yang_seonhyeHW5

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Question 1

$$var(\hat{y}_*) = var(x_*\hat{\beta}) = var(x_*^{\mathbf{T}}(\mathbf{X^TX})^{-1}\mathbf{X^T}y)$$

so we get

$$\mathbf{X}_*^{\mathbf{T}} var(\hat{\beta})x_*$$

and knowing that

$$var(\hat{\beta}) = \sigma^2(\mathbf{X^TX})^{-1}$$

we get

$$x_*^{\mathbf{T}}\sigma^2(\mathbf{X^TX})^{-1}x_*$$

Question 2

$$var(\hat{y}) = \sigma^2 \ \mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}$$

$$var(\hat{y}) = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}y$$

Therefore,

$$var(\mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}y) = \mathbf{H}var(y)\mathbf{H}$$

Where

$$\mathbf{H} = \mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}$$

We know that

$$var(y) = \sigma^2\mathbf{I}$$

and we also know that H is symmetric, so we can do:

$$\mathbf{H}var(y)\mathbf{H} = var(y)\mathbf{HH}$$
$$var(y)\mathbf{H} = \sigma^2\mathbf{H}$$

$$var(\hat{y}) = \sigma^2\mathbf{X}(\mathbf{X^TX})^{-1}\mathbf{X^T}$$

Question 3

We know that:

$$\hat{\epsilon} = y - \hat{y}$$

So we can substitute $y = \mathbf{X}\beta + \epsilon$ and $\hat{y} = \mathbf{X}\hat{\beta}$, resulting in:

$$\hat{\epsilon} = \mathbf{X}\beta + \epsilon - (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T y)$$

Further substituting in $y = \mathbf{X}\beta + \epsilon$, we get:

$$\hat{\epsilon} = \mathbf{X}\beta + \epsilon - (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T(\mathbf{X}\beta + \epsilon))$$

We can then factor:

$$\hat{\epsilon} = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathbf{X}\beta + \epsilon)$$

And then redistributing:

$$\hat{\epsilon} = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathbf{X}\beta) + (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)\epsilon$$

We can then notice that $(\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathbf{X}\beta) = 0$, so we get:

$$\hat{\epsilon} = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)\epsilon$$

Question 4

Proving $\mathbf{H}^T = \mathbf{H}$:

$$\begin{gathered}
\mathbf{H}^T = (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)^T \\
\mathbf{H}^T = (\mathbf{X}^T)^T((\mathbf{X}^T\mathbf{X})^{-1})^T\mathbf{X}^T \\
\mathbf{H}^T = \mathbf{X}((\mathbf{X}^T\mathbf{X})^T)^{-1}\mathbf{X}^T \\
\mathbf{H}^T = \mathbf{X}(\mathbf{X}^T(\mathbf{X}^T)^T)^{-1}\mathbf{X}^T \\
\mathbf{H}^T = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
\mathbf{H}^T = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T = \mathbf{H}
\end{gathered}$$

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Proving $\mathbf{HH} = \mathbf{H}$

$$\begin{gathered}
\mathbf{HH} = (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) \\
\mathbf{HH} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}(\mathbf{X}^T\mathbf{X})(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T \\
\mathbf{HH} = (\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T) = \mathbf{H}
\end{gathered}$$

This is because we know that $(\mathbf{X}^T\mathbf{X})^{-1}(\mathbf{X}^T\mathbf{X}) = \mathbf{I}$

Question 5

We know that $\hat{\epsilon} = (\mathbf{I} - \mathbf{H})\epsilon$, where $\mathbf{H} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T$, so:

$$var(\hat{\epsilon}) = var((\mathbf{I} - \mathbf{H})\epsilon)$$

Then:

$$\begin{gathered}
var(\hat{\epsilon}) = (\mathbf{I} - \mathbf{H})var(\epsilon)(\mathbf{I} - \mathbf{H}) \\
var(\hat{\epsilon}) = (\mathbf{I} - \mathbf{H})(\sigma^2\mathbf{I})(\mathbf{I} - \mathbf{H}) \\
var(\hat{\epsilon}) = \sigma^2(\mathbf{I} - \mathbf{H}) \\
var(\hat{\epsilon}) = (\mathbf{I} - \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)
\end{gathered}$$