4D-Var Analysis of Lorenz-63 Model

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Strong Constraint 4D-Var

- Find a solution to the ODE system that best fits the data
- Inherent assumption that the model is perfect if we have the perfect initial condition
- Find the appropriate initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$ that produces a solution $\mathbf{x}(t)$ that is "closest" to the data over the entire time interval

Strong Constraint 4D-Var

Minimize

$$J(\mathbf{x}_0) = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}_0^b)^T (P_0^b)^{-1} (\mathbf{x}_0 - \mathbf{x}_0^b) + \frac{1}{2}\sum_{k=0}^{K} (\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k)^T R_k^{-1} (\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k)$$

subject to

$$\mathbf{x}_{k+1} = \mathscr{M}_{k+1}(\mathbf{x}_k), \quad k = 0, 1, \ldots, K-1$$

First Variation

$$\delta J = \left(\nabla_{\mathbf{x}_0} J\right)^T \delta \mathbf{x}_0$$

The perturbation of the initial condition is propogated through the model using the tangent linear equation

$$\delta \mathbf{x}_{k+1} = M_{k+1} \delta \mathbf{x}_k$$

M_{k+1} is the Jacobian of $\mathscr{M}_{k+1}$

First Variation

$$\delta J = (\mathbf{x} - \mathbf{x}_0^b)^T (P_0^B)^{-1} \delta\mathbf{x}_0 + \sum_{k=0}^{K} (\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k)^T R_k^{-1} H_k \delta x_k$$

H_k is the Jacobian of $\mathscr{H}_k$

Augmented Cost Functional

$$
\begin{aligned}
\delta J = & (\mathbf{x} - \mathbf{x}_0^b)^T (P_0^B)^{-1} \delta \mathbf{x}_0 \\
& + \sum_{k=0}^{K} (\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k)^T R_k^{-1} H_k \delta x_k \\
& - \sum_{k=0}^{K} \mathbf{p}_k^T (\delta \mathbf{x}_k - M_k \delta \mathbf{x}_{k-1})
\end{aligned}
$$

Augmented Cost Functional

$$
\begin{aligned}
\delta J = & \left[(P_0^B)^{-1}(\mathbf{x}_0 - \mathbf{x}_0^b) + H_0^T R_0^{-1}(\mathscr{H}_0(\mathbf{x}_0) - \mathbf{y}_0) + M_0^T \mathbf{p}_1\right] \delta \mathbf{x}_0 \\
& + \left[\sum_{k=1}^{K-1} H_k^T R_k^{-1}(\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k) - \mathbf{p}_k + M_k^T \mathbf{p}_{k+1}\right] \delta \mathbf{x}_k \\
& + \left[H_K^T R_k^{-1}(\mathscr{H}_K(\mathbf{x}_K) - \mathbf{y}_K) - \mathbf{p}_K\right] \delta \mathbf{x}_K
\end{aligned}
$$

Adjoint Equation

$$
\begin{aligned}
\mathbf{p}_K =& H_K^T \mathbf{R}_k^{-1}(\mathscr{H}_K(\mathbf{x}_k) - \mathbf{y}_k) \\
\mathbf{p}_k =& H_k^T R_k^{-1}(\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k) + M_k^T \mathbf{p}_{k+1}, \\
& k = K-1, K-2, \ldots, 1 \\
\mathbf{p}_0 =& (P_0^B)^{-1}(\mathbf{x}_0 - \mathbf{x}_0^b) + H_0^T R_0^{-1}(\mathscr{H}_0(\mathbf{x}_0) - \mathbf{y}_0) + M_0^T \mathbf{p}_1
\end{aligned}
$$

As a result,

$$\delta J = \mathbf{p}_0^T \delta \mathbf{x}_0$$

so therefore

$$\nabla_{x_0} J = \mathbf{p}_0$$

Innovation Vector

$$d_k = H_k^T R_k^{-1}(\mathscr{H}_k(\mathbf{x}_k) - \mathbf{y}_k)$$

$$\delta J = \delta J^b + \delta J^o$$

$$\delta J^b = (P_0^B)^{-1}(\mathbf{x}_0 - \mathbf{x}_0^b)$$

$$\delta J^o = d_0 + M_0^T(d_1 + M_1^T(d_2 + M_2^T(\dots M_{K-1}^T(d_K))))$$

Algorithm Summary

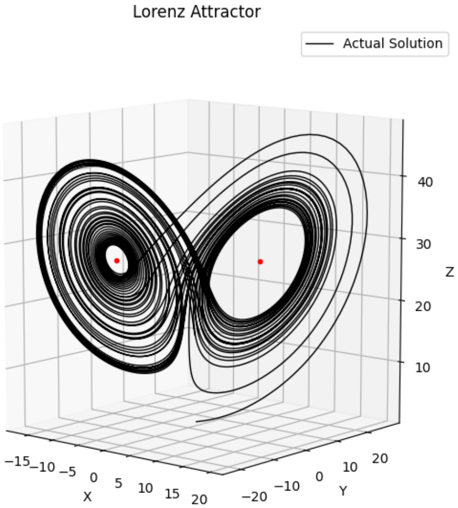
1. Given a guess for the initial condition, integrate the solution of the ODE to obtain $\mathbf{x}_k$.
2. Determine the needed Jacobian matrices M_k and H_k at each observation timestep, as well as the innovation terms d_k.
3. Construct the gradient δJ^o by working backward in time
4. Construct the full gradient δJ by adding δJ_o to δJ^b.
5. Perform gradient descent to find the analyzed initial condition $\mathbf{x}_0^a$ that minimizes the cost functional at this iteration.
6. Repeat steps 1-5 with this new initial condition until some sort of convergence is achieved.

