yang_seonhyeHW7

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04/02/2019

Question 1

The assumptions of the Gauss-Markov theorem are:

$$GM_1 : E(\epsilon) = 0$$
$$GM_2 : var(\epsilon) = \theta^2 I$$

Question 2

In order to show that the $\hat{\beta}$ is unbias we would want the following result:

$$E[\hat{\beta}] = \beta$$
$$\hat{\beta} = (X^T X)^{-1} X^T y$$
$$E[(X^T X)^{-1} X^T y]$$

$$(X^T X)^{-1} X^T$$

is a constant therefore we can take it out and it becomes

$$(X^T X)^{-1} X^T E[y]$$

$$E[y] = X\beta$$
$$(X^T X)^{-1} X^T X\beta$$
$$(X^T X)^{-1} X^T X$$

is an identity matrix I Therefore,

$$E[\hat{\beta}] = \beta$$

Question 3

In order to show that $\hat{y}$ is an unbias estimator of $E[y]$, recall that $\hat{y} = X\hat{\beta}$

$$E(\hat{y}) = y$$
$$E(\hat{y}) = E(X\hat{\beta})$$
$$E(\hat{y}) = XE(\hat{\beta})$$
$$X(X^T X)^{-1} X^T E[y]$$

$$X(X^T X)^{-1} X^T$$

is an identity matrix I

$$E[y] = X\beta$$
$$y = X\beta$$

Therefore, $E(\hat{y}) = y$ and $\hat{y}$ is an unbias estimator of $E[y]$

Question 4

We know that $b_1 = \frac{y_2 - y_1}{x_2 - x_1}$. We also know that (x_1, y_1) and (x_2, y_2) are the first two points of our dataset, When we set out to show that $E(b_1) = \beta_1$, we first find $E(b_1)$:

$$E(b_1) = E(\frac{y_2 - y_1}{x_2 - x_1})$$
$$E(b_1) = E(\frac{\beta_0 + \beta_1 x_2 + \epsilon_2 - \beta_0 - \beta_1 x_1 - \epsilon_1}{x_2 - x_1})$$
$$E(b_1) = E(\frac{\beta_1 x_2 + \epsilon_2 - \beta_1 x_1 - \epsilon_1}{x_2 - x_1})$$
$$E(b_1) = E(\frac{\beta_1(x_2 - x_1) + \epsilon_2 - \epsilon_1}{x_2 - x_1})$$

$$E(b_1) = E(\frac{\beta_1(x_2 - x_1)}{x_2 - x_1}) + E(\frac{\epsilon_2 - \epsilon_1}{x_2 - x_1})$$
$$E(b_1) = E(\beta_1) + E(\frac{\epsilon_2 - \epsilon_1}{x_2 - x_1})$$
$$E(b_1) = \beta_1 + E(\frac{\epsilon_2 - \epsilon_1}{x_2 - x_1})$$
$$E(b_1) = \beta_1 + \frac{E(\epsilon_2) - E(\epsilon_1)}{x_2 - x_1}$$
$$E(b_1) = \beta_1 + \frac{0}{x_2 - x_1}$$
$$E(b_1) = \beta_1$$

Thus $E(b_1) = \beta_1$.

Question 5

We know that b_1:

$$b_1 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$b_1 = \frac{\beta_0 + \beta_1 x_2 + \epsilon_2 - \beta_0 - \beta_1 x_1 - \epsilon_1}{x_2 - x_1}$$
$$b_1 = \frac{\beta_1 x_2 + \epsilon_2 - \beta_1 x_1 - \epsilon_1}{x_2 - x_1}$$
$$b_1 = \frac{\beta_1(x_2 - x_1) + \epsilon_2 - \epsilon_1}{x_2 - x_1}$$

And knowing that $var(a\mathbf{X}) = a^2 var(\mathbf{X})$:

$$var(b_1) = var(\frac{\beta_1(x_2 - x_1)}{x_2 - x_1}) + var(\frac{\epsilon_2 - \epsilon_1}{x_2 - x_1})$$

$$var(b_1) = 0 + var(\frac{\epsilon_2 - \epsilon_1}{x_2 - x_1})$$

$$var(b_1) = \frac{var(\epsilon_2) + var(\epsilon_1)}{(x_2 - x_1)^2}$$

Also knowing that $var(\epsilon) = \sigma^2$:

$$var(b_1) = \frac{2\sigma^2}{(x_2 - x_1)^2}$$