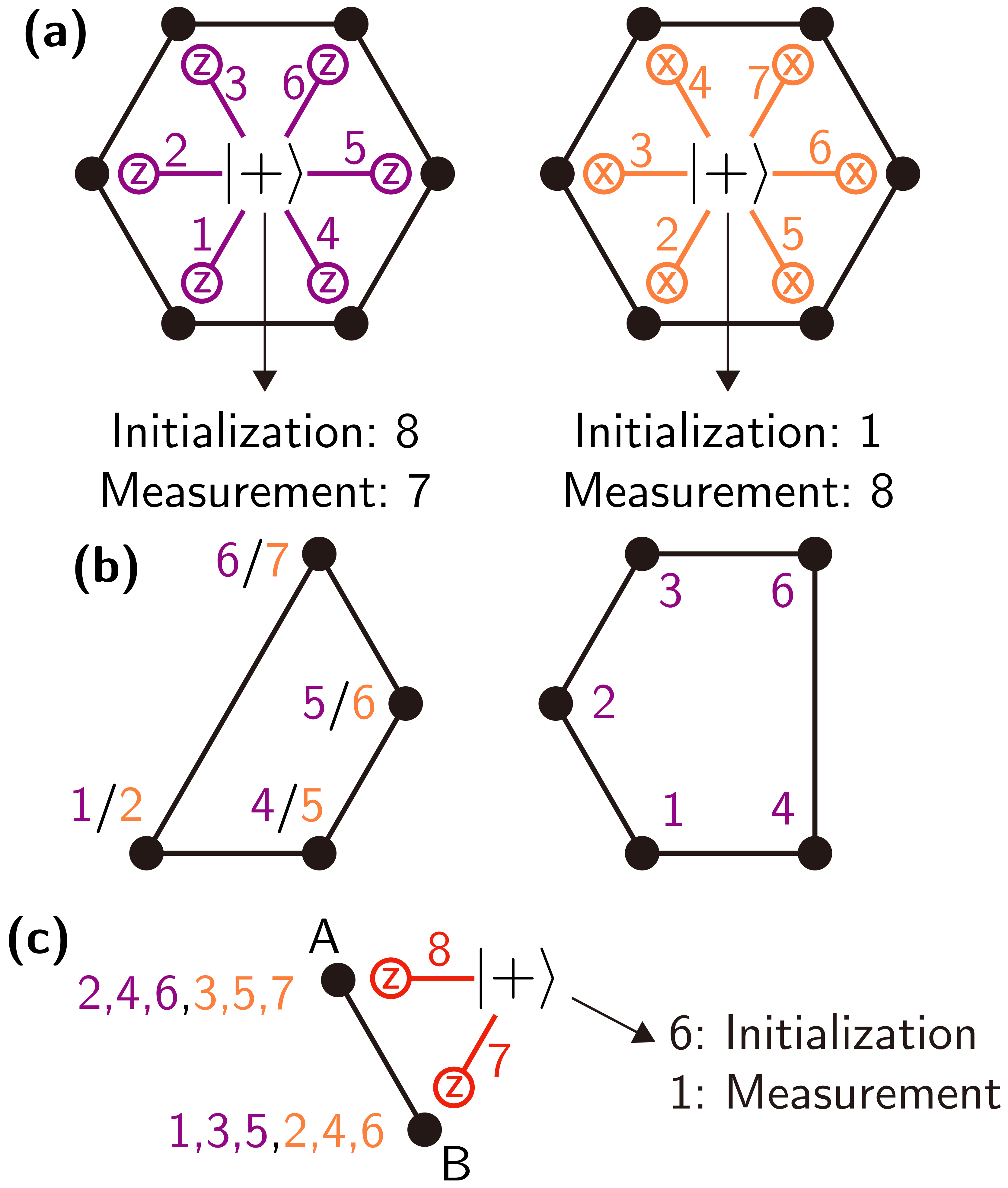
Q&A

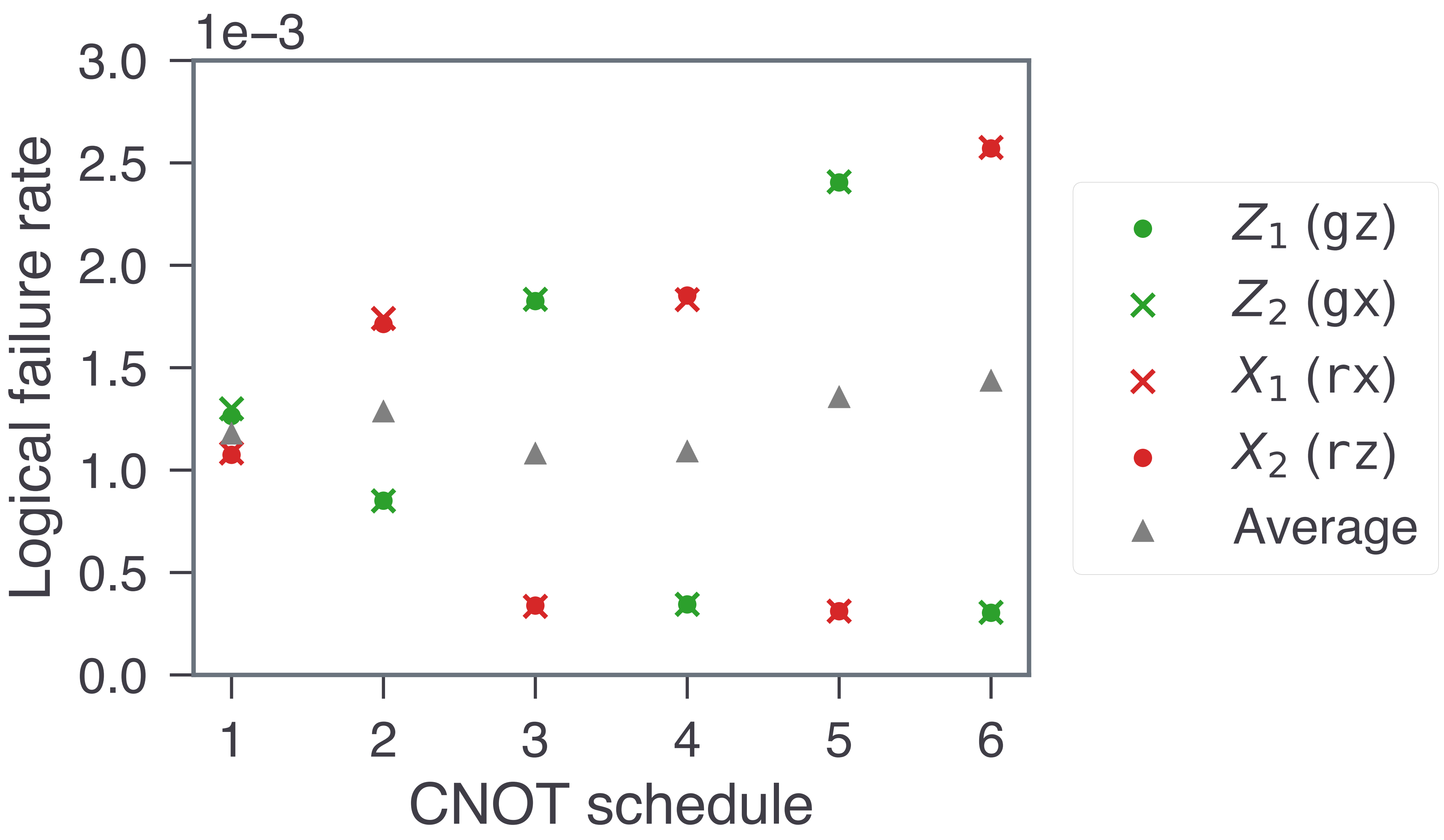
MSD Scheme Performance Table

Scheme	Output infidelity q_{dist}	Failure rate $1 - q_{\text{succ}}$	Space cost n (Qubits)	Time cost t (Time steps)	Effective spacetime cost (nt/q_{succ})	T_{m}	T_{intv}	T_{idle}
(a) $p = 5 \times 10^{-4}$								
sng-$(11, 8, 6, 5)$	1.52×10^{-5}	4.71×10^{-2}	833	384	3.36×10^{5}			
sng-$(19, 10, 12, 7)$	1.02×10^{-7}	1.99×10^{-2}	2401	512	1.25×10^{6}			
cmb-$(23, 14, 16, 7, 3, 4, 5.03)$	1.13×10^{-9}	3.69×10^{-3}	5347	759	4.08×10^{6}	2	11.9	0.6
cmb-$(31, 18, 20, 11, 3, 3, 10.05)$	1.11×10^{-12}	1.00×10^{-4}	8825	1298	1.15×10^{7}	2	20.3	1.2
cmb-$(41, 22, 28, 13, 3, 4, 13.41)$	1.09×10^{-15}	3.02×10^{-5}	1.59×10^{4}	1348	2.14×10^{7}	2	21.1	1.1
cmb-$(49, 30, 34, 15, 5, 4, 18.12)$	1.24×10^{-18}	4.09×10^{-6}	2.52×10^{4}	2272	5.73×10^{7}	3	35.5	3.6
cmb-$(59, 34, 40, 19, 5, 6, 23.23)$	1.06×10^{-21}	4.19×10^{-7}	4.02×10^{4}	2378	9.57×10^{7}	2	37.2	3.0