Lorenz-63 model

$$\frac{dx}{dt} = -\sigma(x - y)$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = xy - \beta z$$

$$\sigma = 10,\ \rho = 28,\ \beta = 8/3$$

Lorenz-63 model



Lorenz-63 model

$$M_{k:k+1} = \begin{pmatrix} 1-\sigma\Delta t & \sigma\Delta t & 0 \\ (\rho - z_k)\Delta t & 1-\Delta t & -x_k\Delta t \\ y_k\Delta t & x_k\Delta t & 1-\beta\Delta t \end{pmatrix}$$

$$H_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Sample code output

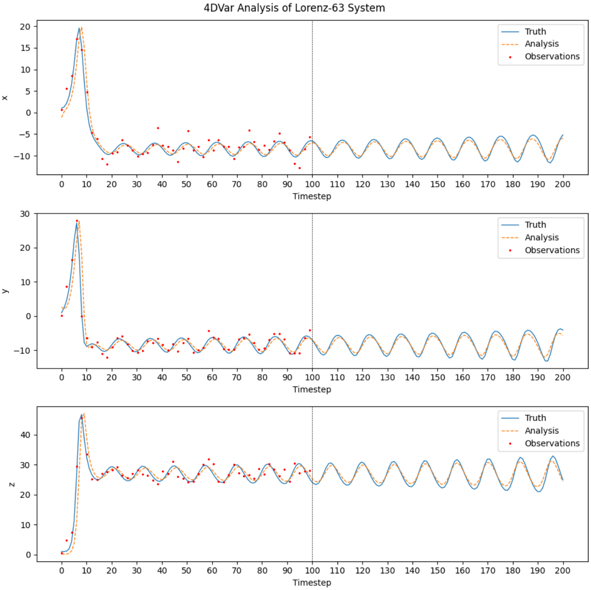
ITERATION: 2500
L2-NORM OF GRADIENT: 0.7509
NEW x_0^a: [0.0763 4.6638 1.3606]

ITERATION: 3000
L2-NORM OF GRADIENT: 0.6895
NEW x_0^a: [-0.0069 4.4343 1.0889]

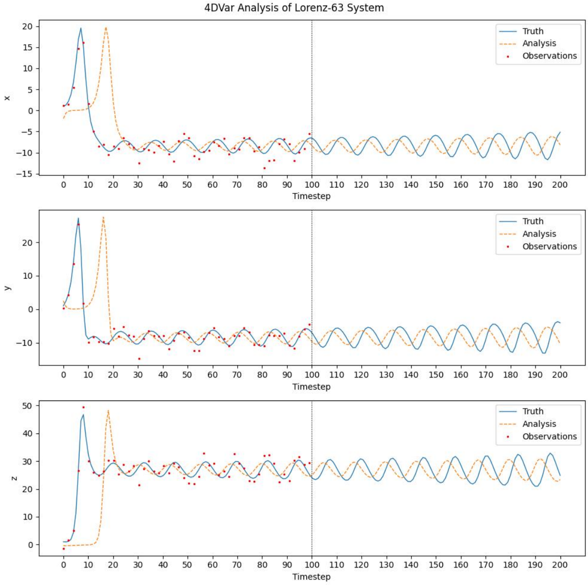
ITERATION: 3500
L2-NORM OF GRADIENT: 0.6565
NEW x_0^a: [-0.0976 4.2081 0.8534]

ITERATION: 4000
L2-NORM OF GRADIENT: 0.5758
NEW x_0^a: [-0.1896 3.9960 0.6533]

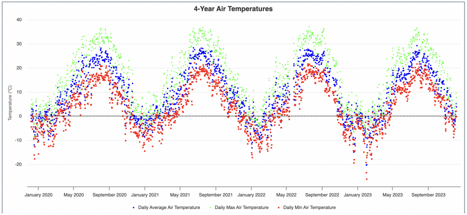
$$B = 2I, R = 5I, x_0^b = [0.7, 1.5, 0.2]$$

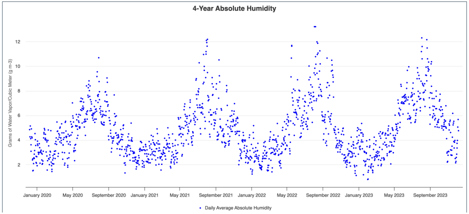


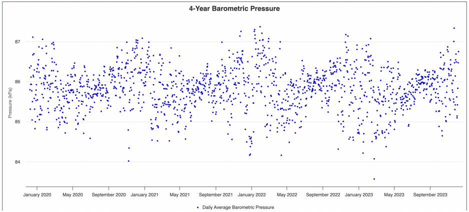
$$B = 7I, R = 5I, x_0^b = [0.2, 5.5, 3.2]$$



USU Climate Data







4D-Var Assimilation