cmb-$(63, 40, 44, 19, 5, 8, 23.23)$	7.75×10^{-22}	4.21×10^{-7}	6.05×10^{4}	2073	1.25×10^{8}	2	32.4	2.2
(b) $p = 10^{-3}$								
sng-$(19, 8, 12, 7)$	1.21×10^{-5}	7.17×10^{-2}	2265	512	1.25×10^{6}			
sng-$(25, 12, 16, 11)$	1.03×10^{-6}	3.77×10^{-2}	4181	768	3.34×10^{6}			
cmb-$(29, 16, 20, 9, 3, 5, 6.09)$	1.04×10^{-7}	1.34×10^{-2}	9081	925	8.52×10^{6}	1	14.5	0.7
cmb-$(39, 22, 26, 13, 3, 4, 7.60)$	1.03×10^{-9}	1.96×10^{-3}	1.50×10^{4}	1391	2.10×10^{7}	2	21.7	1.4
cmb-$(51, 28, 36, 15, 3, 4, 10.66)$	1.00×10^{-11}	8.37×10^{-4}	2.60×10^{4}	1595	4.16×10^{7}	2	24.9	1.5
cmb-$(63, 36, 46, 17, 5, 6, 16.75)$	1.02×10^{-13}	1.95×10^{-4}	4.58×10^{4}	3020	1.38×10^{8}	2	47.2	5.1
cmb-$(71, 36, 48, 23, 5, 8, 20.62)$	1.01×10^{-15}	2.60×10^{-5}	6.70×10^{4}	3513	2.35×10^{8}	2	54.9	5.4
cmb-$(81, 44, 58, 23, 5, 8, 20.62)$	2.00×10^{-16}	2.65×10^{-5}	9.07×10^{4}	3513	3.19×10^{8}	2	54.9	5.4

More on Growing

Syndrome extraction





Memory & Stability Simulations